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OUTER FACTORIZATIONS IN ONE AND SEVERAL VARIABLES

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In memory of Marvin Rosenblum

Abstract. A multivariate version of Rosenblum's Fejér-Riesz theorem on outer factorization of trigonometric polynomials with operator coefficients is considered. Due to a simplification of the proof of the single variable case, new necessary and sufficient conditions for the multivariable outer factorization problem are formulated and proved.

1. Introduction

The Fejér-Riesz theorem for trigonometric polynomials $q(z) = \sum_{i=-n}^{n} q_i z^i$ over $\mathbb{C}$ states that $q(z) \geq 0$, $z \in \mathbb{T}$, if and only if there exists a polynomial $p(z) = \sum_{i=0}^{n} p_i z^i$ so that $q(z) = |p(z)|^2$, $z \in \mathbb{T}$. In addition, one may choose p to be void of roots inside the open unit circle $\mathbb{D}$ (that is, p is *outer*). Though simple to state and prove (use the fundamental theorem of algebra; see, for example, [15]), the lemma has many useful applications; for example, in filter design, H^∞ control, and wavelet theory. The first generalizations of the lemma involved matrix-valued trigonometric polynomials ([16], [12]) and subsequently operator-valued trigonometric polynomials (in [11] a compactness condition appears, in [17] the general operator case is done).

The present paper grew out of an interest in a multivariate analog of the Fejér-Riesz theorem. As is well known, extensions of such results to several variables are far from straightforward. One of the earliest efforts in this direction is Hilbert's well-known observation that not all nonnegative polynomials in several real variables are necessarily sums of squares of polynomials. While Hilbert's result concerns polynomials on $\mathbb{R}^d$, a similar phenomenon occurs in the setting of trigonometric polynomials on the d-torus $\mathbb{T}^d$, where $\mathbb{T} = \{z \in \mathbb{C} \ : \ |z| = 1\}$. Indeed, it follows from Hilbert's result (see [3] and [19]) that a trigonometric polynomial $q(z)$ in d variables of degree $(n_1, \ldots, n_d)$ that takes on nonnegative values on $\mathbb{T}^d$, is not necessarily of the form

$$q(z) = \sum_{i=1}^{k} |p_i(z)|^2, z \in \mathbb{T}^d, \tag{1.1}$$

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where p_i are polynomials of degree $(n_1, \dots, n_d)$. It turns out (see [5]; see also [14], [13]) that checking whether q can be factored in this way is a semidefinite feasibility problem. In this paper, we investigate which multivariable trigonometric polynomials are single squares; that is, we would like $k = 1$ in the above representation (1.1) with similar restrictions on the degree of the polynomial in the factorization.

As might be expected, not putting any restrictions on the degrees of the polynomials p_i in (1.1) enables more nonnegative trigonometric polynomials to be factored. In fact, in [8] it was shown that any strictly positive trigonometric polynomial (i.e., $q(z) > 0$, $z \in \mathbb{T}^d$) allows a representation (1.1), where p_i are polynomials of potentially very high degree. This in turn relates to factorization of real polynomials as sums of squares of rational functions with fixed denominators [8]. An important tool in [8] is the use of Schur complements. Inspired by this we also use the Schur complement as our main tool. This allows for a very simple proof of Rosenblum's operator-valued Fejér-Riesz theorem. The main observation in this proof is that the sequence of finitely supported Schur complements of a banded positive semidefinite Toeplitz operator has a very simple inheritance structure (see Proposition 3.1), a result that was not yet observed in [8]. In fact, beyond a certain matrix size (determined by the number of nonzero diagonals) as the Schur complement is increased one dimension in size, it is constructed by bordering the previous Schur complement with the coefficients of the underlying trigonometric polynomial. Recognizing this, the task becomes how to determine the multivariate analog of this inheritance structure. Clearly, there are now many canonical shifts; how does one use these? As we will see, for the multivariate trigonometric polynomial to have an outer factorization of the required type, the Schur complement of the corresponding Toeplitz operator needs to decompose in a certain way. Subsequently, to obtain the next Schur complement, the different terms in this decomposition need to be shifted in different ways. Bordering the result with the coefficients of the trigonometric polynomial then yields the next Schur complement. The precise statement is given in Theorem 4.3. In [10] a two-variable Fejér-Riesz result was obtained for the case when the trigonometric polynomial is strictly positive and scalar valued. The approach there was completely different, as the factorization result was obtained as a corollary of a result on bivariate autoregressive filters.

The paper is organized as follows. In Section 2 we derive several useful new properties of Schur complements. In Section 3 we use these newly-observed properties to provide easy proofs for Rosenblum's version of the operator-valued Fejér-Riesz theorem and the existence of inner-outer factorizations. In Section 4 the multivariate case is addressed.

2. Auxiliary results on Schur complements

Recall that if $\mathcal{H}_1$ and $\mathcal{H}_2$ are Hilbert spaces and

$$M = \begin{pmatrix} A & B \\ B^* & C \end{pmatrix} : \mathcal{H}_1 \oplus \mathcal{H}_2 \to \mathcal{H}_1 \oplus \mathcal{H}_2 \tag{2.1}$$

is a positive semidefinite operator, then there exists a unique contraction $G : \overline{\operatorname{ran}}\,(C) \to \overline{\operatorname{ran}}\,(A)$ such that $B = A^{1/2} G C^{1/2}$. The *Schur complement S of M supported on* $\mathcal{H}_1$ is defined to be the positive semidefinite operator $A^{1/2}(1 - GG^*)A^{1/2}$.

An alternative way to define the Schur complement of M supported on $\mathcal{H}_1$ is via

$$\langle Sf, f\rangle = \inf \left\{ \langle \begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix}, \begin{pmatrix} f \\ g \end{pmatrix} \rangle \ : \ g \in \mathcal{H}_2 \right\};$$

that is, it is the largest positive semidefinite operator which may be subtracted from A in (2.1) such that the resulting operator matrix remains positive semidefinite.

A few words are required on our notation. We will typically number rows and columns of an $n \times n$ matrix with $0, \dots, n-1$. For $\Lambda \subseteq \{0, \dots, n-1\}$ and an $n \times n$ operator matrix M, we write $S(M; \Lambda)$ (or $S(\Lambda)$ when there is no chance of confusion) for the Schur complement supported on rows and columns labeled by elements of Λ. It is usual to view $S(\Lambda)$ as an $m \times m$ matrix, where $m = \operatorname{card} \Lambda$, however it is often useful to take $S(\Lambda)$ to be an $n \times n$ matrix. If $\Lambda = \{n_0, \dots, n_{m-1}\}$, then this is done by putting the (j,k) entry of $S(\Lambda)$ as an $m \times m$ matrix into the (n_j, n_k) place and padding with zeros. We use the same notation for both versions of the Schur complement, since it should be clear from the context which we are using. Finally, as a further bit of notational convenience, we write $S(M;k)$ (or $S(k)$) when $\Lambda = \{0, \dots, k\}$.

Lemma 2.1. *Let $M : \mathcal{H}_1 \oplus \mathcal{H}_2 \to \mathcal{H}_1 \oplus \mathcal{H}_2$ be factored as*

$$M = \begin{pmatrix} A & B \\ B^* & C \end{pmatrix} = \begin{pmatrix} P^* & Q^* \\ 0 & R^* \end{pmatrix} \begin{pmatrix} P & 0 \\ Q & R \end{pmatrix}, \tag{2.2}$$

where $P : \mathcal{H}_1 \to \mathcal{K}_1$, $Q : \mathcal{H}_1 \to \mathcal{K}_2$, and $R : \mathcal{H}_2 \to \mathcal{K}_2$. Then $S(0)$ equals P^*P if and only if $\operatorname{ran} Q \subseteq \overline{\operatorname{ran}}\, R$. Furthermore, for any P such that $P^*P = S(0)$ and any R such that $R^*R = C$, there is a Q such that (2.2) holds.

Proof. Since $C = R^*R$, there is an isometry $V : \overline{\operatorname{ran}}\, R \to \overline{\operatorname{ran}}\, C^{1/2}$ such that $C^{1/2} = VR$. Clearly $\left(\begin{smallmatrix} A & B \\ B^* & C \end{smallmatrix}\right) \geq 0$, so there is a contraction $G : \overline{\operatorname{ran}}\, C^{1/2} \to \overline{\operatorname{ran}}\, A^{1/2}$ with $B = A^{1/2} G C^{1/2}$, and consequently $B = A^{1/2} G V R$. We also have $B = Q^* R$, so the assumption that $\operatorname{ran} Q \subseteq \overline{\operatorname{ran}}\, R$ implies that $A^{1/2} G V = Q^*$. Moreover, since $VV^* = 1_{\overline{\operatorname{ran}}\, C^{1/2}}$, we get that $A^{1/2} G = Q^* V^*$. We calculate the Schur complement

$$\begin{aligned} S(0) &= A^{1/2}(1 - GG^*)A^{1/2} = A - Q^* V^* V Q = A - Q^* 1_{\overline{\operatorname{ran}}\, R^{1/2}} Q \\ &= A - Q^*Q = P^*P + Q^*Q - Q^*Q = P^*P. \end{aligned}$$

Conversely assume that $P^*P = S(0)$. It follows from (2.2) that

$$\begin{pmatrix} A & B \\ B^* & C \end{pmatrix} - \begin{pmatrix} P^*P & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} Q^*Q & Q^*R \\ R^*Q & R^*R. \end{pmatrix}. \tag{2.3}$$

If we set V_R, V_Q to be the inclusions of $\overline{\operatorname{ran}}\, R$ and $\overline{\operatorname{ran}}\, Q$ into $\mathcal{K}_2$, respectively, then $Q^*R = Q^*GR$, where $G = V_Q^* V_R$. By construction the Schur complement of the right side of (2.3) is zero, which implies that

$$0 = 1_{\overline{\operatorname{ran}}\, Q} - GG^* = 1_{\overline{\operatorname{ran}}\, Q} - V_Q^* V_R V_R^* V_Q = V_Q^*(1 - P_{\overline{\operatorname{ran}}\, R}) V_Q,$$

where $P_{\overline{\operatorname{ran}}\, R}$ is the orthogonal projection onto $\overline{\operatorname{ran}}\, R$. Thus $P_{\overline{\operatorname{ran}}\, R} | \overline{\operatorname{ran}}\, Q = 1_{\overline{\operatorname{ran}}\, Q}$, and hence $\operatorname{ran} Q \subseteq \operatorname{ran} P_{\overline{\operatorname{ran}}\, R} = \overline{\operatorname{ran}}\, R$.

Finally, suppose $P^*P = S(0)$ and $R^*R = C$. Then $A - P^*P \geq 0$ and $B = A^{1/2} G R$ for some contraction $G : \overline{\operatorname{ran}}\, R \to \overline{\operatorname{ran}}\, A^{1/2}$. Hence

$$P^*P = A^{1/2}(1 - GG^*)A^{1/2}$$

and there exists D_G such that $D_G D_G^* = 1 - GG^*$ and $P = A^{1/2} D_G$. Then setting $Q^* = A^{1/2} G$, we have $M = \begin{pmatrix} P^* & Q^* \\ 0 & R^* \end{pmatrix} \begin{pmatrix} P & 0 \\ Q & R \end{pmatrix}$, and $\operatorname{ran} Q \subseteq \operatorname{ran} G^* \subseteq \overline{\operatorname{ran}}\, R$. $\square$

Lemma 2.2. *Suppose*

$$M = \begin{pmatrix} A & B & C \\ B^* & D & E \\ C^* & E^* & F \end{pmatrix} = \begin{pmatrix} P^* & Q^* & R^* \\ 0 & S^* & T^* \\ 0 & 0 & U^* \end{pmatrix} \begin{pmatrix} P & 0 & 0 \\ Q & S & 0 \\ R & T & U \end{pmatrix}, \tag{2.4}$$

where M is acting on $\mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3$. Then

$$S(1) - S(0) = \begin{pmatrix} Q^* \\ S^* \end{pmatrix} \begin{pmatrix} Q & S \end{pmatrix} \tag{2.5}$$

if and only if

$$\operatorname{ran} Q \subseteq \overline{\operatorname{ran}}\, S \qquad \text{and} \qquad \operatorname{ran} T \subseteq \overline{\operatorname{ran}}\, U. \tag{2.6}$$

Furthermore there exists a factorization of M as in (2.4) *with the factors operators on $\mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3$ such that* (2.5) *and* (2.6) *hold,* $\begin{pmatrix} P^* & Q^* \\ 0 & S^* \end{pmatrix} \begin{pmatrix} P & 0 \\ Q & S \end{pmatrix} = S(1)$ *and* $P^*P = S(0) = S(S(1); 0)$.

Proof. To begin with, suppose (2.5) holds. Then if $\tilde{P}^* \tilde{P} = S(0)$, we have

$$S(1) = \begin{pmatrix} \tilde{P}^* & Q^* \\ 0 & S^* \end{pmatrix} \begin{pmatrix} \tilde{P} & 0 \\ Q & S \end{pmatrix}.$$

As $U^*U = F$, by Lemma 2.1 there exist $\tilde{R}$ and $\tilde{T}$ such that

$$M = \begin{pmatrix} \tilde{P}^* & Q^* & \tilde{R}^* \\ 0 & S^* & \tilde{T}^* \\ 0 & 0 & U^* \end{pmatrix} \begin{pmatrix} \tilde{P} & 0 & 0 \\ Q & S & 0 \\ \tilde{R} & \tilde{T} & U \end{pmatrix}.$$

By Lemma 2.1

$$\operatorname{ran} \begin{pmatrix} Q \\ \tilde{R} \end{pmatrix} \subseteq \overline{\operatorname{ran}} \begin{pmatrix} S & 0 \\ \tilde{T} & U \end{pmatrix}$$

and

$$\operatorname{ran} \begin{pmatrix} \tilde{R} & \tilde{T} \end{pmatrix} \subseteq \overline{\operatorname{ran}}\, U.$$

Hence $\operatorname{ran} \tilde{T} \subseteq \overline{\operatorname{ran}}\, U$ and so

$$\overline{\operatorname{ran}} \begin{pmatrix} S & 0 \\ \tilde{T} & U \end{pmatrix} = \overline{\operatorname{ran}}\, S \oplus \overline{\operatorname{ran}}\, U.$$

Thus $\operatorname{ran} Q \subseteq \overline{\operatorname{ran}}\, S$.

Next observe that $D = S^*S + T^*T = S^*S + \tilde{T}^*\tilde{T}$ and so there is an isometry $V_T : \overline{\operatorname{ran}}\, T \to \overline{\operatorname{ran}}\, \tilde{T}$ such that $T^* = \tilde{T}^* V_T^*$. Also $\operatorname{ran} \tilde{T} \subseteq \overline{\operatorname{ran}}\, U$ implies that $\operatorname{ran} V_T \subseteq \overline{\operatorname{ran}}\, U$. Thus V_T is an isometry from $\overline{\operatorname{ran}}\, T$ into $\overline{\operatorname{ran}}\, U$. But $U^*T = E^* = U^*\tilde{T} = U^*V_T T$, so $V_T = 1_{\overline{\operatorname{ran}}\, T}$ and $\operatorname{ran} T \subseteq \overline{\operatorname{ran}}\, U$.

Now conversely assume we have a factorization of M as in (2.4), where (2.6) holds. Set

$$L = \begin{pmatrix} D & E \\ E^* & F \end{pmatrix} = \begin{pmatrix} S^* & T^* \\ 0 & U^* \end{pmatrix} \begin{pmatrix} S & 0 \\ T & U \end{pmatrix}.$$

Using Lemma 2.1, suppose $\tilde{G}$ is any other operator matrix satisfying $\tilde{G}^*\tilde{G} = M$ with

$$\tilde{G} = \begin{pmatrix} \tilde{P} & 0 & 0 \\ \tilde{Q} & \tilde{S} & 0 \\ \tilde{R} & \tilde{T} & U \end{pmatrix}, \tag{2.7}$$

where

$$S(1) = \begin{pmatrix} \tilde{P}^* & \tilde{Q}^* \\ 0 & \tilde{S}^* \end{pmatrix} \begin{pmatrix} \tilde{P} & 0 \\ \tilde{Q} & \tilde{S} \end{pmatrix}$$

and $\tilde{P}$ chosen so that $S(S(1);0) = \tilde{P}^*\tilde{P}$. Note that

$$L = \begin{pmatrix} \tilde{S}^* & \tilde{T}^* \\ 0 & U^* \end{pmatrix} \begin{pmatrix} \tilde{S} & 0 \\ \tilde{T} & U \end{pmatrix}.$$

Since by assumption $\operatorname{ran} T \subseteq \overline{\operatorname{ran}}\, U$, we have $S^*S = S(L;\{1\}) \geq \tilde{S}^*\tilde{S}$. On the other hand, since

$$S(1) \geq \begin{pmatrix} P^* & Q^* \\ 0 & S^* \end{pmatrix} \begin{pmatrix} P & 0 \\ Q & S \end{pmatrix},$$

we also have $\tilde{S}^*\tilde{S} \geq S^*S$. Hence $\tilde{S}^*\tilde{S} = S^*S$. Thus $VS = \tilde{S}$ for some isometry $V : \overline{\operatorname{ran}}\, S \to \overline{\operatorname{ran}}\, \tilde{S}$. Since we have chosen $S(S(1);0) = \tilde{P}^*\tilde{P}$, by Lemma 2.1 $\operatorname{ran} \tilde{Q} \subseteq \overline{\operatorname{ran}}\, \tilde{S}$. Moreover,

$$0 \leq \begin{pmatrix} \tilde{P}^*\tilde{P} + \tilde{Q}^*\tilde{Q} & \tilde{Q}^*S \\ S^*\tilde{Q} & S^*S \end{pmatrix} - \begin{pmatrix} P^*P + Q^*Q & Q^*S \\ S^*Q & S^*S \end{pmatrix}$$

and $\tilde{S}^*\tilde{S} \geq S^*S$ imply that $0 = \tilde{Q}^*\tilde{S} - Q^*S = (\tilde{Q}^*V - Q^*)S$. As $\operatorname{ran} Q \subseteq \overline{\operatorname{ran}}\, S$ it follows that $\tilde{Q}^*V = Q^*$. Thus, in particular, $\tilde{Q}^*\tilde{Q} = Q^*Q$. But then we obtain that

$$\tilde{P}^*\tilde{P} \geq P^*P. \tag{2.8}$$

Observe that (2.8) will be true no matter what the original factorization of M in (2.4) is as long as the range conditions in (2.6) are satisfied.

Now instead consider the factorization $M = G'^*G'$, where

$$G' = \begin{pmatrix} P' & 0 & 0 \\ Q' & S & 0 \\ R' & T & U \end{pmatrix}$$

with $P'^*P' = S(0)$. Such a factorization is possible by Lemma 2.1. Since by assumption $\operatorname{ran} T \subseteq \overline{\operatorname{ran}}\, U$, we have

$$\overline{\operatorname{ran}} \begin{pmatrix} S & 0 \\ T & U \end{pmatrix} \subseteq \overline{\operatorname{ran}}\, S \oplus \overline{\operatorname{ran}}\, U.$$

Also by Lemma 2.1 then,

$$\operatorname{ran} \begin{pmatrix} Q' \\ R' \end{pmatrix} \subseteq \overline{\operatorname{ran}}\, S \oplus \overline{\operatorname{ran}}\, U,$$

and hence $\operatorname{ran} Q' \subseteq \overline{\operatorname{ran}}\, S$. So the conditions in (2.6) are satisfied for this factorization, and hence as noted above, we must have $\tilde{P}^*\tilde{P} \geq P'^*P'$. But by definition of the Schur complement, $P'^*P' \geq \tilde{P}^*\tilde{P}$, so we have equality. Consequently, (2.5) holds.

Finally, using Lemma 2.1, there is a factorization

$$L = \begin{pmatrix} S^* & T^* \\ 0 & U^* \end{pmatrix} \begin{pmatrix} S & 0 \\ T & U \end{pmatrix},$$

where $\operatorname{ran} T \subseteq \overline{\operatorname{ran}}\, U \subseteq \mathcal{H}_3$, so that $\overline{\operatorname{ran}} \left(\begin{smallmatrix} S & T \\ 0 & U \end{smallmatrix}\right) = \overline{\operatorname{ran}}\, S \oplus \overline{\operatorname{ran}}\, U \subseteq \mathcal{H}_2 \oplus \mathcal{H}_3$. Again by Lemma 2.1, there exists $P : \mathcal{H}_1 \to \mathcal{H}_1$ such that $P^*P = S(0)$ and (2.4) holds. Consequently $\operatorname{ran} \left(\begin{smallmatrix} Q \\ R \end{smallmatrix}\right) \subseteq \overline{\operatorname{ran}}\, S \oplus \overline{\operatorname{ran}}\, U$, giving $\operatorname{ran} Q \subseteq \overline{\operatorname{ran}}\, S$.

It is now clear that the factorization in (2.4) with these choices of P, Q, R, S, T and U satisfies the last statement of the theorem. □

Corollary 2.3. *Let $M \geq 0$ be an $n \times n$ operator matrix, $J \subseteq K \subseteq \{0 \dots n-1\}$. Then*

$$S(J) = S(S(K); J). \tag{2.9}$$

Proof. Let $I_1 = J$, $I_2 = K \setminus J$ and $I_3 = \{0, \dots, n-1\} \setminus K$. Writing M as a 3×3 block matrix with respect to the partition

$$\{0, \dots, n-1\} = I_1 \cup I_2 \cup I_3,$$

the corollary follows directly from Lemma 2.2. □

Corollary 2.4. *Given $M \geq 0$ an $n \times n$ operator matrix, there is a factorization $M = P^*P$, where*

$$P = \begin{pmatrix} P_{00} & 0 & \cdots & \cdots & 0 \\ P_{10} & P_{11} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ P_{n-1,0} & P_{n-1,1} & \cdots & \cdots & P_{n-1,n-1} \end{pmatrix},$$

where $\overline{\operatorname{ran}}\, P = \overline{\operatorname{ran}}\, P_{00} \oplus \dots \oplus \overline{\operatorname{ran}}\, P_{n-1,n-1}$ and such that if P_k is the truncation of P to the upper left $(k+1) \times (k+1)$ corner, then $S(k) = P_k^* P_k$, $k = 0, \dots, n$.

The above result also appears in [6].

Lemma 2.5. *Let*

$$\begin{pmatrix} P^* & Q^* \\ 0 & R^* \end{pmatrix} \begin{pmatrix} P & 0 \\ Q & R \end{pmatrix} = \begin{pmatrix} \tilde{P}^* & \tilde{Q}^* \\ 0 & \tilde{R}^* \end{pmatrix} \begin{pmatrix} \tilde{P} & 0 \\ \tilde{Q} & \tilde{R} \end{pmatrix}, \tag{2.10}$$

and suppose that $\operatorname{ran} Q \subseteq \overline{\operatorname{ran}}\, R$. Then there is a unique isometry

$$\begin{pmatrix} V_{11} & 0 \\ V_{21} & V_{22} \end{pmatrix}$$

acting on $\overline{\operatorname{ran}}\, P \oplus \overline{\operatorname{ran}}\, R$ so that

$$\begin{pmatrix} \tilde{P} & 0 \\ \tilde{Q} & \tilde{R} \end{pmatrix} = \begin{pmatrix} V_{11} & 0 \\ V_{21} & V_{22} \end{pmatrix} \begin{pmatrix} P & 0 \\ Q & R \end{pmatrix}. \tag{2.11}$$

Proof. It is a standard result that $A^*A = B^*B$ if and only there exist an isometry $V : \overline{\operatorname{ran}}\, B \to \overline{\operatorname{ran}}\, A$ so that $VB = A$. The operator V is uniquely determined by setting $V(Bx) = Ax$ for every $Bx \in \operatorname{ran} B$, and extending V to $\overline{\operatorname{ran}}\, B$ by continuity. Thus (2.10) implies the existence of an isometry $V = (V_{ij})_{i,j=1}^2$ satisfying

$$\begin{pmatrix} \tilde{P} & 0 \\ \tilde{Q} & \tilde{R} \end{pmatrix} = \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix} \begin{pmatrix} P & 0 \\ Q & R \end{pmatrix}. \tag{2.12}$$

It remains to show that $V_{12} = 0$. Note that (2.12) implies that $V_{22}R = \tilde{R}$. Combining this with (2.10) we get that

$$R^*R = \tilde{R}^*\tilde{R} = R^*V_{22}^*V_{22}R,$$

and thus

$$R^*(I_{\overline{\operatorname{ran}}\, R} - V_{22}^*V_{22})R = 0. \tag{2.13}$$

As $\operatorname{ran} Q \subseteq \overline{\operatorname{ran}}\, R$ we have that

$$\overline{\operatorname{ran}} \begin{pmatrix} P & 0 \\ Q & R \end{pmatrix} = \overline{\operatorname{ran}}\, P \oplus \overline{\operatorname{ran}}\, R.$$

Thus V_{22} and V_{12} act on $\overline{\operatorname{ran}}\, R$. From (2.13) we now obtain that V_{22} is an isometry on $\overline{\operatorname{ran}}\, R$. But then, since V is an isometry, we must have that $V_{12} = 0$. $\square$

In the following lemma we consider a positive semidefinite operator on $\mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3 \oplus \mathcal{H}_4$, the $\mathcal{H}_k$'s Hilbert spaces.

Lemma 2.6. *Let $A = (A_{ij})_{i,j=1}^4 \geq 0$. Then we have that*

$$[S(2)]_{21} = 0 \tag{2.14}$$

if and only if

$$S(2) = S(1) + S(\{0,2\}) - S(0). \tag{2.15}$$

Proof. The direction (2.15) $\Rightarrow$ (2.14) is trivial.

By Corollary 2.4, there is a lower triangular 3×3 operator matrix

$$P = \begin{pmatrix} P_{00} & 0 & 0 \\ P_{10} & P_{11} & 0 \\ P_{20} & P_{21} & P_{22} \end{pmatrix}$$

such that $S(2) = P^*P$, $S(1) = \begin{pmatrix} P_{00}^* & P_{10}^* \\ 0 & P_{11}^* \end{pmatrix} \begin{pmatrix} P_{00} & 0 \\ P_{10} & P_{11} \end{pmatrix}$, and $S(0) = P_{00}^*P_{00}$. Also, $\operatorname{ran} P_{10} \subseteq \overline{\operatorname{ran}}\, P_{11}$ and $\operatorname{ran} P_{20}, \operatorname{ran} P_{21} \subseteq \overline{\operatorname{ran}}\, P_{22}$. Thus $[S(2)]_{21} = 0$ is equivalent to $P_{21} = 0$. Interchanging the order of rows 1 and 2 and columns 1 and 2, we have

$$S(\{0,2,1\}) = \begin{pmatrix} P_{00}^* & P_{20}^* & P_{10}^* \\ 0 & P_{22}^* & 0 \\ 0 & 0 & P_{11}^* \end{pmatrix} \begin{pmatrix} P_{00} & 0 & 0 \\ P_{20} & P_{22} & 0 \\ P_{10} & 0 & P_{11} \end{pmatrix}.$$

Since $\operatorname{ran} \begin{pmatrix} P_{10} & 0 \end{pmatrix} \subseteq \overline{\operatorname{ran}}\, P_{11}$, by Lemma 2.1,

$$S(\{0,2\}) = \begin{pmatrix} P_{00}^* & P_{20}^* \\ 0 & P_{22}^* \end{pmatrix} \begin{pmatrix} P_{00} & 0 \\ P_{20} & P_{22} \end{pmatrix}.$$

A direct calculation verifies the equality in (2.15). $\square$

By relabeling and grouping as we did in the proof of Corollary 2.3, we obtain the following.

Corollary 2.7. *Suppose $M \geq 0$ is an $n \times n$ operator matrix, $K \cup J = N \subseteq \{0, \dots n-1\}$. Then*

$$S(N) = S(K) + S(J) - S(K \cap J) \tag{2.16}$$

if and only if

$$\begin{gathered}[S(N)]_{k,j} = 0, \\ (k,j) \in (N \times N) \setminus ((K \times K) \cup (J \times J)) \\ = [(K \setminus (K \cap J)) \times (J \setminus (K \cap J))] \cup [(J \setminus (K \cap J) \times (K \setminus (K \cap J))]. \end{gathered} \tag{2.17}$$

3. One variable outer and inner-outer factorization

In this section we will provide new proofs for several one-variable factorization results. These proofs are based on the properties of Schur complements.

Given a Hilbert space $\mathcal{H}$ let $H^2_{\mathcal{H}}(\mathbb{D})$ denote the Hardy space of $\mathcal{H}$-valued functions analytic in the unit disk with square integrable boundary values. These functions will be identified with their boundary values whenever convenient. Given a pair of Hilbert spaces $\mathcal{H}$, $\mathcal{K}$, let $\mathbf{L}(\mathcal{H}, K)$ stand for the Banach space of bounded operators acting $\mathcal{H} \to \mathcal{K}$. We will write $\mathbf{L}(\mathcal{H})$ instead of $\mathbf{L}(\mathcal{H}, \mathcal{H})$. As usual, $H^\infty_{\mathbf{L}(\mathcal{H},\mathcal{K})}(\mathbb{D})$ stands for the set of all bounded analytic $\mathbf{L}(\mathcal{H}, K)$-valued functions on $\mathbb{D}$. With the operator-valued function $F \in H^\infty_{\mathbf{L}(\mathcal{H},\mathcal{K})}(\mathbb{D})$, we associate the operator $M_F : H^2_{\mathcal{H}}(\mathbb{D}) \to H^2_{\mathcal{K}}(\mathbb{D})$ of multiplication by F; that is, $M_F g(z) = F(z)g(z)$. The function F is called *outer* if the corresponding multiplication operator M_F has dense range in $H^2_{\mathcal{M}}(\mathbb{D})$ for some subspace $\mathcal{M}$ of $\mathcal{K}$, and this reduces to the usual definition when $\mathcal{H}$ and $\mathcal{K}$ are $\mathbb{C}$. For $Q \in L^\infty_{\mathbf{L}(\mathcal{H})}(\mathbb{T})$, we consider the Toeplitz operator $T_Q : H^2_{\mathcal{H}}(\mathbb{D}) \to H^2_{\mathcal{H}}(\mathbb{D})$ defined via $T_Q f = \Pi_+(Qf)$, where Π_+ the projection is from $L^2_{\mathcal{H}}(\mathbb{T})$ onto $H^2_{\mathcal{H}}(\mathbb{D})$. We shall often represent T_Q via the Toeplitz operator matrix

$$T_Q \equiv \begin{pmatrix} Q_0 & Q_{-1} & \cdots & \\ Q_1 & \ddots & \ddots & \\ \vdots & \ddots & & \end{pmatrix}, \tag{3.1}$$

where we make the obvious identification of $f(z) = \sum_0^\infty f_k z^k \in H^2_{\mathcal{H}}(\mathbb{D})$ with $\operatorname{col}(f_k)_0^\infty \in \ell^2_{\mathcal{H}}(\mathbb{N}_0)$, where $f_j \in \mathcal{H}$ and $\|f\| := \sqrt{\sum_{j=0}^\infty \|f_j\|^2} < \infty$. We view the matrix as an operator matrix with rows and columns indexed by $\mathbb{N}_0 = \{0, 1, \dots\}$. In addition, we shall often use the identification

$$T_Q = \begin{pmatrix} Q_0 & \operatorname{row}(Q_{-i})_{i\geq 1} \\ \operatorname{col}(Q_i)_{i\geq 1} & T_Q \end{pmatrix}. \tag{3.2}$$

In other words, the operator $L : zH^2_{\mathcal{H}}(\mathbb{D}) \to zH^2_{\mathcal{H}}(\mathbb{D})$ defined by $(Lf)(z) = z\Pi_+(Qz^{-1}f)$ will at times be identified with T_Q. Following the notation from the previous section, for $\Lambda \subset \mathbb{N}_0$ we let $S(T_Q; \Lambda)$ (or $S(\Lambda)$ when no confusion is possible) denote the Schur complement of T_Q supported on rows and columns indexed by Λ. In addition, $S(k)$ is shorthand for $S(\{0, \dots, k\})$.

We first address the effect that the Toeplitz structure has on the Schur complements.

Proposition 3.1. *Consider the positive semidefinite Toeplitz operator*

$$T_Q = (Q_{i-j})_{i,j=0}^{\infty}$$

acting on $\ell^2_{\mathcal{H}}(\mathbb{N}_0)$. Then the Schur complements $S(m)$ of T_Q satisfy the recurrence relation

$$S(m) = \begin{pmatrix} A & B^* \\ B & S(m-1) \end{pmatrix}, \tag{3.3}$$

for an appropriate choice of $A : \mathcal{H} \to \mathcal{H}$ and $B : \mathcal{H} \to \mathcal{H}^m$. When $Q_j = 0, j \geq m+1$, then $A = Q_0$ and $B = \mathrm{col}(Q_i)_{i=1}^m$.

Proof. By the definition of Schur complement

$$T_Q - \begin{pmatrix} S(m) & 0 \\ 0 & 0 \end{pmatrix} \geq 0. \tag{3.4}$$

Let us write

$$S(m) = \begin{pmatrix} A & B^* \\ B & C \end{pmatrix} : \mathcal{H} \oplus \mathcal{H}^m \to \mathcal{H} \oplus \mathcal{H}^m.$$

Leaving out row and column 0 in (3.4) yields

$$T_Q - \begin{pmatrix} C & 0 \\ 0 & 0 \end{pmatrix} \geq 0,$$

where we used identification (3.2). This shows that $C \leq S(m-1)$. On the other hand, leaving out row and columns $1, \ldots, m$ in (3.4) yields

$$\begin{pmatrix} Q_0 - A & \mathrm{row}(Q_j^*)_{j \geq m+1} \\ \mathrm{col}(Q_j)_{j \geq m+1} & T_Q \end{pmatrix} \geq 0.$$

Hence

$$A \leq S\left(\begin{pmatrix} Q_0 & \mathrm{row}(Q_j^*)_{j \geq m+1} \\ \mathrm{col}(Q_j)_{j \geq m+1} & T_Q \end{pmatrix}; 0 \right) =: \tilde{A}.$$

Note that when $Q_j = 0$, $j \geq m+1$, we have that $\tilde{A} = Q_0$. Now consider the operator matrix

$$\begin{pmatrix} Q_0 - \tilde{A} & X & \mathrm{row}(Q_j^*)_{j \geq m+1} \\ X^* & (Q_{i-j})_{i,j=1}^m - S(m-1) & (Q_{i-j})_{i=1,j=m+1}^{m+1,\ \infty} \\ \mathrm{col}(Q_j)_{j \geq m+1} & (Q_{i-j})_{i=m+1,j=1}^{\infty,\ m} & T_Q \end{pmatrix}. \tag{3.5}$$

The existence of an operator X making this into a positive semidefinite matrix is a variant of a standard operator matrix completion problem, and by [1] (see also, e.g., Theorem XVI.3.1 in [9] or [2]), there is always such an X. Note that when $\tilde{A} = Q_0$ we have necessarily that $X = 0$. As (3.5) is positive semidefinite we obtain that

$$\begin{pmatrix} \tilde{A} & \mathrm{row}(Q_j^*)_{j=1}^m - X \\ \mathrm{col}(Q_j)_{j=1}^m - X^* & S(m-1) \end{pmatrix} \leq S(m) = \begin{pmatrix} A & B^* \\ B & C \end{pmatrix}.$$

This implies that $\tilde{A} \leq A$ and $S(m-1) \leq C$. As we also had that $A \leq \tilde{A}$ and $C \leq S(m-1)$, the equalities $A = \tilde{A}$ and $C = S(m-1)$ follow. This yields (3.3). Moreover, when $Q_j = 0$ for $j \geq m+1$, we have that $\tilde{A} = Q_0$ and $X = 0$, and thus $B = \mathrm{col}(Q_i)_{i=1}^m$. $\square$

Remark. Note that the proof shows that the operator A in (3.3) is given by

$$A = S\left(\begin{pmatrix} Q_0 & \operatorname{row}(Q_j^*)_{j\geq m+1} \\ \operatorname{col}(Q_j)_{j\geq m+1} & T_Q \end{pmatrix}; 0\right).$$

Because of the inheritance principle observed in Proposition 3.1, the Schur complements of a Toeplitz operator allow a stationary UL Cholesky decomposition.

Corollary 3.2. *Consider the positive semidefinite Toeplitz operator*

$$T_Q = (Q_{i-j})_{i,j=0}^{\infty}$$

acting on $\ell^2_{\mathcal{H}}(\mathbb{N}_0)$. *Then there exist operators* $F_0, F_1, \dots$ *with* $F_i : \mathcal{H} \to \overline{\operatorname{ran}}\, F_0 \subseteq \mathcal{H}$ *so that the Schur complements* $S(m)$ *of* T_Q *satisfy*

$$S(m) = \begin{pmatrix} F_0^* & \cdots & F_m^* \\ & \ddots & \vdots \\ & & F_0^* \end{pmatrix} \begin{pmatrix} F_0 & & \\ \vdots & \ddots & \\ F_m & \cdots & F_0 \end{pmatrix}, \qquad m \geq 0. \tag{3.6}$$

Proof. We prove this by induction. When $m = 0$ we may for instance choose $F_0 = (S(0))^{1/2}$. It follows from Proposition 3.1 that $(S(m))_{m,m} = (S(m-1))_{m-1,m-1} = F_0^* F_0$, where in the last step we used the induction hypothesis. By Corollary 2.3 we have that $S(m-1) = S(S(m); m-1)$, and thus by Lemma 2.1 with

$$P = \begin{pmatrix} F_0 & & \\ \vdots & \ddots & \\ F_{m-1} & \cdots & F_0 \end{pmatrix}, \qquad R = F_0,$$

there exist $\begin{pmatrix} G_m & \cdots & G_1 \end{pmatrix}$ so that

$$S(m) = \begin{pmatrix} F_0^* & \cdots & F_{m-1}^* & G_m^* \\ & \ddots & \vdots & \vdots \\ & & F_0^* & G_1^* \\ & & & F_0^* \end{pmatrix} \begin{pmatrix} F_0 & & & \\ \vdots & \ddots & & \\ F_{m-1} & \cdots & F_0 & \\ G_m & \cdots & G_1 & F_0 \end{pmatrix}, \tag{3.7}$$

and $\operatorname{ran} \begin{pmatrix} G_m & \cdots & G_1 \end{pmatrix} \subseteq \overline{\operatorname{ran}}\, F_0$. Comparing (3.7) with (3.3) along with the induction hypothesis yields

$$\begin{aligned} &\begin{pmatrix} F_0^* & \cdots & F_{m-2}^* & G_{m-1}^* \\ & \ddots & \vdots & \vdots \\ & & F_0^* & G_1^* \\ & & & F_0^* \end{pmatrix} \begin{pmatrix} F_0 & & & \\ \vdots & \ddots & & \\ F_{m-2} & \cdots & F_0 & \\ G_{m-1} & \cdots & G_1 & F_0 \end{pmatrix} = S(m-1) \\ &= \begin{pmatrix} F_0^* & \cdots & F_{m-2}^* & F_{m-1}^* \\ & \ddots & \vdots & \vdots \\ & & F_0^* & F_1^* \\ & & & F_0^* \end{pmatrix} \begin{pmatrix} F_0 & & & \\ \vdots & \ddots & & \\ F_{m-2} & \cdots & F_0 & \\ F_{m-1} & \cdots & F_1 & F_0 \end{pmatrix}, \end{aligned}$$

and thus

$$F_0^* \begin{pmatrix} G_{m-1} & \cdots & G_1 \end{pmatrix} = F_0^* \begin{pmatrix} F_{m-1} & \cdots & F_1 \end{pmatrix}.$$

As $\operatorname{ran} \begin{pmatrix} G_{m-1} & \cdots & G_1 \end{pmatrix} \subseteq \overline{\operatorname{ran}}\, F_0$ and $\operatorname{ran} \begin{pmatrix} F_{m-1} & \cdots & F_1 \end{pmatrix} \subseteq \overline{\operatorname{ran}}\, F_0$, it follows that $G_j = F_j$, $j = 1, \dots, m-1$. By setting $F_m := G_m$, we obtain the result. $\square$

Before we come to our main items, let us develop some equivalent statements for outerness that follow directly from the Schur complement results. Analogously to (3.2), we shall use the identification

$$T_F = \begin{pmatrix} F_0 & 0 \\ \operatorname{col}(F_j)_{j\geq 1} & T_F \end{pmatrix}. \tag{3.8}$$

Theorem 3.3. *Let $F \in H^\infty_{\mathbf{L}(\mathcal{H},\mathcal{K})}(\mathbb{D})$. Denote the Taylor coefficients of F by F_j, $j \geq 0$. The following are equivalent:*

(i) *F is outer;*
(ii) *$\overline{ran}\, M_F = H^2_{\overline{ran}\, F_0}(\mathbb{D})$;*
(iii) *$ran\ \operatorname{col}(F_j)_{j\geq 1} \subset \overline{ran}\, T_F$;*
(iv) *$S(T_F^* T_F; 0) = F_0^* F_0$;*
(v) *for some $k \in \mathbb{N}_0$ we have that*

$$S(T_F^* T_F; k) = \begin{pmatrix} F_0^* & \cdots & F_k^* \\ & \ddots & \vdots \\ & & F_0^* \end{pmatrix} \begin{pmatrix} F_0 & & \\ \vdots & \ddots & \\ F_k & \cdots & F_0 \end{pmatrix}; \tag{3.9}$$

(vi) *for all $k \in \mathbb{N}_0$ equality (3.9) holds.*

Proof. Clearly (ii) implies (i). For the implication (i) $\Rightarrow$ (ii), observe that if $\overline{\operatorname{ran}}\, M_F = H^2_{\mathcal{M}}(\mathbb{D})$, then $P_0(\overline{\operatorname{ran}}\, T_F) = \mathcal{M}$, where P_0 is the projection $F \mapsto F_0$. But when $h \in H^2_{\mathcal{M}}(\mathbb{D})$ we have that $P_0(Fh) = F_0 h(0) \in \operatorname{ran} F_0$. Moreover, since we may let $h(0)$ range over all elements in $\mathcal{H}$, we obtain that $\mathcal{M} = \overline{\operatorname{ran}}\, F_0$.

For (ii) $\Rightarrow$ (iii), note that given (ii) we get that

$$\operatorname{ran}\operatorname{col}(F_j)_{j\geq 1} \subset (I - P_0)\ell^2_{\overline{\operatorname{ran}}\, F_0}(\mathbb{N}_0) \cong H^2_{\overline{\operatorname{ran}}\, F_0}(\mathbb{D}) = \overline{\operatorname{ran}}\, T_F.$$

Here we used the identification of a multiplication operator and its corresponding Toeplitz operator.

Next consider (iii) $\Rightarrow$ (ii). If $h_0 \in \mathcal{H}$ we have that $\operatorname{col}(F_j h_0)_{j\geq 1} \in \overline{\operatorname{ran}}\, T_F$. Thus there exist g_i so that $\lim_{i\to\infty} T_F g_i = \operatorname{col}(F_j h_0)_{j\geq 1}$. But then

$$\begin{pmatrix} F_0 & 0 \\ \operatorname{col}(F_j)_{j\geq 1} & T_F \end{pmatrix} \begin{pmatrix} h_0 \\ -g_i \end{pmatrix} \to \begin{pmatrix} F_0 h_0 \\ 0 \end{pmatrix}.$$

Thus $F_0 h_0 \in \overline{\operatorname{ran}}\, M_F$. As $\overline{\operatorname{ran}}\, M_F$ is closed under multiplication with z, we get that $H^2_{\overline{\operatorname{ran}}\, F_0}(\mathbb{D}) \subseteq \overline{\operatorname{ran}}\, M_F$. The inclusion $\overline{\operatorname{ran}}\, M_F \subseteq H^2_{\overline{\operatorname{ran}}\, F_0}(\mathbb{D})$ follows as (iii) implies $\operatorname{ran} F_j \subseteq \overline{\operatorname{ran}}\, \begin{pmatrix} F_{j-1} & \cdots & F_0 \end{pmatrix}$, $j \geq 1$, which in turn implies $\operatorname{ran} F_j \subseteq \overline{\operatorname{ran}}\, F_0$.

For the implication (iii) $\Rightarrow$ (vi), note that (iii) implies that $\operatorname{ran}\operatorname{col}(F_j)_{j\geq k} \subset \overline{\operatorname{ran}}\, T_F$ for all $k \geq 1$. But then (vi) follows immediately from Lemma 2.1.

The implications (vi) $\Rightarrow$ (iv) $\Rightarrow$ (v) are trivial.

For (v) to (iii) use Lemma 2.1. $\square$

We are now ready to give a simple proof for the operator-valued Fejér-Riesz theorem. The original proof is due to Rosenblum [17].

Theorem 3.4 ([17])**.** *Let $Q_j : \mathcal{H} \to \mathcal{H}$, $j = -m, \ldots, m$, be Hilbert space operators so that $Q(z) := \sum_{j=-m}^m Q_j z^j \geq 0$, $z \in \mathbb{T}$. Then there exists an outer operator polynomial $P(z) = \sum_{j=0}^m P_j z^j$ with $P_j \in \mathbf{L}(\mathcal{H})$, $j = 0, \ldots, m$, so that $Q(z) = P(z)^* P(z)$, $z \in \mathbb{T}$.*

Proof. Let

$$Y = (Y_{ij})_{i,j=0}^m := S(m) - S(m-1),$$

where $S(m-1)$ is viewed as an operator on $\mathcal{H}^{m+1}$ (with last row and column equal to 0). By Corollary 3.2 there exist operators $P_i : \mathcal{H} \to \mathcal{H}$ with $\operatorname{ran} P_i \subseteq \overline{\operatorname{ran}}\, P_0$ so that

$$Y = \begin{pmatrix} P_m^* \\ \vdots \\ P_0^* \end{pmatrix} \begin{pmatrix} P_m & \cdots & P_0 \end{pmatrix}.$$

Put $P(z) = \sum_{j=0}^m P_j z^j$, $Z_m = \begin{pmatrix} z^m & \cdots & 1 \end{pmatrix}^T$. Then, since in Proposition 3.1 we have that $A = Q_0$ and $B = \operatorname{col}(Q_i)_{i=1}^m$, we get that

$$\begin{aligned} P(z)^* P(z) &= Z_m^* Y Z_m \\ &= z^{*m}\Big(\sum_{j=-m}^{-1} Q_j z^j + Q_0 + \sum_{j=1}^{m} Q_j z^{*(-j)}\Big) z^m + Z_{m-1}^* S(m-1) Z_{m-1} \\ &\quad - \overline{z} Z_{m-1}^* S(m-1) z Z_{m-1}, \end{aligned}$$

where the first term equals $Q(z)$ and last two terms cancel when $z \in \mathbb{T}$.

Finally, in order to see that $P(z)$ is outer, use the equivalence (i) $\Leftrightarrow$ (iv) in Theorem 3.3 and the fact that

$$S(T_P^* T_P; 0) = S(T_Q; 0) = S(S(m); 0) = P_0^* P_0.$$

□

Next we will show how Lemma 2.5 leads to the existence of inner-outer factorizations for operator-valued polynomials. Recall that $A \in H^\infty_{\mathbf{L}(\mathcal{H},K)}(\mathbb{T})$ is *inner* if the multiplication operator $M_A : H^2_{\mathcal{H}}(\mathbb{T}) \to H^2_{\mathcal{K}}(\mathbb{T})$ with symbol A is a partial isometry.

Theorem 3.5 (Existence of inner-outer factorization)**.** *Let $A \in H^\infty_{\mathbf{L}(\mathcal{H},K)}(\mathbb{D})$. Then there exists an outer function F and an inner function V, so that $A = VF$.*

Proof. Consider the Toeplitz operator $T_Q := T_A^* T_A$. By Corollary 3.2 there exist $F_j : \mathcal{H} \to \overline{\operatorname{ran}}\, F_0 \subseteq \mathcal{H}$ so that

$$S(m) = \begin{pmatrix} F_0^* & \cdots & F_m^* \\ & \ddots & \vdots \\ & & F_0^* \end{pmatrix} \begin{pmatrix} F_0 & & \\ \vdots & \ddots & \\ F_m & \cdots & F_0 \end{pmatrix} =: \mathcal{F}(m)^* \mathcal{F}(m), \qquad m \geq 0. \tag{3.10}$$

Note also that

$$\mathcal{A}(m)^* \mathcal{A}(m) := \begin{pmatrix} A_0^* & \cdots & A_m^* \\ & \ddots & \vdots \\ & & A_0^* \end{pmatrix} \begin{pmatrix} A_0 & & \\ \vdots & \ddots & \\ A_m & \cdots & A_0 \end{pmatrix} \leq S(m). \tag{3.11}$$

Consider the sequence of operators

$$\begin{pmatrix} \mathcal{F}(m) & 0 \\ 0 & 0 \end{pmatrix} \tag{3.12}$$

acting on $\ell^2_{\mathcal{H}}(\mathbb{N}_0)$. As $\|S(m)\| \leq \|T_A^* T_A\|$, it follows that (3.12) is a bounded sequence of operators, and therefore has a subsequence that converges to T_F, say, in the weak-$*$ topology. But then we must have that

$$T_F = \begin{pmatrix} F_0 & & & \\ F_1 & F_0 & & \\ \vdots & \ddots & \ddots & \end{pmatrix}.$$

Also, $\mathcal{A}(m)$ converges to T_A in the weak-$*$ topology. But now (3.11) and $\begin{pmatrix} S(m) & 0 \\ 0 & 0 \end{pmatrix} \leq T_A^* T_A$ yield that $T_A^* T_A \leq T_F^* T_F \leq T_A^* T_A$. Thus $T_A^* T_A = T_F^* T_F$, or equivalently, $A(z)^* A(z) = F(z)^* F(z)$ a.e on $\mathbb{T}$, where $F(z) = F_0 + zF_1 + \ldots$. As $S(T_F^* T_F; 0) = F_0^* F_0$ it follows by Theorem 3.3 that F is outer.

Next, notice that we may write

$$T_F = \begin{pmatrix} F_0 & 0 \\ \operatorname{col}(F_j)_{j\geq 1} & T_F \end{pmatrix}, \qquad T_A = \begin{pmatrix} A_0 & 0 \\ \operatorname{col}(A_j)_{j\geq 1} & T_A \end{pmatrix}.$$

Moreover, since F is outer we have that $\operatorname{ran}\operatorname{col}(F_j)_{j\geq 1} \subset \overline{\operatorname{ran}}\, T_F$. By Lemma 2.5 there exists a unique isometry

$$\tilde{V} = \begin{pmatrix} V_{11} & 0 \\ V_{21} & V_{22} \end{pmatrix}$$

acting on $\overline{\operatorname{ran}}\, F_0 \oplus \overline{\operatorname{ran}}\, T_F$ so that

$$\begin{pmatrix} A_0 & 0 \\ \operatorname{col}(A_j)_{j\geq 1} & T_A \end{pmatrix} = \begin{pmatrix} V_{11} & 0 \\ V_{21} & V_{22} \end{pmatrix} \begin{pmatrix} F_0 & 0 \\ \operatorname{col}(F_j)_{j\geq 1} & T_F \end{pmatrix}.$$

Since V_{22} is an isometry and satisfies $T_A = V_{22} T_F$ we obtain by the uniqueness statement in Lemma 2.5 that $\tilde{V} = V_{22}$. But that implies that $\tilde{V}$ must be of the form $\tilde{V} = (V_{i-j})_{i,j\geq 0}$ with $V_k = 0$ for $k < 0$. Thus $\tilde{V} = T_V$ and $V(z) = V_0 + zV_1 + z^2 V_2 + \ldots$ is inner. $\square$

Next we provide new proofs to some more of the various equivalent characterizations that exist for outer functions (see [18]), and obtain a few new ones as well.

Theorem 3.6. *Let $F \in H^\infty_{\mathbf{L}(\mathcal{H},\mathcal{K})}(\mathbb{D})$. Denote the Taylor coefficients of F by F_j, $j \geq 0$. The following are equivalent:*

(i) *F is outer.*

(ii) *For any $z \in \mathbb{D}$ and $G \in H^\infty_{\mathbf{L}(\mathcal{L},\mathcal{K})}(\mathbb{D})$ with $G^*G = F^*F$ a.e. on $\mathbb{T}$,*

$$G(z)^* G(z) \leq F(z)^* F(z), z \in \mathbb{D}.$$

(iii) *There exists $z_0 \in \mathbb{D}$ such that whenever $G \in H^\infty_{\mathbf{L}(\mathcal{L},\mathcal{K})}(\mathbb{D})$ and $G^*G = F^*F$ a.e. on $\mathbb{T}$,*

$$G(z_0)^* G(z_0) \leq F(z_0)^* F(z_0).$$

(iv) *$G \in H^\infty_{\mathbf{L}(\mathcal{L},\mathcal{K})}(\mathbb{D})$ and $G^*G = F^*F$ a.e. on $\mathbb{T}$ implies*

$$G_0^* G_0 \leq F_0^* F_0.$$

(v) *For some $k \in \mathbb{N}_0$ we have that $G \in H^\infty_{\mathbf{L}(\mathcal{L},\mathcal{K})}(\mathbb{D})$ and $G^*G = F^*F$ a.e. on $\mathbb{T}$ implies $\sum_{i=0}^{l} G_i^* G_i \leq \sum_{i=0}^{l} F_i^* F_i$, $l = 0, \ldots, k$, where G_i are the Taylor coefficients of G.*

(vi) *For all $k \in \mathbb{N}_0$ we have that $G \in H^\infty_{\mathbf{L}(\mathcal{L},\mathcal{K})}(\mathbb{D})$ and $G^*G = F^*F$ a.e. on $\mathbb{T}$ implies $\sum_{i=0}^{k} G_i^* G_i \leq \sum_{i=0}^{k} F_i^* F_i$, where G_i are the Taylor coefficients of G.*

(vii) *For some* $k \in \mathbb{N}_0$ *we have that* $G \in H^\infty_{\mathbf{L}(\mathcal{L},\mathcal{K})}(\mathbb{D})$ *and* $G^*G = F^*F$ *a.e. on* $\mathbb{T}$ *implies*

$$\begin{pmatrix} F_0^* & \cdots & F_k^* \\ & \ddots & \vdots \\ & & F_0^* \end{pmatrix} \begin{pmatrix} F_0 & & \\ \vdots & \ddots & \\ F_k & \cdots & F_0 \end{pmatrix} \geq \begin{pmatrix} G_0^* & \cdots & G_k^* \\ & \ddots & \vdots \\ & & G_0^* \end{pmatrix} \begin{pmatrix} G_0 & & \\ \vdots & \ddots & \\ G_k & \cdots & G_0 \end{pmatrix}, \tag{3.13}$$

where G_i *are the Taylor coefficients of* G.

(viii) *For all* $k \in \mathbb{N}_0$ *we have that* $G \in H^\infty_{\mathbf{L}(\mathcal{L},\mathcal{K})}(\mathbb{D})$ *and* $G^*G = F^*F$ *a.e. on* $\mathbb{T}$ *implies* (3.13).

Proof. For any $G \in H^\infty_{\mathbf{L}(\mathcal{L},\mathcal{K})}(\mathbb{D})$ we have that

$$S(T_G^*T_G; k) \geq \begin{pmatrix} G_0^* & \cdots & G_k^* \\ & \ddots & \vdots \\ & & G_0^* \end{pmatrix} \begin{pmatrix} G_0 & & \\ \vdots & \ddots & \\ G_k & \cdots & G_0 \end{pmatrix}. \tag{3.14}$$

Combining this observation with Theorem 3.3 (v) and the fact that $T_F^*T_F = T_G^*T_G$, we immediately obtain the implication (i) $\Rightarrow$ (viii).

The implications (viii) $\Rightarrow$ (iv) $\Rightarrow$ (vii) $\Rightarrow$ (v) $\Rightarrow$ (iv), (viii) $\Rightarrow$ (vi) $\Rightarrow$ (v) and (ii) $\Rightarrow$ (iv) $\Rightarrow$ (iii) are trivial.

For (iv) $\Rightarrow$ (i), let $F = V\tilde{F}$ be an inner-outer factorization of F. Then, by Theorem 3.3 we have that

$$F_0^*F_0 \leq S(T_F^*T_F; 0) = S(T_{\tilde{F}}^*T_{\tilde{F}}; 0) = \tilde{F}_0^*\tilde{F}_0.$$

On the other hand, by (iv) $\tilde{F}_0^*\tilde{F}_0 \leq F_0^*F_0$, and thus equality $F_0^*F_0 = S(T_F^*T_F; 0)$ holds. Again applying Theorem 3.3 gives that F is outer.

For the implication (i) $\Rightarrow$ (ii) fix $z \in \mathbb{D}$ and introduce the Blaschke factor $b_z(w) = -\frac{w-z}{1-\bar{z}w}$, $w \in \mathbb{D}$. Then F is outer if and only if $F \circ b_z$ is (use that the composition operators $g \to g \circ b_z$ and $g \to g \circ b_z^{-1}$ are bounded operators on $H^2_{\mathcal{H}}(\mathbb{D})$ [7, Theorem 3.6]). Moreover $F(w)^*F(w) = G(w)^*G(w)$ a.e. on $\mathbb{T}$ if and only if $F(b_z(w))^*F(b_z(w)) = G(b_z(w))^*G(b_z(w))$ a.e. on $\mathbb{T}$. Since $F \circ b_z$ is outer, by (i) $\Rightarrow$ (iv), we have that

$$F(b_z(0))^*F(b_z(0)) \geq \tilde{G}_0^*\tilde{G}_0$$

for any $\tilde{G}$ such that $\tilde{G}^*\tilde{G} = (F \circ b_z)^*(F \circ b_z)$ a.e. on $\mathbb{T}$. Now putting $G = \tilde{G} \circ b_z^{-1}$ gives that

$$F(z)^*F(z) \geq G(z)^*G(z)$$

for any G with $G^*G = F^*F$ a.e. on $\mathbb{T}$. Since $z \in \mathbb{D}$ was arbitrary, the result follows.

As (iv) $\Rightarrow$ (i) holds, it follows that if F satisfies (iii), then $F \circ b_{z_0}$ is outer. But then F is outer as well. This proves (iii) $\Rightarrow$ (i). $\square$

4. Multivariate outer polynomials

With the ideas from the previous section we now present a multivariate operator-valued version of the Fejér-Riesz lemma. As mere positive semidefiniteness on the d-torus does not suffice, an additional condition is required for Q to allow an "outer" factorization. This additional condition on Q is given in terms of Schur complements of T_Q, the Toeplitz operator on $H^2_{\mathcal{H}}(\mathbb{D}^d)$ with symbol Q.

In order to state the result precisely we need some additional notation. For $z = (z_1, \ldots, z_d) \in \mathbb{T}^d$ and $k = (k_1, \ldots, k_d) \in \mathbb{Z}^d$ define $z^k := z_1^{k_1} \cdots z_d^{k_d}$. In this case $z^{*k} = \overline{z}^k = z^{-k}$. We write 0 for $(0, \ldots, 0)$. For set $A, B \subseteq \mathbb{Z}^d$ we denote $A - B = \{a - b \,:\, a \in A, b \in B\}$. For matrices labeled by elements of $\mathbb{Z}^d$ we fix the ordering as lexicographical. Since this is a total ordering, various results from the first section on Schur complements readily translate to this setting. As before, we use the notation $S(T_Q; \Lambda)$ (or simply $S(\Lambda)$ when no confusion is likely) to indicate a Schur complement of T_Q supported in rows and columns $\Lambda \subseteq \mathbb{N}_0^d$. In the same manner as when we labeled matrices using elements of $\mathbb{N}_0$, we sometimes pad Schur complements with zeros. In this way for example, if $\Lambda_2 \subseteq \Lambda_1$, then $S(\Lambda_1) - S(\Lambda_2)$ makes sense. Finally, we need the projections Π_K, $K \subseteq \mathbb{N}_0^d$, on $H^2_{\mathcal{M}}(\mathbb{D}^d)$ defined by

$$\Pi_K \left(\sum_{k \in \mathbb{N}_0^d} h_k z^k \right) = \sum_{k \in K} h_k z^k.$$

Theorem 4.1. *Let $K = \prod_{i=1}^d \{0, \ldots, n_i\}$ and let $Q_k : \mathcal{H} \to \mathcal{H}$, $k \in K - K$, be Hilbert space operators so that $Q(z) := \sum_{k \in K-K} Q_k z^k \geq 0, z \in \mathbb{T}^d$. Furthermore, let $n = (n_1, \ldots, n_d)$ and let Z_K be the column matrix $(z^{n-k})_{k \in K}$. The following are equivalent:*

(i) *there exists an operator polynomial $P(z) = \sum_{k \in K} P_k z^k$ with $P_j \in \mathbf{L}(\mathcal{H})$, $j \in K$, so that $Q(z) = P(z)^* P(z), z \in \mathbb{T}^d$, and*

$$ran\,(\Pi_{\mathbb{N}_0^d \setminus K}\, T_P\, \Pi_{\{n\}}) \subseteq \overline{ran}\,(\Pi_{\mathbb{N}_0^d \setminus K}\, T_P\, \Pi_{\mathbb{N}_0^d \setminus K}), \tag{4.1}$$

and

$$ran\, P_k \subseteq \overline{ran}\, P_0, \qquad k \in K. \tag{4.2}$$

(ii) *The operator*

$$Y := S(K) - S(K \setminus \{n\})$$

satisfies

$$Z_K^* Y Z_K = Q(z), \qquad z \in \mathbb{T}^d. \tag{4.3}$$

Proof. Suppose (ii) holds. By Lemma 2.2 there exist $P_k \in \mathbf{L}(\mathcal{H})$, $k \in K$ such that with $P_K = \mathrm{row}(P_k)_{k \in K}$, $Y = P_K^* P_K$. Defining $P(z) = P_K Z_K = \sum_{k \in K} P_k z^k$, we obtain from (4.3) that $P(z)^* P(z) = Q(z), z \in \mathbb{T}^d$. But then $T_Q = T_P^* T_P$. View this factorization of T_Q with respect to the decomposition

$$\mathrm{ran}\, \Pi_{K \setminus \{n\}} \oplus \mathrm{ran}\, \Pi_{\{n\}} \oplus \mathrm{ran}\, \Pi_{\mathbb{N}_0 \setminus K}, \tag{4.4}$$

in which respect T_P is a 3×3 lower triangular operator matrix. We are now exactly in the situation of Lemma 2.2 with $T^* = \Pi_{\mathbb{N}_0^d \setminus K}\, T_P\, \Pi_{\{n\}}, U^* = \Pi_{\mathbb{N}_0^d \setminus K}\, T_P\, \Pi_{\mathbb{N}_0^d \setminus K}, Q^* = \Pi_{\{n\}}\, T_P\, \Pi_{K \setminus \{n\}}$, and $S^* = \Pi_{\{n\}}\, T_P\, \Pi_{\{n\}} = P_0$. Since (2.5) in Lemma 2.2 holds, we obtain (2.6) of Lemma 2.2, which directly translates into the conditions in (i).

For the converse, assume (i). Again, consider the factorization $T_Q = T_P^* T_P$ with T_P a lower triangular 3×3 matrix with respect to the decomposition in (4.4). By the equivalence of (2.5) and (2.6) in Lemma 2.2, we have $Y = P_K^* P_K$, where $P_K = \mathrm{row}(P_k)_{k \in K}$. Set $P(z) = P_K Z_K = \sum_{k \in K} P_k z^k$. Then $Q(z) = P(z)^* P(z), z \in \mathbb{T}^d$. ☐

The notion of "outerness" of the factor P is given above in equations (4.1) and (4.2). These conditions reduce in the one-variable case to condition (iii) in Theorem 3.3. Clearly, there are many other, perhaps more natural, ways of generalizing the notion of outerness to the multivariable case (see, for example, [4]). For instance, the condition $\overline{\text{ran}}\, T_P = H^2_{\mathcal{M}}(\mathbb{D}^d)$ or the condition that

$$P(z)^*P(z) \geq L(z)^*L(z), z \in \mathbb{D}^d,$$

for all $L(z)$ with $P(z)^*P(z) = L(z)^*L(z), z \in \mathbb{T}^d$, are both options. How all these different notions relate to one another remains to be investigated in future work.

Recall from [10] the following result regarding stable factorization (factorizations in terms of polynomials void of zeros in $\overline{\mathbb{D}}^2$) of a strictly positive scalar-valued trigonometric polynomial.

Theorem 4.2 ([10])**.** *Let $K = \{0, \ldots, n_1\} \times \{0, \ldots, n_2\}$ and let $Q_k : \mathcal{H} \to \mathcal{H}$, $k \in K - K$, be scalar valued so that $Q(z) := \sum_{k \in K-K} Q_k z^k > 0, z \in \mathbb{T}^2$. Then there exists a scalar-valued polynomial $P(z) = \sum_{k \in K} P_k z^k$ so that $Q(z) = |P(z)|^2$, $z \in \mathbb{T}^2$, and $P(z) \neq 0$, $z \in \overline{\mathbb{D}}^2$, if and only if*

$$(\Pi_{K\setminus\{(n_1,n_2)\}} T_{Q^{-1}} \Pi_{K\setminus\{(n_1,n_2)\}})^{-1}$$

has zero entries in locations (k,l) where $k \in \{1, \ldots, n_1\} \times \{0\}$ and $l \in \{0\} \times \{1, \ldots, n_2\}$.

The conditions in Theorem 4.1 and 4.2 are quite different. The following theorem, which gives necessary conditions on the Schur complement in the form of the existence of a decomposition, relates better to Theorem 4.2, as the condition on the Schur complement implies the necessity of some entries in the Schur complement being zero.

Theorem 4.3. *Let $K = \{0, \ldots, n_1\} \times \{0, \ldots, n_2\}$ and let $Q_k : \mathcal{H} \to \mathcal{H}$, $k \in K - K$, be Hilbert space operators so that $Q(z) := \sum_{k \in K-K} Q_k z^k \geq 0, z \in \mathbb{T}^2$. Put*

$$\begin{aligned} S_1 &= S(T_Q; \{0, \ldots, n_1 - 1\} \times \{0, \ldots, n_2\}), \\ S_2 &= S(T_Q; \{0, \ldots, n_1\} \times \{0, \ldots, n_2 - 1\}), \\ S_0 &= S(T_Q; \{0, \ldots, n_1 - 1\} \times \{0, \ldots, n_2 - 1\}). \end{aligned}$$

Suppose that $Q(z) = P(z)^*P(z)$, $z \in \mathbb{T}^2$, where $P(z) = \sum_{k \in K} P_k z^k$, $P_k : \mathcal{H} \to \mathcal{H}$, satisfies

$$\text{ran}\Pi_{\mathbb{N}_0^2\setminus\widetilde{K}} T_P \Pi_{\widetilde{K}} \subset \overline{\text{ran}}\Pi_{\mathbb{N}_0^2\setminus\widetilde{K}} T_P \Pi_{\mathbb{N}_0^2\setminus\widetilde{K}}, \tag{4.5}$$

for $\widetilde{K} = \{0, \ldots, n_1 - 1\} \times \{0, \ldots, n_2 - 1\}, \{0, \ldots, n_1 - 1\} \times \{0, \ldots, n_2\}, \{0, \ldots, n_1\} \times \{0, \ldots, n_2 - 1\}, K \setminus \{(n_1, n_2)\}$ and K. Then

$$S(T_Q; K \setminus \{(n_1, n_2)\}) = S_1 + S_2 - S_0 \tag{4.6}$$

and

$$S(T_Q; K) = T_Q - T_1(T_Q - S_1)T_1^* - T_2(T_Q - S_2)T_2^* + T_1T_2(T_Q - S_0)T_2^*T_1^*, \tag{4.7}$$

where T_i is the Toeplitz operator corresponding to the multiplication operator $M_i : H^2_{\mathcal{H}}(\mathbb{D}^2) \to H^2_{\mathcal{H}}(\mathbb{D}^2)$ with symbol m_i, where $m_i(z) = z_i$, $i = 1, 2$.

Conversely, suppose that (4.6) and (4.7) hold. Then there exists an operator-valued polynomial $P(z) = \sum_{k \in K} P_k z^k : \mathcal{H} \to \mathcal{H}$ so that $Q(z) = P(z)^*P(z)$, $z \in \mathbb{T}^2$ and (4.1) and (4.2) hold.

Proof. Since $Q(z) = P(z)^*P(z)$, $z \in \mathbb{T}^2$, we have that $T_Q = T_P^*T_P$. Let $\widetilde{K} = \{0,\dots,p_1\} \times \{0,\dots,p_2\}$ with $p_i \in \{n_i, n_i - 1\}$, $i = 1,2$, or $\widetilde{K} = K \setminus \{(n_1,n_2)\}$, and view the equation $T_Q = T_P^*T_P$ with respect to the decomposition

$$\mathrm{ran}\Pi_{\widetilde{K}} \oplus \mathrm{ran}\Pi_{\mathbb{N}_0^2\setminus\widetilde{K}}.$$

Since (4.5) holds true we have by Lemma 2.1 that

$$S(T_Q;\widetilde{K}) = \Pi_{\widetilde{K}}T_P^*\Pi_{\widetilde{K}}T_P\Pi_{\widetilde{K}}.$$

This now yields expressions for all operators in (4.6) and (4.7) in terms of P. It is now straightforward to check that (4.6) and (4.7) hold. For illustration purposes let us write out the equalities in the operators in case that $n_1 = n_2 = 1$: here we have that $S_0 = L_0^*L_0$, $S_1 = L_1^*L_1$, $S_2 = L_2^*L_2$, $S(T_Q; K \setminus \{(1,1)\}) = L_3^*L_3$, $S(T_Q;K) = L_4^*L_4$, where

$$L_0 = \begin{pmatrix} P_{00} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad L_1 = \begin{pmatrix} P_{00} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ P_{10} & 0 & P_{00} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$L_2 = \begin{pmatrix} P_{00} & 0 & 0 & 0 \\ P_{01} & P_{00} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad L_3 = \begin{pmatrix} P_{00} & 0 & 0 & 0 \\ P_{01} & P_{00} & 0 & 0 \\ P_{10} & 0 & P_{00} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$L_4 = \begin{pmatrix} P_0 & 0 & 0 & 0 \\ P_{01} & P_{00} & 0 & 0 \\ P_{10} & 0 & P_{00} & 0 \\ P_{11} & P_{10} & P_{01} & P_{00} \end{pmatrix},$$

and

$$Y_0 := T_1T_QT_1^* - T_2T_QT_2^* + T_1T_2T_QT_2^*T_1^* = \begin{pmatrix} Q_{00} & Q_{01}^* & Q_{10}^* & Q_{11}^* \\ Q_{01} & 0 & Q_{1,-1}^* & 0 \\ Q_{10} & Q_{1,-1} & 0 & 0 \\ Q_{11} & 0 & 0 & 0 \end{pmatrix}, \tag{4.8}$$

the operators being restricted to rows and columns indexed by $\{(0,0),(0,1),(1,0),(1,1)\}$, as these contain all the nonzero entries. The operators T_1 and T_2 restricted to this part correspond to

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 & 0 & 0 \\ I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 \end{pmatrix},$$

respectively. Formulas (4.6) and (4.7) now follow directly. The computations for the case $n_1n_2 > 1$ are similar.

For the converse we apply Theorem 4.1. Using (4.6) and (4.7) we find that Y in Theorem 4.1 equals

$$Y = Y_0 - (S_1 - T_1S_1T_1^*) - (S_2 - T_2S_2T_2^*) + (S_0 - T_1T_2S_0T_2^*T_1^*)$$

yielding that

$$Z_K^* Y Z_K = Q(z) - (1 - |z_1|^2)(Z_K^* S_1 Z_K) - (1 - |z_2|^*)(Z_K^* S_2 Z_K) + (1 - |z_1 z_2|^2)(Z_K^* S_0 Z_K).$$

Thus for $(z_1, z_2) \in \mathbb{T}^2$ we obtain equality (4.3). The conclusion now follows from Theorem 4.1. $\square$

Note that Theorem 4.3 is not an if and only if statement due to the different "outerness" requirements on P: in one direction the outerness requirement is (4.5) while in the other direction it is (4.1) and (4.2). We suspect that these two outerness requirements are different, though we have not constructed an example showing this.

Note too that (4.6) implies that $S(T_Q; K \setminus \{(n_1, n_2)\})$ has zeros in locations (k, l), where $k \in \{1, \ldots, n_1\} \times \{0\}$ and $l \in \{0\} \times \{1, \ldots, n_2\}$.

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