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k-HYPONORMALITY OF FINITE RANK PERTURBATIONS OF UNILATERAL WEIGHTED SHIFTS

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ABSTRACT. In this paper we explore finite rank perturbations of unilateral weighted shifts W_α. First, we prove that the subnormality of W_α is never stable under nonzero finite rank perturbations unless the perturbation occurs at the zeroth weight. Second, we establish that 2-hyponormality implies positive quadratic hyponormality, in the sense that the Maclaurin coefficients of $D_n(s) := \det P_n\left[(W_\alpha + sW_\alpha^2)^*, W_\alpha + sW_\alpha^2\right] P_n$ are nonnegative, for every $n \geq 0$, where P_n denotes the orthogonal projection onto the basis vectors $\{e_0, \cdots, e_n\}$. Finally, for α strictly increasing and W_α 2-hyponormal, we show that for a small finite-rank perturbation α' of α, the shift $W_{\alpha'}$ remains quadratically hyponormal.

1. Introduction

Let $\mathcal{H}$ and $\mathcal{K}$ be complex Hilbert spaces, let $\mathcal{L}(\mathcal{H}, \mathcal{K})$ be the set of bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$ and write $\mathcal{L}(\mathcal{H}) := \mathcal{L}(\mathcal{H}, \mathcal{H})$. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be normal if $T^*T = TT^*$, hyponormal if $T^*T \geq TT^*$, and subnormal if $T = N|_{\mathcal{H}}$, where N is normal on some Hilbert space $\mathcal{K} \supseteq \mathcal{H}$. If T is subnormal, then T is also hyponormal. Recall that given a bounded sequence of positive numbers $\alpha : \alpha_0, \alpha_1, \cdots$ (called *weights*), the (*unilateral*) *weighted shift* W_α associated with α is the operator on $\ell^2(\mathbb{Z}_+)$ defined by $W_\alpha e_n := \alpha_n e_{n+1}$ for all $n \geq 0$, where $\{e_n\}_{n=0}^\infty$ is the canonical orthonormal basis for ℓ^2. It is straightforward to check that W_α can never be *normal*, and that W_α is *hyponormal* if and only if $\alpha_n \leq \alpha_{n+1}$ for all $n \geq 0$. The Bram-Halmos criterion for subnormality states that an operator T is subnormal if and only if

$$\sum_{i,j} (T^i x_j, T^j x_i) \geq 0$$

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for all finite collections $x_0, x_1, \cdots, x_k \in \mathcal{H}$ ([2], [4, II.1.9]). It is easy to see that this is equivalent to the following positivity test:

$$\begin{pmatrix} I & T^* & \dots & T^{*k} \\ T & T^*T & \dots & T^{*k}T \\ \vdots & \vdots & \ddots & \vdots \\ T^k & T^*T^k & \dots & T^{*k}T^k \end{pmatrix} \geq 0 \qquad \text{(for all } k \geq 1). \tag{1.1}$$

Condition (1.1) provides a measure of the gap between hyponormality and subnormality. In fact, the positivity condition (1.1) for $k = 1$ is equivalent to the hyponormality of T, while subnormality requires the validity of (1.1) for all k. Let $[A, B] := AB - BA$ denote the commutator of two operators A and B, and define T to be *k-hyponormal* whenever the $k \times k$ operator matrix

$$M_k(T) := ([T^{*j}, T^i])_{i,j=1}^k \tag{1.2}$$

is positive. An application of the Choleski algorithm for operator matrices shows that the positivity of (1.2) is equivalent to the positivity of the $(k+1) \times (k+1)$ operator matrix in (1.1); the Bram-Halmos criterion can then be rephrased to say that T is subnormal if and only if T is k-hyponormal for every $k \geq 1$ ([16]).

Recall ([1], [16], [5]) that $T \in \mathcal{L}(\mathcal{H})$ is said to be *weakly k-hyponormal* if

$$LS(T, T^2, \cdots, T^k) := \left\{ \sum_{j=1}^{k} \alpha_j T^j : \alpha = (\alpha_1, \cdots, \alpha_k) \in \mathbb{C}^k \right\}$$

consists entirely of hyponormal operators, or equivalently, $M_k(T)$ is *weakly positive*, i.e. ([16]),

$$\left(M_k(T) \begin{pmatrix} \lambda_0 x \\ \vdots \\ \lambda_k x \end{pmatrix}, \begin{pmatrix} \lambda_0 x \\ \vdots \\ \lambda_k x \end{pmatrix} \right) \geq 0 \qquad \text{for } x \in \mathcal{H} \text{ and } \lambda_0, \cdots, \lambda_k \in \mathbb{C}. \tag{1.3}$$

If $k = 2$, then T is said to be *quadratically hyponormal*, and if $k = 3$, then T is said to be *cubically hyponormal*. Similarly, $T \in \mathcal{L}(\mathcal{H})$ is said to be *polynomially hyponormal* if $p(T)$ is hyponormal for every polynomial $p \in \mathbb{C}[z]$. It is known that k-hyponormal $\Rightarrow$ weakly k-hyponormal, but the converse is not true in general.

The classes of (weakly) k-hyponormal operators have been studied in an attempt to bridge the gap between subnormality and hyponormality ([7], [8], [10], [11], [12], [14], [16], [19], [22]). The study of this gap has been only partially successful. For example, such a gap is not yet well described for Toeplitz operators on the Hardy space of the unit circle; in fact, even subnormality for Toeplitz operators has not been characterized (cf. [20], [6]). For weighted shifts, positive results appear in [7] and [12], although no concrete example of a weighted shift which is polynomially hyponormal and not subnormal has yet been found (the existence of such weighted shifts was established in [17] and [18]).

In the present paper we renew our efforts to help describe the above-mentioned gap between subnormality and hyponormality, with particular emphasis on polynomial hyponormality. We focus on the class of unilateral weighted shifts, and initiate a study of how the above-mentioned notions behave under finite perturbations of the weight sequence. We first obtain the following three concrete results.

(i) the subnormality of W_α is never stable under nonzero finite rank perturbations unless the perturbation is confined to the zeroth weight (Theorem 2.1);

(ii) 2-hyponormality implies *positive quadratic hyponormality*, in the sense that the Maclaurin coefficients of $D_n(s) := \det P_n\,[(W_\alpha + sW_\alpha^2)^*,\, W_\alpha + sW_\alpha^2]\,P_n$ are nonnegative, for every $n \geq 0$, where P_n denotes the orthogonal projection onto the basis vectors $\{e_0, \cdots, e_n\}$ (Theorem 2.2); and

(iii) if α is strictly increasing and W_α is 2-hyponormal, then for α' a small perturbation of α, the shift $W_{\alpha'}$ remains positively quadratically hyponormal (Theorem 2.3).

Along the way we establish two related results, each of independent interest:

(iv) an integrality criterion for a subnormal weighted shift to have an n-step subnormal extension (Theorem 6.1); and

(v) a proof that the sets of k-hyponormal and weakly k-hyponormal operators are closed in the strong operator topology (Proposition 6.7).

2. Statement of main results

C. Berger's characterization of subnormality for unilateral weighted shifts (cf. [21], [4, III.8.16]) states that W_α is subnormal if and only if there exists a Borel probability measure μ (the so-called Berger measure of W_α) supported in $[0, ||W_\alpha||^2]$, with $||W_\alpha||^2 \in \operatorname{supp}\mu$, such that

$$\gamma_n = \int t^n d\mu(t) \quad \text{for all } n \geq 0.$$

Given an initial segment of weights $\alpha : \alpha_0, \cdots, \alpha_m$, the sequence $\hat{\alpha} \in \ell^\infty(\mathbb{Z}_+)$ such that $\hat{\alpha}_i = \alpha_i$ $(i = 0, \cdots, m)$ is said to be *recursively generated* by α if there exist $r \geq 1$ and $\varphi_0, \cdots, \varphi_{r-1} \in \mathbb{R}$ such that

$$\gamma_{n+r} = \varphi_0\gamma_n + \cdots + \varphi_{r-1}\gamma_{n+r-1} \quad (\text{for all } n \geq 0), \tag{2.1}$$

where $\gamma_0 := 1$, $\gamma_n := \alpha_0^2 \cdots \alpha_{n-1}^2$ $(n \geq 1)$. In this case $W_{\hat{\alpha}}$ with weights $\hat{\alpha}$ is said to be *recursively generated*. If we let

$$g(t) := t^r - \left(\varphi_{r-1}t^{r-1} + \cdots + \varphi_0\right), \tag{2.2}$$

then g has r distinct real roots $0 \leq s_0 < \cdots < s_{r-1}$ ([11, Theorem 3.9]). Let

$$V := \begin{pmatrix} 1 & 1 & \dots & 1 \\ s_0 & s_1 & \dots & s_{r-1} \\ \vdots & \vdots & & \vdots \\ s_0^{r-1} & s_1^{r-1} & \dots & s_{r-1}^{r-1} \end{pmatrix}$$

and let

$$\begin{pmatrix} \rho_0 \\ \vdots \\ \rho_{r-1} \end{pmatrix} := V^{-1} \begin{pmatrix} \gamma_0 \\ \vdots \\ \gamma_{r-1} \end{pmatrix}.$$

If the associated recursively generated weighted shift $W_{\hat{\alpha}}$ is subnormal, then its Berger measure is of the form

$$\mu := \rho_0\delta_{s_0} + \cdots + \rho_{r-1}\delta_{r-1}.$$

For example, given $\alpha_0 < \alpha_1 < \alpha_2$, $W_{(\alpha_0,\alpha_1,\alpha_2)^\wedge}$ is the recursive weighted shift whose weights are calculated according to the recursive relation

$$\alpha_{n+1}^2 = \varphi_1 + \varphi_0 \frac{1}{\alpha_n^2}, \tag{2.3}$$

where

$$\varphi_0 = -\frac{\alpha_0^2\alpha_1^2(\alpha_2^2-\alpha_1^2)}{\alpha_1^2-\alpha_0^2} \quad \text{and} \quad \varphi_1 = \frac{\alpha_1^2(\alpha_2^2-\alpha_0^2)}{\alpha_1^2-\alpha_0^2}. \tag{2.4}$$

In this case, $W_{(\alpha_0,\alpha_1,\alpha_2)^\wedge}$ is subnormal with 2–atomic Berger measure. Let $W_{x\,(\alpha_0,\alpha_1,\alpha_2)^\wedge}$ denote the weighted shift whose weight sequence consists of the initial weight x followed by the weight sequence of $W_{(\alpha_0,\alpha_1,\alpha_2)^\wedge}$.

By the Density Theorem ([11, Theorem 4.2 and Corollary 4.3]), we know that if W_α is a subnormal weighted shift with weights $\alpha = \{\alpha_n\}$ and $\epsilon > 0$, then there exists a nonzero compact operator K with $||K|| < \epsilon$ such that $W_\alpha + K$ is a recursively generated subnormal weighted shift; in fact $W_\alpha + K = W_{\widehat{\alpha^{(m)}}}$ for some $m \geq 1$, where $\alpha^{(m)} : \alpha_0, \cdots, \alpha_m$. The following result shows that K cannot generally be taken to be of finite rank.

Theorem 2.1 (Finite Rank Perturbations of Subnormal Shifts)**.** *If W_α is a subnormal weighted shift, then there exists no nonzero finite rank operator F ($\neq cP_{\{e_0\}}$) such that $W_\alpha + F$ is a subnormal weighted shift. Concretely, suppose W_α is a subnormal weighted shift with weight sequence $\alpha = \{\alpha_n\}_{n=0}^\infty$ and assume $\alpha' = \{\alpha_n'\}$ is a nonzero perturbation of α in a finite number of weights except the initial weight. Then $W_{\alpha'}$ is not subnormal.*

We next consider the self-commutator $[(W_\alpha + s\,W_\alpha^2)^*, W_\alpha + s\,W_\alpha^2]$. Let W_α be a hyponormal weighted shift. For $s \in \mathbb{C}$, we write

$$D(s) := [(W_\alpha + s\,W_\alpha^2)^*, W_\alpha + s\,W_\alpha^2]$$

and we let

$$\begin{aligned} D_n(s) &:= P_n[(W_\alpha + s\,W_\alpha^2)^*, W_\alpha + s\,W_\alpha^2]P_n \\ &= \begin{pmatrix} q_0 & \bar r_0 & 0 & \dots & 0 & 0 \\ r_0 & q_1 & \bar r_1 & \dots & 0 & 0 \\ 0 & r_1 & q_2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & q_{n-1} & \bar r_{n-1} \\ 0 & 0 & 0 & \dots & r_{n-1} & q_n \end{pmatrix}, \end{aligned} \tag{2.5}$$

where P_n is the orthogonal projection onto the subspace generated by $\{e_0, \cdots, e_n\}$,

$$\begin{cases} q_n := u_n + |s|^2 v_n, \\ r_n := s\sqrt{w_n}, \\ u_n := \alpha_n^2 - \alpha_{n-1}^2, \\ v_n := \alpha_n^2\alpha_{n+1}^2 - \alpha_{n-1}^2\alpha_{n-2}^2, \\ w_n := \alpha_n^2(\alpha_{n+1}^2 - \alpha_{n-1}^2)^2, \end{cases} \tag{2.6}$$

and, for notational convenience, $\alpha_{-2} = \alpha_{-1} = 0$. Clearly, W_α is quadratically hyponormal if and only if $D_n(s) \geq 0$ for all $s \in \mathbb{C}$ and all $n \geq 0$. Let $d_n(\cdot) :=$

$\det(D_n(\cdot))$. Then d_n satisfies the following 2–step recursive formula:

$$d_0 = q_0, \quad d_1 = q_0 q_1 - |r_0|^2, \quad d_{n+2} = q_{n+2} d_{n+1} - |r_{n+1}|^2 d_n. \tag{2.7}$$

If we let $t := |s|^2$, we observe that d_n is a polynomial in t of degree $n+1$, and if we write $d_n \equiv \sum_{i=0}^{n+1} c(n,i)t^i$, then the coefficients $c(n,i)$ satisfy a double-indexed recursive formula, namely

$$\begin{aligned} &c(n+2,i) = u_{n+2}\, c(n+1,i) + v_{n+2}\, c(n+1,i-1) - w_{n+1}\, c(n,i-1), \\ &c(n,0) = u_0 \cdots u_n, \quad c(n,n+1) = v_0 \cdots v_n, \quad c(1,1) = u_1 v_0 + v_1 u_0 - w_0 \end{aligned} \tag{2.8}$$

($n \geq 0$, $i \geq 1$). We say that W_α is *positively quadratically hyponormal* if $c(n,i) \geq 0$ for every $n \geq 0$, $0 \leq i \leq n+1$ (cf. [9]). Evidently, positively quadratically hyponormal $\Longrightarrow$ quadratically hyponormal. The converse, however, is not true in general (cf. [3]).

The following theorem establishes a useful relation between 2-hyponormality and positive quadratic hyponormality.

Theorem 2.2. *Let $\alpha \equiv \{\alpha_n\}_{n=0}^\infty$ be a weight sequence and assume that W_α is 2-hyponormal. Then W_α is positively quadratically hyponormal. More precisely, if W_α is 2-hyponormal, then*

$$c(n,i) \;\geq\; v_0 \cdots v_{i-1} u_i \cdots u_n \qquad (n \geq 0,\ 0 \leq i \leq n+1). \tag{2.9}$$

In particular, if α is strictly increasing and W_α is 2-hyponormal, then the Maclaurin coefficients of $d_n(t)$ are positive for all $n \geq 0$.

If W_α is a weighted shift with weight sequence $\alpha = \{\alpha_n\}_{n=0}^\infty$, then the *moments* of W_α are usually defined by $\beta_0 := 1$, $\beta_{n+1} := \alpha_n \beta_n$ ($n \geq 0$) [23]; however, we prefer to reserve this term for the sequence $\gamma_n := \beta_n^2$ ($n \geq 0$). A criterion for k-hyponormality can be given in terms of these moments ([7, Theorem 4]): if we build a $(k+1)\times(k+1)$ Hankel matrix $A(n;k)$ by

$$A(n;k) := \begin{pmatrix} \gamma_n & \gamma_{n+1} & \cdots & \gamma_{n+k} \\ \gamma_{n+1} & \gamma_{n+2} & \cdots & \gamma_{n+k+1} \\ \vdots & \vdots & & \vdots \\ \gamma_{n+k} & \gamma_{n+k+1} & \cdots & \gamma_{n+2k} \end{pmatrix} \quad (n \geq 0), \tag{2.10}$$

then

$$W_\alpha \text{ is } k\text{-hyponormal} \iff A(n;k) \geq 0 \quad (n \geq 0). \tag{2.11}$$

In particular, for α strictly increasing, W_α is 2-hyponormal if and only if

$$\det \begin{pmatrix} \gamma_n & \gamma_{n+1} & \gamma_{n+2} \\ \gamma_{n+1} & \gamma_{n+2} & \gamma_{n+3} \\ \gamma_{n+2} & \gamma_{n+3} & \gamma_{n+4} \end{pmatrix} \geq 0 \quad (n \geq 0). \tag{2.12}$$

One might conjecture that if W_α is a k-hyponormal weighted shift whose weight sequence is strictly increasing, then W_α remains weakly k-hyponormal under a small perturbation of the weight sequence. We will show below that this is true for $k=2$ (Theorem 2.3).

In [12, Theorem 4.3], it was shown that the gap between 2-hyponormality and quadratic hyponormality can be detected by unilateral shifts with a weight sequence $\alpha : \sqrt{x}, (\sqrt{a}, \sqrt{b}, \sqrt{c})^\wedge$. In particular, there exists a maximum value $H_2 \equiv H_2(a,b,c)$

of x that makes $W_{\sqrt{x},(\sqrt{a},\sqrt{b},\sqrt{c})^\wedge}$ 2-hyponormal; H_2 is called the *modulus* of 2-hyponormality (cf. [12]). Any value of $x > H_2$ yields a non-2-hyponormal weighted shift. However, if $x - H_2$ is small enough, $W_{\sqrt{x},(\sqrt{a},\sqrt{b},\sqrt{c})^\wedge}$ is still quadratically hyponormal. The following theorem shows that, more generally, for finite rank perturbations of weighted shifts with strictly increasing weight sequences, there always exists a gap between 2-hyponormality and quadratic hyponormality.

Theorem 2.3 (Finite Rank Perturbations of 2-Hyponormal Shifts)**.** *Let* $\alpha = \{\alpha_n\}_{n=0}^\infty$ *be a strictly increasing weight sequence. If* W_α *is* 2*-hyponormal, then* W_α *remains positively quadratically hyponormal under a small nonzero finite rank perturbation of* α*.*

3. Proof of Theorem 2.1

Proof of Theorem 2.1*.* It suffices to show that if T is a weighted shift whose restriction to $\bigvee\{e_n, e_{n+1}, \cdots\}$ $(n \geq 2)$ is subnormal, then there is at most one α_{n-1} for which T is subnormal.

Let $W := T|_{\bigvee\{e_{n-1},e_n,e_{n+1},\cdots\}}$ and $S := T|_{\bigvee\{e_n,e_{n+1},\cdots\}}$, where $n \geq 2$. Then W and S have weights $\alpha_k(W) := \alpha_{k+n-1}$ and $\alpha_k(S) := \alpha_{k+n}$ $(k \geq 0)$. Thus the corresponding moments are related by the equation

$$\gamma_k(S) = \alpha_n^2 \cdots \alpha_{n+k-1}^2 = \frac{\gamma_{k+1}(W)}{\alpha_{n-1}^2}.$$

We now adapt the proof of [7, Proposition 8]. Suppose S is subnormal with associated Berger measure μ. Then $\gamma_k(S) = \int_0^{||T||^2} t^k\, d\mu$. Thus W is subnormal if and only if there exists a probability measure ν on $[0, ||T||^2]$ such that

$$\frac{1}{\alpha_{n-1}^2}\int_0^{||T||^2} t^{k+1}\, d\nu(t) = \int_0^{||T||^2} t^k\, d\mu(t) \quad \text{for all } k \geq 0,$$

which readily implies that $t\, d\nu = \alpha_{n-1}^2\, d\mu$. Thus W is subnormal if and only if the formula

$$d\nu := \lambda \cdot \delta_0 + \frac{\alpha_{n-1}^2}{t} d\mu \tag{3.1}$$

defines a probability measure for some $\lambda \geq 0$, where δ_0 is the point mass at the origin. In particular $\frac{1}{t} \in L^1(\mu)$ and $\mu(\{0\}) = 0$ whenever W is subnormal. If we repeat the above argument for W and $V := T|_{\bigvee\{e_{n-2},e_{n-1},\cdots\}}$, then we should have that $\nu(\{0\}) = 0$ whenever V is subnormal. Therefore we can conclude that if V is subnormal, then $\lambda = 0$, and hence

$$d\nu = \frac{\alpha_{n-1}^2}{t} d\mu. \tag{3.2}$$

Thus we have

$$1 = \int_0^{||T||^2} d\nu(t) = \alpha_{n-1}^2 \int_0^{||T||^2} \frac{1}{t} d\mu(t),$$

so that

$$\alpha_{n-1}^2 = \left(\int_0^{||T||^2} \frac{1}{t} d\mu(t)\right)^{-1}, \tag{3.3}$$

which implies that α_{n-1} is determined uniquely by $\{\alpha_n, \alpha_{n+1}, \cdots\}$ whenever T is subnormal. This completes the proof. $\square$

Theorem 2.1 says that a nonzero finite rank perturbation of a subnormal shift is never subnormal unless the perturbation occurs at the initial weight. However, this is not the case for k-hyponormality. To see this we use a close relative of the Bergman shift B_+ (whose weights are given by $\alpha = \{\sqrt{\frac{n+1}{n+2}}\}_{n=0}^\infty$); it is well known that B_+ is subnormal.

Example 3.1. For $x > 0$, let T_x be the weighted shift whose weights are given by

$$\alpha_0 := \sqrt{\frac{1}{2}}, \quad \alpha_1 := \sqrt{x}, \quad \text{and} \quad \alpha_n := \sqrt{\frac{n+1}{n+2}} \ (n \geq 2).$$

Then we have:

(i) T_x is subnormal $\Longleftrightarrow x = \frac{2}{3}$;

(ii) T_x is 2-hyponormal $\Longleftrightarrow \frac{63-\sqrt{129}}{80} \leq x \leq \frac{24}{35}$.

Proof. Assertion (i) follows from Theorem 2.1. For assertion (ii) we use (2.12): T_x is 2-hyponormal if and only if

$$\det \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{2}x \\ \frac{1}{2} & \frac{1}{2}x & \frac{3}{8}x \\ \frac{1}{2}x & \frac{3}{8}x & \frac{3}{10}x \end{pmatrix} \geq 0 \quad \text{and} \quad \det \begin{pmatrix} \frac{1}{2} & \frac{1}{2}x & \frac{3}{8}x \\ \frac{1}{2}x & \frac{3}{8}x & \frac{3}{10}x \\ \frac{3}{8}x & \frac{3}{10}x & \frac{1}{4}x \end{pmatrix} \geq 0,$$

or equivalently, $\frac{63-\sqrt{129}}{80} \leq x \leq \frac{24}{35}$. $\square$

For perturbations of recursive subnormal shifts of the form $W_{(\sqrt{a},\sqrt{b},\sqrt{c})^\wedge}$, subnormality and 2-hyponormality coincide.

Theorem 3.2. *Let $\alpha = \{\alpha_n\}_{n=0}^\infty$ be recursively generated by $\sqrt{a}, \sqrt{b}, \sqrt{c}$. If T_x is the weighted shift whose weights are given by $\alpha_x : \alpha_0, \cdots, \alpha_{j-1}, \sqrt{x}, \alpha_{j+1}, \cdots$, then we have*

$$T_x \text{ is subnormal} \Longleftrightarrow T_x \text{ is 2-hyponormal} \Longleftrightarrow \begin{cases} x = \alpha_j^2 & \text{if } j \geq 1; \\ x \leq a & \text{if } j = 0. \end{cases}$$

Proof. Since α is recursively generated by $\sqrt{a}, \sqrt{b}, \sqrt{c}$, we have that $\alpha_0^2 = a$, $\alpha_1^2 = b$, $\alpha_2^2 = c$,

$$\alpha_3^2 = \frac{b(c^2 - 2ac + ab)}{c(b-a)}, \quad \text{and}$$
$$\alpha_4^2 = \frac{bc^3 - 4abc^2 + 2ab^2c + a^2bc - a^2b^2 + a^2c^2}{(b-a)(c^2 - 2ac + ab)}. \tag{3.4}$$

Case 1 ($j = 0$): It is evident that T_x is subnormal if and only if $x \leq a$. For 2-hyponormality observe by (2.12) that T_x is 2-hyponormal if and only if

$$\det \begin{pmatrix} 1 & x & bx \\ x & bx & bcx \\ bx & bcx & \alpha_3^2 bcx \end{pmatrix} \geq 0,$$

or equivalently, $x \leq a$.

Case 2 ($j \ge 1$): Without loss of generality we may assume that $j = 1$ and $a = 1$. Thus $\alpha_1 = \sqrt{x}$. Then by Theorem 2.1, T_x is subnormal if and only if $x = b$. On the other hand, by (2.12), T_x is 2-hyponormal if and only if

$$\det \begin{pmatrix} 1 & 1 & x \\ 1 & x & cx \\ x & cx & \alpha_3^2 cx \end{pmatrix} \ge 0 \quad \text{and} \quad \det \begin{pmatrix} 1 & x & cx \\ x & cx & \alpha_3^2 cx \\ cx & \alpha_3^2 cx & \alpha_3^2\alpha_4^2 cx \end{pmatrix} \ge 0.$$

Thus a direct calculation with the specific forms of α_3, α_4 given in (3.4) shows that T_x is 2-hyponormal if and only if $(x - b)\left(x - \frac{b(c^2-2c+b)}{b-1}\right) \le 0$ and $x \le b$. Since $b \le \frac{b(c^2-2c+b)}{b-1}$, it follows that T_x is 2-hyponormal if and only if $x = b$. This completes the proof. □

4. Proof of Theorem 2.2

With the notation in (2.6), we let

$$p_n := u_n\, v_{n+1} - w_n \qquad (n \ge 0).$$

We then have:

Lemma 4.1. *If $\alpha \equiv \{\alpha_n\}_{n=0}^\infty$ is a strictly increasing weight sequence, then the following statements are equivalent:*

(i) *W_α is 2-hyponormal;*
(ii) $\alpha_{n+1}^2(u_{n+1} + u_{n+2})^2 \le u_{n+1}v_{n+2} \quad (n \ge 0)$*;*
(iii) $\frac{\alpha_n^2}{\alpha_{n+2}^2}\frac{u_{n+2}}{u_{n+3}} \le \frac{u_{n+1}}{u_{n+2}} \quad (n \ge 0)$*;*
(iv) $p_n \ge 0 \quad (n \ge 0)$.

Proof. This follows from a straightforward calculation. □

Proof of Theorem 2.2. If α is not strictly increasing, then α is flat, by the argument of [7, Corollary 6], i.e., $\alpha_0 = \alpha_1 = \alpha_2 = \cdots$. Then

$$D_n(s) = \begin{pmatrix} \alpha_0^2+|s|^2\alpha_0^4 & \bar{s}\alpha_0^3 \\ s\alpha_0^3 & |s|^2\alpha_0^4 \end{pmatrix} \oplus 0_\infty \tag{4.1}$$

(cf. (2.5)), so that (2.9) is evident. Thus we may assume that α is strictly increasing, so that $u_n > 0$, $v_n > 0$ and $w_n > 0$ for all $n \ge 0$. Recall that if we write $d_n(t) := \sum_{i=0}^{n+1} c(n,i)t^i$, then the $c(n,i)$'s satisfy the following recursive formulas (cf. (2.8)):

$$\begin{aligned} c(n+2,i) &= u_{n+2}\, c(n+1,i) + v_{n+2}\, c(n+1,i-1) \\ &\qquad - w_{n+1}\, c(n,i-1) \quad (n \ge 0,\ 1 \le i \le n). \end{aligned} \tag{4.2}$$

Also, $c(n, n+1) = v_0 \cdots v_n$ (again by (2.8)) and $p_n := u_n v_{n+1} - w_n \ge 0$ $(n \ge 0)$, by Lemma 4.1. A straightforward calculation shows that

$$\begin{aligned} d_0(t) &= u_0 + v_0\, t; \\ d_1(t) &= u_0u_1 + (v_0u_1 + p_0)\, t + v_0v_1\, t^2; \\ d_2(t) &= u_0u_1u_2 + (v_0u_1u_2 + u_0p_1 + u_2p_0)\, t \\ &\qquad + (v_0v_1u_2 + v_0p_1 + v_2p_0)\, t^2 + v_0v_1v_2\, t^3. \end{aligned} \tag{4.3}$$

Evidently,

$$c(n,\, i) \ge 0 \qquad (0 \le n \le 2,\ 0 \le i \le n+1). \tag{4.4}$$

Define

$$\beta(n,i) := c(n,i) - v_0 \cdots v_{i-1} u_i \cdots u_n \qquad (n \geq 1,\ 1 \leq i \leq n).$$

For every $n \geq 1$, we now have

$$c(n,i) = \begin{cases} u_0 \cdots u_n \geq 0 & (i=0), \\ v_0 \cdots v_{i-1} u_i \cdots u_n \ + \ \beta(n,i) & (1 \leq i \leq n), \\ v_0 \cdots v_n \geq 0 & (i = n+1). \end{cases} \tag{4.5}$$

For notational convenience we let $\beta(n,0) := 0$ for every $n \geq 0$. □

Claim 1. *For* $n \geq 1$,

$$c(n,\, n) \geq u_n\, c(n-1,\, n) \geq 0. \tag{4.6}$$

Proof of Claim 1. We use mathematical induction. For $n = 1$,

$$c(1,1) = v_0 u_1 + p_0 \ \geq \ u_1\, c(0,1) \geq 0,$$

and

$$\begin{aligned} c(n+1,n+1) &= u_{n+1}\, c(n,n+1) + v_{n+1}\, c(n,n) - w_n c(n-1,n) \\ &\geq u_{n+1}\, c(n,n+1) + v_{n+1}\, u_n c(n-1,n) - w_n\, c(n-1,n) \\ &\quad \text{(by the inductive hypothesis)} \\ &= u_{n+1}\, c(n,n+1) + p_n\, c(n-1,n) \\ &\geq u_{n+1}\, c(n,n+1), \end{aligned}$$

which proves Claim 1.

Claim 2. *For* $n \geq 2$,

$$\beta(n,\, i) \geq u_n\, \beta(n-1,i) \geq 0 \qquad (0 \leq i \leq n-1). \tag{4.7}$$

Proof of Claim 2. We use mathematical induction. If $n = 2$ and $i = 0$, this is trivial. Also,

$$\beta(2,1) = u_0\, p_1 + u_2\, p_0 = u_0\, p_1 + u_2\, \beta(1,1) \geq u_2\, \beta(1,1) \geq 0.$$

Assume that (4.7) holds. We shall prove that

$$\beta(n+1,\, i) \geq u_{n+1}\, \beta(n,i) \geq 0 \qquad (0 \leq i \leq n).$$

For,

$$\begin{aligned} \beta(n+1,i) + v_0 \cdots v_{i-1} u_i \cdots u_{n+1} &= c(n+1,i) \qquad \text{(by (4.2))} \\ &= u_{n+1} c(n,i) + v_{n+1} c(n,i-1) - w_n c(n-1,i-1) \\ &= u_{n+1} \Big(\beta(n,i) + v_0 \cdots v_{i-1} u_i \cdots u_n \Big) \\ &\quad + v_{n+1} \Big(\beta(n,i-1) + v_0 \cdots v_{i-2} u_{i-1} \cdots u_n \Big) \\ &\quad - w_n \Big(\beta(n-1,i-1) + v_0 \cdots v_{i-2} u_{i-1} \cdots u_{n-1} \Big), \end{aligned}$$

so that

$$\begin{aligned}\beta(n+1,i) &= u_{n+1}\beta(n,i)+v_{n+1}\beta(n,i-1)-w_n\beta(n-1,i-1)\\ &\quad + v_0\cdots v_{i-2}u_{i-1}\cdots u_{n-1}\,(u_nv_{n+1}-w_n)\\ &= u_{n+1}\beta(n,i)+v_{n+1}\beta(n,i-1)-w_n\beta(n-1,i-1)\\ &\quad + (v_0\cdots v_{i-2}u_{i-1}\cdots u_{n-1})\,p_n\\ &\ge u_{n+1}\beta(n,i)+v_{n+1}u_n\beta(n-1,i-1)-w_n\beta(n-1,i-1)\\ &\qquad \text{(by the inductive hypothesis and Lemma 4.1;}\\ &\qquad \text{observe that } i-1\le n-1 \text{, so (4.7) applies)}\\ &= u_{n+1}\beta(n,i)+p_n\,\beta(n-1,i-1)\\ &\ge u_{n+1}\,\beta(n,i),\end{aligned}$$

which proves Claim 2.

By Claim 2 and (4.5), we can see that $c(n,i)\ge 0$ for all $n\ge 0$ and $1\le i\le n-1$. Therefore (4.4), (4.5), Claim 1 and Claim 2 imply

$$c(n,i) \;\ge\; v_0\cdots v_{i-1}u_i\cdots u_n \qquad (n\ge 0,\ 0\le i\le n+1).$$

This completes the proof.

5. Proof of Theorem 2.3

To prove Theorem 2.3 we need:

Lemma 5.1 ([15, Lemma 2.3])**.** *Let $\alpha\equiv\{\alpha_n\}_{n=0}^{\infty}$ be a strictly increasing weight sequence. If W_α is 2-hyponormal, then the sequence of quotients*

$$\Theta_n := \frac{u_{n+1}}{u_{n+2}} \qquad (n\ge 0) \tag{5.1}$$

is bounded away from 0 and from ∞. More precisely,

$$1\le\Theta_n\le\frac{u_1}{u_2}\left(\frac{||W_\alpha||^2}{\alpha_0\alpha_1}\right)^2 \quad \textit{for sufficiently large } n. \tag{5.2}$$

In particular, $\{u_n\}_{n=0}^{\infty}$ is eventually decreasing.

Proof of Theorem 2.3. By Theorem 2.2, W_α is *strictly* positively quadratically hyponormal, in the sense that all coefficients of $d_n(t)$ are *positive* for all $n\ge 0$. Note that finite rank perturbations of α affect a finite number of values of $u_n,\ v_n$ and w_n. More concretely, if α' is a perturbation of α in the weights $\{\alpha_0,\cdots,\alpha_N\}$, then $u_n,\ v_n,\ w_n$ and p_n are invariant under α' for $n\ge N+3$. In particular, $p_n\ge 0$ for $n\ge N+3$.

Claim 1. *For $n\ge 3,\ 0\le i\le n+1$,*

$$\begin{aligned}&c(n,i)=u_n\,c(n-1,i)+p_{n-1}\,c(n-2,i-1)\\ &\qquad +\sum_{k=4}^{n}p_{k-2}\left(\prod_{j=k}^{n}v_j\right)c(k-3,\,i-n+k-2)+v_n\cdots v_3\,\rho_{i-n+1},\end{aligned} \tag{5.3}$$

where

$$\rho_{i-n+1} = \begin{cases} 0 & (i < n-1), \\ u_0 p_1 & (i = n-1), \\ v_0 p_1 + v_2 p_0 & (i = n), \\ v_0 v_1 v_2 & (i = n+1) \end{cases}$$

(cf. [12, Proof of Theorem 4.3]*).*

Proof of Claim 1. We use induction. For $n = 3,\ 0 \le i \le 4$,

$$\begin{aligned} c(3,i) &= u_3\, c(2,\, i) + v_3\, c(2,\, i-1) - w_2\, c(1,\, i-1) \\ &= u_3\, c(2,i) + v_3 \Big(u_2\, c(1, i-1) + v_2\, c(1,\, i-2) - w_1\, c(0,\, i-2) \Big) \\ &\quad - w_2\, c(1, i-1) \\ &= u_3\, c(2,i) + p_2\, c(1, i-1) + v_3 \Big(v_2\, c(1, i-2) - w_1\, c(0, i-2) \Big) \\ &= u_3\, c(2,i) + p_2\, c(1, i-1) + v_3\, \rho_{i-2}, \end{aligned}$$

where by (4.3),

$$\rho_{i-2} = \begin{cases} 0 & (i < 2), \\ u_0 p_1 & (i = 2), \\ v_0 p_1 + v_2 p_0 & (i = 3), \\ v_0 v_1 v_2 & (i = 4). \end{cases}$$

Now,

$$\begin{aligned} c(n+1,i) &= u_{n+1} c(n,i) + v_{n+1} c(n, i-1) - w_n c(n-1, i-1) \\ &= u_{n+1} c(n,i) + v_{n+1} \Bigg(u_n c(n-1, i-1) + p_{n-1} c(n-2, i-2) \\ &\quad + \sum_{k=4}^{n} p_{k-2} \left(\prod_{j=k}^{n} v_j \right) c(k-3, i-n+k-3) + v_n \cdots v_3 \rho_{i-n} \Bigg) \\ &\quad - w_n\, c(n-1, i-1) \\ &= u_{n+1} c(n,i) + p_n c(n-1, i-1) + v_{n+1} p_{n-1} c(n-2, i-2) \\ &\quad + v_{n+1} \sum_{k=4}^{n} p_{k-2} \left(\prod_{j=k}^{n} v_j \right) c(k-3, i-n+k-3) + v_{n+1} \cdots v_3 \rho_{i-n} \\ &\qquad \text{(by the inductive hypothesis)} \\ &= u_{n+1} c(n,i) + p_n c(n-1, i-1) \\ &\quad + \sum_{k=4}^{n+1} p_{k-2} \left(\prod_{j=k}^{n+1} v_j \right) c(k-3, i-n+k-3) + v_{n+1} \cdots v_3 \rho_{i-n}, \end{aligned}$$

which proves Claim 1.

Write $u_n', v_n', w_n', p_n', \rho_n'$, and $c'(\cdot,\cdot)$ for the entities corresponding to α'. If $p_n > 0$ for every $n = 0, \cdots, N+2$, then in view of Claim 1, we can choose a small perturbation such that $p_n' > 0$ $(0 \le n \le N+2)$ and therefore $c'(n,i) > 0$ for all $n \ge 0$ and $0 \le i \le n+1$, which implies that $W_{\alpha'}$ is also positively quadratically hyponormal. If instead $p_n = 0$ for some $n = 0, \cdots, N+2$, careful inspection of (5.3) reveals that without loss of generality we may assume $p_0 = \cdots = p_{N+2} = 0$. By Theorem 2.2, we have that for a sufficiently small perturbation α' of α,

$$c'(n,i) > 0 \quad (0 \le n \le N+2,\ 0 \le i \le n+1) \quad \text{and} \quad c'(n,n+1) > 0 \quad (n \ge 0). \tag{5.4}$$

Write

$$k_n := \frac{v_n}{u_n} \qquad (n = 2, 3, \cdots).$$

Claim 2. *$\{k_n\}_{n=2}^{\infty}$ is bounded.*

Proof of Claim 2. Observe that

$$\begin{aligned} k_n = \frac{v_n}{u_n} &= \frac{\alpha_n^2\alpha_{n+1}^2 - \alpha_{n-1}^2\alpha_{n-2}^2}{\alpha_n^2 - \alpha_{n-1}^2} \\ &= \alpha_n^2 + \alpha_{n-1}^2 + \alpha_n^2\frac{\alpha_{n+1}^2 - \alpha_n^2}{\alpha_n^2 - \alpha_{n-1}^2} + \alpha_{n-1}^2\frac{\alpha_{n-1}^2 - \alpha_{n-2}^2}{\alpha_n^2 - \alpha_{n-1}^2}. \end{aligned} \tag{5.5}$$

Therefore if W_α is 2-hyponormal, then by Lemma 5.1, the sequences

$$\left\{\frac{\alpha_{n+1}^2 - \alpha_n^2}{\alpha_n^2 - \alpha_{n-1}^2}\right\}_{n=2}^{\infty} \quad \text{and} \quad \left\{\frac{\alpha_{n-1}^2 - \alpha_{n-2}^2}{\alpha_n^2 - \alpha_{n-1}^2}\right\}_{n=2}^{\infty}$$

are both bounded, so that $\{k_n\}_{n=2}^{\infty}$ is bounded. This proves Claim 2.

Write $k := \sup_n k_n$. Without loss of generality we assume $k < 1$ (this is possible from the observation that $c\alpha$ induces $\{c^2 k_n\}$). Choose a sufficiently small perturbation α' of α such that if we let

$$h := \sup_{\substack{0\le \ell\le N+2\\ 0\le m\le 1}} \left| \sum_{k=4}^{N+4} p_{k-2}' \left(\prod_{j=k}^{N+3} v_j'\right) c'(k-3,\ \ell) + v_{N+3}' \cdots v_3'\, \rho_m' \right|, \tag{5.6}$$

then

$$c'(N+3,\ i) - \frac{1}{1-k}h \ > \ 0 \qquad (0 \le i \le N+3) \tag{5.7}$$

(this is always possible because, by Theorem 2.2, we can choose a sufficiently small $|p_i'|$ such that

$$c'(N+3, i) > v_0 \cdots v_{i-1}u_i \cdots u_{N+3} - \epsilon \quad \text{and} \quad |h| < (1-k)\big(v_0 \cdots v_{i-1}u_i \cdots u_{N+3} - \epsilon\big)$$

for any small $\epsilon > 0$).

Claim 3. *For $j \geq 4$ and $0 \leq i \leq N+j$,*

$$c'(N+j,\, i) \;\geq\; u_{N+j}\cdots u_{N+4}\left(c'(N+3,\; i) - \sum_{n=1}^{j-3} k^n\, h\right). \tag{5.8}$$

Proof of Claim 3. We use induction. If $j = 4$, then by Claim 1 and (5.6),

$$\begin{aligned}
c'(N+4,\, i) &= u'_{N+4}c'(N+3,i) + p'_{N+3}c'(N+2,i-1)\\
&\quad + v'_{N+4}\sum_{k=4}^{N+4} p'_{k-2}\left(\prod_{j=k}^{N+3} v'_j\right) c'(k-3, i-N+k-6)\\
&\quad + v'_{N+4}\cdots v'_3\rho'_{i-(N+3)}\\
&\geq u'_{N+4}c'(N+3,i) + p'_{N+3}c'(N+2,i-1) - v'_{N+4}h\\
&\geq u_{N+4}\big(c'(N+3,i) - k_{N+4}h\big)\\
&\geq u_{N+4}\big(c'(N+3,i) - k\,h\big),
\end{aligned}$$

because $u'_{N+4} = u_{N+4},\; v'_{N+4} = v_{N+4}$ and $p'_{N+3} = p_{N+3} \geq 0$. Now suppose (5.8) holds for some $j \geq 4$. By Claim 1, we have that for $j \geq 4$,

$$\begin{aligned}
c'(N+j+1,i) &= u'_{N+j+1}c'(N+j,\, i) + p'_{N+j}c(N+j-1,\, i-1)\\
&\quad + \sum_{k=4}^{N+j+1} p'_{k-2}\left(\prod_{j=k}^{N+j+1} v'_j\right) c'(k-3, i-N+k-j-3)\\
&\quad + v'_{N+j+1}\cdots v'_3\rho'_{i-(N+j)}\\
&= u'_{N+j+1}c'(N+j,\, i) + p'_{N+j}c(N+j-1,\, i-1)\\
&\quad + \sum_{k=N+5}^{N+j+1} p'_{k-2}\left(\prod_{j=k}^{N+j+1} v'_j\right) c'(k-3, i-N+k-j-3)\\
&\quad + \sum_{k=4}^{N+4} p'_{k-2}\left(\prod_{j=k}^{N+j+1} v'_j\right) c'(k-3, i-N+k-j-3)\\
&\quad + v'_{N+j+1}\cdots v'_3\rho'_{i-(N+j)}.
\end{aligned}$$

Since $p'_n = p_n > 0$ for $n \geq N+3$ and $c'(n,\ell) > 0$ for $0 \leq n \leq N+j$ by the inductive hypothesis, it follows that

$$p'_{N+j}c(N+j-1,\, i-1) + \sum_{k=N+5}^{N+j+1} p'_{k-2}\left(\prod_{j=k}^{N+j+1} v'_j\right) c'(k-3, i-N+k-j-3) \geq 0. \tag{5.9}$$

By the inductive hypothesis and (5.9),

$$
\begin{aligned}
&c'(N+j+1,i)\\
&\geq u'_{N+j+1}c'(N+j,\,i)+\sum_{k=4}^{N+4}p'_{k-2}\left(\prod_{j=k}^{N+j+1}v'_j\right)c'(k-3,i-N+k-j-3)\\
&\quad+v'_{N+j+1}\cdots v'_3\rho'_{i-(N+j)}\\
&\geq u_{N+j+1}u_{N+j}\cdots u_{N+4}\left(c'(N+3,i)-\sum_{n=1}^{j-3}k^nh\right)\\
&\quad+v_{N+j+1}v_{N+j}\cdots v_{N+4}\left(\sum_{k=4}^{N+4}p'_{k-2}\left(\prod_{j=k}^{N+3}v'_j\right)c'(k-3,i-N+k-j-3)\right.\\
&\qquad\qquad\left.+v'_{N+3}\cdots v'_3\rho'_{i-(N+j)}\right)\\
&\geq u_{N+j+1}u_{N+j}\cdots u_{N+4}\left(c'(N+3,i)-\sum_{n=1}^{j-3}k^nh\right)-v_{N+j+1}v_{N+j}\cdots v_{N+4}\,h\\
&=u_{N+j+1}u_{N+j}\cdots u_{N+4}\left(c'(N+3,i)-\sum_{n=1}^{j-3}k^nh-k_{N+j+1}k_{N+j}\cdots k_{N+4}\,h\right)\\
&\geq u_{N+j+1}u_{N+j}\cdots u_{N+4}\left(c'(N+3,i)-\sum_{n=1}^{j-2}k^nh\right),
\end{aligned}
$$

which proves Claim 3.

Since $\sum_{n=1}^{j}k^n<\frac{1}{1-k}$ for every $j>1$, it follows from Claim 3 and (5.7) that

$$c'(N+j,\,i)>0\quad\text{for } j\geq 4 \text{ and } 0\leq i\leq N+j. \tag{5.10}$$

It thus follows from (5.4) and (5.10) that $c'(n,i)>0$ for every $n\geq 0$ and $0\leq i\leq n+1$. Therefore $W_{\alpha'}$ is also positively quadratically hyponormal. This completes the proof. □

Corollary 5.2. *Let W_α be a weighted shift such that $\alpha_{j-1}<\alpha_j$ for some $j\geq 1$, and let T_x be the weighted shift with weight sequence*

$$\alpha_x:\alpha_0,\cdots,\alpha_{j-1},x,\alpha_{j+1},\cdots.$$

Then $\{x:T_x$ is 2-hyponormal$\}$ is a proper closed subset of $\{x:\ T_x$ is quadratically hyponormal$\}$ whenever the latter set is nonempty.

Proof. Write

$$H_2:=\{x:T_x \text{ is 2-hyponormal}\}.$$

Without loss of generality, we can assume that H_2 is nonempty, and that $j=1$. Recall that a 2-hyponormal weighted shift with two equal weights is of the form $\alpha_0=\alpha_1=\alpha_2=\cdots$ or $\alpha_0<\alpha_1=\alpha_2=\alpha_3=\cdots$. Let $x_m:=\inf H_2$. By Proposition 6.7 below, T_{x_m} is hyponormal. Then $x_m>\alpha_0$. By assumption, $x_m<\alpha_2$. Thus $\alpha_0,x_m,\alpha_2,\alpha_3,\cdots$ is strictly increasing. Now we apply Theorem 2.3 to obtain x' such that $\alpha_0<x'<x_m$ and $T_{x'}$ is quadratically hyponormal. However $T_{x'}$ is not 2-hyponormal by the definition of x_m. The proof is complete. □

The following question arises naturally:

Question 5.3. Let α be a strictly increasing weight sequence and let $k \geq 3$. If W_α is a k-hyponormal weighted shift, does it follow that W_α is weakly k-hyponormal under a small perturbation of the weight sequence?

6. Other related results

6.1. Subnormal extensions. Let $\alpha : \alpha_0, \alpha_1, \cdots$ be a weight sequence, let $x_i > 0$ for $1 \leq i \leq n$, and let $(x_n, \cdots, x_1)\alpha : x_n, \cdots, x_1, \alpha_0, \alpha_1, \cdots$ be the augmented weight sequence. We say that $W_{(x_n,\cdots,x_1)\alpha}$ is an *extension* (or *n-step extension*) of W_α. Observe that

$$W_{(x_n,\cdots,x_1)\alpha}|_{\bigvee\{e_n, e_{n+1},\cdots\}} \cong W_\alpha.$$

The hypothesis $F \neq c\, P_{\{e_0\}}$ in Theorem 2.1 is essential. Indeed, there exist infinitely many one-step subnormal extensions of a subnormal weighted shift whenever one such extension exists. Recall ([7, Proposition 8]) that if W_α is a weighted shift whose restriction to $\bigvee\{e_1, e_2, \cdots\}$ is subnormal with associated measure μ, then W_α is subnormal if and only if

(i) $\frac{1}{t} \in L^1(\mu)$;

(ii) $\alpha_0^2 \leq \left(||\frac{1}{t}||_{L^1(\mu)}\right)^{-1}$.

Also note that there may not exist any one-step subnormal extension of the subnormal weighted shift: for example, if W_α is the Bergman shift, then the corresponding Berger measure is $\mu(t) = t$, and hence $\frac{1}{t}$ is not integrable with respect to μ; therefore W_α does not admit any subnormal extension. A similar situation arises when μ has an atom at $\{0\}$.

More generally we have:

Theorem 6.1 (Subnormal Extensions)**.** *Let W_α be a subnormal weighted shift with weights $\alpha : \alpha_0, \alpha_1, \cdots$ and let μ be the corresponding Berger measure. Then $W_{(x_n,\cdots,x_1)\alpha}$ is subnormal if and only if*

(i) $\frac{1}{t^n} \in L^1(\mu)$;

(ii) $x_j = \left(\frac{||\frac{1}{t^{j-1}}||_{L^1(\mu)}}{||\frac{1}{t^j}||_{L^1(\mu)}}\right)^{\frac{1}{2}}$ *for* $1 \leq j \leq n-1$;

(iii) $x_n \leq \left(\frac{||\frac{1}{t^{n-1}}||_{L^1(\mu)}}{||\frac{1}{t^n}||_{L^1(\mu)}}\right)^{\frac{1}{2}}$.

In particular, if we put

$$S := \{(x_1, \cdots, x_n) \in \mathbb{R}^n : W_{(x_n,\cdots,x_1)\alpha} \ \text{ is subnormal}\},$$

then either $S = \emptyset$ or S is a line segment in $\mathbb{R}^n$.

Proof. Write $W_j := W_{(x_n,\cdots,x_1)\alpha}|_{\bigvee\{e_{n-j}, e_{n-j+1},\cdots\}}$ $(1 \leq j \leq n)$ and hence $W_n = W_{(x_n,\cdots,x_1)\alpha}$. By the argument used to establish (3.2) we have that W_1 is subnormal with associated measure ν_1 if and only if

(i) $\frac{1}{t} \in L^1(\mu)$;

(ii) $d\nu_1 = \frac{x_1^2}{t} d\mu$, or equivalently, $x_1^2 = \left(\int_0^{||W_\alpha||^2} \frac{1}{t}\, d\mu(t)\right)^{-1}$.

Inductively W_{n-1} is subnormal with associated measure ν_{n-1} if and only if

(i) W_{n-2} is subnormal;

(ii) $\frac{1}{t^{n-1}} \in L^1(\mu)$;

(iii) $d\nu_{n-1} = \frac{x_{n-1}^2}{t} d\nu_{n-2} = \cdots = \frac{x_{n-1}^2 \cdots x_1^2}{t^{n-1}} d\mu$, or equivalently,

$$x_{n-1}^2 = \frac{\int_0^{||W_\alpha||^2} \frac{1}{t^{n-2}}\, d\mu(t)}{\int_0^{||W_\alpha||^2} \frac{1}{t^{n-1}}\, d\mu(t)}.$$

Therefore W_n is subnormal if and only if

(i) W_{n-1} is subnormal;
(ii) $\frac{1}{t^n} \in L^1(\mu)$;
(iii) $x_n^2 \le \left(\int_0^{||W_\alpha||^2} \frac{1}{t} d\nu_{n-1}\right)^{-1} = \left(\int_0^{||W_\alpha||^2} \frac{x_{n-1}^2 \cdots x_1^2}{t^n} d\mu(t)\right)^{-1}$
$= \frac{\int_0^{||W_\alpha||^2} \frac{1}{t^{n-1}}\, d\mu(t)}{\int_0^{||W_\alpha||^2} \frac{1}{t^n}\, d\mu(t)}$.

□

Corollary 6.2. *If W_α is a subnormal weighted shift with associated measure μ, there exists an n-step subnormal extension of W_α if and only if $\frac{1}{t^n} \in L^1(\mu)$.*

For the next result we refer to the notation in (2.1) and (2.2).

Corollary 6.3. *A recursively generated subnormal shift with $\varphi_0 \neq 0$ admits an n-step subnormal extension for every $n \geq 1$.*

Proof. The assumption about φ_0 implies that the zeros of $g(t)$ are positive, so that $s_0 > 0$. Thus for every $n \geq 1$, $\frac{1}{t^n}$ is integrable with respect to the corresponding Berger measure $\mu = \rho_0 \delta_{s_0} + \cdots + \rho_{r-1}\delta_{s_{r-1}}$. By Corollary 6.2, there exists an n-step subnormal extension. □

We need not expect that for arbitrary recursively generated shifts, 2-hyponormality and subnormality coincide as in Theorem 3.2. For example, if $\alpha : \sqrt{\frac{1}{2}}, \sqrt{x}, (\sqrt{3}, \sqrt{\frac{10}{3}}, \sqrt{\frac{17}{5}})^\wedge$, then by (2.12) and Theorem 6.1,

(i) T_x is 2-hyponormal $\iff 4 - \sqrt{6} \leq x \leq 2$;
(ii) T_x is subnormal $\iff x = 2$.

A straightforward calculation shows, however, that T_x is 3-*hyponormal* if and only if $x = 2$; for,

$$A(0;3) := \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{2}x & \frac{3}{2}x \\ \frac{1}{2} & \frac{1}{2}x & \frac{3}{2}x & 5x \\ \frac{1}{2}x & \frac{3}{2}x & 5x & 17x \\ \frac{3}{2}x & 5x & 17x & 58x \end{pmatrix} \geq 0 \iff x = 2.$$

This behavior is typical of general recursively generated weighted shifts: we show in [13] that subnormality is equivalent to k-hyponormality for some $k \geq 2$.

6.2. Convexity and closedness. Next, we will show that canonical rank-one perturbations of k-hyponormal weighted shifts which preserve k-hyponormality form a convex set. To see this we need an auxiliary result.

Lemma 6.4. *Let $I = \{1, \cdots, n\} \times \{1, \cdots, n\}$ and let J be a symmetric subset of I. Let $A = (a_{ij}) \in M_n(\mathbb{C})$ and let $C = (c_{ij}) \in M_n(\mathbb{C})$ be given by*

$$c_{ij} = \begin{cases} c\, a_{ij} & \text{if } (i,j) \in J \\ a_{ij} & \text{if } (i,j) \in I \setminus J \end{cases} \qquad (c > 0).$$

If A and C are positive semidefinite, then $B = (b_{ij}) \in M_n(\mathbb{C})$ defined by

$$b_{ij} = \begin{cases} b\, a_{ij} & \text{if } (i,j) \in J \\ a_{ij} & \text{if } (i,j) \in I \setminus J \end{cases} \qquad (b \in [1,c] \text{ or } [c,1])$$

is also positive semidefinite.

Proof. Without loss of generality we may assume $c > 1$. If $b = 1$ or $b = c$, the assertion is trivial. Thus we assume $1 < b < c$. The result is now a consequence of the following observation. If $[D]_{(i,j)}$ denotes the (i,j)-entry of the matrix D, then

$$\begin{aligned}\left[\frac{c-b}{c-1}\left(A + \frac{b-1}{c-b}C\right)\right]_{(i,j)} &= \begin{cases} \frac{c-b}{c-1}\left(1 + \frac{b-1}{c-b}c\right) a_{ij} & \text{if } (i,j) \in J, \\ \frac{c-b}{c-1}\left(1 + \frac{b-1}{c-b}\right) a_{ij} & \text{if } (i,j) \in I \setminus J, \end{cases} \\ &= \begin{cases} b\, a_{ij} & \text{if } (i,j) \in J, \\ a_{ij} & \text{if } (i,j) \in I \setminus J, \end{cases} \\ &= [B]_{(i,j)},\end{aligned}$$

which is positive semidefinite because positive semidefinite matrices in $M_n(\mathbb{C})$ form a cone. □

An immediate consequence of Lemma 6.4 is that positivity of a matrix forms a convex set with respect to a fixed diagonal location; i.e., if

$$A_x = \begin{pmatrix} * & * & * \\ & x & * \\ * & * & * \end{pmatrix},$$

then $\{x : A_x \text{ is positive semidefinite}\}$ is convex.

We now have:

Theorem 6.5. *Let $\alpha = \{\alpha_n\}_{n=0}^{\infty}$ be a weight sequence, let $k \geq 1$, and let $j \geq 0$. Define $\alpha^{(j)}(x) : \alpha_0, \cdots, \alpha_{j-1}, x, \alpha_{j+1}, \cdots$. Assume W_α is k-hyponormal and define*

$$\Omega_\alpha^{k,j} := \{x : \ W_{\alpha^{(j)}(x)} \text{ is } k\text{-hyponormal}\}.$$

Then $\Omega_\alpha^{k,j}$ is a closed interval.

Proof. Suppose $x_1, x_2 \in \Omega_\alpha^{k,j}$ with $x_1 < x_2$. Then by (2.11), the $(k+1) \times (k+1)$ Hankel matrix

$$A_{x_i}(n;k) := \begin{pmatrix} \gamma_n & \gamma_{n+1} & \cdots & \gamma_{n+k} \\ \gamma_{n+1} & \gamma_{n+2} & \cdots & \gamma_{n+k+1} \\ \vdots & \vdots & & \vdots \\ \gamma_{n+k} & \gamma_{n+k+1} & \cdots & \gamma_{n+2k} \end{pmatrix} \qquad (n \geq 0;\ i = 1,2)$$

is positive, where A_{x_i} corresponds to $\alpha^{(j)}(x_i)$. We must show that $tx_1 + (1-t)x_2 \in \Omega_\alpha^{k,j}$ $(0 < t < 1)$, i.e.,

$$A_{tx_1+(1-t)x_2}(n;k) \ \geq\ 0 \quad (n \geq 0,\ 0 < t < 1).$$

Observe that it suffices to establish the positivity of the $2k$ Hankel matrices corresponding to $\alpha^{(j)}(tx_1 + (1-t)x_2)$ such that $tx_1 + (1-t)x_2$ appears as a factor in at least one entry but not in every entry. A moment's thought reveals that without loss of generality we may assume $j = 2k$. Observe that

$$A_{z_1}(n;k) - A_{z_2}(n;k) = (z_1^2 - z_2^2)\, H(n;k)$$

for some Hankel matrix $H(n;k)$. For notational convenience, we abbreviate $A_z(n;k)$ as A_z. Then

$$A_{tx_1+(1-t)x_2} = \begin{cases} t^2 A_{x_1} + (1-t)^2 A_{x_2} + 2t(1-t) A_{\sqrt{x_1 x_2}} & \text{for } 0 \le n \le 2k, \\ \left(t + (1-t)\frac{x_2}{x_1}\right)^2 A_{x_1} & \text{for } n \ge 2k+1. \end{cases}$$

Since $A_{x_1} \ge 0$, $A_{x_2} \ge 0$ and $A_{\sqrt{x_1 x_2}}$ have the form described by Lemma 6.4 and since $x_1 < \sqrt{x_1 x_2} < x_2$, it follows from Lemma 6.4 that $A_{\sqrt{x_1 x_2}} \ge 0$. Thus evidently, $A_{tx_1+(1-t)x_2} \ge 0$, and therefore $tx_1 + (1-t)x_2 \in \Omega_\alpha^{k,j}$. This shows that $\Omega_\alpha^{k,j}$ is an interval. The closedness of the interval follows from Proposition 6.7 below. $\square$

In [17] and [18], it was shown that there exists a nonsubnormal polynomially hyponormal operator. Also in [22], it was shown that there exists a nonsubnormal polynomially hyponormal operator if and only if there exists one which is also a weighted shift. However, no concrete weighted shift has yet been found. As a strategy for finding such a shift, we would like to suggest the following:

Question 6.6. Does it follow that the polynomial hyponormality of a weighted shift is stable under small perturbations of the weight sequence?

If the answer to Question 6.6 were affirmative then we would easily find a polynomially hyponormal nonsubnormal (even non-2-hyponormal) weighted shift; for example, if

$$\alpha : 1, \sqrt{x}, (\sqrt{3}, \sqrt{\frac{10}{3}}, \sqrt{\frac{17}{5}})^\wedge$$

and T_x is the weighted shift associated with α, then by Theorem 3.2, T_x is subnormal $\Leftrightarrow x = 2$, whereas T_x is polynomially hyponormal $\Leftrightarrow 2 - \delta_1 < x < 2 + \delta_2$ for some $\delta_1, \delta_2 > 0$ provided the answer to Question 6.6 is yes; therefore for sufficiently small $\epsilon > 0$,

$$\alpha_\epsilon : 1, \sqrt{2+\epsilon}, (\sqrt{3}, \sqrt{\frac{10}{3}}, \sqrt{\frac{17}{5}})^\wedge$$

would induce a non-2-hyponormal polynomially hyponormal weighted shift.

The answer to Question 6.6 for weak k-hyponormality is negative. In fact we have:

Proposition 6.7. (i) *The set of k-hyponormal operators is sot-closed.*

(ii) *The set of weakly k-hyponormal operators is sot-closed.*

Proof. Suppose $T_\eta \in \mathcal{L}(\mathcal{H})$ and $T_\eta \to T$ in *sot*. Then, by the Uniform Boundedness Principle, $\{||T_\eta||\}_\eta$ is bounded. Thus $T_\eta^{*i} T_\eta^j \to T^{*i} T^j$ in *sot* for every i, j, so that $M_k(T_\eta) \to M_k(T)$ in *sot* (where $M_k(T)$ is as in (1.2)).

(i) In this case $M_k(T_\eta) \ge 0$ for all η, so $M_k(T) \ge 0$, i.e., T is k-hyponormal.

(ii) Here, $M_k(T_\eta)$ is weakly positive for all η. By (1.3), $M_k(T)$ is also weakly positive, i.e., T is weakly k-hyponormal. $\square$

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