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ON THE MOD p COHOMOLOGY OF $BPU(p)$

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Abstract. We study the mod p cohomology of the classifying space of the projective unitary group $PU(p)$. We first prove that conjectures due to J.F. Adams and Kono and Yagita (1993) about the structure of the mod p cohomology of the classifying space of connected compact Lie groups hold in the case of $PU(p)$. Finally, we prove that the classifying space of the projective unitary group $PU(p)$ is determined by its mod p cohomology as an unstable algebra over the Steenrod algebra for $p > 3$, completing previous work by Dwyer, Miller and Wilkerson (1992) and Broto and Viruel (1998) for the cases $p = 2, 3$.

1. Introduction

Compact Lie groups provide an example of the classical mathematical maxim: "the richer the mathematical structure of an object, the more rigid it is". For example the structure of a connected compact Lie group can be completely recovered (up to local isomorphism) from the Dynkin diagram or a maximal torus normalizer [9].

In homotopy theory, one expects the rigidity in the structure of a compact Lie group G to be inherited by the classifying space BG and "related structures". Indeed, in the appropriate homotopical setting of p-compact groups [13], maximal torus normalizers do characterize the isomorphism type of BG, at least at odd primes [3].

Our aim here is to study the mod p cohomology of BG, namely $H^*(BG; \mathbb{F}_p)$, and to prove several conjectures in the case when $G = PU(p)$, the projective unitary group obtained as the quotient of the unitary group of rank p, $U(p)$, by the subgroup $\{\mathrm{Diag}(\alpha, \ldots, \alpha) \mid \alpha \in S^1\}$ of diagonal matrices.

In [13, Theorem 1.1], it is shown that $H^*(BG; \mathbb{F}_p)$ is a Noetherian algebra for any compact connected Lie group G, so by [31, Theorem 1.4] (or directly [30, Theorem 6.2]) we know that the kernel of the natural map

$$H^*(BG; \mathbb{F}_p) \longrightarrow \lim_{\substack{\longleftarrow \\ \mathcal{A}_p(G)}} H^*(BE; \mathbb{F}_p), \tag{1}$$

where $\mathcal{A}_p(G)$ stands for the Quillen category of elementary abelian p-subgroups of G [30, 31, 17, 12], contains only nilpotent elements. For $p > 2$, a stronger conjecture was made by Adams. We say that the mod p cohomology of the space

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BG is detected by elementary abelian p-subgroups if the natural map (1) is a monomorphism.

Conjecture 1.1 (J.F. Adams)**.** *Let G be a compact connected Lie group, and let p be an odd prime. Then the mod p cohomology of BG is detected by elementary abelian p-subgroups.*

Conjecture 1.1 trivially holds in the p-torsion-free cases (see [3, Theorem 12.1]). In the case of torsion, only a few examples, all of them for $p = 3$, are known: F_4 [6, Teorema 5], E_6 [25] and $PU(3)$ [16, Theorem 3.3]. Our first result generalizes the last reference, and we prove (in Theorem 2.5):

Theorem A. *For every odd prime p, the group $PU(p)$ verifies Conjecture* 1.1 *at p, i.e. $H^*(BPU(p); \mathbb{F}_p)$ is detected by elementary abelian p-subgroups.*

Knowledge of the structure of $H^*(BG; \mathbb{F}_p)$ plays an important role in studying other generalized cohomologies of BG as is shown in [16]. Understanding Milnor primitive operations (see Section 3) is a crucial step in the use of the Atiyah-Hirzebruch spectral sequence [20, p. 496], and this leads to a new conjecture [16, Conjecture 5]:

Conjecture 1.2 (Kono-Yagita)**.** *Let G be a connected compact Lie group, and let Q_m denote the Milnor primitive operators. Then for each odd-dimensional element $x \in H^*(BG; \mathbb{F}_p)$, there is i such that $Q_m x \neq 0$ for all $m \geq i$.*

Our second result generalizes the case of $PU(3)$ shown in [16], and we prove (in Theorem 3.2):

Theorem B. *For every odd prime p, the group $PU(p)$ verifies Conjecture* 1.2 *at p, i.e. for each odd-dimensional element $x \in H^*(BPU(p); \mathbb{F}_p)$, there is i such that $Q_m x \neq 0$ for all $m \geq i$, where Q_m are the Milnor primitive operators.*

Remark 1.3. We note that while the proofs of Conjectures 1.1 and 1.2 in previously-known cases involve a precise understanding of the cohomology rings involved, i.e. generators and relations, we prove Theorems A and B by geometrical methods, without using any information about the algebra structure of $H^*BPU(p)$.

The structure of $H^*(BG; \mathbb{F}_p)$ is very particular, and one might expect any space X with the same mod p cohomology to be closely related to BG. This idea is captured in the next conjecture [28, Conjecture 4.4]:

Conjecture 1.4. *Let G be a compact connected Lie group, and let X be a p-complete space such that $H^*(X; \mathbb{F}_p) \cong H^*(BG; \mathbb{F}_p)$ as algebras over the mod p Steenrod algebra $\mathcal{A}_p$. Then $X \simeq BG^\wedge_p$.*

The first evidence for Conjecture 1.4 was provided by Dwyer, Miller and Wilkerson [10] who settled $G = SU(2) = S^3$ at $p = 2$. In [11], the same authors settled the case when p does not divide the order of the Weyl group of G. Notbohm [26] proved Conjecture 1.4 when p divides the order of the Weyl group of G, but BG has no p-torsion. When p-torsion exists, there are only a few known results [7, 33, 34, 35]. In Section 4, we prove:

Theorem C. *Let X be a p-complete space such that $H^*(X; \mathbb{F}_p) \cong H^*(BPU(p); \mathbb{F}_p)$ as an unstable algebra over the Steenrod algebra $\mathcal{A}_p$. Then X is homotopy equivalent to $BPU(p)^\wedge_p$.*

Notation. Here, all spaces are assumed to have the homotopy type of CW-complexes, and "completion" means Bousfield-Kan completion [5]. For a given space X, we write H^*X ($\widetilde{H}^*X$) for the (reduced) mod p cohomology $H^*(X;\mathbb{F}_p)$ and $X_p^\wedge$ for the Bousfield-Kan $(\mathbb{Z}_p)_\infty$-completion or p-completion of the space X. Throughout this paper, p is an odd prime number unless otherwise stated. Given a group G and a $\mathbb{Z}G$-module M, we write $\mathcal{H}^*(G;M)$ for the cohomology of G with (twisted) coefficients in M. The acronym Bss denotes Bockstein spectral sequence. We assume that the reader is familiar with Lannes' theory [18].

2. Adams' conjecture

In this section we prove Adams' conjecture (Conjecture 1.1) for the group $PU(p)$ at the prime $p > 2$. Given a connected Lie group G, $T(G) \subset G$ denotes a maximal torus and $N(G) \subset G$ denotes its normalizer. The p-normalizer of the maximal torus $T(G)$, namely $N_p(G) \subset N(G)$, is defined as the preimage of a p-Sylow subgroup in the Weyl group of G, $W_G = N(G)/T(G)$. If X is a subgroup of G, i_X denotes the inclusion morphism $X \hookrightarrow G$.

Let ω be a primitive p-th root of unity, and consider the following matrices in $SU(p)$:

- $A = \mathrm{Diag}(\omega, \ldots, \omega)$,
- $B = \mathrm{Diag}(1, \omega, \omega^2, \ldots, \omega^{p-1})$,
- P is the permutation matrix corresponding to the cycle $(1, \ldots, p) \in \Sigma_p$.

Our first result in this section describes some cohomological properties of $N_p(U(p))$.

Lemma 2.1. *The cohomology $H^*BN_p(U(p))$ is detected by the elementary abelian subgroups $V_t = (\mathbb{Z}/p)^p \subset T(U(p))$, the maximal elementary abelian toral subgroup, and $V_n = \langle A, P\rangle$. Moreover, if $y \in H^*BN_p(U(p))$ is not detected by V_n (thus detected by V_t), then y is a permanent cycle in the Bockstein spectral sequence associated to $H^*BN_p(U(p))$.*

Proof. Since $N_p(U(p)) \cong S^1 \wr \mathbb{Z}/p$, by [1, Lemma 4.4] we know that $H^*BN_p(U(p))$ is detected by the subgroups $T(U(p))$ and $\widetilde{V_n} = \langle Z(U(p)), P\rangle \cong S^1 \times \mathbb{Z}/p$, and therefore by the elementary abelian subgroups V_t and V_n defined above.

Now, let $y \in H^*BN_p(U(p))$ be such that $Bi^*_{V_n}(y) = 0$, so that $Bi^*_{V_t}(y) \neq 0$. Therefore $Bi^*_{\widetilde{V_n}}(y) = 0$, $Bi^*_{T(U(p))}(y) \neq 0$, and y is even dimensional. If y is not a permanent cycle in the Bss associated to $H^*BN_p(U(p))$, then there exists $r > 0$ such that one of the following hold:

- $y = \beta_r x$ for some $x \in H^*BN_p(U(p))$. Comparing the $(r+1)$-stage of the Bss's of $H^*BT(U(p))$ and $H^*BN_p(U(p))$, we see that the trivial class, represented by y, is mapped to the non-trivial class represented by $Bi^*_{T(U(p))}(y)$, which is impossible.
- $\beta_r y = x \neq 0$ for some $x \in H^*BN_p(U(p))$. Then, x is odd dimensional and so $Bi^*_{T(U(p))}(x) = 0$, hence $Bi^*_{\widetilde{V_n}}(x) \neq 0$. Comparing the r-stage of the Bss's of $H^*B\widetilde{V_n}$ and $H^*BN_p(U(p))$ we see that the non-trivial class represented by $Bi^*_{\widetilde{V_n}}(x)$ must be a cycle, but every odd-dimensional class in $H^*B\widetilde{V_n}$ has a non-trivial Bockstein and so cannot be a cycle in any stage of the Bss.

Since none of the above holds, y must be a permanent cycle in the Bss associated to $H^*BN_p(U(p))$. □

We now compare $N_p(PU(p))$ and $N_p(SU(p))$.

Lemma 2.2. *The groups $N_p(PU(p))$ and $N_p(SU(p))$ are isomorphic.*

Proof. Note first that $N_p(PU(p)) = N_p(SU(p))/\{\mathrm{Diag}(\alpha,\dots,\alpha) \mid \alpha \in S^1\}$. Now, every element in $N_p(SU(p))$ can be written in a unique way as $\mathrm{Diag}(z_1,\dots,z_p)P^i$, where P is the permutation matrix corresponding to the cycle $(1,\dots,p) \in \Sigma_p$. Then $\varphi\colon N_p(PU(p)) \longrightarrow N_p(SU(p))$, given by

$$\varphi([\mathrm{Diag}(z_1,\dots,z_p)P^i]) = \mathrm{Diag}(\frac{z_1}{z_2},\dots,\frac{z_{p-1}}{z_p},\frac{z_p}{z_1})P^i,$$

provides the desired isomorphism. □

We now prove Conjecture 1.1 for $N_p(SU(p))$:

Lemma 2.3. *The cohomology $H^*BN_p(SU(p))$ is detected by elementary abelian subgroups $V_{st} = (\mathbb{Z}/p)^{(p-1)} \subset T(SU(p))$, the maximal elementary abelian toral subgroup, and $V_n = \langle A, P\rangle$.*

Proof. According to Lemma 2.1, $H^*BN_p(U(p))$ is detected by the elementary abelian subgroups $V_t, V_n \subset N_p(U(p))$. Now the fibration

$$S^1 \longrightarrow BSU(p) \longrightarrow BU(p)$$

restricts to a fibration

$$S^1 \longrightarrow BN_p(SU(p)) \xrightarrow{Bj} BN_p(U(p)), \tag{2}$$

whose Gysin sequence is [20, Example 5.C]

$$\cdots \to H^nBN_p(U(p)) \xrightarrow{\gamma} H^{n+2}BN_p(U(p)) \xrightarrow{Bj^*} H^{n+2}BN_p(SU(p)) \xrightarrow{d} H^{n+11}BN_p(U(p)) \to \cdots$$

where γ is multiplication by the two-dimensional class $c_2 \in H^2BN_p(U(p))$ that classifies the fibration (2).

Let $x \in H^*BN_p(SU(p))$ and consider the following cases:

<u>$d(x) \neq 0$</u>. Let V be either V_t or V_n detecting $d(x)$, and define $V' = V \cap N_p(SU(p))$. Then V' is either V_{st} or V_n and it appears in the fibration

$$S^1 \longrightarrow BV' \longrightarrow B\langle V, Z(U(p))\rangle.$$

Comparing the Gysin sequence of the latter fibration with that of (2) we observe that V' detects the element x.

<u>$d(x) = 0$</u>. Thus $x = Bj^*(y)$ for some $y \in H^*BN_p(U(p))$. Let V be the elementary abelian subgroup detecting y, and let $V \cap N_p(SU(p)) \xrightarrow{k} V$ be the inclusion. We consider the following cases:

- If $Bk^*Bi_V^*(y) \neq 0$ (which always happens if $V = V_n$), then x is detected by $V \cap N_p(SU(p))$, that is, by V_{st} or V_n.
- If $Bk^*Bi_V^*(y) = 0$ (thus $V = V_t$). Then we may assume that y is not detected by V_n. By Lemma 2.1, y is a permanent cycle in the Bss associated to $H^*BN_p(U(p))$, hence y is the mod p reduction of an integral class $\bar{y} \in H^*(BN_p(U(p)); \mathbb{Z}_p^\wedge)$. As $Bk^*Bi_{V_t}^*(y) = 0$, then $Bi_{T(SU(p))}^*Bi_{T(U(p))}^*(y) = 0$.

Now, considering $\mathbb{Q}_p^\wedge$-coefficients, $Bi^*_{T(SU(p))}Bi^*_{T(U(p))}(\bar{y}\otimes_{\mathbb{Q}}1)=0$, and comparing the Gysin sequence of the fibration

$$S^1 \longrightarrow BT(SU(p)) \xrightarrow{Bj} BT(U(p))$$

with that of (2), we observe that $Bi^*_{T(U(p))}(\bar{y}\otimes_{\mathbb{Q}}1)$ is a multiple of

$$Bi^*_{T(U(p))}(\bar{c_2}\otimes_{\mathbb{Q}}1),$$

where our original c_2 is the mod p reduction of the integral class $\bar{c_2}\in H^2(BN_p(U(p));\mathbb{Z}_p^\wedge)$. But

$$H^*_{\mathbb{Q}_p^\wedge}BN_p(U(p)) \overset{Bi^*_{T(U(p))}}{\cong} \big(H^*_{\mathbb{Q}_p^\wedge}BT(U(p))\big)^{\mathbb{Z}/p},$$

hence there exists an integral class $\bar{z}\in H^*(BN_p(U(p));\mathbb{Z}_p^\wedge)$ such that $\bar{z}\bar{c_2}\otimes_{\mathbb{Q}}1=\bar{y}\otimes_{\mathbb{Q}}1$. If z denotes the mod p reduction of the class $\bar{z}$, then there exists $a\in\mathbb{F}_p$ such that $Bi^*_{T(U(p))}(y-azc_2)=0$, hence $\tilde{y}\overset{def}{=}y-azc_2$ is detected by V_n. Moreover $Bj^*(\tilde{y})=x$, hence applying the previous case x is detected by V_n.

In all cases, then, x is detected by either V_{st} or V_n. □

An easy consequence of the previous lemmas is

Lemma 2.4. *The mod p cohomology of $BN(PU(p))$ is detected by the elementary abelian p-subgroups $V_{pt}=(\mathbb{Z}/p)^{(p-1)}\subset T(PU(p))$ and $V_{pn}=\langle[B],[P]\rangle$.*

Proof. Combining Lemmas 2.2 and 2.3 we obtain that $H^*BN_p(PU(p))$ is detected by the elementary abelian p-subgroups defined above. Then, as the index

$$[N(PU(p))\colon N_p(PU(p))]=(p-1)!$$

is nonzero in $\mathbb{F}_p$, the transfer argument [36, Lemma 6.7.17] shows that

$$H^*BN(PU(p)) \longrightarrow H^*BN_p(PU(p))$$

is a monomorphism. Therefore $H^*BN(PU(p))$ is also detected by elementary abelian p-subgroups. □

Finally,

Theorem 2.5. *The mod p cohomology of $BPU(p)$ is detected by the elementary abelian p-subgroups $V_{pt}=(\mathbb{Z}/p)^{(p-1)}\subset T(PU(p))$ and $V_{pn}=\langle[B],[P]\rangle$.*

Proof. According to [4, §6], the Euler characteristic $\chi\big(PU(p)/N(PU(p))\big)$ is 1, hence $H^*BPU(p)\longrightarrow H^*BN(PU(p))$ is a monomorphism by the transfer argument [13, Theorem 9.13]. As $H^*BN(PU(p))$ is detected by the elementary abelian subgroups V_{pt} and V_{pn} by previous lemma, $H^*BPU(p)$ is as well. □

Remark 2.6. According to [8, Corollary 3.4] or [3, Theorem 9.1], the group $PU(p)$ contains exactly two conjugacy classes of maximal elementary abelian subgroups. Therefore, the subgroups V_{pt} and V_{pn} are the representatives of those two conjugacy classes.

The following series of lemmas describe the interplay between the cohomology of $BPU(p)$ and that of BG when G is one of the subgroups described in this section.

Lemma 2.7. *$\widetilde{H}^{\leq 3}BPU(p)=\mathbb{F}_p\{y_2\}\oplus\mathbb{F}_p\{y_3\}$, where $y_3=\beta y_2\neq 0$, $|y_2|=2$ and $|y_3|=3$.*

Proof. The space $BPU(p)$ is 1-connected and therefore $H_2(BPU(p);\mathbb{Z}) \cong \pi_1 PU(p) = \mathbb{Z}/p$. Then, by the Universal Coefficient Theorem for cohomology [19, Theorem 4.3 in p. 163] we obtain $H^1BPU(p) = 0$. We now consider the Serre spectral sequence for the fibration

$$B\mathbb{Z}/p \longrightarrow BSU(p) \longrightarrow BPU(p)$$

that converges to $H_*(BSU(p);\mathbb{Z})$, thus $E^\infty_{3,0} = 0$. There are only two possible non-trivial differentials starting from $E^*_{3,0}$. The first one, $d_2\colon E^2_{3,0} \longrightarrow E^2_{1,1}$, is trivial, since $E^2_{1,1} = H_1(BPU(p); H_1(B\mathbb{Z}/p,\mathbb{Z})) = H_1(BPU(p);\mathbb{Z}/p) = 0$, and also the second one, $d_3\colon E^3_{3,0} \longrightarrow E^3_{0,2}$, vanishes, since $E^2_{0,2} = H_0(BPU(p); H_2(B\mathbb{Z}/p,\mathbb{Z})) = H_1(BPU(p); 0) = 0$ and then $E^3_{0,2} = 0$, too. Hence $E^2_{3,0} = H_3(BPU(p);\mathbb{Z})$ is trivial. Therefore the Universal Coefficient Theorem for cohomology and the description of the Bockstein morphism [20, p. 455] imply the statement. □

Lemma 2.8. *Set $V = (\mathbb{Z}/p)^2$ and let $H^*BPU(p) \xrightarrow{\psi} H^*BV$ be a morphism of unstable Steenrod algebras, such that $\psi H^{odd}BPU(p) \neq 0$. Then ψ is completely determined by $\psi(y_2)$, where $y_2 \in H^2BPU(p)$ is the class defined in Lemma* 2.7.

Proof. Recall that $H^*BV = E(u_1,u_2) \otimes \mathbb{F}_p[v_1,v_2]$. According to Lannes' theory [18, Théorème 3.1.1] and [14, Theorem 1.1], $\psi = Bi^*$ for some group morphism $V \xrightarrow{i} PU(p)$. As $Bi^*H^{odd}BPU(p) = \psi H^{odd}BPU(p) \neq 0$, then i cannot factor through $T(PU(p))$ (otherwise $Bi^*H^{odd}BPU(p) = 0$), and therefore $i(V)$ equals V_{pn} up to conjugation. Hence $\psi = Bi^* = Bf^*Bi^*_{V_{pn}}$ for some $f \in \mathrm{GL}_2(p)$ and, in view of Theorem 2.5, $\psi|_{H^{odd}BPU(p)}$ is a monomorphism.

Now, using the description of $H^*BPU(p)$ in Lemma 2.7, $0 \neq \psi(y_3) = \psi(\beta y_2) = \beta\psi(y_2)$ implies $\psi(y_2) \neq 0$. Moreover, $N_{PU(p)}(V_{pn})/V_{pn} = \mathrm{SL}_2(p)$ [8, Lemma 4.1], and therefore $Bf^*Bi^*_{V_{pn}}$ depends only on the class $[f] \in GL_2(p)/SL_2(p) \cong \mathbb{F}_p^*$. But the latter group acts faithfully on $(H^2BV)^{\mathrm{SL}_2(p)} = \mathbb{F}_p\{u_1u_2\} = \mathbb{F}_p\{Bi^*_{V_{pn}}(y_2)\}$ by scalar multiplication, so the class $[f]$ is determined by $\psi(y_2)$. □

Lemma 2.9. *If $p > 3$, then $H^nBN(PU(p)) \cong H^nBPU(p)$ for $n \leq 3$.*

Proof. $H^*BPU(p)$ is a summand of $H^*BN(PU(p))$, by a standard transfer argument, and therefore we just need to check that the Poincaré series of $H^*BPU(p)$ and $H^*BN(PU(p))$ agree in degrees ≤ 3. The low-dimensional cohomology of $BN(PU(p))$ can be easily computed by means of the Serre spectral sequence associated to the fibration

$$BT(PU(p)) \longrightarrow BN(PU(p)) \longrightarrow BW_{PU(p)}.$$

Note that $H^*BW_{PU(p)} = H^*B\Sigma_p = (H^*B\mathbb{Z}/p)^{\mathbb{Z}/(p-1)} = E(a_{2p-3}) \otimes \mathbb{F}_p[b_{2p-2}]$, hence $\widetilde{H}^{\leq 3}BW_{PU(p)} = 0$ for $p > 3$. Moreover $H^*BT(PU(p))$ is concentrated in even degrees, hence the non-trivial groups of total degree at most 3 in the spectral sequence are

$$E_2^{0,2} = \mathcal{H}^0\big(W_{PU(p)}; H^2BT(PU(p))\big) = \big(H^2BT(PU(p))\big)^{W_{PU(p)}} = \mathbb{Z}/p$$

and

$$E_2^{1,2} = \mathcal{H}^1\big(W_{PU(p)}; H^2BT(PU(p))\big).$$

In order to calculate the latter group we use the cohomology long sequence associated to the short exact sequence of coefficients

$$0 \longrightarrow H^2BT(PU(p)) \longrightarrow H^2BT(U(p)) \longrightarrow H^2BS^1 \longrightarrow 0.$$

Note also that

$$H^0\big(W(PU(p)); H^2BT(PU(p))\big) = \big(H^2BT(PU(p))\big)^{W(PU(p))} \cong \big(H^2BT(U(p))\big)^{W(U(p))} = H^0\big(W(U(p)); H^2BT(U(p))\big),$$

and $\mathcal{H}^1\big(W_{U(p)}; H^2BT(U(p))\big) \cong H^1(\Sigma_{p-1}; \mathbb{Z}/p) = 0$ by Shapiro's lemma [36, Section 6.3]. Therefore $\mathcal{H}^1\big(W(PU(p)); H^2BT(PU(p))\big) \cong H^0(\Sigma_p; \mathbb{Z}/p) = \mathbb{Z}/p$, where the isomorphism is induced by the connecting morphism, and the Poincaré series of $H^*BPU(p)$ and $H^*BN(PU(p))$ agree in degrees ≤ 3. □

The last lemma in this section provides a characterization of the homomorphism $Bi^*_{N(PU(p))}$. If X is a subgroup of $N(PU(p))$, j_X denotes the inclusion morphism $X \hookrightarrow N(PU(p))$.

Lemma 2.10. *Let $H^*BPU(p) \xrightarrow{a} H^*BN(PU(p))$ be any homomorphism of algebras over the Steenrod algebra. If $Bj^*_{T(PU(p))}a = Bi^*_{T(PU(p))}$, then $a = Bi^*_{N(PU(p))}$.*

Proof. Since $H^*BN(PU(p))$ is detected by V_{pt} and V_{pn} (Lemma 2.4), it is enough to prove that $Bj^*_V a = Bj^*_V Bi^*_N$ for $V = V_{pt}$ and V_{pn}.

By hypothesis, the composition

$$H^*BPU(p) \xrightarrow{a} H^*BN(PU(p)) \xrightarrow{Bj^*_{T(PU(p))}} H^*BT(PU(p))$$

is the same as $Bi^*_{T(PU(p))}$. Therefore $Bj^*_{V_{pt}}a = Bj^*_{V_{pt}}Bi^*_N$.

Now consider the case of V_{pn}. According to Lemma 2.8, it is enough to check that $Bj^*_V a(y_2) = Bj^*_V Bi^*_{N(PU(p))}(y_2)$ for $y_2 \in H^2BPU(p)$ as defined in Lemma 2.7.

Recall from Lemma 2.7 that the class $y_2 \in H^2BPU(p)$ is the mod p reduction of the dual class representing $H_2(BPU(p);\mathbb{Z}) = \pi_1(BPU(p))$. As

$$\pi_1BT(PU(p)) \xrightarrow{\pi_1Bi_{T(PU(p))}} \pi_1BPU(p)$$

is surjective [23, Corollary 5.6], then

$$H_2(BT(PU(p));\mathbb{Z}) \xrightarrow{H_2Bi_{T(PU(p))}} H_2(BPU(p);\mathbb{Z})$$

is too, and the class y_2 is detected by $V_{pt} \subset T(PU(p))$. According to the previous case, $Bj^*_{V_{pt}}a(y_2) = Bj^*_{V_{pt}}Bi^*_{N(PU(p))}(y_2)$ and since

$$Bi^*_{N(PU(p))}\colon H^2BN(PU(p)) \cong H^2BPU(p) = \mathbb{F}_p\{y_2\}$$

by Lemma 2.9, then $a(y_2) = Bi^*_{N(PU(p))}(y_2)$ and $Bj^*_V a(y_2) = Bj^*_V Bi^*_{N(PU(p))}(y_2)$. □

3. Kono-Yagita's conjecture

Here we provide a proof of Theorem B (see Theorem 3.2) using Theorem A. Recall that for an odd prime p, the Milnor primitive operators are inductively defined as $Q_0 = \beta$ and $Q_{n+1} = \mathcal{P}^{p^n}Q_n - Q_n\mathcal{P}^{p^n}$, where β and $\mathcal{P}^j$ are the Bockstein and the j-th Steenrod power, respectively. These operators are derivations [21, Remark after Lemma 9], that is,

$$Q_n(xy) = Q_n(x)y + (-1)^{|x|}xQ_n(y). \tag{3}$$

We first show that Conjecture 1.2 holds for rank two elementary abelian groups:

Lemma 3.1. *Let x be an odd-dimensional element in $H^*B(\mathbb{Z}/p)^2 = E(x_1, x_2) \otimes \mathbb{F}_p[y_1, y_2]$. Then there exists an $i > 0$ such that $Q_m x$ is not trivial for all $m > i$.*

Proof. First note that $Q_n x_j = y_j^{p^n}$ and $Q_n y_j = 0$ for $j = 1, 2$. Now, if x is odd dimensional, then $x = x_1 f + x_2 g$, where $f, g \in \mathbb{F}_p[y_1, y_2]$. If $Q_n x$ is non-trivial for all n, the lemma holds. So, let i be an integer such that $Q_i x = 0$. Using the formula (3), $Q_i x = y_1^{p^i} f + y_2^{p^i} g = 0$ and therefore there exists $h \in \mathbb{F}_p[y_1, y_2]$ such that $f = y_2^{p^i} h$ and $g = -y_1^{p^i} h$. For $m > i$ we have that

$$Q_m x = y_1^{p^m} f + y_2^{p^m} g = y_1^{p^m} y_2^{p^i} h - y_2^{p^m} y_1^{p^i} h = (y_1^{p^m - p^i} - y_2^{p^m - p^i}) y_1^{p^i} y_2^{p^i} h$$

is non-trivial. □

We complete the proof of Theorem B with

Theorem 3.2. *For each odd-dimensional element $x \in H^*BPU(p)$, there exists an $i > 0$ such that such that $Q_m x \neq 0$ for all $m \geq i$.*

Proof. Let x be an odd-dimensional element in $H^*BPU(p)$. By Theorem 2.5, $Bi_V^*(x)$ is non-trivial for $i_V \colon V \longrightarrow PU(p)$, where V is either V_{pt} or V_{pn}. But $i_{V_{pt}}$ factors through a maximal torus $i_T \colon T(PU(p)) \longrightarrow PU(p)$, and $H^*BT(PU(p))$ is concentrated in even degrees, so $Bi^*_{V_{pt}}$ is trivial on elements of odd degree. Therefore $Bi^*_{V_{pn}}(x)$ is a non-trivial odd-dimensional element in H^*BV_{pn}. As $V_{pn} \cong (\mathbb{Z}/p)^2$, the previous lemma implies that there exists $i > 0$ such that for all $m > i$, $Q_m Bi^*_{V_{pn}}(x) = Bi^*_{V_{pn}}(Q_m x)$ is non-trivial. Thus for all $m > i$, $Q_m x$ is non-trivial. □

4. Cohomological uniqueness

In this section we prove Theorem C. If $p = 2$, then $PU(2) = SO(3)$, and the theorem is known [10]. If $p = 3$ the theorem is proved in [7]. Therefore it only remains to prove Theorem C when $p > 3$. In what follows, X is a p-complete space, such that there exists an isomorphism $\phi \colon H^*BPU(p) \cong H^*X$ as an unstable algebra over the Steenrod algebra $\mathcal{A}_p$, for $p > 3$.

The idea is to construct a homotopy equivalence $BPU(p)^\wedge_p \longrightarrow X$ by means of the cohomology decomposition of $BPU(p)$ given by p-stubborn subgroups [15].

Recall that for a compact Lie group G, a subgroup $P \subset G$ is called p-stubborn [15, p. 186] if the following conditions hold:

- The connected component of P is a torus and $\pi_0 P$ is a p-group.
- The quotient group $N_G(P)/P$ is finite and possesses no non-trivial normal p-subgroups.

Then, if $\mathcal{R}_p(G)$ denotes the full subcategory of the orbit category of G whose objects are the homogeneous spaces G/P where $P \subset G$ is p-stubborn, the natural map

$$\operatorname*{hocolim}_{G/P \in \mathcal{R}_p(G)} EG/P \longrightarrow BG$$

induces an isomorphism in homology with $\mathbb{Z}_{(p)}$-coefficients [15, Theorem 4].

The p-stubborn subgroups of $PU(p)$ are described in the next proposition.

Proposition 4.1. *The group $PU(p)$ contains exactly three p-stubborn subgroups up to conjugation:*

(1) *the maximal torus $T \overset{def}{:=} T(PU(p))$, where $N_{PU(p)}T/T \cong \Sigma_p$,*
(2) *the p-normalizer $N_p \overset{def}{:=} N_p(PU(p))$ of the maximal torus, where $N_{PU(p)}N_p/N_p \cong \mathbb{Z}/(p-1)$, and*
(3) *the group V_{pn} defined in Section 2, where $N_{PU(p)}V_{pn}/V_{pn} \cong \mathrm{SL}_2(p)$.*

Proof. By [15, Proposition 1.6], $P \subset SU(p)$ is a p-stubborn subgroup if and only if $P/(P \cap Z)$ is a p-stubborn subgroup of $PU(p)$, where $Z \cong \mathbb{Z}/p$ is the center of $SU(p)$. Finally, [29, Theorems 6, 8 & 10] describe all the conjugacy classes of p-stubborn groups in $SU(p)$, yielding the desired result. $\square$

Let $\widetilde{\mathcal{R}}_p(PU(p))$ be the full subcategory of $\mathcal{R}_p(PU(p))$ with only three objects: $PU(p)/T$, $PU(p)/N_p$, and $PU(p)/V_{pn}$.

Remark 4.2. Note that N_p contains just one subgroup T, and also just one conjugacy class of rank two elementary p-subgroups not contained in T, represented by V_{pn}. Therefore every morphism in $\widetilde{\mathcal{R}}_p(PU(p))$ consists in the composition of an automorphism and an inclusion.

Our strategy is to construct a homotopy commutative diagram (Lemma 4.4)

$$\{EG/P \simeq BP\}_{PU(p)/P \in \widetilde{\mathcal{R}}_p(PU(p))} \xrightarrow{f_P} X$$

which can be lifted to the topological category (after Proposition 4.6), so that we can recover $BPU(p)$ (up to p-completion) as a hocolim.

As every p-stubborn $P \subset PU(p)$ which $PU(p)/P \in \widetilde{\mathcal{R}}_p(PU(p))$ appears as a subgroup of $N \overset{def}{:=} N(PU(p))$, we first construct a map $BN \longrightarrow X$.

Theorem 4.3. *There exists a map $f_N \colon BN \longrightarrow X$ such that the diagram*

$$(4) \qquad \begin{array}{ccc} & H^*BN & \\ {\scriptstyle Bi_N^*}\nearrow & & \nwarrow{\scriptstyle f_N^*} \\ H^*BPU(p) & \underset{\cong}{\xrightarrow{\ \phi\ }} & H^*X \end{array}$$

commutes.

Proof. Let $i_{V_{pt}} \colon V_{pt} \longrightarrow T \longrightarrow PU(p)$ be the standard inclusion. By Lannes' theory [18, Théorème 3.1.1], there exists a map $f_{V_{pt}} \colon BV_{pt} \longrightarrow X$ such that $f_{V_{pt}}^* = Bi_{V_{pt}}^* \phi^{-1} \colon H^*X \longrightarrow H^*BV_{pt}$. By [18, Proposition 3.4.6],

$$T_{Bi_{V_{pt}}^*}^{V_{pt}} H^*BPU(p)_p^\wedge \cong H^* \operatorname{Map}(BV_{pt}, BPU(p)_p^\wedge)_{Bi_{V_{pt}}}.$$

Since

$$\operatorname{Map}(BV_{pt}, BPU(p)_p^\wedge)_{Bi_{V_{pt}}} \simeq BC_{PU(p)}(V_{pt})_p^\wedge \simeq BT_p^\wedge,$$

where $C_{PU(p)}(V_{pt})$ denotes the centralizer ([14], [27]), it follows that

$$T_{f_{V_{pt}}^*}^{V_{pt}} H^*X \cong T_{Bi_{V_{pt}}^*}^{V_{pt}} H^*BPU(p) \cong H^*BT_p^\wedge.$$

Because $T_{f^*_{V_{pt}}}^{V_{pt}} H^*X$ is zero in dimension 1, we can use [18, Théorème 3.2.1.] and obtain

$$T_{f^*_{V_{pt}}}^{V_{pt}} H^*X \cong H^* \operatorname{Map}(BV_{pt}, X)_{f_{V_{pt}}}.$$

Hence the mapping space $\operatorname{Map}(BV_{pt}, X)_{f_{V_{pt}}}$ has the same cohomology ring as $BT_p^\wedge$. The mapping space $\operatorname{Map}(BV_{pt}, X)_{f_{V_{pt}}}$ is p-complete [18, Proposition 3.4.4], so $BT_p^\wedge \simeq \operatorname{Map}(BV_{pt}, X)_{f_{V_{pt}}}$.

Now, the standard action of $W_{PU(p)} = \Sigma_p$ on T restricts to an action on V_{pt}, which induces an action of Σ_p on $\operatorname{Map}(BV_{pt}, X)$. If $\sigma \in \Sigma_p$, then $Bi_{V_{pt}} \simeq Bi_{V_{pt}}\sigma$, and therefore

$$f^*_{V_{pt}} = Bi^*_{V_{pt}}\phi^{-1} = \sigma^* Bi^*_{V_{pt}}\phi^{-1} = \sigma^* f^*_{V_{pt}},$$

and by Lannes' theory [18, Théorème 3.1.1], $f_{V_{pt}} \simeq f_{V_{pt}}\sigma$. This means that Σ_p acts on $\operatorname{Map}(BV_{pt}, X)_{f_{V_{pt}}}$.

Now consider the space $Y = \operatorname{Map}(BV_{pt}, X)_{f_{V_{pt}}} \times_{\Sigma_p} E\Sigma_p$ which fits in the fibration

$$\operatorname{Map}(BV_{pt}, X)_{f_{V_{pt}}} \longrightarrow Y \longrightarrow B\Sigma_p.$$

Fibrations with fiber $\operatorname{Map}(BV_{pt}, X)_{f_{V_{pt}}}$ and base $B\Sigma_p$ with the given Σ_p-action on the fiber are classified by [26, Lemma 3.13(1)]

$$\begin{aligned}\mathcal{H}^{n+1}(B\Sigma_p; \pi_n(\operatorname{Map}(BV_{pt}, X)_{f_{V_{pt}}})) &= \mathcal{H}^3(B\Sigma_p; \pi_2(\operatorname{Map}(BV_{pt}, X)_{f_{V_{pt}}}))\\ &\cong \mathcal{H}^3(B\Sigma_p; (\mathbb{Z}_p^\wedge)^{p-1}),\end{aligned}$$

as $\operatorname{Map}(BV_{pt}, X)_{f_{V_{pt}}} \simeq BT_p^\wedge \simeq K((\mathbb{Z}_p^\wedge)^{p-1}, 2)$. According to [2, Theorem 3.6], this group is trivial (recall that $p \geq 5$) which shows that $Y \simeq BN_p^\circ$, the fiberwise p-completion of BN.

Let $f_N\colon \operatorname{Map}(BV_{pt}, X)_{f_{V_{pt}}} \times_{\Sigma_p} E\Sigma_p \longrightarrow X$ denote the Borel construction of the evaluation map. We have to prove that the diagram (4) commutes, that is, that $f_N^*\phi = Bi_N^*$. But by construction, the composition

$$H^*BPU(p) \xrightarrow{f_N^*\phi} H^*BN \xrightarrow{Bi^*} H^*BT$$

is the same as Bi_T^*, and therefore by Lemma 2.10 we obtain $f_N^*\phi = Bi_N^*$. □

Now define maps $f_P\colon EPU(p)/P \simeq BP \xrightarrow{Bi_P} BN \xrightarrow{f_N} X$ for $P = T$, N_p, and V_{pn}. This gives rise to a diagram

$$\{EG/P \simeq BP\}_{PU(p)/P \in \widetilde{\mathcal{R}}_p(PU(p))} \xrightarrow{f_P} X. \tag{5}$$

The next lemma shows that diagram (5) commutes up to homotopy.

Lemma 4.4. *For every two objects $PU(p)/P$ and $PU(p)/Q$ in $\widetilde{\mathcal{R}}_p(PU(p))$ and morphism $c_g \in \operatorname{Mor}(PU(p)/P, PU(p)/Q)$, the diagram*

$$\begin{array}{ccc} BP & \xrightarrow{f_P} & X \\ {\scriptstyle Bc_g}\downarrow & & \| \\ BQ & \xrightarrow{f_Q} & X \end{array}$$

commutes up to homotopy.

Proof. Fix a pair of objects $PU(p)/P$ and $PU(p)/Q$ in $\widetilde{\mathcal{R}}_p(PU(p))$ and morphism $c_g \in \mathrm{Mor}(PU(p)/P, PU(p)/Q)$. We analyze the different cases.

If $P = N_p$, then also $Q = N_p$ since N_p is a maximal p-stubborn. Moreover, T is the connected component of N_p, hence $c_g(N_p) = N_p$ implies $c_g(T) = T$, and therefore $g \in N$. That is, the diagram

$$\begin{array}{ccccc} BP & \rightarrowtail & BN & \xrightarrow{f_N} & X \\ {\scriptstyle Bc_g}\downarrow & & \| & & \| \\ BP & \rightarrowtail & BN & \xrightarrow{f_N} & X \end{array}$$

commutes up to homotopy.

If $P = T$, then Q is either T or N_p. In both cases $c_g(T) = T$, so $g \in N$ and again we get a commutative diagram up to homotopy as in the previous case.

Finally, if $P = V_{pn}$, then $Bi^*_{V_{pn}} = Bc^*_g Bi^*_Q$ since $Bi_{V_{pn}} \simeq Bi_Q Bc_g$, and therefore

$$f^*_{V_{pn}} = Bi^*_{V_{pn}} \phi^{-1} = Bc^*_g Bi^*_Q \phi^{-1} = Bc^*_g f^*_Q.$$

By Lannes' theory [18, Théorème 3.1.1], $f_{V_{pn}} \simeq f_Q Bc_g$, which finishes the proof. $\square$

The diagram (5) commutes only up to homotopy, hence we do not know if the collection of maps $\{f_P\}_{PU(p)/P \in \widetilde{\mathcal{R}}_p(PU(p))}$ induces a map

$$\operatorname*{hocolim}_{PU(p)/P \in \widetilde{\mathcal{R}}_p(PU(p))} EPU(p)/P \longrightarrow X.$$

The obstructions lie in the groups

$$\varprojlim^i_{\widetilde{\mathcal{R}}_p(PU(p))} \pi_j(\mathrm{Map}(BP, X)_{f_P}),$$

where $\lim^i$ is the i-th derived functor of the inverse limit functor ([5] and [37]). Now we will prove that all obstruction groups are trivial.

Let

$$Pi^X_j, \Pi^{PU(p)}_j \colon \widetilde{\mathcal{R}}_p(PU(p)) \longrightarrow \mathcal{A}b$$

be functors defined by

$$\Pi^X_j(PU(p)/P) = \pi_j(\mathrm{Map}(BP, X)_{f_P}),$$
$$\Pi^{PU(p)}_j(PU(p)/P) = \pi_j(\mathrm{Map}(BP, BPU(p)^\wedge_p)_{(Bi_P)^\wedge_p}),$$

where $\mathcal{A}b$ is the category of abelian groups. Note that $\mathrm{Map}(BP, BPU(p)^\wedge_p)_{(Bi_P)^\wedge_p} \cong BZ(P)^\wedge_2$ [15, Theorem 3.2] and therefore $\Pi^{PU(p)}_1(PU(p)/P)$ is well defined. In the next lemma, we also show that $\Pi^X_1(PU(p)/P)$ is well defined.

Lemma 4.5. *There exists a natural transformation $\mathcal{T} \colon \Pi^{PU(p)}_j \longrightarrow \Pi^X_j$ which is an equivalence.*

Proof. Let P be either the maximal torus T or the p-normalizer N_p, and let E be V_{pt}. Consider $\widetilde{BE} = EP/E$, where EP is the total space of the fibration

$$P \longrightarrow EP \longrightarrow BP.$$

Then $\widetilde{BE} \simeq BE$ and $\widetilde{BE}$ carries a free (P/E)-action. For any space Y, on which P/E acts trivially, we have $\mathrm{Map}(BP, Y) \simeq \mathrm{Map}(\widetilde{BE}, Y)^{h(P/E)}$.

We apply Lannes' T functor to the diagram

$$\begin{array}{ccc} & H^*BN & \\ {\scriptstyle Bi_N^*}\nearrow & & \nwarrow{\scriptstyle f_N^*} \\ H^*BPU(p) & & H^*X \end{array} \tag{6}$$

to obtain

$$\begin{array}{ccc} & T^E_{Bj_E^*}H^*BN & \\ \nearrow & & \nwarrow \\ T^E_{Bi_E^*}H^*BPU(p) & & T^E_{f_E^*}H^*X \end{array}$$

From [18, Théorème 3.4.5] and [14, Theorem 1.1], it follows that

$$T^E_{Bj_E^*}H^*BN \cong H^*BC_N(E) = H^*BT,$$
$$T^E_{Bi_E^*}H^*BPU(p) \cong H^*BC_{PU(p)}(E) = H^*BT,$$

and the left-hand map in the above diagram is an isomorphism. Because $T^E_{f_E^*}H^*X \cong T^E_{Bi_E^*}H^*BPU(p) \cong H^*BT$, it is zero in degree 1, hence by [18, Théorème 3.2.1.], $T^E_{f_E^*}H^*X \cong H^*\operatorname{Map}(BE,X)_{f_E}$ and the right-hand map in the diagram is also an isomorphism. We conclude that in the diagram

$$\begin{array}{ccc} & \operatorname{Map}(\widetilde{BE}, BN^\wedge_p)_{(Bj_E)^\wedge_p} & \\ \swarrow & & \searrow \\ \operatorname{Map}(\widetilde{BE}, BPU(p)^\wedge_p)_{(Bi_E)^\wedge_p} & & \operatorname{Map}(\widetilde{BE}, X)_{f_E} \end{array}$$

both maps are (P/E)-equivariant mod p equivalences. Taking homotopy fixed points we obtain the following diagram:

$$\begin{array}{ccc} & \operatorname{Map}(\widetilde{BE}, BN^\wedge_p)^{h(P/E)}_{(Bj_E)^\wedge_p} & \\ \swarrow & & \searrow \\ \operatorname{Map}(\widetilde{BE}, BPU(p)^\wedge_p)^{h(P/E)}_{(Bi_E)^\wedge_p} & & \operatorname{Map}(\widetilde{BE}, X)^{h(P/E)}_{f_E} \end{array}$$

where both maps are mod p equivalences (since an equivariant mod p equivalence between 1-connected spaces induces a mod p equivalence between the homotopy fixed-point sets). Using $\operatorname{Map}(BP,\cdot) \simeq \operatorname{Map}(\widetilde{BE},\cdot)^{h(P/E)}$, we obtain mod p equivalences

$$\begin{array}{ccc} & \operatorname{Map}(BP, BN^\wedge_p)_{(Bj_P)^\wedge_p} & \\ {\scriptstyle Bi_N\circ -}\swarrow & & \searrow{\scriptstyle Bf_N\circ -} \\ \operatorname{Map}(BP, BPU(p)^\wedge_p)_{(Bi_P)^\wedge_p} & & \operatorname{Map}(BP, X)_{f_P} \end{array} \tag{7}$$

Let us consider the remaining case $P = V_{pn}$. Applying Lannes' functor to diagram (6) yields

$$\begin{array}{ccc} & T^P_{Bj^*_P} H^* BN & \\ \nearrow & & \nwarrow \\ T^P_{Bi^*_P} H^* BPU(p) & & T^P_{f^*_P} H^* X \end{array}$$

From [18, Théorème 3.4.5], we obtain

$$T^P_{Bj^*_P} H^* BN \cong H^* BC_N(P) = H^* BP,$$
$$T^P_{Bi^*_P} H^* BPU(p) \cong H^* BC_{PU(p)}(P) = H^* BP,$$

and the left-hand map is an isomorphism. Since $T^P_{f^*_P} H^* X$ is free in dimension ≤ 2, it follows by [18, Théorème 3.2.4] that $T^P_{f^*_P} H^* X \cong H^* \operatorname{Map}(BP, X)_{f_P}$, so that the right-hand map is an isomorphism. Thus, both maps in the diagram (7) are also mod p equivalences when $P = V_{pn}$.

We have shown that in all cases ($P = N_p$, T, or V_{pn}) the maps in diagram (7) are mod p equivalences. This provides a homotopy equivalence

$$\operatorname{Map}(BP, BPU(p)^\wedge_p)_{(Bi_P)^\wedge_p} \longrightarrow \operatorname{Map}(BP, X)_{f_P}$$

since these are p-complete spaces. To see that this homotopy equivalence is natural, we have to show that the diagram

(8)

$$\begin{array}{ccccc} \operatorname{Map}(BP, BPU(p)^\wedge_p)_{(Bi_P)^\wedge_p} & \xleftarrow{Bi_N \circ -} & \operatorname{Map}(BP, BN^\wedge_p)_{(Bj_P)^\wedge_p} & \xrightarrow{f_N \circ -} & \operatorname{Map}(BP, X)_{f_P} \\ \downarrow{\scriptstyle -\circ Bc_g} & & & & \downarrow{\scriptstyle -\circ Bc_g} \\ \operatorname{Map}(BQ, BPU(p)^\wedge_p)_{(Bi_Q)^\wedge_p} & \xleftarrow{Bi_N \circ -} & \operatorname{Map}(BQ, BN^\wedge_p)_{(Bj_Q)^\wedge_p} & \xrightarrow{f_N \circ -} & \operatorname{Map}(BQ, X)_{f_Q} \end{array}$$

commutes for every pair of objects $PU(p)/P$ and $PU(p)/Q$ in $\widetilde{\mathcal{R}}_p(PU(p))$ and morphism $c_g \in \operatorname{Mor}(PU(p)/P, PU(p)/Q)$. Since every morphism in $\widetilde{\mathcal{R}}_p(PU(p))$ consists of an automorphism composed with an inclusion (Remark 4.2), and inclusions obviously make commutative the diagram (8), it is enough to consider $Q = P$ (thus $g \in N_{PU(p)}(P)$). The argument is similar to that in the proof on Lemma 4.4:

- If $P = N_p$ or T, then $g \in N$ and therefore

$$\operatorname{Map}(BP, BN^\wedge_p)_{(Bj_P)^\wedge_p} \xrightarrow{-\circ Bc_g} \operatorname{Map}(BP, BN^\wedge_p)_{(Bj_P)^\wedge_p}$$

  closes the diagram (8) (recall $Q = P$) and shows it is commutative.
- Assume now that $P = V_{pn}$, and let (Z, h) denote either $(BPU(p)^\wedge_p, (Bi_P)^\wedge_p)$, $(BN^\wedge_p, (Bj_P)^\wedge_p)$ or (X, f_P). Then the adjoint of the map

$$BP \times BP \xrightarrow{B\mu} BP \xrightarrow{h} Z,$$

  where μ is the multiplication in P, provides a map $BP \xrightarrow{ad_Z} \operatorname{Map}(BP, Z)_h$ such that composition with the evaluation map $\operatorname{Map}(BP, Z)_h \xrightarrow{ev} Z$ recovers the original h. Therefore, the map ad_Z is the homotopy equivalence

$\mathrm{Map}(BP, Z)_h \simeq BP$ constructed above, and the diagram

$$\begin{array}{ccccc}
BP & = & BP & = & BP \\
\downarrow{\scriptstyle ad_{BPU(p)^\wedge_p}} & & \downarrow{\scriptstyle ad_{BN^\wedge_p}} & & \downarrow{\scriptstyle ad_X} \\
\mathrm{Map}(BP, BPU(p)^\wedge_p)_{(Bi_P)^\wedge_p} & \xleftarrow{Bi_N \circ -} & \mathrm{Map}(BP, BN^\wedge_p)_{(Bj_P)^\wedge_p} & \xrightarrow{f_N \circ -} & \mathrm{Map}(BP, X)_{f_P}
\end{array}$$

clearly commutes. Now note that $B\mu \circ (Bc_g \times Bc_g) = Bc_g \circ B\mu$, and $h \circ Bc_g = h$ (by Lemma 4.4 in the case $Z = X$, obvious if $Z = BPU(p)$). Then $ad_Z \circ Bc_{g^{-1}} = (- \circ Bc_g) \circ ad_Z$, where $Z = X$ or $PU(p)$, and taking adjoints transforms diagram (8) into the diagram (recall $Q = P$)

$$\begin{array}{ccccc}
BP & = & BP & = & BP \\
\downarrow{\scriptstyle Bc_{g^{-1}}} & & & & \downarrow{\scriptstyle Bc_{g^{-1}}} \\
BP & = & BP & = & BP
\end{array}$$

which is clearly commutative.

□

Proposition 4.6. *For all $i, j \geq 1$,*

$$\lim_{\overleftarrow{\tilde{\mathcal{R}}_p(PU(p))}}{}^{i} \, \pi_j(\mathrm{Map}(BP, X)_{f_P}) = 0.$$

Proof. By the previous lemma,

$$\lim_{\overleftarrow{\tilde{\mathcal{R}}_p(PU(p))}}{}^{i} \, \pi_j(\mathrm{Map}(BP, X)_{f_P}) = \lim_{\overleftarrow{\tilde{\mathcal{R}}_p(PU(p))}}{}^{i} \, \pi_j(\mathrm{Map}(BP, BPU(p)^\wedge_p)_{(Bi_P)^\wedge_p}),$$

so the proof reduces to showing that the latter group is trivial. But this follows from [15, Proposition 5.6] since,

- $PU(p)$ is centerfree,
- if $P \subset PU(p)$ is p-stubborn and does not contain a maximal torus, then $P = V_{pn}$ up to conjugation and $N_{PU(p)}P/P \cong \mathrm{SL}_2(p)$ by Proposition 4.1, and
- $\Lambda(\mathrm{SL}_2(p), (\mathbb{Z}/p)^2) = 0$ by [15, Proposition 6.3]. □

Because all obstructions vanish, there exists a map $f\colon BPU(p)^\wedge_p \longrightarrow X$. By construction of the map f, the diagram

$$\begin{array}{ccc}
 & (BN_p)^\wedge_p & \\
{\scriptstyle Bi_N}\swarrow & & \searrow{\scriptstyle f_N} \\
BPU(p)^\wedge_p & \xrightarrow{\quad f \quad} & X
\end{array}$$

commutes. The Euler characteristic $\chi(PU(p)/N_p) \neq 0 \mod p$, hence a transfer argument shows that Bi_N^* is a monomorphism. By Theorem 4.3, f_N^* is also a monomorphism. Therefore, f^* is a monomorphism and, because $H^*BPU(p) \cong H^*X$ is finite dimensional in each degree, f^* is an isomorphism. This shows that f is a homotopy equivalence and finishes the proof of Theorem C.

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