Abstract. In this paper we explore finite rank perturbations of unilateral weighted shifts $W_\alpha$. First, we prove that the subnormality of $W_\alpha$ is never stable under nonzero finite rank perturbations unless the perturbation occurs at the zeroth weight. Second, we establish that 2-hyponormality implies positive quadratic hyponormality, in the sense that the Maclaurin coefficients of $D_n(s) := \det P_n [(W_\alpha + sW_2^2)^*, (W_\alpha + sW_2^2)] P_n$ are nonnegative, for every $n \geq 0$, where $P_n$ denotes the orthogonal projection onto the basis vectors $\{e_0, \ldots, e_n\}$. Finally, for $\alpha$ strictly increasing and $W_\alpha$ 2-hyponormal, we show that for a small finite-rank perturbation $\alpha'$ of $\alpha$, the shift $W_{\alpha'}$ remains quadratically hyponormal.

1. Introduction

Let $\mathcal{H}$ and $\mathcal{K}$ be complex Hilbert spaces, let $\mathcal{L}(\mathcal{H}, \mathcal{K})$ be the set of bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$ and write $\mathcal{L}(\mathcal{H}) := \mathcal{L}(\mathcal{H}, \mathcal{H})$. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be normal if $T^*T = TT^*$, hyponormal if $T^*T \geq TT^*$, and subnormal if $T = N|_H$, where $N$ is normal on some Hilbert space $\mathcal{K} \supseteq \mathcal{H}$. If $T$ is subnormal, then $T$ is also hyponormal. Recall that given a bounded sequence of positive numbers $\alpha : a_0, a_1, \cdots$ (called weights), the (unilateral) weighted shift $W_\alpha$ associated with $\alpha$ is the operator on $\ell^2(\mathbb{Z}_+)$ defined by $W_\alpha e_n := a_n e_{n+1}$ for all $n \geq 0$, where $\{e_n\}_{n=0}^\infty$ is the canonical orthonormal basis for $\ell^2$. It is straightforward to check that $W_\alpha$ can never be normal, and that $W_\alpha$ is hyponormal if and only if $a_n \leq a_{n+1}$ for all $n \geq 0$. The Bram-Halmos criterion for subnormality states that an operator $T$ is subnormal if and only if

$$\sum_{i,j} \langle T^i x_j, T^j x_i \rangle \geq 0$$

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for all finite collections $x_0, x_1, \cdots, x_k \in H$ ([2, 4 II.1.9]). It is easy to see that this is equivalent to the following positivity test:

\[
\begin{pmatrix}
I & T^* & \cdots & T^{*k} \\
T & T^*T & \cdots & T^{*k}T \\
\vdots & \vdots & \ddots & \vdots \\
T^k & T^kT & \cdots & T^kT^k
\end{pmatrix} \geq 0 \quad \text{(for all } k \geq 1). 
\]

Condition (1.1) provides a measure of the gap between hyponormality and subnormality. In fact, the positivity condition (1.1) for $k = 1$ is equivalent to the hyponormality of $T$, while subnormality requires the validity of (1.1) for all $k$. Let $[A, B] := AB - BA$ denote the commutator of two operators $A$ and $B$, and define $T$ to be $k$-hyponormal whenever the $k \times k$ operator matrix

\[
M_k(T) := ([T^i, T^j])_{i,j=1}^k
\]

is positive. An application of the Choleski algorithm for operator matrices shows that the positivity of (1.2) is equivalent to the positivity of the $(k + 1) \times (k + 1)$ operator matrix in (1.1); the Bram-Halmos criterion can then be rephrased to say that $T$ is subnormal if and only if $T$ is $k$-hyponormal for every $k \geq 1$ ([16]).

Recall ([1, 16, 5]) that $T \in \mathcal{L}(H)$ is said to be weakly $k$-hyponormal if

\[
LS(T, T^2, \cdots, T^k) := \left\{ \sum_{j=1}^k \alpha_j T^j : \alpha = (\alpha_1, \cdots, \alpha_k) \in \mathbb{C}^k \right\}
\]

consists entirely of hyponormal operators, or equivalently, $M_k(T)$ is weakly positive, i.e. ([16]),

\[
\begin{pmatrix}
M_k(T) & \lambda_0 x \\
\vdots & \vdots \\
\lambda_k x & \lambda_k x
\end{pmatrix} \geq 0 \quad \text{for } x \in H \text{ and } \lambda_0, \cdots, \lambda_k \in \mathbb{C}.
\]

If $k = 2$, then $T$ is said to be quadratically hyponormal, and if $k = 3$, then $T$ is said to be cubically hyponormal. Similarly, $T \in \mathcal{L}(H)$ is said to be polynomially hyponormal if $p(T)$ is hyponormal for every polynomial $p \in \mathbb{C}[z]$. It is known that $k$-hyponormal $\Rightarrow$ weakly $k$-hyponormal, but the converse is not true in general.

The classes of (weakly) $k$-hyponormal operators have been studied in an attempt to bridge the gap between subnormality and hyponormality ([7, 8, 10, 11, 12, 13, 16, 19, 22]). The study of this gap has been only partially successful. For example, such a gap is not yet well described for Toeplitz operators on the Hardy space of the unit circle; in fact, even subnormality for Toeplitz operators has not been characterized (cf. [20, 6]). For weighted shifts, positive results appear in [17] and [12], although no concrete example of a weighted shift which is polynomially hyponormal and not subnormal has yet been found (the existence of such weighted shifts was established in [17] and [18]).

In the present paper we renew our efforts to help describe the above-mentioned gap between subnormality and hyponormality, with particular emphasis on polynomial hyponormality. We focus on the class of unilateral weighted shifts, and initiate a study of how the above-mentioned notions behave under finite perturbations of the weight sequence. We first obtain the following three concrete results.
The subnormality of \( W_\alpha \) is never stable under nonzero finite rank perturbations unless the perturbation is confined to the zeroth weight (Theorem 2.1); 2-hyponormality implies positive quadratic hyponormality, in the sense that the Maclaurin coefficients of \( D_n(s) := \det P_n [(W_\alpha + sW_\alpha^2)^*, W_\alpha + sW_\alpha^2] P_n \) are nonnegative, for every \( n \geq 0 \), where \( P_n \) denotes the orthogonal projection onto the basis vectors \( \{e_0, \ldots, e_n\} \) (Theorem 2.2); and if \( \alpha \) is strictly increasing and \( W_\alpha \) is 2-hyponormal, then for a small perturbation of \( \alpha \), the shift \( W_\alpha' \) remains positively quadratically hyponormal (Theorem 2.3).

Along the way we establish two related results, each of independent interest:

(iv) an integrality criterion for a subnormal weighted shift to have an \( n \)-step subnormal extension (Theorem 6.1); and

(v) a proof that the sets of \( k \)-hyponormal and weakly \( k \)-hyponormal operators are closed in the strong operator topology (Proposition 6.7).

2. Statement of main results

C. Berger's characterization of subnormality for unilateral weighted shifts (cf. [21], [III.8.16]) states that \( W_\alpha \) is subnormal if and only if there exists a Borel probability measure \( \mu \) (the so-called Berger measure of \( W_\alpha \)) supported in \([ 0, ||W_\alpha||^2 \] , with \( ||W_\alpha||^2 \in \text{supp} \mu \), such that
\[
\gamma_n = \int t^n d\mu(t) \quad \text{for all } n \geq 0.
\]

Given an initial segment of weights \( \alpha : \alpha_0, \ldots, \alpha_m \), the sequence \( \hat{\alpha} \in \ell^\infty(\mathbb{Z}_+) \) such that \( \hat{\alpha}_i = \alpha_i \) \( (i = 0, \ldots, m) \) is said to be recursively generated by \( \alpha \) if there exist \( r \geq 1 \) and \( \varphi_0, \ldots, \varphi_{r-1} \in \mathbb{R} \) such that
\[
(2.1) \quad \gamma_{n+r} = \varphi_0 \gamma_n + \cdots + \varphi_{r-1} \gamma_{n+r-1} \quad \text{(for all } n \geq 0),
\]
where \( \gamma_0 := 1, \gamma_n := \alpha_0^2 \cdots \alpha_{n-1}^2 (n \geq 1) \). In this case \( W_\alpha \) with weights \( \hat{\alpha} \) is said to be recursively generated. If we let
\[
(2.2) \quad g(t) := t^r - (\varphi_{r-1} t^{r-1} + \cdots + \varphi_0),
\]
then \( g \) has \( r \) distinct real roots \( 0 \leq s_0 < \cdots < s_{r-1} \) ([11, Theorem 3.9]). Let
\[
V := \begin{pmatrix}
1 & 1 & \cdots & 1 \\
\vdots & \vdots & & \vdots \\
1 & & & \\
s_0 & s_1 & \cdots & s_{r-1}
\end{pmatrix}
\]
and let
\[
\begin{pmatrix}
\rho_0 \\
\vdots \\
\rho_{r-1}
\end{pmatrix} := V^{-1} \begin{pmatrix}
\gamma_0 \\
\vdots \\
\gamma_{r-1}
\end{pmatrix}.
\]
If the associated recursively generated weighted shift \( W_\hat{\alpha} \) is subnormal, then its Berger measure is of the form
\[
\mu := \rho_0 \delta_{s_0} + \cdots + \rho_{r-1} \delta_{s_{r-1}}.
\]
For example, given $\alpha_0 < \alpha_1 < \alpha_2$, $W_{(\alpha_0, \alpha_1, \alpha_2)^\wedge}$ is the recursive weighted shift whose weights are calculated according to the recursive relation

\begin{equation}
\alpha_{n+1}^2 = \varphi_1 + \varphi_0 \frac{1}{\alpha_n^2},
\end{equation}

where

\begin{equation}
\varphi_0 = -\frac{\alpha_0^2 \alpha_1^2 (\alpha_2^2 - \alpha_1^2)}{\alpha_1^2 - \alpha_0^2} \quad \text{and} \quad \varphi_1 = \frac{\alpha_1^2 (\alpha_2^2 - \alpha_1^2)}{\alpha_2^2 - \alpha_0^2}.
\end{equation}

In this case, $W_{(\alpha_0, \alpha_1, \alpha_2)^\wedge}$ is subnormal with 2–atomic Berger measure. Let $W_{x(\alpha_0,\alpha_1,\alpha_2)^\wedge}$ denote the weighted shift whose weight sequence consists of the initial weight $x$ followed by the weight sequence of $W_{(\alpha_0, \alpha_1, \alpha_2)^\wedge}$.

By the Density Theorem ([11, Theorem 4.2 and Corollary 4.3]), we know that if $W_{\alpha}$ is a subnormal weighted shift with weights $\alpha = \{\alpha_n\}$ and $\epsilon > 0$, then there exists a nonzero compact operator $K$ with $||K|| < \epsilon$ such that $W_{\alpha} + K$ is a recursively generated subnormal weighted shift; in fact $W_{\alpha} + K = W_{\alpha(\epsilon)}$ for some $m \geq 1$, where $\alpha(\epsilon) : \alpha_0, \ldots, \alpha_m$. The following result shows that $K$ cannot generally be taken to be of finite rank.

**Theorem 2.1** (Finite Rank Perturbations of Subnormal Shifts). If $W_{\alpha}$ is a subnormal weighted shift, then there exists no nonzero finite rank operator $F \neq cP(e_0)$ such that $W_{\alpha} + F$ is a subnormal weighted shift. Concretely, suppose $W_{\alpha}$ is a subnormal weighted shift with weight sequence $\alpha = \{\alpha_n\}_{n=0}^\infty$ and assume $\alpha' = \{\alpha_n'\}$ is a nonzero perturbation of $\alpha$ in a finite number of weights except the initial weight. Then $W_{\alpha'}$ is not subnormal.

We next consider the self-commutator $[(W_{\alpha} + s W_{\alpha}^2)^*, W_{\alpha} + s W_{\alpha}^2]$. Let $W_{\alpha}$ be a hyponormal weighted shift. For $s \in \mathbb{C}$, we write

\[ D(s) := [(W_{\alpha} + s W_{\alpha}^2)^*, W_{\alpha} + s W_{\alpha}^2] \]

and we let

\[ D_n(s) := P_n[(W_{\alpha} + s W_{\alpha}^2)^*, W_{\alpha} + s W_{\alpha}^2]P_n \]

\begin{equation}
\begin{pmatrix}
q_0 & \bar{r}_0 & 0 & \cdots & 0 & 0 \\
r_0 & q_1 & \bar{r}_1 & \cdots & 0 & 0 \\
0 & r_1 & q_2 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & q_{n-1} & \bar{r}_{n-1} \\
0 & 0 & 0 & \cdots & r_{n-1} & q_n
\end{pmatrix},
\end{equation}

where $P_n$ is the orthogonal projection onto the subspace generated by $\{e_0, \ldots, e_n\}$,

\begin{equation}
\begin{cases}
q_n := u_n + |s|^2 v_n, \\
r_n := s \sqrt{w_n}, \\
u_n := \alpha_n^2 - \alpha_{n-1}^2, \\
v_n := \alpha_n^2 \alpha_{n+1}^2 - \alpha_{n-1}^2 \alpha_{n-2}^2, \\
w_n := \alpha_n^2 (\alpha_{n+1}^2 - \alpha_{n-1}^2)^2,
\end{cases}
\end{equation}

and, for notational convenience, $\alpha_{-2} = \alpha_{-1} = 0$. Clearly, $W_{\alpha}$ is quadratically hyponormal if and only if $D_n(s) \geq 0$ for all $s \in \mathbb{C}$ and all $n \geq 0$. Let $d_n(\cdot) := \ldots$
Theorem 2.2. Let $\alpha \equiv \{\alpha_n\}_{n=0}^\infty$ be a weight sequence and assume that $W_\alpha$ is $2$-hyponormal. Then $W_\alpha$ is positively quadratically hyponormal. More precisely, if $W_\alpha$ is $2$-hyponormal, then
\begin{equation}
\alpha(n, i) \geq v_0 \cdots v_{i-1} u_i \cdots u_n \quad (n \geq 0, \ 0 \leq i \leq n + 1).
\end{equation}

In particular, if $\alpha$ is strictly increasing and $W_\alpha$ is $2$-hyponormal, then the Maclaurin coefficients of $d_n(t)$ are positive for all $n \geq 0$.

If $W_\alpha$ is a weighted shift with weight sequence $\alpha = \{\alpha_n\}_{n=0}^\infty$, then the moments of $W_\alpha$ are usually defined by $\beta_0 := 1, \beta_{n+1} := \alpha_n \beta_n \ (n \geq 0)$ [23]; however, we prefer to reserve this term for the sequence $\gamma_n := \beta_n^2 \ (n \geq 0)$. A criterion for $k$-hyponormality can be given in terms of these moments ([17 Theorem 4]): if we build a $(k+1) \times (k+1)$ Hankel matrix $A(n; k)$ by
\begin{equation}
A(n; k) := \begin{pmatrix}
\gamma_n & \gamma_{n+1} & \cdots & \gamma_{n+k} \\
\gamma_{n+1} & \gamma_{n+2} & \cdots & \gamma_{n+k+1} \\
\vdots & \vdots & \ddots & \vdots \\
\gamma_{n+k} & \gamma_{n+k+1} & \cdots & \gamma_{n+2k}
\end{pmatrix} \quad (n \geq 0),
\end{equation}
then
\begin{equation}
W_\alpha \text{ is } k \text{-hyponormal } \iff A(n; k) \geq 0 \quad (n \geq 0).
\end{equation}
In particular, for $\alpha$ strictly increasing, $W_\alpha$ is $2$-hyponormal if and only if
\begin{equation}
\det \begin{pmatrix}
\gamma_n & \gamma_{n+1} & \gamma_{n+2} \\
\gamma_{n+1} & \gamma_{n+2} & \gamma_{n+3} \\
\gamma_{n+2} & \gamma_{n+3} & \gamma_{n+4}
\end{pmatrix} \geq 0 \quad (n \geq 0).
\end{equation}

One might conjecture that if $W_\alpha$ is a $k$-hyponormal weighted shift whose weight sequence is strictly increasing, then $W_\alpha$ remains weakly $k$-hyponormal under a small perturbation of the weight sequence. We will show below that this is true for $k = 2$ (Theorem 2.3).

In [12 Theorem 4.3], it was shown that the gap between $2$-hyponormality and quadratic hyponormality can be detected by unilateral shifts with a weight sequence $\alpha : \sqrt{a}, (\sqrt{a}, \sqrt{b}, \sqrt{c})^\infty$. In particular, there exists a maximum value $H_2 \equiv H_2(a, b, c)$
of $x$ that makes $W_{\sqrt{2}(\sqrt{3},\sqrt{5})}$ 2-hyponormal; $H_2$ is called the modulus of 2-hyponormality (cf. [12]). Any value of $x > H_2$ yields a non-2-hyponormal weighted shift. However, if $x - H_2$ is small enough, $W_{\sqrt{2}(\sqrt{3},\sqrt{5})}$ is still quadratically hyponormal. The following theorem shows that, more generally, for finite rank perturbations of weighted shifts with strictly increasing weight sequences, there always exists a gap between 2-hyponormality and quadratic hyponormality.

**Theorem 2.3** (Finite Rank Perturbations of 2-Hyponormal Shifts). Let $\alpha = \{\alpha_n\}_{n=0}^\infty$ be a strictly increasing weight sequence. If $W_\alpha$ is $2$-hyponormal, then $W_\alpha$ remains positively quadratically hyponormal under a small nonzero finite rank perturbation of $\alpha$.

### 3. Proof of Theorem 2.1

**Proof of Theorem 2.1.** It suffices to show that if $T$ is a weighted shift whose restriction to $\{e_n, e_{n+1}, \ldots\}$ ($n \geq 2$) is subnormal, then there is at most one $\alpha_{n-1}$ for which $T$ is subnormal.

Let $W := T|_{\{e_n, e_{n+1}, \ldots\}}$ and $S := T|_{\{e_n, e_{n+1}, \ldots\}}$, where $n \geq 2$. Then $W$ and $S$ have weights $\alpha_n(W) := \alpha_{k+n-1}$ and $\alpha_k(S) := \alpha_{k+n}$ ($k \geq 0$). Thus the corresponding moments are related by the equation

$$\gamma_k(S) = \alpha_n^2 \cdots \alpha_{n+k-1}^2 = \frac{\gamma_{k+1}(W)}{\alpha_{n-1}^2}.$$  

We now adapt the proof of [7, Proposition 8]. Suppose $S$ is subnormal with associated Berger measure $\mu$. Then $\gamma_k(S) = \int_0^{||T||^2} t^k d\mu$. Thus $W$ is subnormal if and only if there exists a probability measure $\nu$ on $[0, ||T||^2]$ such that

$$\frac{1}{\alpha_{n-1}^2} \int_0^{||T||^2} t^{k+1} d\nu(t) = \int_0^{||T||^2} t^k d\mu(t) \quad \text{for all } k \geq 0,$$

which readily implies that $t d\nu = \alpha_{n-1}^2 d\mu$. Thus $W$ is subnormal if and only if the formula

$$d\nu := \lambda \cdot \delta_0 + \frac{\alpha_{n-1}^2}{t} d\mu$$

defines a probability measure for some $\lambda \geq 0$, where $\delta_0$ is the point mass at the origin. In particular $\frac{1}{t} \in L^1(\mu)$ and $\mu(\{0\}) = 0$ whenever $W$ is subnormal. If we repeat the above argument for $W$ and $V := T|_{\{e_{n-2}, e_{n-1}, \ldots\}}$, then we should have that $\nu(\{0\}) = 0$ whenever $V$ is subnormal. Therefore we can conclude that if $V$ is subnormal, then $\lambda = 0$, and hence

$$d\nu = \frac{\alpha_{n-1}^2}{t} d\mu.$$  

Thus we have

$$1 = \int_0^{||T||^2} d\nu(t) = \alpha_{n-1}^2 \int_0^{||T||^2} \frac{1}{t} d\mu(t),$$

so that

$$\alpha_{n-1}^2 = \left(\int_0^{||T||^2} \frac{1}{t} d\mu(t)\right)^{-1}.$$
which implies that \( \alpha_{n-1} \) is determined uniquely by \( \{ \alpha_n, \alpha_{n+1}, \ldots \} \) whenever \( T \) is subnormal. This completes the proof. \( \square \)

Theorem 2.1 says that a nonzero finite rank perturbation of a subnormal shift is never subnormal unless the perturbation occurs at the initial weight. However, this is not the case for \( k \)-hyponormality. To see this we use a close relative of the Bergman shift \( B_+ \) (whose weights are given by \( \alpha = \{ \sqrt{\frac{n+1}{n+2}} \}_{n=0}^\infty \)); it is well known that \( B_+ \) is subnormal.

**Example 3.1.** For \( x > 0 \), let \( T_x \) be the weighted shift whose weights are given by
\[
\alpha_0 := \sqrt{\frac{1}{2}}, \quad \alpha_1 := \sqrt{x}, \quad \text{and} \quad \alpha_n := \sqrt{\frac{n+1}{n+2}} \quad (n \geq 2).
\]
Then we have:

(i) \( T_x \) is subnormal \( \iff \) \( x = \frac{2}{3} \);

(ii) \( T_x \) is 2-hyponormal \( \iff \) \( \frac{63-\sqrt{129}}{80} \leq x \leq \frac{24}{35} \).

**Proof.** Assertion (i) follows from Theorem 2.1. For assertion (ii) we use (2.12): \( T_x \) is 2-hyponormal if and only if
\[
\det \begin{pmatrix} 1 & \frac{1}{2} x & \frac{1}{3} x \\ \frac{1}{2} x & \frac{1}{3} x & \frac{3}{8} x \\ \frac{1}{3} x & \frac{3}{8} x & \frac{3}{10} x \end{pmatrix} \geq 0 \quad \text{and} \quad \det \begin{pmatrix} \frac{1}{2} x & \frac{3}{8} x & \frac{3}{10} x \\ \frac{3}{8} x & \frac{3}{10} x & \frac{3}{12} x \\ \frac{3}{10} x & \frac{3}{12} x & \frac{3}{14} x \end{pmatrix} \geq 0,
\]
or equivalently, \( \frac{63-\sqrt{129}}{80} \leq x \leq \frac{24}{35} \). \( \square \)

For perturbations of recursive subnormal shifts of the form \( W_{(\sqrt{a}, \sqrt{b}, \sqrt{c})^n} \), subnormality and 2-hyponormality coincide.

**Theorem 3.2.** Let \( \alpha = \{ \alpha_n \}_{n=0}^\infty \) be recursively generated by \( \sqrt{a}, \sqrt{b}, \sqrt{c} \). If \( T_x \) is the weighted shift whose weights are given by \( \alpha_x : \alpha_0, \ldots, \alpha_{j-1}, \sqrt{x}, \alpha_{j+1}, \ldots \), then we have
\[
T_x \text{ is subnormal} \iff T_x \text{ is 2-hyponormal} \iff \begin{cases} x = \alpha_j^2 & \text{if } j \geq 1; \\ x \leq a & \text{if } j = 0. \end{cases}
\]

**Proof.** Since \( \alpha \) is recursively generated by \( \sqrt{a}, \sqrt{b}, \sqrt{c} \), we have that \( \alpha_0^2 = a, \ \alpha_1^2 = b, \ \alpha_2^2 = c \),
\[
\begin{align*}
\alpha_3^2 &= \frac{b(c^2 - 2ac + ab)}{c(b - a)}, \quad \text{and} \\
\alpha_4^2 &= \frac{bc^3 - 4abc^2 + 2ab^2c + a^2bc - a^2b^2 + a^2c^2}{(b - a)(c^2 - 2ac + ab)}.
\end{align*}
\]

**Case 1 \((j = 0)\):** It is evident that \( T_x \) is subnormal if and only if \( x \leq a \). For 2-hyponormality observe by (2.12) that \( T_x \) is 2-hyponormal if and only if
\[
\det \begin{pmatrix} 1 & x & bx \\ x & bx & bcx \\ bx & bcx & \alpha_3^2 bcx \end{pmatrix} \geq 0,
\]
or equivalently, \( x \leq a \).
Case 2 ($j \geq 1$): Without loss of generality we may assume that $j = 1$ and $a = 1$. Thus $a_1 = \sqrt{x}$. Then by Theorem 2.1, $T_x$ is subnormal if and only if $x = b$. On the other hand, by (2.12), $T_x$ is 2-hyponormal if and only if
\[
\det \begin{pmatrix} 1 & x & cx \\ x & cx & \alpha_2^2 cx \\ cx & \alpha_2^2 cx & \alpha_3^2 cx \end{pmatrix} \geq 0.
\]
Thus a direct calculation with the specific forms of $\alpha_3, \alpha_4$ given in (3.4) shows that $T_x$ is 2-hyponormal if and only if $(x - b) \left( x - \frac{b(c^2 - 2c + b)}{b^2 - 1} \right) \leq 0$ and $x \leq b$. Since $b \leq \frac{b(c^2 - 2c + b)}{b^2 - 1}$, it follows that $T_x$ is 2-hyponormal if and only if $x = b$. This completes the proof. \hfill \Box

4. Proof of Theorem 2.2

With the notation in (2.6), we let
\[
p_n := u_n v_{n+1} - w_n \quad (n \geq 0).
\]
We then have:

**Lemma 4.1.** If $\alpha \equiv \{\alpha_n\}_{n=0}^{\infty}$ is a strictly increasing weight sequence, then the following statements are equivalent:

(i) $W_\alpha$ is 2-hyponormal;
(ii) $\alpha_{n+1}^2 (u_{n+1} + u_{n+2})^2 \leq u_{n+1} v_{n+2}$ \quad $(n \geq 0)$;
(iii) $\frac{\alpha_n^2}{\alpha_{n+2}^2} \frac{u_{n+2}}{u_{n+3}} \leq \frac{u_{n+1}}{u_{n+2}}$ \quad $(n \geq 0)$;
(iv) $p_n \geq 0$ \quad $(n \geq 0)$.

**Proof.** This follows from a straightforward calculation. \hfill \Box

**Proof of Theorem 2.2.** If $\alpha$ is not strictly increasing, then $\alpha$ is flat, by the argument of [7] Corollary 6, i.e., $\alpha_0 = \alpha_1 = \alpha_2 = \cdots$. Then
\[
D_n(s) = \left( \frac{\alpha_n^2 + |s|^2 \alpha_0^2}{\alpha_0^2} \right) \oplus 0_{\infty}
\]
(cf. (2.5)), so that (2.9) is evident. Thus we may assume that $\alpha$ is strictly increasing, so that $u_n > 0$, $v_n > 0$ and $w_n > 0$ for all $n \geq 0$. Recall that if we write $d_n(t) := \sum_{i=0}^{n+1} c(n, i) t^i$, then the $c(n, i)$'s satisfy the following recursive formulas (cf. (2.8)):
\[
c(n + 2, i) = u_{n+2} c(n + 1, i) + v_{n+2} c(n + 1, i - 1) + w_{n+1} c(n, i - 1) \quad (n \geq 0, 1 \leq i \leq n).
\]

Also, $c(n, n+1) = v_0 \cdots v_n$ (again by (2.8)) and $p_n := u_n v_{n+1} - w_n \geq 0$ \quad $(n \geq 0)$, by Lemma 4.1. A straightforward calculation shows that
\[
d_0(t) = u_0 + v_0 t;
\]
\[
d_1(t) = u_0 u_1 + (v_0 u_1 + p_0) t + v_0 v_1 t^2;
\]
\[
d_2(t) = u_0 u_1 u_2 + (v_0 u_1 u_2 + u_0 p_1 + w_2 p_0) t + (v_0 v_1 u_2 + v_0 p_1 + v_2 p_0) t^2 + v_0 v_1 v_2 t^3.
\]

Evidently,
\[
c(n, i) \geq 0 \quad (0 \leq n \leq 2, 0 \leq i \leq n+1).
\]
Define
\[ \beta(n, i) := c(n, i) - v_0 \cdots v_{i-1} u_i \cdots u_n \quad (n \geq 1, 1 \leq i \leq n). \]

For every \( n \geq 1 \), we now have
\[ c(n, i) = \begin{cases} u_0 \cdots u_n \geq 0 & (i = 0), \\ v_0 \cdots v_{i-1} u_i \cdots u_n + \beta(n, i) & (1 \leq i \leq n), \\ v_0 \cdots v_n \geq 0 & (i = n + 1). \end{cases} \]

For notational convenience we let \( \beta(n, 0) := 0 \) for every \( n \geq 0 \).

Claim 1. For \( n \geq 1 \),
\[ c(n, n) \geq u_n c(n - 1, n) \geq 0. \]

Proof of Claim 1. We use mathematical induction. For \( n = 1 \),
\[ c(1, 1) = v_0 u_1 + p_0 \geq u_1 c(0, 1) \geq 0, \]
and
\[ c(n + 1, n + 1) = u_{n+1} c(n, n + 1) + v_{n+1} c(n, n) - w_n c(n - 1, n) \]
\[ \geq u_{n+1} c(n, n + 1) + v_{n+1} u_n c(n - 1, n) - w_n c(n - 1, n) \]
(by the inductive hypothesis)
\[ = u_{n+1} c(n, n + 1) + p_n c(n - 1, n) \]
\[ \geq u_{n+1} c(n, n + 1), \]
which proves Claim 1.

Claim 2. For \( n \geq 2 \),
\[ \beta(n, i) \geq u_n \beta(n - 1, i) \geq 0 \quad (0 \leq i \leq n - 1). \]

Proof of Claim 2. We use mathematical induction. If \( n = 2 \) and \( i = 0 \), this is trivial. Also,
\[ \beta(2, 1) = u_0 p_1 + u_2 p_0 = u_0 p_1 + u_2 \beta(1, 1) \geq u_2 \beta(1, 1) \geq 0. \]

Assume that (4.7) holds. We shall prove that
\[ \beta(n + 1, i) \geq u_{n+1} \beta(n, i) \geq 0 \quad (0 \leq i \leq n). \]

For,
\[ \beta(n + 1, i) + v_0 \cdots v_{i-1} u_i \cdots u_{n+1} = c(n + 1, i) \quad \text{(by (4.2))} \]
\[ = u_{n+1} c(n, i) + v_{n+1} c(n, i - 1) - w_n c(n - 1, i - 1) \]
\[ = u_{n+1} \left( \beta(n, i) + v_0 \cdots v_{i-1} u_i \cdots u_n \right) + v_{n+1} \left( \beta(n, i - 1) + v_0 \cdots v_{i-2} u_i \cdots u_{n-1} \right) - w_n \left( \beta(n - 1, i - 1) + v_0 \cdots v_{i-2} u_i \cdots u_{n-1} \right), \]
so that
\[
\beta(n+1,i) = u_{n+1}\beta(n,i) + v_{n+1}\beta(n,i-1) - w_n\beta(n-1,i-1) \\
+ v_0 \cdots v_{i-2} u_{i-1} \cdots u_{n-1} (u_n v_{n+1} - w_n) \\
= u_{n+1}\beta(n,i) + v_{n+1}\beta(n,i-1) - w_n\beta(n-1,i-1) \\
+ (v_0 \cdots v_{i-2} u_{i-1} \cdots u_{n-1}) p_n \\
\geq u_{n+1}\beta(n,i) + v_{n+1} u_n\beta(n-1,i-1) - w_n\beta(n-1,i-1) \\
\quad \text{(by the inductive hypothesis and Lemma 4.1;)} \\
\text{observe that } i - 1 \leq n - 1, \text{ so (4.7) applies) } \\
= u_{n+1}\beta(n,i) + p_n\beta(n-1,i-1) \\
\geq u_{n+1}\beta(n,i),
\]
which proves Claim 2.

By Claim 2 and (4.5), we can see that \(c(n,i) \geq 0\) for all \(n \geq 0\) and \(1 \leq i \leq n - 1\). Therefore (4.4), (4.5), Claim 1 and Claim 2 imply
\[
c(n,i) \geq v_0 \cdots v_{i-1} u_i \cdots u_n \quad (n \geq 0, \, 0 \leq i \leq n + 1).
\]
This completes the proof.

5. Proof of Theorem 2.3

To prove Theorem 2.3 we need:

**Lemma 5.1 (\cite{15} Lemma 2.3).** Let \(\alpha \equiv \{\alpha_n\}_{n=0}^\infty\) be a strictly increasing weight sequence. If \(W_\alpha\) is 2-hyponormal, then the sequence of quotients
\[
\Theta_n := \frac{u_{n+1}}{u_{n+2}} \quad (n \geq 0)
\]
is bounded away from 0 and from \(\infty\). More precisely,
\[
1 \leq \Theta_n \leq \frac{u_1}{u_2} \left(\frac{|W_\alpha|^{2}}{\alpha_0 \alpha_1}\right)^{2} \quad \text{for sufficiently large } n.
\]
In particular, \(\{u_n\}_{n=0}^\infty\) is eventually decreasing.

**Proof of Theorem 2.3.** By Theorem 2.2, \(W_\alpha\) is strictly positively quadratically hyponormal, in the sense that all coefficients of \(d_n(t)\) are positive for all \(n \geq 0\). Note that finite rank perturbations of \(\alpha\) affect a finite number of values of \(u_n, v_n\) and \(w_n\). More concretely, if \(\alpha'\) is a perturbation of \(\alpha\) in the weights \(\{\alpha_0, \cdots, \alpha_N\}\), then \(u_n, v_n, w_n\) and \(p_n\) are invariant under \(\alpha'\) for \(n \geq N + 3\). In particular, \(p_n \geq 0\) for \(n \geq N + 3\).

**Claim 1.** For \(n \geq 3, \, 0 \leq i \leq n + 1,\)
\[
c(n,i) = u_n c(n-1,i) + p_{n-1} c(n-2,i-1) \\
+ \sum_{k=4}^{n} \frac{p_k}{2} \left(\prod_{j=k}^{n} v_j\right) c(k-3,i-n+k-2) + v_n \cdots v_3 \rho_{i-n+1},
\]

where

\[ \rho_{i-n+1} = \begin{cases} 
0 & (i < n - 1), \\
u_0 p_1 & (i = n - 1), \\
v_0 p_1 + v_2 p_0 & (i = n), \\
v_0 v_1 v_2 & (i = n + 1) 
\end{cases} \]

(cf. [12] Proof of Theorem 4.3).

**Proof of Claim 1.** We use induction. For \( n = 3, \ 0 \leq i \leq 4, \)

\[
c(3, i) = u_3 c(2, i) + v_3 c(2, i - 1) - w_2 c(1, i - 1)
\]

\[
= u_3 c(2, i) + v_3 \left( u_2 c(1, i - 1) + v_2 c(1, i - 2) - w_1 c(0, i - 2) \right) \\
- w_2 c(1, i - 1)
\]

\[
= u_3 c(2, i) + p_2 c(1, i - 1) + v_3 \left( v_2 c(1, i - 2) - w_1 c(0, i - 2) \right)
\]

\[
= u_3 c(2, i) + p_2 c(1, i - 1) + v_3 \rho_{i-2},
\]

where by (4.3),

\[
\rho_{i-2} = \begin{cases} 
0 & (i < 2), \\
u_0 p_1 & (i = 2), \\
v_0 p_1 + v_2 p_0 & (i = 3), \\
v_0 v_1 v_2 & (i = 4). 
\end{cases}
\]

Now,

\[
c(n + 1, i) = u_{n+1} c(n, i) + v_{n+1} c(n, i - 1) - w_n c(n - 1, i - 1)
\]

\[
= u_{n+1} c(n, i) + v_{n+1} \left( u_n c(n - 1, i - 1) + p_{n-1} c(n - 2, i - 2) \\
\sum_{k=4}^{n} p_{k-2} \left( \prod_{j=k}^{n} v_j \right) c(k - 3, i - n + k - 3) + v_n \cdots v_3 \rho_{i-n} \right)
\]

\[
- w_n c(n - 1, i - 1)
\]

\[
= u_{n+1} c(n, i) + p_n c(n - 1, i - 1) + v_{n+1} p_{n-1} c(n - 2, i - 2)
\]

\[
+ v_{n+1} \sum_{k=4}^{n} p_{k-2} \left( \prod_{j=k}^{n} v_j \right) c(k - 3, i - n + k - 3) + v_{n+1} \cdots v_3 \rho_{i-n}
\]

(by the inductive hypothesis)

\[
= u_{n+1} c(n, i) + p_n c(n - 1, i - 1)
\]

\[
+ \sum_{k=4}^{n+1} p_{k-2} \left( \prod_{j=k}^{n+1} v_j \right) c(k - 3, i - n + k - 3) + v_{n+1} \cdots v_3 \rho_{i-n},
\]

which proves Claim 1.
Write $w_n', v_n', w_n', p_n', \rho_n'$, and $c'(\cdot, \cdot)$ for the entities corresponding to $\alpha'$. If $p_n > 0$ for every $n = 0, \cdots, N + 2$, then in view of Claim 1, we can choose a small perturbation such that $p_n' > 0 \ (0 \leq n \leq N + 2)$ and therefore $c'(n, i) > 0$ for all $n \geq 0$ and $0 \leq i \leq n + 1$, which implies that $W_{\alpha'}$ is also positively quadratically hyponormal. If instead $p_n = 0$ for some $n = 0, \cdots, N + 2$, careful inspection of (5.3) reveals that without loss of generality we may assume $p_0 = \cdots = p_N = 0$. By Theorem 2.2, we have that for a sufficiently small perturbation $\alpha'$ of $\alpha$,

\[(5.4) \ c'(n, i) > 0 \ (0 \leq n \leq N + 2, \ 0 \leq i \leq n + 1) \quad \text{and} \quad c'(n, n + 1) > 0 \ (n \geq 0).\]

Write

\[k_n := \frac{v_n}{u_n} \quad (n = 2, 3, \cdots).\]

**Claim 2.** $\{k_n\}_{n=2}^\infty$ is bounded.

**Proof of Claim 2.** Observe that

\[(5.5) \quad k_n = \frac{v_n}{u_n} = \frac{\alpha_n^2 \alpha_{n+1}^2 - \alpha_{n-1}^2 \alpha_n^2}{\alpha_n^2 - \alpha_{n-1}^2} \]

\[= \alpha_n^2 + \alpha_{n-1}^2 + \alpha_n^2 \alpha_{n+1}^2 - \alpha_{n-1}^2 \alpha_n^2 + \alpha_{n-1}^2(\frac{\alpha_n^2 - \alpha_{n-1}^2}{\alpha_n^2 - \alpha_{n-1}^2}).\]

Therefore if $W_\alpha$ is 2-hyponormal, then by Lemma 5.1, the sequences

\[\left\{\frac{\alpha_{n+1}^2 - \alpha_n^2}{\alpha_n^2 - \alpha_{n-1}^2}\right\}_{n=2}^\infty \quad \text{and} \quad \left\{\frac{\alpha_{n-1}^2 - \alpha_n^2}{\alpha_n^2 - \alpha_{n-1}^2}\right\}_{n=2}^\infty\]

are both bounded, so that $\{k_n\}_{n=2}^\infty$ is bounded. This proves Claim 2.

Write $k := \sup_n k_n$. Without loss of generality we assume $k < 1$ (this is possible from the observation that $\alpha$ induces $\{c^2k_n\}$). Choose a sufficiently small perturbation $\alpha'$ of $\alpha$ such that if we let

\[(5.6) \quad h := \sup_{0 \leq \ell \leq N+2} \left| \sum_{k=4}^{N+2} p'_k \left( \prod_{j=k}^{N+3} v'_j \right) c'(k - 3, \ell) + v'_{N+3} \cdots v'_3 \rho'_m \right|,\]

then

\[(5.7) \quad c'(N + 3, i) - \frac{1}{1-k} h > 0 \quad (0 \leq i \leq N + 3)\]

(this is always possible because, by Theorem 2.2, we can choose a sufficiently small $|p'_k|$ such that

\[c'(N+3, i) > v_0 \cdots v_{i-1} u_i \cdots u_{N+3} - \epsilon \quad \text{and} \quad |h| < (1-k)(v_0 \cdots v_{i-1} u_i \cdots u_{N+3} - \epsilon)\]

for any small $\epsilon > 0$).
Claim 3. For \( j \geq 4 \) and \( 0 \leq i \leq N + j \),

\[
(5.8) \quad c'(N + j, i) \geq u_{N + j} \cdots u_{N + 4} \left( c'(N + 3, i) - \sum_{n=1}^{j-3} k^n h \right).
\]

Proof of Claim 3. We use induction. If \( j = 4 \), then by Claim 1 and (5.6),

\[
\begin{align*}
&c'(N + 4, i) = u'_{N + 4} c'(N + 3, i) + p'_{N + 3} c'(N + 2, i - 1) \\
&\quad + v'_{N + 4} \sum_{k=4}^{N+4} p_{k-2}^{'} \left( \prod_{j=k}^{N+3} v'_{j} \right) c'(k - 3, i - N + k - 6) \\
&\quad + v'_{N + 4} \cdots v'_{3} p'_{3} \left( (N + 3) \right) \\
&\geq u'_{N + 4} c'(N + 3, i) + p'_{N + 3} c'(N + 2, i - 1) - v'_{N + 4} h \\
&\geq u_{N + 4} \left( c'(N + 3, i) - k_{N + 4} h \right) \\
&\geq u_{N + 4} \left( c'(N + 3, i) - k h \right),
\end{align*}
\]

because \( u'_{N + 4} = u_{N + 4}, \ v'_{N + 4} = v_{N + 4} \) and \( p'_{N + 3} = p_{N + 3} \geq 0 \). Now suppose (5.8) holds for some \( j \geq 4 \). By Claim 1, we have that for \( j \geq 4 \),

\[
\begin{align*}
&c'(N + j + 1, i) = u'_{N + j + 1} c'(N + j, i) + p'_{N + j} c(N + j - 1, i - 1) \\
&\quad + \sum_{k=4}^{N+j+1} p'_{k-2} \left( \prod_{j=k}^{N+j+1} v'_{j} \right) c'(k - 3, i - N + k - j - 3) \\
&\quad + v'_{N+j+1} \cdots v'_{3} p'_{3} \left( (N + j) \right) \\
&\geq u'_{N + j + 1} c'(N + j, i) + p'_{N + j} c(N + j - 1, i - 1) \\
&\quad + \sum_{k=N+5}^{N+j+1} p'_{k-2} \left( \prod_{j=k}^{N+j+1} v'_{j} \right) c'(k - 3, i - N + k - j - 3) \\
&\quad + \sum_{k=4}^{N+4} p'_{k-2} \left( \prod_{j=k}^{N+j+1} v'_{j} \right) c'(k - 3, i - N + k - j - 3) \\
&\quad + v'_{N+j+1} \cdots v'_{3} p'_{3} \left( (N + j) \right).
\end{align*}
\]

Since \( p'_{n} = p_{n} > 0 \) for \( n \geq N + 3 \) and \( c'(n, \ell) > 0 \) for \( 0 \leq n \leq N + j \) by the inductive hypothesis, it follows that

\[
(5.9) \quad p'_{N+j} c(N+j-1, i-1) + \sum_{k=N+5}^{N+j+1} p'_{k-2} \left( \prod_{j=k}^{N+j+1} v'_{j} \right) c'(k-3, i-N+k-j-3) \geq 0.
\]
By the inductive hypothesis and (5.9),
\[ c'(N + j + 1, i) \]
\[ \geq u'_{N+j+1}c'(N + j, i) + \sum_{k=4}^{N+4} p'_{k-2} \left( \prod_{j=k}^{N+j+1} v'_j \right) c'(k - 3, i - N + k - j - 3) \]
\[ + v'_{N+j+1} \cdots v'_3 \rho'_1(N+j) \]
\[ \geq u_{N+j+1}u_{N+j} \cdots u_{N+4} \left( c'(N + 3, i) - \sum_{n=1}^{j-3} k^n h \right) \]
\[ + v_{N+j+1}v_{N+j} \cdots v_{N+4} \left( \sum_{k=4}^{N+4} p'_{k-2} \left( \prod_{j=k}^{N+3} v'_j \right) c'(k - 3, i - N + k - j - 3) \right) \]
\[ + v'_{N+3} \cdots v'_3 \rho'_1(N+j) \]
\[ \geq u_{N+j+1}u_{N+j} \cdots u_{N+4} \left( c'(N + 3, i) - \sum_{n=1}^{j-3} k^n h \right) - v_{N+j+1}v_{N+j} \cdots v_{N+4} h \]
\[ = u_{N+j+1}u_{N+j} \cdots u_{N+4} \left( c'(N + 3, i) - \sum_{n=1}^{j-3} k^n h - k_{N+j+1}k_{N+j+2} \cdots k_{N+4} h \right) \]
\[ \geq u_{N+j+1}u_{N+j} \cdots u_{N+4} \left( c'(N + 3, i) - \sum_{n=1}^{j-2} k^n h \right), \]
which proves Claim 3.

Since \( \sum_{n=1}^{j} k^n < \frac{1}{1-k} \) for every \( j > 1 \), it follows from Claim 3 and (5.7) that
\[ c'(N + j, i) > 0 \quad \text{for } j \geq 4 \quad \text{and} \quad 0 \leq i \leq N + j. \]

It thus follows from (5.4) and (5.10) that \( c'(n, i) > 0 \) for every \( n \geq 0 \) and \( 0 \leq i \leq n + 1 \). Therefore \( W_{\alpha'} \) is also positively quadratically hyponormal. This completes the proof. \( \square \)

**Corollary 5.2.** Let \( W_{\alpha} \) be a weighted shift such that \( \alpha_{j-1} < \alpha_j \) for some \( j \geq 1 \), and let \( T_x \) be the weighted shift with weight sequence

\[ \alpha_x : \alpha_0, \ldots, \alpha_{j-1}, x, \alpha_{j+1}, \ldots. \]

Then \( \{ x : T_x \text{ is 2-hyponormal} \} \) is a proper closed subset of \( \{ x : T_x \text{ is quadratically hyponormal} \} \) whenever the latter set is nonempty.

**Proof.** Write

\[ H_2 := \{ x : T_x \text{ is 2-hyponormal} \}. \]

Without loss of generality, we can assume that \( H_2 \) is nonempty, and that \( j = 1 \). Recall that a 2-hyponormal weighted shift with two equal weights is of the form \( \alpha_0 = \alpha_1 = \alpha_2 = \cdots \) or \( \alpha_0 < \alpha_1 = \alpha_2 = \cdots \). Let \( x_m := \inf H_2 \). By Proposition 6.7 below, \( T_{x_m} \) is hyponormal. Then \( x_m > \alpha_0 \). By assumption, \( x_m < \alpha_2 \). Thus \( \alpha_0, x_m, \alpha_2, \alpha_3, \cdots \) is strictly increasing. Now we apply Theorem 2.3 to obtain \( x' \) such that \( \alpha_0 < x' < x_m \) and \( T_{x'} \) is quadratically hyponormal. However \( T_{x'} \) is not 2-hyponormal by the definition of \( x_m \). The proof is complete. \( \square \)
The following question arises naturally:

**Question 5.3.** Let $\alpha$ be a strictly increasing weight sequence and let $k \geq 3$. If $W_\alpha$ is a $k$-hyponormal weighted shift, does it follow that $W_\alpha$ is weakly $k$-hyponormal under a small perturbation of the weight sequence?

### 6. Other related results

#### 6.1. Subnormal extensions

Let $\alpha : \alpha_0, \alpha_1, \cdots$ be a weight sequence, let $x_i > 0$ for $1 \leq i \leq n$, and let $(x_n, \cdots, x_1)\alpha : x_n, \cdots, x_1, \alpha_0, \alpha_1, \cdots$ be the augmented weight sequence. We say that $W_{(x_n, \cdots, x_1)\alpha}$ is an extension (or n-step extension) of $W_\alpha$. Observe that

$$W_{(x_n, \cdots, x_1)\alpha} \setminus \{e_{n+1}, \cdots\} \cong W_\alpha.$$  

The hypothesis $F \not= c P_{(0)}$ in Theorem 2.1 is essential. Indeed, there exist infinitely many one-step subnormal extensions of a subnormal weighted shift whenever one such extension exists. Recall ([7, Proposition 8]) that if $W_\alpha$ is a weighted shift whose restriction to $\mathcal{V}\{e_1, e_2, \cdots\}$ is subnormal with associated measure $\mu$, then $W_\alpha$ is subnormal if and only if

1. $t \in L^1(\mu)$;
2. $\alpha_0^2 \leq (||t||_{L^1(\mu)})^{-1}$.

Also note that there may not exist any one-step subnormal extension of the subnormal weighted shift: for example, if $W_\alpha$ is the Bergman shift, then the corresponding Berger measure is $\mu(t) = t$, and hence $t$ is not integrable with respect to $\mu$; therefore $W_\alpha$ does not admit any subnormal extension. A similar situation arises when $\mu$ has an atom at $\{0\}$.

More generally we have:

**Theorem 6.1 (Subnormal Extensions).** Let $W_\alpha$ be a subnormal weighted shift with weights $\alpha : \alpha_0, \alpha_1, \cdots$ and let $\mu$ be the corresponding Berger measure. Then $W_{(x_n, \cdots, x_1)\alpha}$ is subnormal if and only if

1. $\frac{1}{n} \in L^1(\mu)$;
2. $||x_j||_{L^1(\mu)} \leq \left(\frac{1}{n}||x_j||_{L^1(\mu)}\right)^{1/2}$ for $1 \leq j \leq n - 1$;
3. $x_n \leq \left(\frac{1}{n}||x_n||_{L^1(\mu)}\right)^{1/n}$.

In particular, if we put

$$S := \{(x_1, \cdots, x_n) \in \mathbb{R}^n : W_{(x_n, \cdots, x_1)\alpha} \text{ is subnormal}\},$$

then either $S = \emptyset$ or $S$ is a line segment in $\mathbb{R}^n$.

**Proof.** Write $W_j := W_{(x_n, \cdots, x_1)\alpha} \setminus \{e_{n-j}, e_{n-j+1}, \cdots\} \ (1 \leq j \leq n)$ and hence $W_0 = W_{(x_n, \cdots, x_1)\alpha}$. By the argument used to establish (3.2) we have that $W_1$ is subnormal with associated measure $\nu_1$ if and only if

1. $\frac{1}{n} \in L^1(\mu)$;
2. $d\nu_1 = \frac{1}{n} d\mu$, or equivalently, $x_1^2 = \left(\int_0^{||W_1||^2} \frac{1}{t} \mu(t)\right)^{-1}$.

Inductively $W_{n-1}$ is subnormal with associated measure $\nu_{n-1}$ if and only if

1. $W_{n-2}$ is subnormal;
2. $\frac{1}{n-1} \in L^1(\mu)$;
Corollary 6.2. If $W_\alpha$ is a subnormal weighted shift with associated measure $\mu$, there exists an $n$-step subnormal extension of $W_\alpha$ if and only if $\frac{1}{t} \in L^1(\mu)$.

For the next result we refer to the notation in (2.1) and (2.2).

Corollary 6.3. A recursively generated subnormal shift with $\varphi_0 \neq 0$ admits an $n$-step subnormal extension for every $n \geq 1$.

Proof. The assumption about $\varphi_0$ implies that the zeros of $g(t)$ are positive, so that $s_0 > 0$. Thus for every $n \geq 1$, $\frac{1}{t}$ is integrable with respect to the corresponding Berger measure $\mu = \rho_0 \delta_{s_0} + \cdots + \rho_r \delta_{s_{r-1}}$. By Corollary 6.2, there exists an $n$-step subnormal extension. □

We need not expect that for arbitrary recursively generated shifts, 2-hyponormality and subnormality coincide as in Theorem 3.2. For example, if $\alpha : \sqrt{2}, \sqrt{-2}, (\sqrt{3}, \sqrt{-2}, \sqrt{-3})^\wedge$, then by (2.12) and Theorem 6.1,

(i) $T_x$ is 2-hyponormal $\iff \sqrt{4 - \sqrt{6}} \leq x \leq 2$;

(ii) $T_x$ is subnormal $\iff x = 2$.

A straightforward calculation shows, however, that $T_x$ is 3-hyponormal if and only if $x = 2$; for,

$$A(0; 3) := \begin{pmatrix} 1 & \frac{1}{2} x & \frac{1}{2} x & \frac{3}{2} x & \frac{5}{2} x \\ \frac{1}{2} x & \frac{1}{2} x & \frac{3}{2} x & \frac{5}{2} x \\ \frac{3}{2} x & \frac{5}{2} x & 17x & 58x \\ \frac{3}{2} x & \frac{5}{2} x & 17x & 58x \end{pmatrix} \succeq 0 \iff x = 2.$$

This behavior is typical of general recursively generated weighted shifts: we show in [E3] that subnormality is equivalent to $k$-hyponormality for some $k \geq 2$.

6.2. Convexity and closedness. Next, we will show that canonical rank-one perturbations of $k$-hyponormal weighted shifts which preserve $k$-hyponormality form a convex set. To see this we need an auxiliary result.

Lemma 6.4. Let $I = \{1, \ldots, n\} \times \{1, \ldots, n\}$ and let $J$ be a symmetric subset of $I$. Let $A = (a_{ij}) \in M_n(\mathbb{C})$ and let $C = (c_{ij}) \in M_n(\mathbb{C})$ be given by

$$c_{ij} = \begin{cases} c a_{ij} & \text{if } (i, j) \in J \\ a_{ij} & \text{if } (i, j) \in I \setminus J \end{cases} (c > 0).$$
If $A$ and $C$ are positive semidefinite, then $B = (b_{ij}) \in M_n(\mathbb{C})$ defined by

$$b_{ij} = \begin{cases} \frac{b}{c}a_{ij} & \text{if } (i, j) \in J \\ a_{ij} & \text{if } (i, j) \in I \setminus J \end{cases} \quad (b \in [1, c] \text{ or } [c, 1])$$

is also positive semidefinite.

**Proof.** Without loss of generality we may assume $c > 1$. If $b = 1$ or $b = c$, the assertion is trivial. Thus we assume $1 < b < c$. The result is now a consequence of the following observation. If $[D]_{(i,j)}$ denotes the $(i,j)$-entry of the matrix $D$, then

$$\begin{bmatrix} c - b \\ c - 1 \end{bmatrix} \begin{bmatrix} A + b - 1 \\ c - b \end{bmatrix} = \begin{cases} \frac{c-b}{c-1} \left( 1 + \frac{b-1}{c-b}c \right) a_{ij} & \text{if } (i, j) \in J, \\ \frac{c-b}{c-1} \left( 1 + \frac{b-1}{c-b}c \right) a_{ij} & \text{if } (i, j) \in I \setminus J, \\ b \cdot a_{ij} & \text{if } (i, j) \in J, \\ a_{ij} & \text{if } (i, j) \in I \setminus J, \\ = [B]_{(i,j)} \end{cases},$$

which is positive semidefinite because positive semidefinite matrices in $M_n(\mathbb{C})$ form a cone.

An immediate consequence of Lemma 6.4 is that positivity of a matrix forms a convex set with respect to a fixed diagonal location; i.e., if

$$A_x = \begin{pmatrix} * & * & * \\ & x & * \\ * & * & * \end{pmatrix},$$

then $\{x : A_x \text{ is positive semidefinite}\}$ is convex.

We now have:

**Theorem 6.5.** Let $\alpha = \{\alpha_n\}_{n=0}^{\infty}$ be a weight sequence, let $k \geq 1$, and let $j \geq 0$. Define $\alpha^{(j)}(x) : \alpha_0, \ldots, \alpha_j, x, \alpha_{j+1}, \ldots$. Assume $W_\alpha$ is $k$-hyponormal and define

$$\Omega_{\alpha}^{k,j} := \{x : W_{\alpha^{(j)}(x)} \text{ is } k\text{-hyponormal}\}.$$

Then $\Omega_{\alpha}^{k,j}$ is a closed interval.

**Proof.** Suppose $x_1, x_2 \in \Omega_{\alpha}^{k,j}$ with $x_1 < x_2$. Then by (2.11), the $(k+1) \times (k+1)$ Hankel matrix

$$A_{x_i}(n; k) := \begin{pmatrix} \gamma_n & \gamma_{n+1} & \cdots & \gamma_{n+k} \\ \gamma_{n+1} & \gamma_{n+2} & \cdots & \gamma_{n+k+1} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{n+k} & \gamma_{n+k+1} & \cdots & \gamma_{n+2k} \end{pmatrix} \quad (n \geq 0; \ i = 1, 2)$$

is positive, where $A_{x_i}$ corresponds to $\alpha^{(j)}(x_i)$. We must show that $tx_1 + (1-t)x_2 \in \Omega_{\alpha}^{k,j}$ $(0 < t < 1)$, i.e.,

$$A_{tx_1 + (1-t)x_2}(n; k) \geq 0 \quad (n \geq 0, \ 0 < t < 1).$$

Observe that it suffices to establish the positivity of the $2k$ Hankel matrices corresponding to $\alpha^{(j)}(tx_1 + (1-t)x_2)$ such that $tx_1 + (1-t)x_2$ appears as a factor in at least one entry but not in every entry. A moment’s thought reveals that without loss of generality we may assume $j = 2k$. Observe that

$$A_{z_1}(n; k) - A_{z_2}(n; k) = (z_1^2 - z_2^2) H(n; k)$$
for some Hankel matrix $H(n;k)$. For notational convenience, we abbreviate $A_k(n;k)$ as $A_k$. Then

$$A_{tx_1+(1-t)x_2} = \begin{cases} t^2 A_{x_1} + (1-t)^2 A_{x_2} + 2t(1-t)A_{x_1x_2} & \text{for } 0 \leq n \leq 2k; \\ (t + (1-t)\frac{x_1}{x_2})^2 A_{x_1} & \text{for } n \geq 2k + 1. \end{cases}$$

Since $A_{x_1} \geq 0$, $A_{x_2} \geq 0$ and $A_{x_1x_2}$ have the form described by Lemma 6.4 and since $x_1 < \sqrt{x_1x_2} < x_2$, it follows from Lemma 6.4 that $A_{\sqrt{x_1x_2}} \geq 0$. Thus evidently, $A_{tx_1+(1-t)x_2} \geq 0$, and therefore $tx_1 + (1-t)x_2 \in \Omega_{\alpha}^{k,j}$. This shows that $\Omega_{\alpha}^{k,j}$ is an interval. The closedness of the interval follows from Proposition 6.7 below.

In [17] and [18], it was shown that there exists a nonsubnormal polynomially hyponormal operator. Also in [22], it was shown that there exists a nonsubnormal polynomially hyponormal operator if and only if there exists one which is also a weighted shift. However, no concrete weighted shift has yet been found. As a strategy for finding such a shift, we would like to suggest the following:

**Question 6.6.** Does it follow that the polynomial hyponormality of a weighted shift is stable under small perturbations of the weight sequence?

If the answer to Question 6.6 were affirmative then we would easily find a polynomially hyponormal nonsubnormal (even non-2-hyponormal) weighted shift; for example, if

$$\alpha : 1, \sqrt{x}, (\sqrt{3}, \sqrt{\frac{10}{3}}, \sqrt{\frac{17}{5}}),$$

and $T_\alpha$ is the weighted shift associated with $\alpha$, then by Theorem 3.2, $T_\alpha$ is subnormal if $x = 2$, whereas $T_\alpha$ is polynomially hyponormal if $2 - \delta_1 < x < 2 + \delta_2$ for some $\delta_1, \delta_2 > 0$ provided the answer to Question 6.6 is yes; therefore for sufficiently small $\epsilon > 0$,

$$\alpha_\epsilon : 1, \sqrt{x+\epsilon}, (\sqrt{3}, \sqrt{\frac{10}{3}}, \sqrt{\frac{17}{5}}),$$

would induce a non-2-hyponormal polynomially hyponormal weighted shift.

The answer to Question 6.6 for weak $k$-hyponormality is negative. In fact we have:

**Proposition 6.7.** (i) The set of $k$-hyponormal operators is sot-closed.

(ii) The set of weakly $k$-hyponormal operators is sot-closed.

**Proof.** Suppose $T_\eta \in \mathcal{L}(H)$ and $T_\eta \rightharpoonup T$ in sot. Then, by the Uniform Boundedness Principle, $\{(|T_\eta|)|_{\eta}\}$ is bounded. Thus $T_\eta^* T_\eta^* \rightarrow T^* T$ in sot for every $i, j$, so that $M_k(T_\eta) \rightharpoonup M_k(T)$ in sot (where $M_k(T)$ is as in (1.2)).

(i) In this case $M_k(T_\eta) \geq 0$ for all $\eta$, so $M_k(T) \geq 0$, i.e., $T$ is $k$-hyponormal.

(ii) Here, $M_k(T_\eta)$ is weakly positive for all $\eta$. By (1.3), $M_k(T)$ is also weakly positive, i.e., $T$ is weakly $k$-hyponormal.  

**References**


Department of Mathematics, University of Iowa, Iowa City, Iowa 52242
E-mail address: rcurto@math.uiowa.edu

Department of Mathematics, Seoul National University, Seoul 151-742, Korea
E-mail address: wylee@math.snu.ac.kr