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ON THE ABSOLUTELY CONTINUOUS SPECTRUM OF ONE-DIMENSIONAL QUASI-PERIODIC SCHRÖDINGER OPERATORS IN THE ADIABATIC LIMIT

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Abstract. In this paper we study the spectral properties of families of quasi-periodic Schrödinger operators on the real line in the adiabatic limit in the case when the adiabatic iso-energetic curves are extended along the position direction. We prove that, in energy intervals where this is the case, most of the spectrum is purely absolutely continuous in the adiabatic limit, and that the associated generalized eigenfunctions are Bloch-Floquet solutions.

Résumé. Cet article est consacré à l'étude du spectre de certaines familles d'équations de Schrödinger quasi-périodiques sur l'axe réel lorsque les variétés iso-énergetiques adiabatiques sont étendues dans la direction des positions. Nous démontrons que, dans un intervalle d'énergie où ceci est le cas, le spectre est dans sa majeure partie purement absolument continu et que les fonctions propres généralisées correspondantes sont des fonctions de Bloch-Floquet.

0. Introduction

In this paper, we analyze the spectrum of the ergodic family of Schrödinger equations

$$H_{z,\varepsilon}\psi = -\frac{d^2}{dx^2}\psi(x) + (V(x-z) + W(\varepsilon x))\psi(x) = E\psi(x), \quad x \in \mathbb{R}, \tag{0.1}$$

where $V(x)$ and $W(\xi)$ are periodic, and ε is chosen so that the potential $V(\cdot - z) + W(\varepsilon\cdot)$ is quasi-periodic. Our aim is to study the spectral properties of $H_{z,\varepsilon}$ in the limit as $\varepsilon \to 0$. In the paper [11], we have studied this operator near the bottom of the spectrum when W is the cosine. In the present paper, we consider a rather general W but in a different energy region. We are interested in the spectrum situated in the "middle" of a spectral band of the "unperturbed" periodic operator

$$H_0\psi(x) = -\psi''(x) + V(x)\psi(x). \tag{0.2}$$

More precisely, we assume that the set $E - W(\mathbb{R})$ lies inside a spectral band of (0.2); see Figure 1, left part. Such energy regions always exist as the length of the bands of the spectrum of H_0 increases with energy (see e.g. [8, 13]). We roughly prove that,

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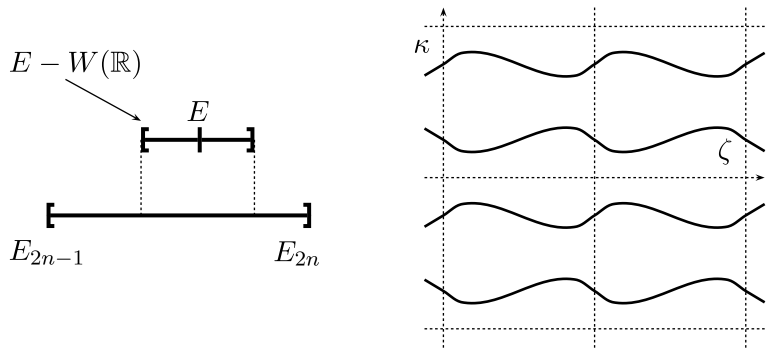

FIGURE 1. The band center

for ε sufficiently small, in the corresponding energy region, most of the spectrum of $H_{z,\varepsilon}$ is absolutely continuous, and that the corresponding generalized eigenfunctions are Bloch-Floquet solutions of the same functional structure as in [7, 9].

In [9], Eliasson has considered one-dimensional Schrödinger operators with (analytic) almost periodic potentials. He has proved that, in the high energy region, the spectrum is absolutely continuous, and also that, if the almost periodic potential is small, all the spectrum is absolutely continuous. The novelty here is that we do not need the energy to be large or $V + W$ to be small. Our result holds in the "middle" of any spectral band of H_0 as soon as the amplitude of W is smaller than the width of the band and ε is sufficiently small. In our case, V only needs to be square integrable. To the best of our knowledge, this and our previous paper [11] contain the first results on the nature of the spectrum of quasi-periodic Schrödinger operators with potentials of low regularity.

For small ε, our results give a qualitative criterion for the existence of absolutely continuous spectrum. Our assumption on the relative position of the spectral bands of (0.2) and of the interval $E-W(\mathbb{R})$ admits a geometric interpretation. It is known (see, for example, [11, 3]) that for adiabatic problems of the kind studied in the present paper, the "effective" Hamiltonian obtained by taking $\mathcal{E}(\kappa)$, the dispersion relation of the periodic Schrödinger operator H_0, as kinetic energy and the potential $W(\zeta)$ as potential energy, $H(\kappa,\zeta) = \mathcal{E}(\kappa) + W(\zeta)$, plays an important part. Semi-classical "wisdom" says that one should consider the iso-energy curves $\mathcal{E}(\kappa)+W(\zeta) = E$. Our assumption means that these curves are extended along the position axis; see Figure 1, right part. From the quantum physicist's point of view, existence of such iso-energy curves corresponds to extended states: a semi-classical particle should "live" near these curves, hence be "extended" in the position variable; see [16]. We justify these heuristics.

1. THE RESULTS

1.1. **The assumptions.** We assume that

- V and W are periodic,

$$V(x+1) = V(x), \quad W(x+2\pi) = W(x), \quad x \in \mathbb{R}; \tag{1.1}$$

- ε is fixed positive number;
- V is real valued and locally square integrable;

- W is real on $\mathbb{R}$ and analytic in a neighborhood of $\mathbb{R}$;
- z is a real parameter indexing the equations of the family.

1.2. **Periodic Schrödinger operator.** To formulate our results, we need to recall some information about the periodic Schrödinger operator (0.2). Its spectrum on $L^2(\mathbb{R})$ is absolutely continuous and consists of *spectral bands*, i.e. intervals $[E_1, E_2]$, $[E_3, E_4]$, ..., $[E_{2n+1}, E_{2n+2}]$, ..., of the real axis such that

$$E_1 < E_2 \le E_3 < E_4 \le \cdots < E_{2n} \le E_{2n+1} < E_{2n+2} \le \dots, \tag{1.2}$$

$$E_n \to +\infty, \quad n \to +\infty. \tag{1.3}$$

The open intervals (E_2, E_3), (E_4, E_5), ..., (E_{2n}, E_{2n+1}), ..., are called the *spectral gaps*. The ends of the bands are eigenvalues of the Schrödinger equation (0.2) with either the periodic or the anti-periodic boundary conditions at the ends of the interval $(0,1)$. Some gaps can be closed (empty). In this case, connected components of the spectrum are unions of spectral bands with common ends.

1.3. **The spectral result.** Let $W_+ = \max_{x\in\mathbb{R}} W(x)$ and $W_- = \min_{x\in\mathbb{R}} W(x)$. For an energy E, consider the "window" $\mathcal{W}(E) = [E - W_+, E - W_-]$. Let $[E_{2n_0-1}, E_{2n_0}]$ be one of the spectral bands of the periodic operator H_0. We analyze the spectrum of the family of equations (0.1) in the middle of this band, i.e. we pick $J \subset \mathbb{R}$, a compact interval, such that

(**A**) $\mathcal{W}(E) \subset]E_{2n_0-1}, E_{2n_0}[$ for all $E \in J$.

Such an interval J exists if $W_+ - W_-$, the "size" of the adiabatic perturbation, is smaller than the size of the spectral band $[E_{2n_0-1}, E_{2n_0}]$.

Our main result is

Theorem 1.1. *Let J be a nonempty closed interval satisfying hypothesis* (A). *Fix $0 < \sigma < 1$. Then, there exists $S > 0$ and $\mathcal{D} \subset (0,1)$, a set of Diophantine numbers, such that*

- *one has*

$$\frac{\text{mes}\,(\mathcal{D} \cap (0,\varepsilon))}{\varepsilon} = 1 + o\left(\varepsilon\lambda^{\sigma}\right) \quad \varepsilon \to +0 \text{ where } \lambda = \exp\left(-\frac{S}{\varepsilon}\right); \tag{1.4}$$

- *for any $\varepsilon \in \mathcal{D}$ sufficiently small, there exists a Borel set $B \subset J$ of small measure*

$$\frac{\text{mes}\,(B)}{\text{mes}\,(J)} = O(\lambda^{\sigma/2}),$$

  such that $J \backslash B$ belongs to the absolutely continuous spectrum of the equation family (0.1)*;*
- *for all $E \in J \setminus B$, there exist two linearly independent Bloch-Floquet solutions $\psi_\pm(x,E)$ of* (0.1) *satisfying*

$$\psi_\pm(x) = e^{\pm ip(E)x} P_\pm(x - z, \varepsilon x, E), \tag{1.5}$$

  where $p(E)$ is a monotonously increasing, Lipschitz continuous function of E, the functions $P_\pm(x,\zeta,E)$ differ by the complex conjugation, $P_- = \overline{P_+}$, and the function P_+ is 1-periodic in x and 2π-periodic in ζ. This function belongs to H^2_{loc} in x and is analytic in ζ in an ε-independent neighborhood of the real line. Moreover, P_+ is a Lipschitz continuous function of E.

1.4. **Comments.** The coefficient λ is exponentially small in ε as $\varepsilon \to 0$. Under our assumptions, the adiabatic perturbation W being quite general, we cannot give its optimal value. For more details, see the remark following Theorem 1.4.

The Bloch-Floquet solutions $\psi_\pm$ described in Theorem 1.1 have the same functional structure as the Bloch-Floquet solutions constructed in [7, 9] for small almost periodic potentials or high energies. The regularity of the solutions

$$e^{\pm ip(E)x}\, P_\pm(x-z, \varepsilon x, E)$$

in the "slow" variable εx is determined by the function W, and, in the "fast" variable $x - z$ by the function V.

Note that the existence of such Bloch-Floquet solutions corresponds to the reducibility of the underlying cocyle (see [1]); in the present paper, this reducibility is proved with minimal assumptions on the regularity of the coefficients of the cocycle.

In a previous paper [11], we have studied equation (0.1) in the case when the potential W is the cosine. In this case too, we have found that some parts of the spectrum are absolutely continuous. Although it was not stated there, the results developed in section 7 of the present paper show that, in [11], the generalized eigenfunctions associated to the absolutely continuous spectrum have the structure described in Theorem 1.1.

Theorem 1.1 follows from the analysis of a finite difference equation, the monodromy equation. The reduction to the monodromy equation is independent of ε. In the adiabatic limit, the monodromy equation takes a simple model form. The condition "$\varepsilon \in \mathcal{D}$" only appears in the analysis of the monodromy equation. To analyze this equation, we use a rather simple but robust version of the KAM theory; the condition "$\varepsilon \in \mathcal{D}$" is used to control the small denominators that appear in the process. One could presumably use a more sophisticated method (along the lines of [9]) to control the small denominators, but this would require much more additional work (as the monodromy does not depend on energy in a simple way) and is outside the scope of the present paper.

1.5. **The monodromy matrix.** First, following section 2.1 of [11], we recall the definition of *the monodromy matrix* and introduce *the monodromy equation*. Then, we describe the asymptotics of a monodromy matrix for (0.1) in the adiabatic case, and, finally, we use this information to explain the heuristics guiding the proof of Theorem 1.1.

1.5.1. *Definition of the monodromy matrix.* Consider $(\psi_{1,2})$, a *consistent basis*, i.e. a basis of solutions of (0.1) whose Wronskian is independent of z and that are 1-periodic in z,

$$\psi_{1,2}(x, z+1) = \psi_{1,2}(x, z), \quad \forall x, z. \tag{1.6}$$

The functions $\psi_{1,2}(x + 2\pi/\varepsilon, z + 2\pi/\varepsilon)$ being solutions of equation (0.1), one can write

$$\Psi(x + 2\pi/\varepsilon, z + 2\pi/\varepsilon) = M(E, z)\Psi(x, z), \quad \Psi = \begin{pmatrix} \psi_1(x,z) \\ \psi_2(x,z) \end{pmatrix}, \tag{1.7}$$

where $M(E, z)$ is a 2×2 matrix with coefficients independent of x. This matrix is called *the monodromy matrix* corresponding to the consistent basis (ψ_1, ψ_2). Let us mention two elementary properties of the monodromy matrix

$$\det M(E, z) \equiv 1, \quad M(E, z+1) = M(E, z), \quad \forall z. \tag{1.8}$$

1.5.2. *The monodromy equation.* Set $h \equiv 2\pi/\varepsilon \mod 1$. Let M be a monodromy matrix corresponding to a consistent basis $(\psi_{1,2})$. Consider the equation

$$F(n+1) = M(E, z+nh)F(n), \quad \forall n \in \mathbb{Z}, \tag{1.9}$$

where F is a function $F : \mathbb{Z} \to \mathbb{C}^2$. Equation (1.9) is the *monodromy equation.*

The main feature of the monodromy equation is that the behavior of its solutions for $n \to \pm\infty$ mimics the behavior of solutions of the input equation (0.1) for $x \to \mp\infty$; see Theorem 7.1.

Moreover, the Bloch-Floquet solutions to the monodromy equation (1.9) are related to the Bloch-Floquet solutions of (0.1); see Theorem 7.2. Here, we state only one of the results given in section 7. Let $\Theta(E, z)$ and $\theta(E, z)$ be the Lyapunov exponents for the equations (0.1) and (1.9), respectively. One has

Theorem 1.2 ([11], Corollary 2.1)**.** *Assume that ψ_1, ψ_2, $d\psi_1/dx$ and $d\psi_2/dx$ are locally bounded in x and z. Then, the Lyapunov exponents $\Theta(E, z)$ and $\theta(E, z)$ satisfy the relation*

$$\Theta(E, z) = \frac{\varepsilon}{2\pi}\,\theta(E, z). \tag{1.10}$$

Remark 1.3. The above definition of the monodromy matrix applies to any quasi-periodic equation with two frequencies. It generalizes the definitions of the monodromy matrix for the one-dimensional periodic differential equations. The passage to the monodromy equation is close to the monodromization idea developed in [6] for difference equations with periodic coefficients. For a detailed discussion, we refer to section 2.1 in [11].

1.6. **Asymptotics of a monodromy matrix.** To describe the asymptotics of a monodromy matrix, we return to the periodic Schrödinger operator (0.2). Consider the Bloch quasi-momentum associated to this operator; see section 2. It is an analytic multi-valued function of the spectral parameter. Its branch points coincide with the ends of the connected components of the spectrum of the periodic operator (0.2). There exists a branch of the Bloch quasi-momentum that conformally maps the upper half complex plane onto the first quadrant cut along finite vertical segments beginning at the points $k = \pi n$, $n = 1, 2, 3 \ldots$. In particular, it maps (E_{2n-1}, E_{2n}), the n-th spectral band of the periodic Schrödinger operator, onto the interval $(\pi(n-1), \pi n)$; and, on the n-th spectral gap, one has $\operatorname{Im} \kappa_p > 0$ and $\operatorname{Re} \kappa_p = \pi n$. We denote this branch by k_p. Now, we define *the phase integral* Φ by

$$\Phi(E) = \int_0^{2\pi} \kappa_{n_0}(\zeta) d\zeta, \tag{1.11}$$

where the function κ_{n_0} is given by

$$\kappa_{n_0}(\varphi) = \begin{cases} k_p(E - W(\varphi)) - \pi(n_0 - 1), & \text{if } n_0 \text{ is odd;} \\ \pi n_0 - k_p(E - W(\varphi)), & \text{if } n_0 \text{ is even.} \end{cases}$$

Under the condition (A), the function Φ is analytic in a neighborhood of J and real-valued for $E \in J$. Moreover, in section 8.3.2, we check that the derivative of Φ does not vanish on J.

As the potential in (0.1) is real, it is possible to construct a monodromy matrix of the form

$$M(z,E) = \begin{pmatrix} a(z,E) & b(z,E) \\ \overline{b(\overline{z},\overline{E})} & \overline{a(\overline{z},\overline{E})} \end{pmatrix}. \tag{1.12}$$

We prove

Theorem 1.4. *Assume the interval J satisfies assumption* (A) *and fix $E_0 \in J$. There exist S_0, a positive number, and V_0, a neighborhood of E_0, such that for ε sufficiently small, in the space of solutions of* (0.1) *there exists a consistent basis for which the monodromy matrix has the following properties:*

- *it is analytic in $(E,z) \in V_0 \times \{|\mathrm{Im}\, z| \le y/\varepsilon\}$ for some $y > 0$ independent of ε;*
- *it is of the form* (1.12)*;*
- *if we represent the coefficient a of the monodromy matrix as*

$$a = a_0 + a_1(z), \quad a_0 = \int_0^1 a(z)dz,$$

then, for $\varepsilon \to 0$, one has

$$\begin{aligned} &a_0 = \exp(i\Phi(E)/\varepsilon + i\Omega(E) + o(1)), \\ &a_1 = o\left(a_0\, e^{-\frac{S_0 - 2\pi\varepsilon|\mathrm{Im}\, z|}{\varepsilon}}\right), \quad |\mathrm{Im}\, z| < y/\varepsilon, \end{aligned} \tag{1.13}$$

where $E \mapsto \Omega(E)$ is a real analytic function in V_0;

- *the coefficient b of the monodromy matrix satisfies the estimate*

$$b = o\left(e^{-\frac{S_0 - 2\pi\varepsilon|\mathrm{Im}\, z|}{\varepsilon}}\right), \quad |\mathrm{Im}\, z| < y/\varepsilon; \tag{1.14}$$

- *the estimates for a and b are uniform in $(E,z) \in V_0 \times \{|\mathrm{Im}\, z| \le y/\varepsilon\}$.*

Theorem 1.4 is proved using the complex WKB method for adiabatic perturbations of periodic Schrödinger equations developed in [10].

Under our assumptions, the adiabatic perturbation W being quite general, we cannot say more about a_1. If one makes more restrictive assumptions, it is possible to get an asymptotic for a_1. The leading term of a_1 is given by $\hat{a}_1(-1)e^{-2i\pi z} + \hat{a}_1(1)e^{2i\pi z}$, the sum of the first two terms of the Fourier series of a_1. The modulus of the Fourier coefficients are exponentially small in ε and can be interpreted as tunneling coefficients measuring the complex tunneling between distinct branches of the iso-energy curve corresponding to Figure 1. This will be analyzed in a subsequent paper.

Note that, as the constant S in Theorem 1.1, one can take any fixed positive number smaller than the constant S_0 in Theorem 1.4. The difference $S_0 - S$ determines the size of the neighborhood of the real line where the functions $P_\pm(x,\zeta,E)$ (defined in Theorem 1.1) are analytic in ζ.

1.6.1. *The monodromy matrix asymptotics and the spectral properties of* (0.1). Let us explain how Theorem 1.4 is used to derive Theorem 1.1. By Theorem 1.4, the monodromy matrix is the form

$$M = \begin{pmatrix} U & 0 \\ 0 & U^* \end{pmatrix} + O(\lambda), \quad U^* = \overline{U(\bar{E})},$$

where U is independent of z,

$$U = e^{\frac{i}{\varepsilon}\Phi(E)+i\Omega(E)+o(1)}, \quad \lambda = e^{-S/\varepsilon}, \quad |\mathrm{Im}\, z| \leq (S_0 - S)/\varepsilon.$$

As we shall see later, the relation $\det M \equiv 1$ implies that $|U| = 1+o(\lambda)$ for $E \in J$. So, up to error terms of order $O(\lambda)$, M is a constant diagonal matrix with diagonal elements of absolute value 1. Now, consider the monodromy equation with the monodromy matrix M. If the error terms could be omitted, one would immediately obtain that, for all $E \in J$, there are bounded solutions of the monodromy equation. To take care of the exponentially small error terms, we apply standard ideas of the spectral KAM theory: we use a simple version (prepared in section 11 in [11]) and construct bounded solutions of the monodromy equation for E outside a Borel set B. B is a countable union of intervals of small total measure; these intervals contain KAM resonances that can be roughly characterized by the "quantization condition"

$$\frac{1}{\varepsilon}\Phi(E) + \Omega(E) = \pi(k \cdot h + l), \quad k, l \in \mathbb{Z}.$$

Having constructed bounded solutions of the monodromy equation outside B, by Theorem 1.2, we conclude that the Lyapunov exponent of the equation family (0.1) is zero on $J \setminus B$. By the Ishii-Pastur-Kotani Theorem [14], this implies that the essential closure of the set $J \setminus B$ belongs to the absolutely continuous spectrum of (0.1). Finally, Theorem 7.2 allows us to analyze the functional structure of the generalized eigenfunctions on $J \setminus B$.

In Theorem 1.1, we have only described the part of the spectrum outside a small set. As said above, this set is related to the KAM resonances for the monodromy equation. We believe that, adapting the techniques developed in [9], one can prove that, in this small set, the spectrum is purely absolutely continuous.

1.6.2. *Outline of the paper.* In section 2, we recall some information on the periodic Schrödinger operator (0.2). Sections 3, 4 and 5 are devoted to the complex WKB method for adiabatic perturbations of periodic Schrödinger operators. In section 6, we prove Theorem 1.4. In section 7, we study how the solutions to equation (0.1) relate to those of the monodromy equation (1.9). Section 8 is devoted to the proof of Theorem 1.1.

2. Periodic Schrödinger operators

In this section, we collect known information (see [15, 8, 12, 13]) about the periodic Schrödinger operator (0.2). We assume that V is a real-valued, 1-periodic, L^2_{loc}-function.

2.1. **Bloch solutions.** Let ψ be a solution of the equation

$$-\frac{d^2}{dx^2}\psi(x) + V(x) = E\psi(x), \quad x \in \mathbb{R}, \tag{2.1}$$

satisfying the relation $\psi(x+1) = \mu\,\psi(x)$, $\forall x \in \mathbb{R}$, with μ independent of x. It is a *Bloch* solution, and μ is the *Floquet multiplier* associated to ψ. Write $\mu = \exp(ik)$; then k is called the *Bloch quasi-momentum*. The Bloch solution ψ can be represented in the form $\psi(x) = e^{ikx}p(x)$, where $x \mapsto p(x)$ is a 1-periodic function.

The spectrum of the periodic Schrödinger operator (0.2) was described in section 1.2. Consider two copies of the complex plane $E \in \mathbb{C}$ cut along the spectral bands of the periodic Schrödinger operator. Paste them together into a Riemann surface Γ with square root branch points. One constructs a Bloch solution $\psi(x,E)$ of equation (2.1) meromorphic on this Riemann surface. It is normalized by the condition $\psi(1,E)\equiv 1$. The poles of this solution are located in the open spectral gaps or at their ends (the closure of each open spectral gap contains precisely one simple pole).

Outside the edges of the spectrum, the two branches $\psi_\pm$ of the Bloch solution are linearly independent solutions of the periodic equation (2.1). On the spectral bands, they differ only by complex conjugation.

2.2. **Bloch quasi-momentum.** The Floquet multiplier $\mu(E)$ associated to $\psi(x,E)$ is also analytic on Γ. The corresponding Bloch quasi-momentum is an analytic multi-valued function of E and has the same branch points as $\psi(x,E)$.

Let D be a simply-connected domain containing no branch points of the Bloch quasi-momentum. On D, fix k_0, an analytic single-valued branch of k. All the other single-valued branches that are analytic in $E\in D$ are described by the formulae

$$k_{\pm,l}(E)=\pm k_0(E)+2\pi l,\quad l\in\mathbb{Z}. \tag{2.2}$$

Given a branch k, analytic on D, we fix $\psi_\pm(x,E)$, two branches of the Bloch solution $\psi(x,E)$, analytic on D so that $\pm k$ be their quasi-momenta. Then, one has

$$\int_0^1 \psi_+(t,E)\psi_-(t,E)dt=-ik'(E)w(E),\quad E\in D, \tag{2.3}$$

where $w(E)$ is the Wronskian of the solutions $\psi_\pm$, i.e. $w(E)=\frac{\partial\psi_+}{\partial x}\psi_- - \frac{\partial\psi_-}{\partial x}\psi_+$.

Consider $\mathbb{C}_+$, the upper half of the complex plane. There exists k_p, an analytic branch of the complex momentum that conformally maps $\mathbb{C}_+$ onto the quadrant $\{\operatorname{Im}k>0,\ \operatorname{Re}k>0\}$ cut along finite vertical slits beginning at the points πl, $l=1,2,3\dots$. The branch k_p is continuous on $\mathbb{C}_+\cup\mathbb{R}$. It is real and monotonically increasing along the spectrum; it maps the spectral band (E_{2n-1},E_{2n}) onto the interval $(\pi(n-1),\pi n)$.

2.3. **Analytic continuation through a connected component of the spectrum.** We denote by $\mathbb{C}_{n_0}$ the complex plane cut along the half-lines $(-\infty,E_{2n_0-1}]$ and $[E_{2n_0},+\infty)$, where E_{2n_0-1} and E_{2n_0} are the ends of the n_0-th spectral band $[E_{2n_0-1},E_{2n_0}]$.

The function k_p, being continuous and real on $[E_{2n_0-1},E_{2n_0}]$, can be analytically continued to a function analytic on $\mathbb{C}_{n_0}$ by the relation

$$k_p(\overline{\zeta})=\overline{k_p(\zeta)},\quad \zeta\in\mathbb{C}_{n_0}. \tag{2.4}$$

For this branch defined on $\mathbb{C}_{n_0}$, we keep the "old" notation k_p. Two branches $\psi_\pm$ of the Bloch solution $\psi(x,E)$ are analytic on $\mathbb{C}_{n_0}$ and satisfy

$$\psi_-(x,\overline{E})=\overline{\psi_+(x,E)},\quad \zeta\in\mathbb{C}_{n_0}. \tag{2.5}$$

They are linearly independent solutions of (2.1) for $E\in\mathbb{C}_{n_0}$. One indexes $\psi_\pm$ so that $\pm k_p$ be their respective quasi-momenta for $E\in\mathbb{C}_{n_0}$.

3. The main theorem of the complex WKB method

In this section, following [10], we briefly describe the main constructions of the complex WKB method for adiabatically perturbed periodic Schrödinger equations

$$-\frac{d^2}{dx^2}\psi(x)+(V(x)+W(\varepsilon x))\psi(x)=E\psi(x), \tag{3.1}$$

where V is a real-valued 1-periodic function of x, and ε is a small positive parameter. The complex WKB method allows us, in particular, to describe the exponentially small effects due to the complex tunneling. In this section, we assume that $V \in L^2_{\mathrm{loc}}$, and that W is analytic in a neighborhood $\mathcal{D}(W)$ of the real line.

3.1. **Additional complex parameter.** To decouple the "slow variable" $\xi = \varepsilon x$ and the "fast variable" x, one introduces an additional parameter ζ so that (3.1) becomes

$$-\frac{d^2}{dx^2}\psi(x) + (V(x) + W(\varepsilon x + \zeta))\psi(x) = E\psi(x), \quad x \in \mathbb{R}. \tag{3.2}$$

Then, one studies solutions of (3.2) on the complex plane of ζ to recover information on their behavior on $\mathbb{R}$. To control the dependence of solutions of equation (3.2) on ζ we assume that they satisfy the *consistency condition*

$$\psi(x+1, \zeta) = \psi(x, \zeta + \varepsilon) \quad \forall \zeta. \tag{3.3}$$

Remark 3.1. Condition (3.3) plays a crucial role for the asymptotic analysis of (3.2). It appears that it leads to the geometric objects and analytic constructions similar to those typical for the classical complex WKB method. Condition (1.6), used to define the monodromy matrix, was also called the "consistency condition". In section 3, 4 and 5, the words "consistency condition" and "consistent" refer to objects satisfying (3.3). Though being of different nature, in the analysis of the family of equations (0.1), conditions (1.6) and (3.3) are related to each other by a change of variables as we shall see in the beginning of section 6.

3.2. **The complex momentum.** The central analytic object of the complex WKB method is the *complex momentum* $\kappa(\zeta)$. It is defined in terms of the Bloch quasi-momentum of (2.1) by the formula

$$\kappa(\zeta) = k(E - W(\zeta)) \tag{3.4}$$

in $\mathcal{D}(W)$, the domain of analyticity of the function W. The complex momentum κ is a multi-valued analytic function. Its branch points are related to the branch points of the quasi-momentum by the relations

$$E_l = E - W(\zeta), \quad l = 1, 2, 3, \ldots, \tag{3.5}$$

where $(E_l)_{l\geq 1}$ are the ends of the spectral gaps of the operator H_0.

We say that a set is *regular* if it is in the domain of analyticity of W and contains no branch points of κ. For regular curves, we also ask that they are simply connected and piecewise C^1. For regular domains, we ask that they are simply connected.

Let D be a regular domain. Then, in D, one can fix κ_0, an analytic branch of κ. By (3.7), all the other analytic branches are described by the formulas

$$\kappa_m^{\pm} = \pm\kappa_0 + 2\pi m, \tag{3.6}$$

where $\pm$ and m are indexing the branches.

3.3. **Canonical domains.** The canonical domain notion is the main geometric notion of the complex WKB method. Let us proceed to the definitions.

A regular curve γ is called *vertical* if it intersects the lines $\{\operatorname{Im}\zeta = \mathrm{Const}\}$ at nonzero angles θ, $0 < \theta < \pi$. Vertical lines are naturally parameterized by $\operatorname{Im}\zeta$.

Let γ be a C^1 regular vertical curve. On γ, fix a continuous branch of the momentum κ. The curve γ is *canonical* if, setting $y = \operatorname{Im}\zeta$ along γ, one has

$$\frac{\partial}{\partial y}\left(\operatorname{Im}\int^{\zeta}\kappa(\zeta)d\zeta\right) > 0 \quad \text{and} \quad \frac{\partial}{\partial y}\left(\operatorname{Im}\int^{\zeta}(\kappa(\zeta)-\pi)d\zeta\right) < 0.$$

Note that canonical lines are stable under small $\mathcal{C}^1$-perturbations.

Definition 3.2. Let K be a regular domain. On K, fix a continuous branch of the quasi-momentum, say κ. The domain K is called *canonical* if it is the union of curves canonical with respect to κ and connecting two points ζ_1 and ζ_2 located on ∂K.

3.4. **Canonical Bloch solutions.** Consider the periodic Schrödinger equation

$$-\frac{d^2}{dx^2}\psi(x) + V(x)\psi(x) = \mathcal{E}\psi(x), \quad \mathcal{E} = E - W(\zeta), \quad x \in \mathbb{R}. \tag{3.7}$$

Here, ζ plays the role of a parameter. The function $\psi(x, E - W(\zeta))$ is a solution of (3.7) meromorphic in ζ; we now construct solutions of (3.7) that are analytic in ζ. Let D be a regular domain. There are two different branches of the function $\psi(x, E - W(\zeta))$ that are meromorphic on D. Denote them by $\psi_\pm(x,\zeta)$. On D, fix κ, an analytic branch of the complex momentum, so that $\pm\kappa(\zeta)$ is the Bloch quasi-momenta of $\psi_\pm$. We can represent $\psi_\pm$ in the form

$$\psi_\pm(x,\zeta) = e^{\pm i\kappa(\zeta)x}p_\pm(x,\zeta), \tag{3.8}$$

where $p_\pm(x,\zeta)$ are 1-periodic functions of x. Let

$$\omega_\pm(\zeta) = -\frac{\int_0^1 p_\mp(x,\zeta)\frac{\partial p_\pm}{\partial\zeta}(x,\zeta)dx}{\int_0^1 p_+(x,\zeta)p_-(x,\zeta)dx}. \tag{3.9}$$

On D, fix a continuous branch of the function $k'(E - W(\zeta))$. One has

Lemma 3.3. *The functions $\omega_\pm$ are meromorphic on D. The set of poles ω_+ and ω_- is the union of the set of poles of $\psi_\pm$ and the set of points where $k'(E-W(\zeta)) = 0$.*

Fix $\zeta_0 \in D$ so that $k'(E - W(\zeta_0)) \neq 0$. In a neighborhood of this point, choose an analytic branch of $q(\zeta) = \sqrt{k'(E - W(\zeta))}$. The functions

$$\Psi_\pm(x,\zeta) = q(\zeta)e^{\int_{\zeta_0}^{\zeta}\omega_\pm(\zeta)d\zeta}\psi_\pm(x,\zeta) \tag{3.10}$$

are called the *canonical Bloch solutions* normalized at ζ_0. They are analytic in D.

The Wronskian of the canonical Bloch solutions is given by

$$w(\Psi_+,\Psi_-) = q^2(\zeta_0)w(\psi_+(x,\zeta_0),\psi_-(x,\zeta_0)). \tag{3.11}$$

As $q^2(\zeta_0) = k'(E - W(\zeta_0)) \neq 0$, the canonical Bloch solutions are linearly independent.

3.5. **The main theorem of the WKB method.** Before stating this result, we note that, in the sequel, C denotes various positive constants independent of ζ, E and ε. Moreover, the terms $o(1)$ tend to 0 as ε tends to 0 uniformly in ζ and E in the domains under consideration.

One has

Theorem 3.4 ([10], Theorem 1.1)**.** *Fix $X > 0$ and $E = E_0 \in \mathbb{C}$. Let K be a bounded canonical domain for the family of equation* (3.2)*, and let κ be the branch of the complex momentum with respect to which K is canonical. For sufficiently small positive ε, there exists a consistent basis $(f_\pm)$ defined for $x \in \mathbb{R}$ and $\zeta \in K$ and having the following properties:*

- *For any fixed $x \in \mathbb{R}$, the functions $f_\pm(x, \zeta)$ are analytic in $\zeta \in K$.*
- *For $-X \leq x \leq X$, and $\zeta \in K$, the functions $f_\pm(x, \zeta)$ have the asymptotic representations*

$$f_\pm(x, \zeta) = e^{\pm \frac{i}{\varepsilon} \int_{\zeta_0}^{\zeta} \kappa d\zeta} (\Psi_\pm(x, \zeta) + o(1)), \quad \varepsilon \to 0. \tag{3.12}$$

  Here, $\Psi_\pm$ are the canonical Bloch solutions corresponding to the domain K, normalized at ζ_0 and indexed so that $\kappa(\zeta)$ is the Bloch quasi-momentum corresponding to the solution $\Psi_+(x, \zeta)$.
- *The error estimates in* (3.12) *are uniform in $x \in [-X, X]$ and locally uniform in ζ in the interior of K. Moreover, they may be differentiated once in x without losing the uniformity properties.*

Remark 3.5. The solutions $f_\pm$ are analytic in ζ in $S(K)$, the minimal "horizontal" strip $\{Y_1 < \operatorname{Im} \zeta < Y_2\}$ containing the domain K. This is explicitly checked in section 4.7.2 of [10] where Theorem 3.4 is proved.

One can easily calculate the Wronskian of the solutions $f_\pm(x, \zeta)$

$$w(f_+, f_-) = w(\Psi_+, \Psi_-) + o(1). \tag{3.13}$$

By (3.11), the solutions $f_\pm$ are linearly independent as $k'(E - W(\zeta_0)) \neq 0$.

3.5.1. *Dependence on the spectral parameter and admissible subdomains.* To simplify the statement of Theorem 3.4, we have not considered the dependence of the solutions on the spectral parameter E. Let K be a canonical domain. We call K without the δ-neighborhood of its boundary the *δ-admissible sub-domain* of K. One has

Proposition 3.6 ([11], Proposition 6.1)**.** *In the setting of Theorem* 3.4*, the solutions $f_\pm$ are analytic in E in V_0, a complex neighborhood of E_0 independent of ε. For A, an admissible sub-domain of the canonical domain K, there exists $V_A \subset V_0$, a complex neighborhood of E_0 independent of ε such that the asymptotics* (3.12) *are uniform in $(\zeta, E, x) \in A \times V_A \times [-X, X]$. They are once differentiable in x without losing their uniformity properties.*

3.5.2. *Terminology: Standard behavior.* Fix $E = E_0 \in \mathbb{C}$ and let $D \subset \mathbb{C}$ be a regular domain. Let κ be a branch of the complex momentum continuous in D, and let $\Psi_\pm$ be the canonical Bloch solutions associated to D and indexed so that κ is the quasi-momentum for Ψ_+. We say that a consistent solution f of equation (3.2) has *standard behavior* $f \sim e^{\frac{i}{\varepsilon} \int^{\zeta} \kappa \, d\zeta} \Psi_+$ (or $f \sim e^{-\frac{i}{\varepsilon} \int^{\zeta} \kappa \, d\zeta} \Psi_-$) in D if

- there exists V_0, a neighborhood of E_0 and $X > 0$ such that f is defined and satisfies (3.2) for any $(\zeta, E, x) \in D \times V_0 \times [-X, X]$, and f is analytic for $\zeta \in D$;

- for A, an admissible sub-domain of D, there exists $V_A \subset V_0$, a neighborhood of E_0 such that, for any $(\zeta, E, x) \in A \times V_A \times [-X, X]$, $f = e^{\frac{i}{\varepsilon}\int^{\zeta}\kappa\, d\zeta}\,(\Psi_+ + o\,(1))$ (or $f = e^{-\frac{i}{\varepsilon}\int^{\zeta}\kappa\, d\zeta}\,(\Psi_- + o\,(1))$, respectively) as $\varepsilon \to 0$;
- this asymptotic is uniform in $(\zeta, E, x) \in A \times V_A \times [-X, X]$;
- this asymptotic can be differentiated once in x without losing its uniformity properties.

Given a canonical domain K, Theorem 3.4 establishes the existence of two consistent solutions $f_\pm$ analytic in K and having standard behavior in K.

4. Canonical domains

Here, following section 6.7 in [11], we describe a simple approach to "constructing" canonical domains. Below, we assume that D is a regular domain, and that κ is a branch of the complex momentum analytic in D. A *segment* of a curve is a connected, closed subset of that curve.

1. Let $\gamma \subset D$ be a canonical line. Denote its ends by ζ_1 and ζ_2. Let a domain $K \subset D$ be a canonical domain corresponding to the triple κ, ζ_1 and ζ_2. If $\gamma \in K$, then K is called a canonical domain *enclosing* γ. As any line close enough in C^1-norm to a canonical line γ is canonical, one has

Lemma 4.1. *One can always construct a canonical domain enclosing any given canonical curve.*

Such canonical domains are called *local*.

2. Let $\gamma \subset D$ be a smooth curve. We say that γ is a line *of Stokes type* with respect to κ if, along γ, one has

$$\text{either Im}\left(\int^{\zeta}\kappa d\zeta\right) = \text{Const} \quad \text{or} \quad \text{Im}\left(\int^{\zeta}(\kappa - \pi) d\zeta\right) = \text{Const}.$$

Let $\gamma \subset D$ be a vertical curve. We call γ *pre-canonical* if it consists of a finite union of bounded segments of canonical lines and/or lines of Stokes type.

3. We construct "global" canonical domains by means of

Proposition 4.2 ([11], Proposition 6.3)**.** *Let γ be a canonical line with respect to κ. Assume that $K \subset D$ is a simply-connected domain containing γ (without its ends). The domain K is a canonical domain enclosing γ if it is the union of pre-canonical lines obtained from γ by replacing some of γ's internal segments by pre-canonical lines.*

5. The continuation tools

Given a canonical line, Lemma 4.1 gives us a local canonical domain K. So, by Theorem 3.4, we can construct solutions $f_\pm$ having standard behavior in this domain. Recall that $f_\pm$ are analytic in ζ in $S(K)$, the minimal "horizontal" strip containing the domain K; see Remark 3.5.

To study the asymptotic behavior of $f_\pm$ outside the canonical domain K, we develop two general tools, the Rectangle Lemma and the Adjacent Canonical Domain Principle.

In the sequel, a set is called *constant* if it is independent of ε.

5.1. **Asymptotics of increasing solutions.** Here, we roughly prove that the standard behavior of a solution stays valid along a horizontal line as long as the leading term of the asymptotics is growing along that line.

Fix $\eta_m < \eta_M$. Define $S = \{\zeta \in \mathbb{C} : \ \eta_m \leq \operatorname{Im}\zeta \leq \eta_M\}$. Let γ_1 and γ_2 be two vertical lines such that $\gamma_1 \cap \gamma_2 = \emptyset$. Assume that both lines intersect the strip S at the lines $\operatorname{Im}\zeta = \eta_m$ and $\operatorname{Im}\zeta = \eta_M$, and that γ_1 is situated to the left of γ_2.

Consider the compact set R bounded by γ_1, γ_2 and the boundaries of S. Let $D{=}R \setminus (\gamma_1 \cup \gamma_2)$. One has

Lemma 5.1 (The Rectangle Lemma)**.** *Fix an $E = E_0$. Assume that the "rectangle" R is in constant regular domain. Let f be a consistent solution of* (3.2)*. Then, for sufficiently small ε, one has*

- **1:** *If $\operatorname{Im}\kappa < 0$ in D and if, in a neighborhood of γ_1, f has standard behavior $f \sim e^{\frac{i}{\varepsilon}\int_{\zeta_0}^{\zeta}\kappa d\zeta}\Psi_+$, then this asymptotic remains valid in a constant domain containing the "rectangle" R.*
- **2:** *If $\operatorname{Im}\kappa > 0$ in D and if, in a neighborhood of γ_2, f has standard behavior $f \sim e^{\frac{i}{\varepsilon}\int_{\zeta_0}^{\zeta}\kappa d\zeta}\Psi_+$, then this asymptotic remains valid in a constant domain containing the "rectangle" R.*

Lemma 8.1 in [11] gives a weaker result. Also, a similar result for some difference equations was obtained in [4].

Remark 5.2. The vertical boundaries of R can be lines where $\operatorname{Im}\kappa = 0$. So, Lemma 5.1 says that the asymptotic of a solution stays valid along horizontal lines where it grows and, actually, even somewhat beyond the point where it stops growing.

Proof. We prove only the first statement; the second one is proved in an analogous way.

As a starter let us sketch the proof. First, we prove that there exists a constant finite set of constant open disks $(D_j)_j$ covering R such that, in each of these disks, there exists a basis of solutions $f_\pm^j$ having standard behavior, $f_\pm^j \sim e^{\pm\frac{i}{\varepsilon}\int^{\zeta}\kappa d\zeta}\Psi_\pm$. Next, we express f in terms of this basis of solutions

$$f(x,\zeta) = a_j(\zeta) f_+^j(x,\zeta) + b_j(\zeta) f_-^j(x,\zeta), \zeta \in D_j,$$

and prove that, under the assumptions of Lemma 5.1, the coefficients b_j are small enough. To complete the proof of Lemma 5.1, we compute the asymptotics of a_j.

First step. The first step is a consequence of the following general observation

Lemma 5.3. *Let ζ_* be a regular point. Then, there exists a branch of the complex momentum, say κ_*, and a domain K_* canonical with respect to κ_* that contains ζ_*.*

Proof. It suffices to show that there exists a complex number $z \in \mathbb{C}_+$ and a branch κ_* of the complex momentum such that

$$\operatorname{Im}(\kappa_*(\zeta_*)z) > 0 \text{ and } \operatorname{Im}((\kappa_*(\zeta_*) - \pi)z) < 0. \tag{5.1}$$

Let κ be one of the branches of the complex momentum analytic in V_*, a neighborhood of ζ_*. By (3.6), all the branches of the complex momentum analytic in V_* are described by

$$\kappa_m^\pm = \pm\kappa + 2im\pi. \tag{5.2}$$

We pick a branch, say κ_*, such that

$$0 \le \operatorname{Re} \kappa_*(\zeta_*) \le \pi.$$

There are three possible cases:

(1) If $0 < \operatorname{Re} \kappa_*(\zeta_*) < \pi$, then $z = i$ and κ_* satisfy (5.1).
(2) If $\operatorname{Re} \kappa_*(\zeta_*) = 0$, then $\operatorname{Im}\kappa_*(\zeta_*) \neq 0$ (as ζ_* is not a branch point). So, changing the branch if necessary, we can assume that $\operatorname{Im} \kappa_*(\zeta_*) > 0$ and that $\operatorname{Re} \kappa_*(\zeta_*) = 0$. In this case, we set $z = ie^{-i\epsilon}$. For $\varepsilon > 0$ sufficiently small, z and κ_* satisfy (5.1).
(3) If $\kappa_*(\zeta_*) = \pi$, we can assume that $\operatorname{Im} \kappa_*(\zeta_*) > 0$. In this case, we choose $z = ie^{i\epsilon}$, and, for $\epsilon > 0$ being small enough, (5.1) is satisfied.

This completes the proof of Lemma 5.3. □

As R is compact, using Lemma 5.3, we construct finitely many open disks $(D_j)_j$ covering the rectangle R and such that

(1) each D_j is regular;
(2) for each D_j, there exist solutions $f^j_\pm$ as described in the beginning of the proof of Lemma 5.1;
(3) on each disk, between γ_1 and γ_2, one has $\operatorname{Im} \kappa < 0$.

Second step. Let $\zeta_1 \in R \cap \gamma_1$. Consider the compact interval $I = [\zeta_1, \zeta_2] = R \cap \{\operatorname{Im} \zeta = \operatorname{Im} \zeta_1\}$. It suffices to prove that f has standard behavior in a constant neighborhood of this interval.

Assume that I is covered by constant disks D_j, $j = 0, \dots, J$, such that

(1) $\zeta_1 \in D_0$ and $\zeta_2 \in D_J$,
(2) f has standard behavior in D_0,
(3) for $j = 1, \dots, J$, $D_j \cap D_{j-1} \neq \emptyset$,
(4) the disks D_j with $0 < j < J$ are strictly between γ_1 and γ_2.

Let I_δ be a constant δ-neighborhood of the interval I contained in $\bigcup_{j=0}^J D_j$. Suppose that we have already proved that f has standard behavior in $I_\delta \cap \left(\bigcup_{j=1}^{J_0} D_j\right)$, $1 \le J_0 < J - 1$. Let us show that f has standard behavior in $D_{J_0+1} \cap I_\delta$.

For the sake of simplicity, we omit the index j and write $d = D_{J_0}$, $d_1 = D_{J_0+1}$. The solution f is a linear combination of the solutions $f_\pm$ corresponding to d_1, i.e.

$$f = a\, f_+ + b\, f_-, \quad \zeta \in d_1. \tag{5.3}$$

Here,

$$a(\zeta) = \frac{w(f, f_-)}{w(f_+, f_-)} \quad \text{and} \quad b(\zeta) = \frac{w(f_+, f)}{w(f_+, f_-)}. \tag{5.4}$$

By the consistency condition (3.3), the coefficients a and b are ε-periodic. As usual, $w(f_+, f_-) = w(\Psi_+, \Psi_-) + o(1)$ uniformly on any compact in d_1. The value of the leading term depends on the choice of normalization point for $f_\pm$. We choose the normalization point ζ_0 in $d \cap d_1 \cap I$ so that $q(\zeta_0) \neq 0$. Then, the leading term of $w(f_+, f_-)$ is bounded away from 0; see (3.11).

The solution f is normalized at a point which can be different from ζ_0. Therefore, we represent f as $f = f_0 \hat{f}$, where f_0 is a constant factor depending on the normalization points, and $\hat{f}$ is a solution such that the leading terms of the asymptotics of $\hat{f}$ and f_+ coincide in $d \cap d_1$.

As the leading terms of $\hat{f}$ and f_+ coincide in $d \cap d_1$, and as the leading term of the Wronskian $w(f_+, f_-)$ is non-zero, locally uniformly in $\zeta \in d \cap d_1 \cap I_\delta$, we obtain

$$a(\zeta) = f_0(1 + o(1)). \tag{5.5}$$

Due to the ε-periodicity of a, the asymptotic (5.5) stays valid in I_δ.

As for b, one has $b(\zeta) = f_0 \cdot o\left(e^{\frac{2i}{\varepsilon}\int_{\zeta_0}^{\zeta} \kappa d\zeta}\right)$ locally uniformly in $\zeta \in d \cap d_1 \cap I_\delta$. Due to the ε-periodicity of b, one can write

$$b(\zeta) = f_0 \cdot o\left(e^{\frac{2i}{\varepsilon}\int_{\zeta_0}^{\tilde{\zeta}} \kappa d\zeta}\right), \quad \zeta \in I_\delta,$$

where $\tilde{\zeta} \in d \cap d_1$, $\operatorname{Im} \tilde{\zeta} = \operatorname{Im} \zeta$, and $\operatorname{Re} \zeta - \operatorname{Re} \tilde{\zeta} = 0 \bmod \varepsilon$. This estimate is locally uniform in ζ.

Substituting the estimates obtained for a and b into the representation for f, we get, $\forall \zeta \in d_1 \cap I_\delta$,

$$\begin{aligned} f &= a(\zeta) f_+ + b(\zeta) f_- \\ &= f_0 e^{\frac{i}{\varepsilon}\int_{\zeta_0}^{\zeta} \kappa d\zeta}\left(\Psi_+(x, \zeta) + o(1) + o\left(e^{-\frac{2i}{\varepsilon}\int_{\tilde{\zeta}}^{\zeta} \kappa d\zeta}\right) + o(1)\right). \end{aligned} \tag{5.6}$$

Let $\zeta \in (I_\delta \cap d_1) \setminus d$. As d_1 is to the left of γ_2, in d_1, one has $\operatorname{Im} \kappa < 0$. Hence, $\operatorname{Re}\left(i \int_{\tilde{\zeta}}^{\zeta} \kappa\right) > 0$. This implies that f has standard behavior in $I_\delta \cap d_1$. Clearly, one can apply the above arguments and get (5.6) when $d = D_{J-1}$ and $d_1 = D_J$ (the last disk). Now, in (5.6), $\operatorname{Re}\left(i \int_{\tilde{\zeta}}^{\zeta} \kappa\right) > 0$ for any ζ either in the part of $d_1 \cup I_\delta$ situated to the left of γ_2, or in $\gamma_2 \cap I_\delta$. Moreover, this expression stays non-negative in a small enough (but constant) neighborhood V of $\gamma_2 \cap I_\delta$. This implies that f has standard behavior in I_δ both to the left of γ_2 and in V. This completes the proof of Lemma 5.1. □

5.2. **Estimates of decreasing solutions.** The Rectangle Lemma allows us to "continue" standard behavior as long as its leading term increases along a horizontal line. If the leading term decreases, then, in general, we can only estimate the solution, but not get an asymptotic behavior.

Lemma 5.4. *Fix $E = E_0$. Let ζ_1, ζ_2 be fixed points such that*

(1) $\operatorname{Im} \zeta_1 = \operatorname{Im} \zeta_2$;
(2) $\operatorname{Re} \zeta_1 < \operatorname{Re} \zeta_2$;
(3) *the segment $[\zeta_1, \zeta_2]$ of the line $\operatorname{Im} \zeta = \operatorname{Im} \zeta_1$ is regular.*

Fix a continuous branch of κ on $[\zeta_1, \zeta_2]$. Assume that $Im(\kappa(\zeta)) > 0$ on the segment $[\zeta_1, \zeta_2]$. Let ψ be a consistent solution having in a neighborhood of ζ_1 standard behavior $\psi \sim e^{\frac{i}{\varepsilon}\int_{\zeta_1}^{\zeta} \kappa d\zeta}\Psi_+$.

Then, there exists $C > 0$ such that, for ε sufficiently small, one has

$$\left|\frac{d\psi}{dx}(x, \zeta)\right| + |\psi(x, \zeta)| \le C e^{\frac{1}{\varepsilon}\int_{\zeta_1}^{\zeta} |Im\kappa| d\zeta}, \quad \zeta \in [\zeta_1, \zeta_2], \tag{5.7}$$

uniformly in E in a constant neighborhood of E_0.

Remark 5.5. Note that the leading term of the asymptotics of f is defined and decreases along the segment $[\zeta_1, \zeta_2]$ as ζ increases from ζ_1 to ζ_2.

One proves the "symmetric" statement for the case where $\operatorname{Im} \kappa < 0$ and f has the standard behavior $f \sim e^{\frac{i}{\varepsilon}\int_{\zeta_2}^{\zeta} \kappa d\zeta}\Psi_+$ in a neighborhood of ζ_2.

Proof. In a neighborhood of ζ_1, using Lemma 5.3, we construct two linearly independent solutions $\psi_\pm^{(1)}$ of (3.2) having standard behavior $\psi_\pm^{(1)} \sim e^{\pm \frac{i}{\varepsilon}\int_{\zeta_1}^{\zeta} \kappa d\zeta}\Psi_\pm$. We normalize them at the point ζ_1. By Lemma 5.1, $\psi_-^{(1)}$ has the standard behavior in a constant neighborhood of whole interval $[\zeta_1, \zeta_2]$.

In the same way, starting in a neighborhood of ζ_2, we construct $\psi_+^{(2)}$ a solution of (3.2) having standard behavior $\psi_+^{(2)} \sim e^{\frac{i}{\varepsilon}\int_{\zeta_1}^{\zeta} \kappa d\zeta}\Psi_+$ in a constant neighborhood of $[\zeta_1, \zeta_2]$. We normalize this solution at ζ_1.

As in the proof of Lemma 5.1, we compute the Wronskian of $(\psi_-^{(1)}, \psi_+^{(2)})$ and see that it is bounded away from 0 by a constant independent of ζ and ε. Hence, $\psi_-^{(1)}$ and $\psi_+^{(2)}$ form a basis of solutions. We decompose ψ as

$$\psi = a(\zeta)\psi_+^{(2)} + b(\zeta)\psi_-^{(1)}. \tag{5.8}$$

As in the proof of the Rectangle Lemma, we get

$$a(\zeta) = 1 + o(1), \quad b(\zeta) = o\left(e^{\frac{2i}{\varepsilon}\int_{\zeta_1}^{\zeta} \kappa d\zeta}\right),$$

uniformly in a constant neighborhood of ζ_1. Note that the last estimate implies that $|b| \leq \text{Const}$ in the ε-neighborhood of ζ_1. As a and b are ε-periodic, then

$$a(\zeta) = 1 + o(1), \quad |b(\zeta)| \leq \text{Const}, \tag{5.9}$$

uniformly on $[\zeta_1, \zeta_2]$. As these estimates were obtained using standard behavior, they are uniform for E in a constant neighborhood of E_0. Substituting (5.9) into (5.8), and taking into account the asymptotics of $\psi_+^{(2)}$ and $\psi_-^{(1)}$, we get

$$|\psi| + \left|\frac{d\psi}{dx}\right| \leq \text{Const}\left(e^{-\frac{1}{\varepsilon}\int_{\zeta_1}^{\zeta} \text{Im}\,\kappa d\zeta} + e^{\frac{1}{\varepsilon}\int_{\zeta_1}^{\zeta} \text{Im}\,\kappa d\zeta}\right), \quad \zeta \in [\zeta_1, \zeta_2].$$

This implies (5.7) and completes the proof of Lemma 5.4. □

5.3. **Adjacent Canonical Domain Principle.** The estimate we obtained in Lemma 5.4 can be very far from optimal: the estimate only says that the solution ψ cannot increase faster than $\exp\left(\frac{1}{\varepsilon}\int_{\zeta_1}^{\zeta} |\text{Im}\kappa| d\zeta\right)$ whereas it can, in fact, decrease along $[\zeta_1, \zeta_2]$. Under additional conditions, the following construction yields a better result.

Lemma 5.6 (The Adjacent Canonical Domain Principle)**.** *Assume that a consistent solution f has standard behavior in either the left-hand side or the right-hand side of a constant neighborhood of a vertical curve γ. Assume that γ is canonical with respect to some branch of the complex momentum. Then, f has the standard behavior in any given canonical domain enclosing γ.*

Remark 5.7. In practice, one knows the asymptotic of a solution f, say, to the left of a vertical line γ and needs to compute the asymptotics of f to the right of γ. If γ is canonical, Lemma 5.6 gives the asymptotics of f in any canonical domain enclosing γ and, thus, in its part adjacent to γ to the right of γ.

Proof. Consider M, a canonical domain enclosing the canonical curve γ. Denote by κ the corresponding branch of the complex momentum. Let ζ_m and ζ_M be the lower and the upper ends of γ.

By Theorem 3.4, in M, we construct the corresponding consistent basis of solutions $f_\pm$ with standard asymptotics $f_\pm(x,\zeta,\zeta_0) \sim e^{\pm \frac{i}{\varepsilon}\int_{\zeta_0}^{\zeta} \kappa d\zeta}\Psi_\pm(x,\zeta,\zeta_0)$. Here, we have explicitly indicated the dependence on the normalization point ζ_0. We choose $\zeta_0 \in M$ so that $q(\zeta_0) \neq 0$. There exists $c > 0$ such that, for ε sufficiently small, one has $|w(f_+, f_-)| > c$ locally uniformly in $\zeta \in M$. Inside the domain M, the function f admits the representation

$$f = af_+ + bf_-, \quad a(\zeta) = \frac{w(f, f_-)}{w(f_+, f_-)} \quad \text{and} \quad b(\zeta) = \frac{w(f_+, f)}{w(f_+, f_-)}. \tag{5.10}$$

For any $\delta > 0$ and ε sufficiently small, the functions a and b are ε-periodic and analytic in the strip $\operatorname{Im}\zeta_m + \delta < \operatorname{Im}\zeta < \operatorname{Im}\zeta_M - \delta$.

To prove Lemma 5.6, we need asymptotic estimates of the coefficients a and b. Therefore, we use the standard behavior of f near the curve γ,

$$f \sim e^{\frac{i}{\varepsilon}\int_{\tilde\zeta_0}^{\zeta} \tilde\kappa d\zeta}\, \tilde\Psi_+(x, \zeta, \tilde\zeta_0).$$

Both the branch $\tilde\kappa$ and the normalization point $\tilde\zeta_0$ can be different from the branch κ and the normalization point ζ_0; the indexing of the canonical Bloch solution $\tilde\Psi_+$ is determined by $\tilde\kappa$. We express $\tilde\kappa$ in terms of κ

$$\tilde\kappa(\zeta) = \sigma\kappa(\zeta) + 2\pi n,$$

where σ belongs to $\{-1, +1\}$, and n is an integer.

Let us first discuss the case $\sigma = +1$. Let V be the part of the neighborhood of γ where all the solutions f, f_+ and f_- have standard behavior. We represent f in the form

$$f = \varphi(\zeta)\tilde f, \quad \varphi(\zeta) = f_0 e^{2\pi n(\zeta - \zeta_m)/\varepsilon}, \quad \zeta \in V, \tag{5.11}$$

where f_0 is a constant factor (depending on the normalization points), and $\tilde f$ has standard behavior with the same leading term as f_+ in V. This and the standard behavior of $f_\pm$ imply

$$a = \varphi(\zeta)(1 + o(1)), \quad b = \varphi(\zeta) o\left(e^{\frac{2i}{\varepsilon}\int_{\zeta_0}^{\zeta} \kappa d\zeta}\right). \tag{5.12}$$

To get these estimates, we have used $|w(f_+, f_-)| > c$. The estimates (5.12) are locally uniform in $\zeta \in V$.

Fix $\delta > 0$ sufficiently small. The ε-periodicity of a implies that its asymptotics remain valid and uniform in the whole strip $\operatorname{Im}\zeta_m + \delta \le \operatorname{Im}\zeta \le \zeta_M - \delta$.

Let us now study b in more detail. Below, C denotes a positive constant independent of ε. As b is ε-periodic, the estimate for b from (5.12) implies that, for sufficiently small ε, for $\operatorname{Im}\zeta = \operatorname{Im}\zeta_M - \delta_1$,

$$|b| \le |\varphi(\zeta)|\, Ce^{C\delta/\varepsilon} \left| e^{\frac{2i}{\varepsilon}\int_{\zeta_0}^{\zeta_M} \kappa d\zeta} \right| = |\varphi(\zeta)| Ce^{C\delta/\varepsilon}\, |A|\, |B|, \tag{5.13}$$

and, for $\operatorname{Im}\zeta = \operatorname{Im}\zeta_m + \delta_1$,

$$|b| \le |\varphi(\zeta)|\, Ce^{C\delta/\varepsilon}\, |B|, \tag{5.14}$$

where δ_1 is fixed such that $0 < \delta_1 < \delta$, and

$$A = \exp\left(\frac{2i}{\varepsilon}\int_{\zeta_m,\ \text{along}\ \gamma}^{\zeta_M} \kappa d\zeta\right), \quad B = \exp\left(\frac{2i}{\varepsilon}\int_{\zeta_0}^{\zeta_m} \kappa d\zeta\right).$$

As b is analytic in the strip $\operatorname{Im}\zeta_m < \operatorname{Im}\zeta < \zeta_M$, we can represent b in the form

$$b = b_- + e^{\frac{2\pi i}{\varepsilon}(\zeta - \zeta_m)} b_+,$$

where $b_\pm$ are ε-periodic, b_- is analytic in the half-plane $\operatorname{Im}\zeta < \operatorname{Im}\zeta_M$, and b_+ is analytic in the half-plane $\operatorname{Im}\zeta > \operatorname{Im}\zeta_m$. One can obtain estimates of $b_\pm$ by estimating the Fourier coefficients of b. Estimating the Fourier coefficients with positive indexes by means of (5.14) and the Fourier coefficients with non-positive indexes by means of (5.13), one obtains

$$|b_-| \le |\varphi(\zeta)| C e^{C\delta/\varepsilon} \, |A| \, |B|, \quad \operatorname{Im}\zeta \le \operatorname{Im}\zeta_M - \delta,$$
$$|b_+| \le |\varphi(\zeta)| \, C e^{C\delta/\varepsilon} \, |B|, \quad \operatorname{Im}\zeta \ge \operatorname{Im}\zeta_m + \delta.$$

Therefore, for $\operatorname{Im}\zeta_m + \delta \le \operatorname{Im}\zeta \le \zeta_M - \delta$,

$$|b| \le C|\varphi(\zeta)| \, e^{C\delta/\varepsilon} \, |B| \, \left(|A| + \left| e^{\frac{2\pi i}{\varepsilon}(\zeta - \zeta_m)} \right| \right). \tag{5.15}$$

Now, the representation (5.10), the estimate (5.12) for a and (5.15) for b, and the standard behavior of $f_\pm$ imply that, uniformly in the δ-admissible subdomain of M,

$$\begin{aligned} f = \varphi(\zeta) \, e^{\frac{i}{\varepsilon}\int_{\zeta_0}^{\zeta} \kappa d\zeta} \Big\{ \Psi_+(x,\zeta,\zeta_0) + o(1) + O\left(e^{C\delta/\varepsilon} \, e^{\frac{2i}{\varepsilon}\int_{\zeta}^{\zeta_M} \kappa d\zeta} \right) \\ + O\left(e^{C\delta/\varepsilon} \, e^{-\frac{2i}{\varepsilon}\int_{\zeta_m}^{\zeta} (\kappa - \pi) d\zeta} \right) \Big\}, \end{aligned} \tag{5.16}$$

where we integrate along curves in M. The last two terms in the curly brackets are small. Indeed, as the domain M is canonical, when computing these two terms, we can integrate along a canonical curve going from ζ_m to ζ_M via the point ζ. It follows from the definition of canonical curves that both the exponentials $E_M = \exp\left(-\frac{2i}{\varepsilon}\int_{\zeta_M}^{\zeta} \kappa d\zeta \right)$ and $E_m = \left(-\frac{2i}{\varepsilon}\int_{\zeta_m}^{\zeta} (\kappa - \pi) d\zeta \right)$ are small as $\varepsilon \to 0$. We need only prove that they are sufficiently small to compensate the factor $e^{C\delta/\varepsilon}$. Consider the function $i_M(\zeta) = \operatorname{Im}\left(\int_{\zeta}^{\zeta_M} \kappa d\zeta \right)$ inside M. Clearly, i_M is continuous. As M is canonical, the points ζ and ζ_M can be connected by canonical curves inside M, and, therefore, the values of i_M are positive. So, inside a fixed admissible subdomain, i_M is positive and bounded away from zero by a positive constant (of course, independent of ε). As δ can be chosen arbitrarily small, the expression $e^{C\delta/\varepsilon} \, e^{-\frac{2i}{\varepsilon}\int_{\zeta_M}^{\zeta} \kappa d\zeta}$ can be made uniformly small for all ζ in any fixed admissible subdomain of M. Similarly, one studies the last term in the curly brackets in (5.16). In the result, one sees that, uniformly in any given admissible subdomain of M,

$$f = \varphi(\zeta) \, e^{\frac{i}{\varepsilon}\int_{\zeta_0}^{\zeta} \kappa d\zeta} \left(\Psi_+(x,\zeta,\zeta_0) + o(1) \right).$$

This proves Lemma 5.6 in the case of $\sigma = +1$. The case of $\sigma = -1$ is treated similarly. We note only that, in this case, one starts with the representation

$$f = \hat{\varphi}(\zeta)\hat{f}, \quad \hat{\varphi} = f_0 e^{-2\pi i n(\zeta - \zeta_m)/\varepsilon},$$

where f_0 is constant, and, in V, the solution $\hat{f}$ has standard behavior with the same leading term as f_-. □

6. Asymptotics of the monodromy matrix

This section is devoted to the proof of Theorem 1.4. Therefore, one first performs the change of variable $x := x - z$ and $\zeta := \varepsilon z$ in (0.1). One sees that

- equation (0.1) becomes (3.2);
- the consistency condition (1.6) becomes (3.3);

- if $f_\pm$ is a consistent basis of solutions of (0.1), in the new variables, the definition of the corresponding monodromy matrix becomes

$$F(x,\zeta+2\pi) = \tilde{M}(\zeta)F(x,\zeta), \quad F = \begin{pmatrix} f_+ \\ f_- \end{pmatrix}; \tag{6.1}$$

- the monodromy matrix $\tilde{M}$ is ε-periodic in ζ.

To the "new" equation (3.2), we apply the complex WKB method to calculate the monodromy matrix for a consistent basis constructed by means of Theorem 3.4. Our plan is the following. First, we find a canonical line. Then, in a local canonical domain enclosing this line, using Theorem 3.4, we construct a consistent basis. Then, we apply our continuation tools and find a large enough domain D, where both $F(x,\zeta)$ and $F(x,\zeta+2\pi)$ have standard behavior. At last, we compute the monodromy matrix. To carry out this plan, we begin by discussing properties of the complex momentum in a neighborhood of the real line under the condition (A).

In the sequel we fix $E = E_0$ in the interval J satisfying assumption (A).

6.1. **The complex momentum and a canonical line.** Recall that the complex momentum is related to the Bloch quasi-momentum of the periodic Schrödinger operator (0.2) by the formula (3.4). For real values of E, the pre-image of the spectral axis with respect to the mapping $\mathcal{E}:\ \zeta \mapsto E - W(\zeta)$ is the set $W^{-1}(\mathbb{R})$. This set plays an important part for the analytic properties of the complex momentum.

6.1.1. *The set* $W^{-1}(\mathbb{R})$*.* As W is 2π-periodic and real analytic, the set $W^{-1}(\mathbb{R})$ is a 2π-periodic analytic set symmetric with respect to the real axis. It consists of

(1) the real line;
(2) complex branches beginning at any extremum of W along the real line:
   - each of these branches is an analytic curve beginning at an extremum;
   - at a real extremum, the branches are transversal to the real line, and transversal to each other;
(3) complex branches of $W^{-1}(\mathbb{R})$ separated from the real line.

6.1.2. *The complex momentum.* The real line belongs to the pre-image of the n_0-th spectral band of the periodic Schrödinger operator (0.2) with respect to the mapping $\mathcal{E}:\ \zeta \to E - W(\zeta)$. For $Y > 0$, define the strip $\mathcal{S}_Y = \{-Y \le \operatorname{Im}\zeta \le Y\}$. Let $Z = \mathcal{S}_Y \cap W^{-1}(\mathbb{R})$. We choose Y sufficiently small so that

- the strip $\mathcal{S}_Y$ is contained in the domain of analyticity of W;
- the set Z consists of the real line and of complex lines beginning at the extrema of W situated along the real line;
- these complex lines do not intersect outside $\mathbb{R}$ and are vertical;
- the image of Z by $\mathcal{E}:\zeta \to E - W(\zeta)$ is contained in the n_0-th spectral band and the distance from this image to the ends of the band is positive.

Note that the last condition implies that the branch points of the complex momentum and the poles of the functions $\psi_\pm(x,\zeta)$ and $\omega_\pm(\zeta)$ stay outside $\mathcal{S}_Y$.

Fix a branch of the complex momentum analytic in the strip $\mathcal{S}_Y$ by the formula

$$\kappa_p(\zeta) = k_p(E - W(\zeta)), \tag{6.2}$$

where k_p is the branch of the Bloch quasi-momentum of the periodic Schrödinger operator described in sections 2.2 and 2.3.

Lemma 6.1. *The branch κ_p has the following properties:*

1. *κ_p takes real values if and only if $\zeta \in Z$;*
2. *$\pi(n-1) < \kappa_p(\zeta) < \pi n$ for all $\zeta \in Z$;*
3. *$\kappa_p(\overline{\zeta}) = \overline{\kappa_p(\zeta)}$.*

Proof. Point (1) directly follows from the definitions of κ_p and the properties of k_p (see section 2.3). Point (2) follows from the last property of the strip $\mathcal{S}_Y$. Point (3) holds as κ_p is analytic in $\mathcal{S}_Y$ and takes real values on $\mathbb{R}$. □

6.1.3. *Canonical line.* To find a canonical line, let us discuss the function W. Let $\zeta_0 \in \mathbb{R}$ be one of the minima of W. As W is analytic and non-constant, near ζ_0, one has $W(\zeta) - W(\zeta_0) \sim W_n(\zeta - \zeta_0)^{2n}$, where W_n is positive and n is a positive integer. This implies that there are $2n-1$ complex branches of $W^{-1}(\mathbb{R})$ beginning at ζ_0 and going upwards. One of these lines is orthogonal to $\mathbb{R}$ at ζ_0. Denote it by C_0. The line $\overline{C_0}$, symmetric of C_0 with respect to $\mathbb{R}$, is also a branch of $W^{-1}(\mathbb{R})$. Define

$$\beta = \left(C_0 \cup \{\zeta_0\} \cup \overline{C_0}\right) \cap S_Y. \tag{6.3}$$

Clearly, β is a vertical C^1-curve.

Describe the branch of the complex momentum with respect to which β is canonical. The branches of the complex momentum analytic in $\mathcal{S}_Y$ is described by (3.6). Therefore, the function $\kappa_{n_0}(\zeta)$ defined by

$$\kappa_{n_0} = \begin{cases} \kappa_p - \pi(n_0 - 1), & \text{if } n_0 \text{ is odd}, \\ \pi n_0 - \kappa_p, & \text{if } n_0 \text{ is even}, \end{cases} \tag{6.4}$$

is a branch of the complex momentum analytic in $\mathcal{S}_Y$. In view of point (2) in Lemma 6.1, one has

$$0 < \kappa_{n_0}(\zeta) < \pi, \quad \forall \zeta \in Z. \tag{6.5}$$

As β is vertical, equation (6.5) implies that, along β,

$$\frac{d}{dy}\left(\operatorname{Im}\int^{\zeta} \kappa_{n_0}\, d\zeta\right) > 0, \text{ and } \frac{d}{dy}\left(\operatorname{Im}\int^{\zeta} (\kappa_{n_0} - \pi)\, d\zeta\right) < 0.$$

So, β is canonical with respect to κ_{n_0}.

6.2. **Solution f.** We now define and study one of the solutions of the consistent basis for which we compute the monodromy matrix.

6.2.1. *Local canonical domain.* The curve β being canonical, by Lemma 4.1, there exists a local canonical domain K enclosing β. Using Theorem 3.4, for $\zeta \in K$, we construct f, a solution of equation (3.2), having standard behavior $f \sim \exp\left(\frac{i}{\varepsilon}\int_0^{\zeta} \kappa d\zeta\right)\Psi_+$. In view of Remark 3.5, f is analytic in S_Y.

6.2.2. *Asymptotics of f outside K.* Here, we study the asymptotics of f in the strip $\mathcal{S}_Y$ outside the domain K. We prove

Proposition 6.2. *For any fixed positive $l \in \mathbb{N}$, there exists $Y_l > 0$ such that, for ε sufficiently small, the solution f has the standard behavior in the domain R_l bounded by the lines β, $\beta + 2\pi l$ and $\operatorname{Im}\zeta = \pm Y_l$.*

The rest of this subsection is devoted to the proof of Proposition 6.2. We start with a more detailed discussion of the set Z.

6.2.3. *The structure of the set* Z. Recall that ζ_0 is a minimum of W. Denote by $(\zeta_i)_{i\in\mathbb{Z}}$, the extrema of W on $\mathbb{R}$. We order them increasingly. Clearly, the set of the extrema is 2π-periodic: there exists $l_0 \in \mathbb{N}$ such that $\zeta_{l+l_0} = \zeta_l + 2\pi$ for all $l \in \mathbb{Z}$.

Pick $l \in \mathbb{Z}$. We have, say, n_l branches of the set Z starting at ζ_l and situated in $\mathbb{C}_+$, the upper half plane. These branches are vertical and have only one common point ζ_l. Recall that, at ζ_l, each of these branches is transversal to the real line as well as to all the other branches; moreover, the analyticity of W guarantees that the branches beginning at different extrema of W do not intersect in the strip $\mathcal{S}_Y$. Denote the complex branches of Z beginning at the extrema of W and going upwards by $(C_j)_{j\in\mathbb{Z}}$, so that

- C_0 is the branch used for the construction of the canonical line β;
- in the strip $\{0 < \operatorname{Im}\zeta < Y\}$, the branch C_{j+1} is situated to the right of C_j for all $j \in \mathbb{Z}$.

Of course, some of these branches can have common beginning points (situated on $\mathbb{R}$). The set of lines C_j being 2π-periodic, there exists $j_0 \in \mathbb{Z}$ such that $C_{j+j_0} = C_j + 2\pi$ for all $j \in \mathbb{Z}$.

For $j \in \mathbb{Z}$, we let D_j be the open "rectangle" delimited by the lines C_j, $\operatorname{Im}\zeta = Y$, C_{j+1} and $\mathbb{R}$. The complex branches of Z situated in $\mathbb{C}_-$ can be obtained from the lines $(C_j)_{j\in\mathbb{Z}}$ by symmetry with respect to $\mathbb{R}$. We denote them by $\overline{C_j}$; for $j \in \mathbb{Z}$, the symmetric of D_j with respect to the real line is denoted by $\overline{D_j}$. We finish this subsection by proving

Lemma 6.3. *Let* $E = E_0 \in J$. *Then:*

1. *for all* $j \in \mathbb{Z}$, *the imaginary part of* κ_{n_0} *does not vanish in the rectangles* D_j *and* $\overline{D_j}$;
2. *for all* $j \in \mathbb{Z}$, *in the domains* D_j *and* D_{j+1}, *the imaginary parts of* κ_{n_0} *are of opposite sign;*
3. *for all* $j \in \mathbb{Z}$, *in the domains* D_j *and* $\overline{D_j}$, *the imaginary parts of* κ_{n_0} *are of opposite sign.*

Proof. Point (1) follows from point (1) of Lemma 6.1. Point (3) follows from point (1) and point (3) of Lemma 6.1. Check point (2). By construction of the strip $\mathcal{S}_Y$ (see section 6.1.1) one has

- the line C_{j+1} separating D_j from D_{j+1} is bijectively mapped by the analytic function $\mathcal{E}$ onto an interval $\mathcal{I}$ of $\mathbb{R}$,
- the interval $\mathcal{I}$ is contained in the n_0-th spectral band of the periodic operator (0.2),
- one of the domains D_j and D_{j+1} is mapped into $\mathbb{C}_+$ and the second one is mapped into $\mathbb{C}_- := \{\zeta \in \mathbb{C};\ \operatorname{Im}\zeta < 0\}$.

Point (2) holds as the main branch k_p of the Bloch quasi-momentum maps $\mathbb{C}_+$ into $\mathbb{C}_+$ and $\mathbb{C}_-$ into $\mathbb{C}_-$; see sections 2.2 and 2.3. □

6.2.4. *Proof of Proposition* 6.2. Our main tools are the Rectangle Lemma, Lemma 5.1, and the Adjacent Canonical Domain Principle, Lemma 5.6. Recall that $\operatorname{Im}\kappa_{n_0} \neq 0$ in D_0. First, we treat the case where $\operatorname{Im}\kappa_{n_0} < 0$ in D_0. The proof is made in several steps; at each step, using one of the continuation tools, we justify the asymptotics of f on a larger domain.

1. Let us show that f has standard behavior in the domain D_0. This follows from the Rectangle Lemma. Indeed, pick $\delta > 0$ small and consider the compact

R_δ bounded by the lines β, C_1, $\operatorname{Im}\zeta = \delta$ and $\operatorname{Im}\zeta = Y - \delta$. In D_0, one has $\operatorname{Im}\kappa < 0$, so the "rectangle" R_δ satisfies the conditions of Lemma 5.1. Thus, the solution f, having standard behavior in a neighborhood of R_δ's left boundary, also has standard behavior in a constant neighborhood of R_δ. As $\delta > 0$ is arbitrary, this implies that f has standard behavior in the whole domain D_0.

2. Now, let us study the asymptotics of f in the domain D_1. Note that as $\operatorname{Im}\kappa_{n_0} < 0$ in D_0, by Lemma 6.3, $\operatorname{Im}\kappa_{n_0} > 0$ in D_1. So, we cannot apply the Rectangle Lemma to "continue" the asymptotics of f into D_1. The line C_1 is canonical with respect to κ_{n_0} (for the same reason as C_0 was; see (6.5)). So, by the Adjacent Canonical Domain Principle, the standard behavior for f remains valid in any bounded canonical domain enclosing C_1.

Let us describe a part of a canonical domain enclosing C_1 situated in the domain D_1. Denote by ζ^* the uppermost point of C_1 in $\mathcal{S}_Y$. Let σ be the line of Stokes type starting at ζ^* and defined by $\operatorname{Im}\int_{\zeta^*}^{\zeta}\kappa_{n_0}d\zeta = 0$. One proves

Lemma 6.4. *The line σ enters in D_1 at the point ζ^*; inside D_1, it is vertical and goes downwards. It leaves D_1 at a point $\zeta^{**} \in C_2$ such that $0 < \operatorname{Im}\zeta^{**} < Y$. Let $\tilde{D}_1$ be the domain delimited by the lines σ, $\mathbb{R}$, C_1 and C_2; it is a part of a canonical domain enclosing C_1.*

Proof. First, we check the geometric properties of σ. We identify $\mathbb{C}$ and $\mathbb{R}^2$ in the usual way. Let $\zeta^0 \in C_1$ and $0 < \operatorname{Im}\zeta^0 < Y$. Consider σ_0 a line of Stokes type $\operatorname{Im}\int_{\zeta^0}^{\zeta}\kappa_{n_0}d\zeta = 0$ containing the point ζ^0. This line is an integral curve of the vector field $t(\zeta) = \overline{\kappa_{n_0}(\zeta)}$. As κ_{n_0} is real on C_1, the tangent vector $t(\zeta^0)$ to σ_0 at ζ^0 is horizontal, $\operatorname{Im} t(\zeta^0) = 0$. So, σ_0 intersects C_1 transversally and enters D_1 at ζ^0. In D_1, one has $\operatorname{Im}\kappa_{n_0} > 0$; near C_1, $\operatorname{Re}\kappa_{n_0}$ is positive as it is positive on C_1. Therefore, the line σ_0 is vertical in D_1, and, inside D_1, it goes downwards starting from ζ^0.

As $\kappa_{n_0} \neq 0$ in $\mathcal{S}_Y$ (see (6.5)), the lines of Stokes type $\operatorname{Im}\int^{\zeta}\kappa_{n_0}d\zeta = \text{Const}$ fibrate $\mathcal{S}_Y$, and the observations on σ_0 imply that

- either σ beginning at ζ^* contains a segment of the line $\operatorname{Im}\zeta = \operatorname{Im}\zeta^*$,
- or it enters in D_1 at ζ^*.

The first alternative is impossible; since then, κ_{n_0} must be real on that segment which contradicts the assumptions on the strip $\mathcal{S}_Y$. Therefore, σ enters D_1 at the point ζ^*, and, inside D_1, it is vertical and goes downwards from ζ^*.

Since σ goes downwards, it can leave D_1 only through C_2, C_1 or $\mathbb{R}$. As $\mathbb{R}$ is also a line of Stokes type, if σ intersects $\mathbb{R}$, then the intersection point is a critical point for $\operatorname{Im}\int^{\zeta}\kappa_{n_0}d\zeta$. At this point κ_{n_0} vanishes, which is impossible. So, we need only to check that σ cannot intersect C_1. Assume it intersects C_1 at, say, a point ζ_e. Then, one has

$$0 = \operatorname{Im}\int_{\zeta^*,\text{ along }\sigma}^{\zeta_e}\kappa_{n_0}d\zeta = \operatorname{Im}\int_{\zeta^*,\text{ along }C_1}^{\zeta_e}\kappa_{n_0}d\zeta.$$

But, the last integral is nonzero since $\kappa_{n_0} > 0$ along C_1 and C_1 is vertical. So, σ has to leave D_1 through C_2 at a point ζ^{**} with a positive imaginary part. We have proved all the properties of σ described in Lemma 6.4.

Now, let us check the statement of Lemma 6.4 concerning $\tilde{D}_1$; we use Proposition 4.2. Pick a point $\tilde{\zeta} \in \tilde{D}_1$. Show that there exists a pre-canonical curve contained in $\tilde{D}_1$, connecting two internal points of C_1 and containing $\tilde{\zeta}$.

As above, consider the line of the Stokes type σ_0 starting at $\zeta^0 \in C_1$. If $\zeta^0 = \zeta^*$, then σ_0 coincides with σ. If $\zeta^0 \in \mathbb{R}$, then σ_0 coincides with the real line. Since the lines $\operatorname{Im} \int^\zeta \kappa_{n_0} d\zeta = \text{Const}$ fibrate $\mathcal{S}_Y$, there exists a point ζ^0 such that the line σ_0 contains $\tilde{\zeta}$. We denote this line by β_u and the corresponding starting point by ζ_u. Clearly, the segment of line β_u between ζ_u and $\tilde{\zeta}$ belongs to $\tilde{D}_1$, is vertical, and goes downwards.

Similarly, one proves that there exists a vertical segment β_d of the line of Stokes type $\operatorname{Im} \int_{\tilde{\zeta}}^{\zeta} (\kappa_{n_0} - \pi) d\zeta = 0$ connecting $\tilde{\zeta}$ in $\tilde{D}_1$ with a point $\zeta_d \in C_1$ such that $0 < \zeta_d < \tilde{\zeta}$. First, we pick a point ζ_d inside C_1. Then, we consider the line of Stokes type $\operatorname{Im} \int_{\zeta_d}^{\zeta} (\kappa_{n_0} - \pi) d\zeta = 0$ beginning at this point. This line is transversal to C_1 at ζ_d. It is vertical in D_1 and goes upwards there. If $\operatorname{Im} \zeta_d > \operatorname{Im} \tilde{\zeta}$, then $\tilde{\zeta}$ is below this line. If $\operatorname{Im} \zeta_d \to 0$, then this line approaches $\mathbb{R}$ ($\mathbb{R}$ is a line of Stokes type of the same family). So, there exists ζ_d situated below $\tilde{\zeta}$ and above $\mathbb{R}$ which is connected in D_1 to $\tilde{\zeta}$ by a line of Stokes type $\operatorname{Im} \int_{\zeta_d}^{\zeta} (\kappa_{n_0} - \pi) d\zeta = 0$. This line is denoted by β_d.

The line $\beta_d \cup \beta_u$ is not pre-canonical: it is not vertical at its ends (common points with C_1). But, as the canonical lines are stable under C^1-perturbations, we can deform somewhat the canonical line C_1 near the points ζ_d and ζ_u in such a way that, from $\beta_d \cup \beta_u$ and the deformed segments of C_1, we construct the pre-canonical line containing $\tilde{\zeta}$ and connecting in $\tilde{D}_1$ two internal points of C_1. Since $\tilde{\zeta}$ was an arbitrary point in $\tilde{D}_1$, then, by Proposition 4.2, $\tilde{D}_1$ is a part of a canonical domain enclosing C_1. □

3. The right boundary of $\tilde{D}_1$ is the segment of the line C_2 between the real line and the point ζ^{**} ($0 < \operatorname{Im} \zeta^{**} < Y$). Denote this segment by $\tilde{C}_2$. As C_1, the line $\tilde{C}_2$ is canonical. Consider a local canonical domain (see Lemma 4.1) enclosing $\tilde{C}_2$. By the Adjacent Canonical Domain Principle, f has standard behavior also in this domain.

4. To "continue" the standard behavior of f to the right of $\tilde{C}_2$, one essentially only has to inductively repeat the steps 1–3. In particular, when studying the asymptotic behavior of f in D_2, one considers the open "rectangle" $\tilde{D}_2$ bounded by $\tilde{C}_2$, $\mathbb{R}$, C_3 and the line $\operatorname{Im} \zeta = \operatorname{Im} \zeta^{**}$. As $\tilde{D}_2 \subset D_2$, in $\tilde{D}_2$, one has $\operatorname{Im} \kappa_{n_0} < 0$. To this rectangle, we apply the Rectangle Lemma in the same way as we have applied it to the rectangle D_0 (in step 1), and we see that f has standard behavior in $\tilde{D}_2$.

Continuing the proof inductively, one shows that f has standard behavior in the domain $R_{+,l}$ bounded by the lines C_0, $\mathbb{R}$, $C_0 + 2\pi l$ and $\operatorname{Im} \zeta = Y_{+,l}$, where l is a fixed arbitrary positive integer, and $Y_{+,l}$ is positive and independent of ε.

5. Similarly, one studies the asymptotic behavior of f below the real line. Let us outline one induction step of the proof.

In the domain $\overline{D_0}$, by point (3) of Lemma 6.3, $\operatorname{Im} \kappa_p > 0$. One justifies the standard behavior of f in a subdomain $\tilde{\overline{D_0}}$ of $\overline{D_0}$ by the Adjacent Canonical Domain Principle. The domain $\tilde{\overline{D_0}}$ is bounded by $\overline{C_0}$, $\mathbb{R}$, $\overline{C_1}$ and the line of Stokes type $\operatorname{Im} \int \kappa_{n_0} d\zeta = 0$ staying in $\overline{D_0}$ and connecting the lower end of $\overline{C_0}$ with ζ_{**}, an internal point of $\overline{C_1}$.

The line $\overline{C_1}$ is canonical. By the Adjacent Canonical Domain Principle, the standard behavior remains valid in a local canonical domain enclosing the segment $\tilde{\overline{C_1}}$ of the line $\overline{C_1}$ between $\mathbb{R}$ and ζ_{**}.

In $\overline{D_1}$, one has $\operatorname{Im}\kappa_{n_0} < 0$. Applying the Rectangle Lemma, one justifies the standard behavior in the rectangle bounded by $\overline{C_1}$, $\mathbb{R}$, $\overline{C}_2$ and the line $\operatorname{Im}\zeta = \operatorname{Im}\zeta_{**}$.

Continuing the proof inductively, one shows that f has standard behavior in the domain $R_{-,l}$ bounded by $\overline{C_0}$, $\mathbb{R}$, $\overline{C_0} + 2\pi l$ and the line $\operatorname{Im}\zeta = -Y_{-,l}$, where l is an fixed arbitrary positive integer, and $Y_{-,l}$ is positive and independent of ε.

6. Fix $l \in \mathbb{Z}$. To show that f has standard behavior in a domain R_l as described in Proposition 6.2, we need only to check that it has standard behavior in V, a constant neighborhood of the interval $[\zeta_0, \zeta_0 + 2\pi l] \subset \mathbb{R}$.

Let c_0 be the supremum of the values of c such that the standard behavior is valid in a constant neighborhood of a interval (ζ_0, c) in the half-plane $\{\operatorname{Re}\zeta < c\}$. The standard behavior of f is valid in a constant neighborhood of the point of intersection of the canonical line β and $\mathbb{R}$ (as it is valid in a local canonical domain enclosing β). Therefore, $c_0 > \zeta_0$.

Assume now that $c_0 < \zeta_0 + 2\pi l$. Note that

(1) f has standard behavior in $R_l \setminus [c_0, \zeta_0 + 2\pi l]$,
(2) by Lemma 5.3, there exists a canonical domain containing c_0.

By the Adjacent Canonical Domain Principle, these two observations imply that f has standard behavior in a constant neighborhood of c_0 which is impossible. So, $c_0 \geq \zeta_0 + 2\pi l$. This completes the proof of Proposition 6.2 when $\operatorname{Im}\kappa_p < 0$ in D_0. When $\operatorname{Im}\kappa_{n_0} > 0$, the analysis is similar; we omit it.

6.3. **The consistent basis.** Now, to obtain a consistent basis, we construct another consistent solution of (0.1) in terms of f. In the sequel, for g, a function of the complex variables $(z_1, z_2, \ldots, z_k)$, we define

$$g^*(z_1, z_2, \ldots, z_k) = \overline{g(\overline{z_1}, \overline{z_2}, \ldots, \overline{z_k})}.$$

6.3.1. *The solution f^*.* Recall that f is analytic in a constant neighborhood of E_0. In this neighborhood, consider $f^*(x, \zeta, E)$, where we have indicated explicitly the dependence on E. As $V + W$ is real on $\mathbb{R}$, f^* is a solution of (3.2). The solution f^* also satisfies the consistency condition, and it is analytic in ζ in S_Y.

6.3.2. *The asymptotics of f^*.* Let us write the asymptotics of f explicitly. We assume that f is normalized at ζ_0. Then,

$$\begin{aligned} f(x,\zeta) &= e^{\frac{i}{\varepsilon}\int_{\zeta_0}^{\zeta}\kappa_{n_0}d\zeta}(\Psi_+(x,\zeta) + o(1)), \\ \Psi_+(x,\zeta) &= q(\zeta)e^{\int_{\zeta_0}^{\zeta}\omega_+ d\zeta}\psi_+(x,\zeta). \end{aligned} \tag{6.6}$$

Let us fix a branch of the function $q = \sqrt{k_p'(E - W(\zeta))}$. Recall that $k_p'(\mathcal{E})$ is positive on the n_0-th spectral band. So, we fix a branch of q so that $q(\zeta) > 0$ on $\mathbb{R}$. The asymptotics (6.6) is valid in the domain R_l described by Proposition 6.2. Note that, for real E and ζ,

$$\begin{aligned} \overline{\kappa_{n_0}(\zeta)} &= \kappa_{n_0}(\zeta), \quad \overline{\psi_+(x,\zeta)} = \psi_-(x,\zeta), \\ \overline{\omega_+(\zeta)} &= \omega_-(\zeta), \quad \overline{q(\zeta)} = q(\zeta). \end{aligned} \tag{6.7}$$

The first relation follows from point (3) of Lemma 6.1. The second one holds as, on the spectral band, the branches of the Bloch solution $\psi(x, \mathcal{E})$ differ by complex conjugation. The second relation and the definition of the functions $\omega_\pm$ imply the third one. The last relation follows from our choice of q.

The relations (6.7) and the standard behavior of f imply that f^* has standard behavior

$$f^* = e^{-\frac{i}{\varepsilon}\int_{\zeta_0}^{\zeta}\kappa_{n_0}\,d\zeta}(\Psi_-(x,\zeta)+o(1)),$$

where

$$\Psi_-(x,\zeta) = q(\zeta)e^{\int_{\zeta_0}^{\zeta}\omega_-\,d\zeta}\psi_-(x,\zeta), \quad \zeta \in R_l.$$

6.3.3. *The Wronskian of f and f^*.* Both solutions f and f^* satisfy the consistency condition. To define the monodromy matrix for the pair f and f^*, we have to check that these solutions are linearly independent and that their Wronskian is independent of ζ. The asymptotics of f and f^* imply that

$$w(f,f^*) = w(\Psi_+,\Psi_-) + o(1) \tag{6.8}$$

locally uniformly in $\zeta \in K$. The Wronskian of the canonical Bloch solutions $\Psi_\pm$ is given by (3.11). As $k_p'(\mathcal{E})$ is positive on the n_0-th spectral band, and as $\mathcal{E} = E - W(\zeta_0)$ belongs to this band for $E \in J$, the leading term in (6.8) is nonzero. So, for ε sufficiently small, f and f^* are linearly independent for any ζ in a fixed admissible subdomain of K.

The leading term in (6.8) is independent of ζ, but, the error term in (6.8) can depend on ζ. We modify the solutions f and f^* so that this error term is constant. As both solutions satisfy the consistency condition (3.3), this error term is ε-periodic in ζ. So, we can redefine the solution f by multiplying it by an ε-periodic factor of the form $(1+o(1))$ to get exactly

$$w(f,f^*) = w(\Psi_+,\Psi_-). \tag{6.9}$$

The "new" solutions f and f^* form a consistent basis and, in R_l, they have the same standard behavior as the "old" solutions f and f^*.

6.4. **The monodromy matrix.** We now study the monodromy matrix associated to the basis (f,f^*). It has the form (1.12), where the coefficients a and b are given by

$$a(\zeta) = \frac{w(f(x,\zeta+2\pi),f^*(x,\zeta))}{w(f(x,\zeta),f^*(x,\zeta))}, \quad b(\zeta) = \frac{w(f(x,\zeta),f(x,\zeta+2\pi))}{w(f(x,\zeta),f^*(x,\zeta))}. \tag{6.10}$$

As M is unimodular (see (1.8)), one has

$$a(\zeta)a^*(\zeta) - b(\zeta)b^*(\zeta) = 1. \tag{6.11}$$

Together with the solutions f and f^*, the monodromy matrix is analytic in $\zeta \in R_l$ and in E in a constant neighborhood of E_0. Actually, being periodic, the monodromy matrix is analytic in ζ in the whole strip $|\mathrm{Im}\,\zeta| \le Y_l$. Below, we compute the asymptotics of the coefficients a and b. This computation will complete the proof of Theorem 1.4.

6.4.1. *Coefficient a.* We compute the Wronskian $w(f(x,\zeta+2\pi),f^*(x,\zeta))$. Fix an integer $l \ge 2$. Assume that ζ and $\zeta+2\pi$ belong to R_l. Then, both functions $f(x,\zeta+2\pi)$ and $f^*(x,\zeta)$ have standard behavior. In particular,

$$f(x,\zeta+2\pi) = e^{\frac{i}{\varepsilon}\int_{\zeta_0}^{\zeta+2\pi}\kappa_{n_0}\,d\zeta}\left(q(\zeta+2\pi)e^{\int_{\zeta_0}^{\zeta+2\pi}\omega_+\,d\zeta}\psi_+(x,\zeta+2\pi)+o(1)\right),$$

where we integrate in R_l. To rewrite this representation in a more convenient form, we check that the functions κ_{n_0}, q, ω_+ and ψ_+ are 2π-periodic in $\mathcal{S}_Y$. All of these functions being analytic, it suffices to check their periodicity on $\mathbb{R}$ and for real E. In this case,

- k_n and $\psi_\pm$ are periodic as $\mathcal{E}(\zeta) = E - W(\zeta)$ is 2π-periodic and takes values inside the n_0-th spectral band;
- the periodicity of $\omega_\pm$ follows from their definition and the previous point;
- the function q is periodic as $k_p'(E - W(\zeta))$ is periodic and positive on $\mathbb{R}$.

So, one rewrites the above representation for $f(x, \zeta + 2\pi)$ in the form

$$f(x, \zeta + 2\pi) = c_0 e^{\frac{i}{\varepsilon}\int_{\zeta_0}^{\zeta} \kappa_{n_0} d\zeta} \left(q(\zeta) e^{\int_{\zeta_0}^{\zeta} \omega_+ d\zeta} \psi_+(\zeta) + o(1) \right), \tag{6.12}$$

where we integrate in R_l and have defined

$$\begin{aligned} c_0 &= e^{\frac{i}{\varepsilon}\Phi + i\Omega}, \quad \Omega = -i \int_{\zeta}^{\zeta+2\pi} \omega_+ d\zeta, \\ \Phi &= \int_{\zeta}^{\zeta+2\pi} \kappa_{n_0} d\zeta = \int_0^{2\pi} \kappa_{n_0} d\zeta. \end{aligned} \tag{6.13}$$

Representation (6.12) implies that

$$\begin{aligned} w\left(f(x, \zeta + 2\pi), f^*\right) &= c_0 \left(w(\Psi_+, \Psi_-) + o(1)\right) = c_0(w(f, f^*) + o(1)) \\ &= c_0 w(f, f^*)(1 + o(1)). \end{aligned}$$

Here, we have used relation (6.9) and the fact that $w(\Psi_+, \Psi_-)$ does not vanish and is independent of ε. Now, by (6.10), we obtain the formula $a(\zeta) = c_0(1 + o(1))$.

6.4.2. *Proof of the statements of Theorem* 1.4 *concerning the coefficient* a. Fix $\delta > 0$. Being a consequence of the standard behavior of f and f^*, the asymptotics of a are uniform for ζ in the δ-admissible subdomain of R_l and for E in a constant neighborhood V_δ of E_0.

As a is ε-periodic, its asymptotics are uniform in the strip $|\mathrm{Im}\,\zeta| \le Y_l - \delta$.

Clearly, the zeroth Fourier coefficient of a has the asymptotics $a_0 = c_0(1 + o(1))$. Consider the functions Φ and Ω determining the leading term for the asymptotics of a. As for $E = E_0$, the branch points of κ_{n_0} and the poles of $\omega_\pm$ are outside the strip $\mathcal{S}_Y$, and as they continuously depend on E, the definitions of κ_{n_0} and $\omega_\pm$ imply that Φ and Ω are analytic in E in a constant neighborhood of E_0. Note that, as $\kappa_{n_0}(\zeta)$ takes real values on the real line, Φ is real on J. Check that Ω is real on J. Using (6.7) and the definition of $\omega_\pm$, we get

$$\begin{aligned} \Omega - \overline{\Omega} &= -i \int_0^{2\pi} (\omega_+ + \overline{\omega_+}) d\zeta = -i \int_0^{2\pi} (\omega_+ + \omega_-) d\zeta \\ &= i \int_0^{2\pi} \frac{\partial}{\partial \zeta} \log g(\zeta) d\zeta, \end{aligned}$$

where

$$g(\zeta) = \int_0^1 \psi_+(x, \zeta)\psi_-(x, \zeta) dx.$$

In view of (6.7), g is positive. As it is periodic, we get $\Omega - \overline{\Omega} = 0$. We have checked all the properties of a_0 announced in Theorem 1.4.

The asymptotics of a implies that a_1 satisfies the uniform estimate $a_1 = o(a_0)$ for $(\zeta, E) \in \mathcal{S}_{Y_l - \delta} \times V_\delta$. Therefore, the Fourier coefficients $(\hat{a}_1(n))_n$ of a_1 are exponentially decreasing, i.e. uniformly in $E \in V_\delta$,

$$|\hat{a}_1(n)| \leq C|a_0| \exp\left(-\frac{2\pi|n|}{\varepsilon}(Y_l - \delta)\right), \quad n \neq 0. \tag{6.14}$$

Come back to the variable $z = \zeta/\varepsilon$ of the initial equation (0.1). Fix $y > 0$ independent of ε sufficiently small. By means of (6.14), for sufficiently small ε, we get

$$|a_1| \leq C|a_0| \exp\left(-\frac{2\pi}{\varepsilon}(Y_l - \delta - \varepsilon|\mathrm{Im}\, z|)\right), \quad |\mathrm{Im}\, z| \leq y/\varepsilon.$$

We have checked all the statements of Theorem 1.4 concerning the coefficient a.

6.4.3. *Coefficient* b. Let us prove the estimate (1.14) for the coefficient b. Fix an integer $l \geq 2$. Using the standard behavior of f and (6.10), we get

$$b = o\left(\exp\left(\frac{i}{\varepsilon}\int_{\zeta_0}^{\zeta+2\pi} \kappa_{n_0} d\zeta + \frac{i}{\varepsilon}\int_{\zeta_0}^{\zeta} \kappa_{n_0} d\zeta\right)\right), \quad \zeta, \zeta + 2\pi \in R_l.$$

Consider the integral $I := \int_\zeta^{\zeta+2\pi} \kappa d\zeta$. As κ is 2π-periodic, the integral I is constant, and as κ_{n_0} is real on $\mathbb{R}$, it is real. So,

$$b = o\left(\exp\left(\frac{2i}{\varepsilon}\int_{\zeta_0}^{\zeta} \kappa_{n_0} d\zeta\right)\right), \quad \zeta, \zeta + 2\pi \in R_l. \tag{6.15}$$

Being a consequence of the standard behavior, the estimate (6.15) is uniform in ζ for ζ and $\zeta+2\pi$ in the δ-admissible subdomain of R_l, and for E in $V(\delta)$, a constant neighborhood of E_0. If, in (6.15), we choose ζ in the ε-neighborhood of β, the canonical line used when constructing f, we get

$$b(\zeta) = o\left(\exp\left(\frac{2i}{\varepsilon}\int_{\zeta_0,\ \text{along}\ \beta}^{\hat{\zeta}} \kappa_{n_0} d\zeta\right)\right), \quad \hat{\zeta} \in \beta, \quad \mathrm{Im}\,\hat{\zeta} = \mathrm{Im}\,\zeta. \tag{6.16}$$

As b is ε-periodic, this estimate is uniform in $(\zeta, E) \in \{|\mathrm{Im}\,\zeta| \leq Y_l - \delta\} \times V(\delta)$.

Show that the estimate (6.16) implies (1.14). Recall that $\beta \subset Z$. So, there exists $c > 0$ such that, for $E = E_0$ and $\zeta \in \beta$, one has $c < \kappa_{n_0}(\zeta) < \pi - c$. Therefore, for $\zeta \in \beta$ and $\eta = \mathrm{Im}\,\zeta$,

$$\begin{aligned} c\eta &< \mathrm{Im}\int_{\zeta_0,\ \text{along}\ \beta}^{\zeta} \kappa_{n_0} d\zeta < (\pi - c)\eta, \quad \eta \geq 0, \\ (\pi - c)\eta &< \mathrm{Im}\int_{\zeta_0,\ \text{along}\ \beta}^{\zeta} \kappa_{n_0} d\zeta < c\eta, \quad \eta \leq 0. \end{aligned} \tag{6.17}$$

These estimates also hold in a constant neighborhood of E_0 (with a smaller constant c). Equations (6.16) and (6.17) show that

$$b(\zeta) = o\left(\exp(-\frac{2c}{\varepsilon}(Y_l - \delta))\right), \quad \mathrm{Im}\,\zeta = Y_l - \delta,$$
$$b(\zeta) = o\left(\exp(\frac{2(\pi - c)}{\varepsilon}(Y_l - \delta))\right), \quad \mathrm{Im}\,\zeta = -(Y_l - \delta).$$

Let $(\hat{b}_n)_n$ be the Fourier coefficients b. The last estimates imply that

$$
\text{(6.18)}\qquad
\begin{aligned}
|\hat{b}_n| &\le C \exp\left(-\frac{2\pi(n-1)+2c}{\varepsilon}\,(Y_l-\delta)\right), \quad n>0,\\
|\hat{b}_n| &\le C \exp\left(-\frac{2\pi|n|+2c}{\varepsilon}\,(Y_l-\delta)\right), \quad n<0,
\end{aligned}
$$

uniformly in E in a constant neighborhood of E_0. Come back to the variable $z=\zeta/\varepsilon$ of the initial equation (0.1). Fix $y>0$ independent of ε and sufficiently small. Using (6.18) for sufficiently small ε, uniformly in z and in E in a constant neighborhood of E_0, we get

$$
|b(z)| = o\left(\exp\left(-\frac{2c}{\varepsilon}\,(Y_l-\delta)+2\pi\varepsilon|\operatorname{Im} z|\right)\right), \quad |\operatorname{Im} z|\le y/\varepsilon.
$$

We have proved the statement of Theorem 1.4 concerning the coefficient b. This completes the proof of Theorem 1.4.

7. The monodromy equation

Let $(\psi_{1,2})$ be a consistent basis of solutions of (0.1), and let M be the corresponding monodromy matrix.

7.1. **Behavior at infinity.** Consider the monodromy equation (1.9). As already mentioned, the behavior of its solutions for $n\to\pm\infty$ mimics the behavior of solutions of equation (0.1) for $x\to\mp\infty$. One has

Theorem 7.1 ([11], Theorem 3.1)**.** *Fix $z\in\mathbb{R}$. Then, for χ, a solution of equation* (1.9)*, there exists a unique solution of* (0.1)*, say f, such that*

$$
\text{(7.1)}\qquad
\begin{pmatrix} f(x+2\pi n/\varepsilon, z)\\ f'(x+2\pi n/\varepsilon, z)\end{pmatrix} = M_\psi(x, z-nh)\cdot\sigma\cdot\chi_{-n}, \quad \forall x\in\mathbb{R},\ n\in\mathbb{Z},
$$

where

$$
\text{(7.2)}\qquad
M_\psi = \begin{pmatrix} \psi_1 & \psi_2\\ \frac{d\psi_1}{dx} & \frac{d\psi_2}{dx}\end{pmatrix}, \quad \sigma=\begin{pmatrix} 0 & -1\\ 1 & 0\end{pmatrix}, \quad h=\frac{2\pi}{\varepsilon} \mod (1).
$$

Moreover, reciprocally for f, a solution of (0.1)*, there exists a unique vector χ, a solution of* (1.9)*, satisfying* (7.1)*.*

As $(\psi_{1,2})$ is a consistent basis, $\det M_\psi$ is a nonvanishing constant; thus, Theorem 7.1 immediately implies Theorem 1.2.

7.2. **Bloch-Floquet solutions.** There is also a relation between Bloch-Floquet solutions of the family of equation (0.1) and Bloch-Floquet solutions of the monodromy equation. Let us first define these solutions.

7.2.1. *Bloch-Floquet solutions for difference equations.* Consider the equation

$$
\text{(7.3)}\qquad \chi(x+h)=M(x)\chi(x), \quad \chi(x)\in\mathbb{C}^2, \quad x\in\mathbb{R},
$$

where $x\mapsto M(x)\in SL(2,\mathbb{C})$ is 1-periodic and h is a fixed positive number. The set of solutions of (7.3) is a two-dimensional module over the ring of h-periodic functions. Hence, it is natural to call χ a *Bloch solution* if it satisfies the relation

$$
\text{(7.4)}\qquad \chi(x+1)=\mu(x)\chi(x), \quad x\in\mathbb{R},
$$

where $x \mapsto \mu(x)$ is h-periodic; see [6]. If μ is constant, the Bloch solution can be represented in the form

$$\chi(x) = e^{i\phi x}U(x), \quad x \in \mathbb{R}, \tag{7.5}$$

where ϕ is a constant and $x \mapsto U(x)$ is 1-periodic. In this case, we call ϕ the *quasi-momentum* of χ and U the *periodic component* of χ. The solution χ is called a *Bloch-Floquet* solution of (7.3).

7.2.2. *Bloch-Floquet solutions for differential equations.* Fix $a > 0$ and $b > 0$, two constants, and $(x, y) \mapsto Q(x, y)$, a sufficiently regular function 1-periodic both in x and in y. Consider the differential equation

$$-f''(x) + Q(ax, bx)f(x) = 0, \quad x \in \mathbb{R}. \tag{7.6}$$

Assume that a solution f can be written as

$$f(x) = e^{ipx}P(ax, bx), \quad x \in \mathbb{R}, \tag{7.7}$$

where p is constant and $P(x, y)$ is a function which is 1-periodic in x and in y. We call such a solution a *Bloch-Floquet* solution of (7.6), and we call the constant p its *quasi-momentum*. The first example of Bloch-Floquet solutions of an equation of the form (7.6) for quasi-periodic potentials was constructed in [7].

7.2.3. *Relation between Bloch-Floquet solutions of the family* (0.1) *and of the monodromy equation* (1.9). Now, instead of the monodromy equation (1.9) on $\mathbb{Z}$, we consider its continuous analog (7.3). The matrix M is the monodromy matrix for a consistent basis $(\psi_{1,2})$, and, as before, $h \equiv 2\pi/\varepsilon \mod 1$. Note that if χ satisfies (7.3), then the sequence $(\chi_n)_n$ defined by $\chi_n = \chi(x + nh)$ satisfies (1.9). We prove

Theorem 7.2. *Assume that there exists χ, a Bloch-Floquet solution* (7.5) *of the "continuous" monodromy equation* (7.3). *Let*

$$F(x, z) = M_\psi(x, z)\sigma U(z), \quad x, z \in \mathbb{R}, \tag{7.8}$$

where U is the periodic component of χ, and the matrices M_ψ and σ are as in Theorem 7.1. *Then, $f(x, z)$, the first component of the vector F, is a Bloch-Floquet solution of* (0.1). *It can be written as*

$$f(x, z) = e^{ip(E)x}\, P(x - z, x), \tag{7.9}$$

where $(y, x) \mapsto P(y, x)$ is $2\pi/\varepsilon$-periodic in x and 1-periodic in y. The quasi-momenta of f and χ are related by

$$p(E) = -\frac{\varepsilon h}{2\pi}\, \phi(E). \tag{7.10}$$

Note that between the Bloch-Floquet solutions of difference equations related by the monodromization procedure, there is a relation similar to (7.8); see [5].

Proof. It follows from the definition of the matrix M_ψ that f is a linear combination of ψ_1 and ψ_2. So, it satisfies (0.1). The representation (7.9) and formula (7.10) follow from the next two statements, Proposition 7.3 and Lemma 7.4.

Proposition 7.3. *The solution f satisfies the relations*

$$f(x + a, z + a) = e^{-ih\phi}f(x, z), \quad f(x, z + 1) = f(x, z), \quad x, z \in \mathbb{R}, \tag{7.11}$$

where $a = 2\pi/\varepsilon$ and ϕ is the quasi-momentum of χ.

Proof. As M is the monodromy matrix corresponding to the consistent basis (ψ_1, ψ_2) and as $\det M \equiv 1$, we get

$$\begin{aligned} F(x+a, z+a) &= M_\psi(x,z)\, M^t(z)\, \sigma\, U(z+a) \\ &= M_\psi(x,z)\, \sigma\, M^{-1}(z)\, U(z+a), \end{aligned} \tag{7.12}$$

where ^{t} denotes the transposition. As U is 1-periodic, and $h = a \mod 1$, we have $U(z+a) = U(z+h)$. In view of the representation (7.5) and equation (7.3), one has

$$\begin{aligned} M^{-1}(z)\, U(z+h) = M^{-1}(z)\, \chi(z+h)\, e^{-i\phi\cdot(z+h)} &= \chi(z)\, e^{-i\phi\cdot(z+h)} \\ = U(z)\, e^{-i\phi h}. \end{aligned}$$

Therefore, the right-hand side of (7.12) can be transformed to the form

$$M_\psi(x,z)\, \sigma\, U(z)\, e^{-i\phi h} = F(x,z)\, e^{-i\phi h}.$$

This implies the first of the relations (7.11). Due to the consistency condition, $M_\psi(x,z)$ is 1-periodic in z. Thus, the second equality follows from the 1-periodicity of $U(z)$. □

Lemma 7.4. *Assume that a function g satisfies the relations*

$$g(x+a, z+a) = e^{iq} g(x,z), \quad g(x, z+b) = g(x,z), \quad x, z \in \mathbb{R},$$

with $a > 0$, $b > 0$ and $q \in \mathbb{C}$, three constants. Then it admits the representation

$$g(x,z) = e^{iqx/a} v(x-z, x), \quad x, z \in \mathbb{R},$$

with a function $v(y,x)$ which is a-periodic in x and b-periodic in y.

Proof. One just defines $v(y,x) = e^{-iqx/a} g(x, x-y)$ and checks the periodicity of v. □

8. The spectral results

8.1. **Local version of Theorem 1.1.** Pick $E_0 \in J$, and let V_0 be a complex neighborhood of E_0. Fix S so that

$$0 < S < S_0, \tag{8.1}$$

where S_0 is defined in Theorem 1.4. Fix I, a closed real interval in V_0, and define

$$\lambda = \exp(-S/\varepsilon). \tag{8.2}$$

We prove

Theorem 8.1. *Fix $\sigma \in (0,1)$. There exists $\mathcal{D} \subset (0,1)$, a set of Diophantine numbers such that*

$$\frac{\text{mes}\,(\mathcal{D} \cap (0,\varepsilon))}{\varepsilon} = 1 + o\,(\varepsilon\lambda^\sigma)\,, \quad \varepsilon \to 0, \tag{8.3}$$

and that, for any $\varepsilon \in \mathcal{D}$, the interval I contains absolutely continuous spectrum, and

$$\text{mes}\,(I \cap \Sigma_{\text{ac}}) = \text{mes}\,(I) \cdot (1 + O(\lambda^{\sigma/2})).$$

Here, Σ_{ac} is the absolutely continuous spectrum for the family of equations (0.1). *For $E \in I$, outside a Borel set of measure $O(\lambda^{\sigma/2})$, equation* (0.1) *has solutions $\psi_\pm$ as described in Theorem* 1.1.

As J is compact, Theorem 1.1 immediately follows from Theorem 8.1. To prove Theorem 8.1, we work with (7.3), the continuous analog of the monodromy equation, for the monodromy matrix M described by Theorem 1.4. First, we construct solutions of (7.3) by means of a KAM theory construction described below. Then, we use Theorem 7.2 and Theorem 1.2 to obtain Theorem 8.1. Theorem 1.2 and the Ishii-Pastur-Kotani Theorem allow us to control the location of the absolutely continuous spectrum, and Theorem 7.2 allows us to describe the functional structure of the generalized eigenfunctions.

8.2. **Poor man KAM theory.** We recall a form of KAM theory suited for our purpose; it was developed in section 11 in [11] using standard ideas of KAM theory (see [7, 2]). Let $\mathcal{B}$ be a Borel subset of $\mathbb{R}$. Let S_r be the strip $\{z \in \mathbb{C};\ |\mathrm{Im}\, z| \leq r\}$. For the matrix-valued functions of (φ, z) that are Lipschitz in $\varphi \in \mathcal{B}$ and analytic in z in S_r we introduce the norm

$$\|M\|_{r,\mathcal{B}} := \sup_{\substack{|y|\leq r\\ \varphi\in\mathcal{B}}} \|M(z,\varphi)\| + \sup_{\substack{|y|\leq r\\ \varphi,\varphi'\in\mathcal{B}\\ \varphi\neq\varphi'}} \frac{\|M(z,\varphi)-M(z,\varphi')\|}{|\varphi-\varphi'|}, \quad y = \mathrm{Im}\, z. \tag{8.4}$$

Here, $\|.\|$ is the matrix norm associated to the ℓ^1-norm on $\mathbb{C}^2$. If the matrix M is independent of z, we write $\|M\|_{\mathcal{B}}$ instead of $\|M\|_{r,\mathcal{B}}$. A matrix-valued function $(z, \varphi) \in S_r \times \mathcal{B} \mapsto M(z, \varphi)$ belongs to the class $\mathcal{M}$ if it is analytic and 1-periodic in S_r and is of the form

$$M = \begin{pmatrix} a & b \\ b^* & a^* \end{pmatrix}.$$

Let A and D be two matrices such that

- the matrix D is diagonal,
$$D = D(\varphi) = \begin{pmatrix} d & 0 \\ 0 & \overline{d} \end{pmatrix} \text{ where } d = \exp(i\varphi);$$
- the matrix $A(z, \varphi)$ belongs to $\mathcal{M}$ and satisfies $\|A\|_{r,I} \leq 1$, where $I \subset \mathbb{R}$ is a bounded interval.

For $\Psi(z)$, a 2×2 matrix, consider the equation

$$\Psi(z+h) = (D + \lambda A)\Psi(z), \quad z \in \mathbb{R}, \tag{8.5}$$

where λ is a positive parameter, and h is fixed, $0 < h < 1$.

To solve (8.5) using a KAM-type method, we impose a Diophantine condition on the number h. Fix $0 < \sigma < 1$ and define

$$H_\sigma(\lambda) := \{h \in (0,1);\ \min_{l\in\mathbb{N}} |h - l/k| \geq \lambda^\sigma/k^3,\ k = 1, 2, 3 \dots\}. \tag{8.6}$$

One has

Proposition 8.2 ([11], Proposition 11.1)**.** *Let A and D be chosen as above; assume that* $\det(D + \lambda A) \equiv 1$. *Then there exists $\lambda_0(r, \sigma, I) > 0$ such that, for all $|\lambda| < \lambda_0$ and $h \in H_\sigma(\lambda)$, there exists a Borel set $\Phi_\infty(r, \sigma, I, \lambda) \subset I$ satisfying*

$$\mathrm{mes}\ \Phi_\infty \leq K_0(r, \sigma, I)\lambda^{\sigma/2}, \tag{8.7}$$

outside which, for $\varphi \in \mathcal{B}_\infty = I \setminus \Phi_\infty$, equation (8.5) *has a solution of the form*

$$\Psi(z, \varphi) = U(z, \varphi) \begin{pmatrix} e^{i\varphi_\infty \cdot z/h} & 0 \\ 0 & e^{-i\varphi_\infty \cdot z/h} \end{pmatrix}, \tag{8.8}$$

where

- $U \in \mathcal{M}$ *and* $det(U) = 1$*;*
- U *is defined and analytic for* $|\operatorname{Im} z| < r/2$*, it is* 1*-periodic in* z*, and* $\|U - 1\|_{r/2, \mathcal{B}_\infty} \leq C\lambda^{2-\sigma}$*;*
- φ_∞ *is a real valued Lipschitz continuous function of* φ *and satisfies* $\|\varphi_\infty(\varphi) - \varphi\|_{\mathcal{B}_\infty} \leq 2\lambda$.

Here, λ_0*,* Φ_∞ *and* K_0 *only depend on the arguments indicated explicitly above.*

Remark 8.3. Reading the proof of Proposition 8.2, one easily checks that

(1) all of the constants in Proposition 8.2 depend on the length of I, but not on its position;
(2) if, say, $r \geq 1$, then all these constants can be chosen independent of r.

The set Φ_∞ admits the following description:

$$\Phi_\infty = \bigcup_{j=0}^{\infty} \Phi_j, \quad \Phi_j = \bigcup_{k,l \in \mathbb{Z}} \left\{ \varphi \in I : \ |\alpha_j(\varphi) - hk - l| < \frac{\lambda^\sigma}{k^2} \right\}, \tag{8.9}$$

where the functions α_j are Lipschitz continuous and, for some $C > 0$, satisfy

$$\|\alpha_j - \alpha_{j-1}\|_I \leq \lambda^\sigma (C\lambda^{1-\sigma})^{2^j}, \quad \|\alpha_0(\varphi) - \frac{1}{\pi}\varphi\|_I \leq 2\lambda, \quad j \in \mathbb{N}. \tag{8.10}$$

8.3. **Constructing solutions of the monodromy equation.** First, we check that the monodromy equation can be rewritten in the form required by Proposition 8.2. Then, we check that Proposition 8.2 is applicable for sufficiently small $\varepsilon \in \mathcal{D} \subset (0,1)$ where $\mathcal{D}$ is a set with the properties described above. Last, we use Proposition 8.2 to construct the solutions of the monodromy equation outside a set $E_\infty \subset I$ of Lebesgue measure $O(\lambda^{\sigma/2})$.

8.3.1. *The coefficient* a_0. Here, we check that the zero-th Fourier coefficient of the coefficient $a(z, E)$ of the monodromy matrix admits the representation

$$a_0 = \exp\left(\frac{i}{\varepsilon}\Phi(E) + i\Omega(E) + i\phi(E)\right) + O(e^{\frac{2i}{\varepsilon}\Phi(E)}\lambda^2), \tag{8.11}$$

where $\phi(E)$ is real analytic and $\phi(E) = o(1)$ uniformly in $E \in V_0$, the constant neighborhood of E_0 from Theorem 1.4.

Therefore, we first write $a_0 = \exp\left(\frac{i}{\varepsilon}\Phi(E) + i\Omega(E) + ig(E)\right)$. By Theorem 1.4, the function g is analytic and $g(E) = o(1)$ uniformly in $E \in V_0$. Let $\phi(E) = (g(E) + \overline{g(\overline{E})})/2$, and let $\phi_1(E) = g(E) - \overline{g(\overline{E})}$. The function ϕ is real analytic, and $\phi(E) = o(1)$ uniformly in $E \in V_0$. Hence, it suffices to prove that $\phi_1 = O(a_0\lambda^2)$ uniformly in $E \in V_0$. For complex E, the relation $M(z, E) \equiv 1$ implies that

$$a(z, E)\overline{a(\overline{z}, \overline{E})} = 1 + b(z, E)\overline{b(\overline{z}, \overline{E})}$$

(compare with (6.11)). Now, we use the representation $a(z, E) = a_0(E) + a_1(z, E)$, the fact that Φ and Ω are real analytic, and the estimates (1.13) and (1.14) for a_1 and b from Theorem 1.4 to obtain

$$\exp(g(E) - \overline{g(\overline{E})}) + a_0(E)\overline{a_1(\overline{z}, \overline{E})} + a_1(z, E)\overline{a_0(\overline{E})} = 1 + O(a_0^2\lambda^2), \quad z \in \mathbb{R}.$$

Integrating this formula in z from 0 to 1, and recalling that $\int_0^1 a_1 dz = 0$, we get $\exp(g(E) - \overline{g(\overline{E})}) = 1 + O(a_0^2\lambda^2)$. This implies $\phi_1 = O(a_0^2\lambda^2)$, hence (8.11).

8.3.2. *A new parameterization of the monodromy matrix.* We consider the monodromy matrix as a function of the parameter $\varphi(E) = \Phi(E)/\varepsilon + \Omega(E) + \phi(E)$ instead of the parameter E. Pick $\Delta > 0$ and let I_Δ be the Δ-neighborhood of the interval I in $\mathbb{C}$. We fix Δ so that I_Δ is inside a constant compact subset of V_0. We show that, for Δ sufficiently small but independent of ε, and sufficiently small ε, the mapping $E \mapsto \varphi$ is an analytic isomorphism of I_Δ onto its image.

First, we note that

$$\frac{d\varphi}{dE}(E) = \frac{1}{\varepsilon}\Phi'(E) + O(1), \quad E \in I_\Delta, \tag{8.12}$$

where the estimate of the error term follows from the Cauchy estimates applied to Ω and ϕ. Then, we study Φ'. By (6.4) and (6.2), $\frac{\partial \kappa_{n_0}}{\partial E} = \sigma k_p'(E - W(\zeta))$, where $\sigma \in \{-1, +1\}$ depends only on n_0, and k_p is the main branch of the Bloch quasi-momentum. Therefore,

$$\Phi'(E) = \sigma \int_0^{2\pi} k_p'(E - W(\zeta)) d\zeta. \tag{8.13}$$

Recall that $k'(\mathcal{E})$ is real and does not vanish inside the n_0-th spectral band, and that, for $E \in J$ and for real ζ, $\mathcal{E} = E - W(\zeta)$ is inside this band. Therefore, $\Phi'(E)$ is real and of fixed sign on J. This and (8.12) imply that $\frac{d\varphi}{dE}$ is bounded away from 0 uniformly on I for sufficiently small ε. Therefore, for sufficiently small Δ, and for sufficiently small ε, the function $\varphi(E)$ is an analytic isomorphism of I_Δ onto its image.

Introduce the function $E(\varphi)$ inverse to $\varphi(E)$. It is defined on $\varphi(I_\Delta)$. Note that that $\mathcal{I} = \varphi(I)$ is an interval of the real axis; its length is of order $O(1/\varepsilon)$.

8.3.3. *The monodromy equation.* Let us check that one can apply Proposition 8.2 to equation (7.3). First, we note that $\det M \equiv 1$, and that the monodromy matrix already has the form (1.12). Consider the monodromy matrix as a function of $\varphi \in \varphi(I_\Delta)$. It can be represented in the form (8.5) with the diagonal matrix $D = \begin{pmatrix} e^{i\varphi} & 0 \\ 0 & e^{-i\varphi} \end{pmatrix}$ and the matrix A defined by

$$\lambda A_{11}(z, \varphi) = a_1(z, E) + (a_0(E) - e^{i\varphi}), \quad \lambda A_{12}(z, \varphi) = b(z, E), \quad E = E(\varphi).$$

Let us study the matrix A. As the monodromy matrix, it is analytic in z in the strip $|\operatorname{Im} z| \leq y/\varepsilon$. Fix Y so that $0 < Y \leq S_0 - S$. We now show that $\|A\|_{Y/\varepsilon, \varphi(I)}$ is bounded uniformly in ε. As A is analytic in $(z, \varphi) \in \{|\operatorname{Im} z| \leq y/\varepsilon\} \times \varphi(I_\Delta)$, we have only to check that the elements of A are uniformly bounded in $z \in \{|\operatorname{Im} z| < Y/\varepsilon\}$ and for φ in a constant neighborhood of $\varphi(I)$, i.e. that its elements, considered as functions of (z, E), are uniformly bounded in $\{|\operatorname{Im} z| \leq Y/\varepsilon\} \times [I + D(0, c\varepsilon)]$, where c is a fixed positive constant.

Since $\Phi(E)$ is real analytic in E, in such a neighborhood, $|e^{\frac{i}{\varepsilon}\Phi}|$ is bounded by a constant independent of E and ε. So, by (8.11), $a_0(E) - e^{i\varphi} = O(\lambda^2)$. By Theorem 1.4, one has

$$a_1 = O\left(e^{-\frac{S_0}{\varepsilon} + |\operatorname{Im} z|}\right) = O\left(e^{-(S_0 - Y)/\varepsilon}\right) = O\left(e^{-S/\varepsilon}\right) = O(\lambda).$$

Therefore, $|A_{11}(z, E)| = O(1)$. Similarly, one proves that $|A_{12}| = O(1)$. Both estimates are uniform in $\{|\operatorname{Im} z| \leq Y/\varepsilon\} \times [I + i(0, c\varepsilon)]$. This implies that

$$\|A(z, \varphi)\|_{Y/\varepsilon, \varphi(I)} = O(1).$$

8.4. **Locating the absolutely continuous spectrum.** The length of the interval $\varphi(I)$ is of order $1/\varepsilon$. But, Proposition 8.2 can only be applied to the monodromy equation for φ in an interval of finite length independent of ε. So, we divide $\varphi(I)$ into $L_0(\varepsilon) = O(1/\varepsilon)$ subintervals of unit length and apply Proposition 8.2 to each of them. This can be done since the constants in Proposition 8.2 depend only on the length of the interval of φ, on the size of the band of analyticity of A in z and on the exponent σ. As a result, by Proposition 8.2 and in view of Remark 8.3, one proves that if

- σ is a fixed number in $(0,1)$ and we set $r = Y/\varepsilon$,
- ε is small enough,
- φ is outside a subset $\tilde{\Phi}_\infty$ of $\varphi(I)$ of measure $O(\frac{1}{\varepsilon}\lambda^{\sigma/2})$,
- h belongs to the set $H_\sigma(\lambda)$ defined in (8.6),

then the monodromy equation (7.3) has solutions described in Proposition 8.2.

The sets $\mathcal{D}$ and $H_\sigma(\lambda)$. Define $\mathcal{D}$ as the set $\varepsilon \in (0,1)$ such that $h = \frac{2\pi}{\varepsilon} \mod 1$ belongs to $H_\sigma(\lambda)$. Let us show that $\operatorname{mes}(\mathcal{D})$ satisfies (1.4). One has

$$\operatorname{mes}((0,\varepsilon)\setminus\mathcal{D}) \leq \sum_{n=N(\varepsilon)}^{\infty} \int_{(0,1)\setminus H_\sigma} \frac{2\pi}{(h+n)^2}dh \quad \text{where} \quad H_\sigma = H_\sigma(e^{-\frac{Sn}{2\pi}}),$$

and $N(\varepsilon)$ is equal to the integer part of $2\pi/\varepsilon$. Therefore,

$$\begin{aligned}\operatorname{mes}((0,\varepsilon)\setminus\mathcal{D}) &\leq 2\pi \sum_{n=N(\varepsilon)}^{\infty} \frac{1}{n^2}\sum_{k=1}^{\infty}\sum_{l=0}^{k}\frac{1}{k^3}\exp\left(-\frac{\sigma n S}{2\pi}\right)\\ &\leq C\sum_{n=N(\varepsilon)}^{\infty}\frac{1}{n^2}\exp\left(-\frac{\sigma n S}{2\pi}\right)\\ &\leq C\frac{1}{N(\varepsilon)^2}\exp\left(-\frac{\sigma N(\varepsilon) S}{2\pi}\right) \leq C\varepsilon^2\lambda^\sigma.\end{aligned}$$

This proves (8.3).

The sets $\tilde{\Phi}_\infty$ and E_∞. Now, let $E_\infty = E(\tilde{\Phi}_\infty)$ where the function $E(\varphi)$ is the inverse of $\varphi(E)$. Clearly, $\operatorname{mes}(E_\infty) = \int_{\tilde{\Phi}_\infty}\frac{dE}{d\varphi}d\varphi$. This, the estimate for $\operatorname{mes}(\tilde{\Phi}_\infty)$ and the asymptotics (8.12) imply that $\operatorname{mes}(E_\infty) = O(\lambda^{\sigma/2})$.

The absolutely continuous spectrum. We have shown that, under the assumptions of Theorem 8.1, equation (7.3) has bounded solutions for $E \in I \setminus E_\infty$. Let $\chi(x)$ be a bounded vector solution of equation (7.3). Then, $\chi_k = \chi(z+hk)$ is a bounded vector solution of the monodromy equation. Therefore, the Lyapunov exponent for the monodromy equation is zero on $I \setminus E_\infty$. Theorem 1.2 then implies that the Lyapunov exponent for the family of equations (0.1) is zero on $I \setminus E_\infty$. Applying the Ishii-Pastur-Kotani Theorem, we see that the essential closure of $I \setminus E_\infty$ is in the absolutely continuous spectrum of the equation family (0.1).

8.5. **Bloch-Floquet solutions for** (0.1)**.** To complete the proof of Theorem 8.1, we have only to check the statements on the solutions $\psi_\pm$. We break the proof into a few steps.

1. The monodromy matrix $M(x,z)$ described in Theorem 1.4 corresponds to $(\psi_\pm)$, a consistent basis constructed in terms of the functions $f(x,\zeta)$ and $f^*(x,\zeta)$, solutions of equation (3.2). More precisely, ψ_+ and ψ_- are related to f and f^* by the change of variables described in the beginning of section 6:

$$\psi_+(x,z) = f(x-z,\varepsilon z) \text{ and } \psi_-(x,z) = f^*(x-z,\varepsilon z).$$

The functions $\zeta \mapsto f$ and $\zeta \mapsto f^*$ are analytic in a constant neighborhood of the real axis.

2. Consider the "continuous" monodromy equation (7.3) with the monodromy matrix M described in Theorem 1.4. Let χ_- and χ_+ be the first and second columns of the matrix solution to the monodromy equation (7.3) constructed in Proposition 8.2. By (8.8), they are Bloch-Floquet solutions of (7.3). Their quasi-momenta are equal to $\pm\varphi_\infty(E) = \pm\varphi_\infty(\varphi(E))$. Their periodic components, $U_\pm$, are just the first and second columns of the matrix $U(z,E) = U(z,\varphi(E))$ in (8.8). The function $E \mapsto \varphi(E)$ is defined in section 8.3.2. Hence,

$$\chi_\pm(z) = e^{\mp i\varphi_\infty(E)z} U_\pm(z,E). \tag{8.14}$$

The functions $(z,E) \mapsto U_\pm(z,E)$ are analytic in z in the strip $|\mathrm{Im}\, z| < r/2$ (where $r = Y/\varepsilon$) and Lipschitz continuous for E in $I \setminus E_\infty$. As $U \in \mathcal{M}$, one has

$$U_+ = \sigma_1 U_-^*, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{8.15}$$

3. Define the vectors

$$\Psi_\pm(x,z) = iF(x-z,\varepsilon z)\sigma U_\pm(z), \tag{8.16}$$

where

$$F(x,z) = \begin{pmatrix} f & f^* \\ \frac{df}{dx} & \frac{df^*}{dx} \end{pmatrix} \text{ and } \sigma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

By Theorem 7.2, ψ_+ (resp. ψ_-), the first component of Ψ_+ (resp. Ψ_-), is a Bloch-Floquet solution of (0.1). Moreover, by the same theorem and (8.14), $\Psi_\pm$ admit the representations

$$\Psi_\pm(x,z) = e^{\pm ip(E)x} P_\pm(x-z,\varepsilon x), \tag{8.17}$$

where $p(E) = \dfrac{\varepsilon h}{2\pi}\varphi_\infty(E)$ and $(x,\zeta) \mapsto P_\pm(x,\zeta)$ are 1-periodic in x and 2π-periodic in ζ. This proves the representation (1.5) and the properties of $p(E)$ described in Theorem 1.1.

We also see that $P_\pm$ has the announced periodicity properties and is Lipschitz continuous in E. Let us discuss the regularity of $P_\pm$ in (x,ζ). By (8.16) and (8.17), one has

$$P_\pm(x,\zeta) = ie^{\mp ip(E)x} F(x,\zeta-\varepsilon x)\sigma U_\pm(\zeta/\varepsilon - x). \tag{8.18}$$

This immediately implies that $P_\pm$ are H^2_{loc}-functions in x and are analytic in ζ in a constant neighborhood of the real line.

Finally, check that $\overline{P_+} = P_-$ for z real. By (8.15), (8.18) and the definition of F, one has

$$\overline{P_+} = -ie^{ipx} F\sigma_1\sigma\sigma_1 U_- = ie^{ipx} F\sigma U_- = P_-.$$

This completes the proof of Theorem 1.1. $\square$

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