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ON THE K-THEORY AND TOPOLOGICAL CYCLIC HOMOLOGY OF SMOOTH SCHEMES OVER A DISCRETE VALUATION RING

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Abstract. We show that for a smooth and proper scheme over a henselian discrete valuation ring of mixed characteristic $(0,p)$, the p-adic étale K-theory and p-adic topological cyclic homology agree.

Introduction

In this paper, we consider a smooth scheme X over a henselian discrete valuation ring V of mixed characteristic $(0,p)$. Let $i\colon Y \hookrightarrow X$ denote the inclusion of the special fiber. We have previously showed that the cyclotomic trace

$$K_q^{\mathrm{et}}(Y,\mathbb{Z}/p^v) \xrightarrow{\sim} \mathrm{TC}_q(Y;p,\mathbb{Z}/p^v)$$

defines an isomorphism of the Dwyer-Friedlander étale K-theory of Y with $\mathbb{Z}/p^v$-coefficients and the topological cyclic homology of Y with $\mathbb{Z}/p^v$-coefficients [2]. Starting from this result, we show:

Theorem A. *Suppose that X is smooth and proper over V. Then for all integers q and all $v \geq 1$, the cyclotomic trace*

$$K_q^{\mathrm{et}}(X,\mathbb{Z}/p^v) \xrightarrow{\sim} \mathrm{TC}_q(X;p,\mathbb{Z}/p^v)$$

is an isomorphism.

We briefly outline the proof. The left-hand term in the statement of Theorem A is the abutment of a strongly convergent spectral sequence

$$E_2^{s,t} = H^s(X_{\mathrm{et}}, K_{-t}(-,\mathbb{Z}/p^v)^{\sim}) \Rightarrow K_{-s-t}^{\mathrm{et}}(X,\mathbb{Z}/p^v)$$

from the cohomology of the sheaf on the small étale site X_{et} associated with the presheaf which to $U \to X$ assigns $K_{-t}(U,\mathbb{Z}/p^v)$ [12]. Similarly, there is a strongly convergent spectral sequence

$$E_2^{s,t} = H^s_{\mathrm{cont}}(X_{\mathrm{et}}, \{\mathrm{TC}^m_{-t}(-;p,\mathbb{Z}/p^v)^{\sim}\}_{m\in\mathbb{N}}) \Rightarrow \mathrm{TC}_{-s-t}(X;p,\mathbb{Z}/p^v)$$

from the continuous cohomology of the sheaf on X_{et} associated with the presheaf which to $U \to X$ assigns the pro-abelian group $\{\mathrm{TC}^m_{-t}(U;p,\mathbb{Z}/p^v)\}_{m\in\mathbb{N}}$ [2]. The cyclotomic trace induces a map of spectral sequences and we show that the induced

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map of E_2-terms is an isomorphism. Since V is henselian, the proper base change theorem for étale cohomology implies that the canonical map

$$H^*(X_{\mathrm{et}}, F) \xrightarrow{\sim} H^*(Y_{\mathrm{et}}, i^*F)$$

is an isomorphism, for every sheaf of torsion abelian groups on X_{et}. Hence, it suffices to prove the following result, which is valid also if f is not proper.

Theorem B. *Suppose that X is smooth over V. Then the map of sheaves of pro-abelian groups on the small étale site of Y induced by the cyclotomic trace,*

$$i^* K_q(-, \mathbb{Z}/p^v)^{\sim} \xrightarrow{\sim} \{i^* \operatorname{TC}_q^m(-; p, \mathbb{Z}/p^v)^{\sim}\}_{m\in\mathbb{N}},$$

is an isomorphism, for all integers q and all $v \geq 1$.

The first step in the proof of Theorem B is to show that the sheaves in question depend only on the formal completion of X along Y. The inclusion of the special fiber factors, for all $n \geq 1$, through the surjective closed immersion

$$Y \xhookrightarrow{i_n} X_n = X \times_{\operatorname{Spec} V} \operatorname{Spec} V/p^n V,$$

and we show that in the following diagram of sheaves of pro-abelian groups on the small étale site of Y, the vertical maps are isomorphisms.

$$\begin{array}{ccc} i^* K_q(-, \mathbb{Z}/p^v)^{\sim} & \longrightarrow & \{i^* \operatorname{TC}_q^m(-; p, \mathbb{Z}/p^v)^{\sim}\}_{m\in\mathbb{N}} \\ \downarrow \sim & & \downarrow \sim \\ \{i_n^* K_q(-, \mathbb{Z}/p^v)^{\sim}\}_{n\in\mathbb{N}} & \longrightarrow & \{i_n^* \operatorname{TC}_q^m(-; p, \mathbb{Z}/p^v)^{\sim}\}_{m,n\in\mathbb{N}}. \end{array}$$

The proof for the right-hand map, which is given in section 3, is rather straightforward and valid also for the Zariski topology. The proof for the left-hand map, given in section 1, is an adaptation of the proof by Suslin and Panin [11, 9] of continuity of K-theory for henselian valuation rings.

Theorem C. *Let $\mathcal{O}$ be a local ring such that $(\mathcal{O}, p\mathcal{O})$ is a henselian pair and such that p is a non-zero-divisor. Then for all $q \geq 0$ and all $v \geq 1$, the map induced from the reduction*

$$K_q(\mathcal{O}, \mathbb{Z}/p^v) \xrightarrow{\sim} \{K_q(\mathcal{O}/p^n\mathcal{O}, \mathbb{Z}/p^v)\}_{n\in\mathbb{N}}$$

is an isomorphism of pro-abelian groups.

The requirement that the ring $\mathcal{O}$ be local may be relaxed a bit. It suffices that $\mathcal{O}$ has finite stable rank and that $SK_1(\mathcal{O})$ be zero.

It remains to show that the lower horizontal map in the diagram above is an isomorphism. To this end, we consider the following diagram of sheaves of pro-abelian groups on the small étale site of Y:

$$\begin{array}{ccc} \{i_n^* K_q(-, \mathbb{Z}/p^v)^{\sim}\}_{n\in\mathbb{N}} & \longrightarrow & \{i_n^* \operatorname{TC}_q^m(-; p, \mathbb{Z}/p^v)^{\sim}\}_{m,n\in\mathbb{N}} \\ \downarrow & & \downarrow \\ K_q(-; \mathbb{Z}/p^v)^{\sim} & \longrightarrow & \{\operatorname{TC}_q^m(-; p, \mathbb{Z}/p^v)^{\sim}\}_{m\in\mathbb{N}} \end{array}$$

As mentioned earlier, the lower horizontal map is an isomorphism by the main result of [2]. We remark that this is the only point in the proof of Theorems A and B that the smoothness assumption on X is used. Finally, a variant of a theorem

of McCarthy [8] implies that also the top horizontal map is an isomorphism. The details of the argument are given in section 2.

There is an analog of Theorem B for the Nisnevich topology: Let $W_m\Omega^q_{Y,\log}$ be the logarithmic de Rham-Witt sheaf on Y, and let $\epsilon\colon Y_{\mathrm{et}} \to Y_{\mathrm{Nis}}$ be the canonical map of sites. Then for all integers q and all $v \geq 1$, there is a canonical isomorphism of sheaves of pro-abelian groups on Y_{Nis}:

$$i^* K_q(-, \mathbb{Z}/p^v)^{\sim} \oplus \{R^1\epsilon_* W_m\Omega^{q+1}_{Y,\log}\}_{m\in\mathbb{N}} \xrightarrow{\sim} \{i^* \operatorname{TC}^m_q(-; p, \mathbb{Z}/p^v)^{\sim}\}_{m\in\mathbb{N}}.$$

The second summand on the right vanishes if $q \geq d + r$, where $d = \dim(Y)$ and where r is the cardinality of a p-basis of the residue field of V. It seems reasonable to expect that for these values of q, the cyclotomic trace

$$K_q(X, \mathbb{Z}/p^v) \to \operatorname{TC}_q(X; p, \mathbb{Z}/p^v)$$

is an isomorphism.

Unless otherwise stated, all rings considered in this paper are assumed commutative and unital. Part of the work reported here was done while the authors were visiting the University of Tokyo. It is a pleasure to thank the university and, in particular, Kazuya Kato and Takeshi Saito for their hospitality.

1. K-theory and group homology of henselian pairs

1.1. In this section, we prove Theorem C of the introduction. The proof follows Suslin [11, §3] and we make little claim of originality with the possible exception of Proposition 1.2.2 below, which replaces *loc.cit.*, Corollary 3.4. We first recall the fundamental rigidity result of Gabber [1], which we shall use below.

A pair (R, I) of a ring R and an ideal $I \subset R$ is called *henselian* if for every commutative diagram of rings,

$$\begin{tikzcd} R/I & B \arrow[l, "g"'] \\ R \arrow[u, "\pi"] \arrow[ur, dashed, "h"] & R \arrow[l, equal] \arrow[u, "f"'] \end{tikzcd}$$

with f étale, there exists a map $h : B \to R$ such that $hf = \mathrm{id}_R$ and $\pi h = g$. The following equivalent criterion for (R, I) to be henselian is useful: for every polynomial $f \in R[T]$ and every root $\bar{\alpha} \in R/I$ of the reduction $\bar{f} \in R/I[T]$ such that $\bar{f}'(\bar{\alpha})$ is a unit, there exists a root $\alpha \in R$ of f such that $\alpha + I = \bar{\alpha}$. We note that if (R, I) is henselian and if $J \subset I$ is a subideal, then also (R, J) is henselian.

Suppose that (R, I) is a henselian pair. Then the ideal I is contained in the radical of R, and hence for all $n \geq 1$, the map induced by the reduction

$$\mathrm{GL}_n(R) \twoheadrightarrow \mathrm{GL}_n(R/I)$$

is surjective. We denote the kernel by $\mathrm{GL}_n(R, I)$. We recall from [1] that if (R, I) is a henselian pair, and if p is invertible in R, then for all $q \geq 0$, the map induced from the reduction

$$K_q(R, \mathbb{Z}/p^v) \xrightarrow{\sim} K_q(R/I, \mathbb{Z}/p^v)$$

is an isomorphism. By [11, Corollary 1.6], this is equivalent to the statement that for all $q \geq 1$, the homology group $H_q(\mathrm{GL}(R, I), \mathbb{Z}/p^v)$ vanishes.

1.2. We now assume that (R, pR) is a henselian pair and that p is a non-zero-divisor in R.

Lemma 1.2.1. *Let $a \in R$ be a root of $f \in R[T]$ such that $f'(a)$ is a unit, and let $i \geq 1$ be an integer. Then every $g \in f + p^i R[T]$ has a unique root in $a + p^i R$.*

Proof. We let B denote the ring $(R[T]/(g))[1/f']$. Since $f'(a)$ is a unit, we have an R-algebra homomorphism $\beta\colon B \to R/p^i R$, which maps T to $a + p^i R$. The lemma is equivalent to the statement that in the diagram

$$\begin{tikzcd} R/p^iR & B \arrow[l, "\beta"'] \\ R \arrow[u, "\pi"] \arrow[ur, dashed] & R \arrow[u, "\alpha"'] \arrow[l, equal] \end{tikzcd}$$

a lift exists. Here α is the inclusion of R as the constant polynomials. Since $(R, p^i R)$ is a henselian pair, it suffices to show that α is an étale map. But $g' \in f' + p^i B$ is a unit since p is in the radical. □

Proposition 1.2.2. *Let (X, x) be a pointed topological space, and let $\mathcal{O}_{X,x}^{\mathrm{cont}}$ be the ring of germs of continuous functions defined in a neighborhood of $x \in X$ with values in $R[\frac{1}{p}]$ endowed with the p-adic topology. Let $\mathfrak{a}_x \subset \mathcal{O}_{X,x}^{\mathrm{cont}}$ be the ideal of germs of functions which vanish at x. Then $(\mathcal{O}_{X,x}^{\mathrm{cont}}, \mathfrak{a}_x)$ is a henselian pair.*

Proof. We must show that for every polynomial $\alpha(T) \in \mathcal{O}_{X,x}^{\mathrm{cont}}[T]$ and every simple root $\bar{f} \in \mathcal{O}_{X,x}^{\mathrm{cont}}/\mathfrak{a}_x$ of the reduction $\bar{\alpha}(T) \in (\mathcal{O}_{X,x}^{\mathrm{cont}}/\mathfrak{a}_x)[T]$, there exists a (necessarily simple) root $f \in \mathcal{O}_{X,x}^{\mathrm{cont}}$ of $\alpha(T)$ such that $f + \mathfrak{a}_x = \bar{f}$.

We write $\alpha(T) = g_n T^n + \cdots + g_0$ where $g_i\colon U \to R[\frac{1}{p}]$, $0 \leq i \leq n$, are continuous functions defined on an open neighborhood U of $x \in X$. We may assume, without loss of generality, that the g_i take values in R. For replacing $\alpha(T)$ by $p^r \alpha(T/p^s)$ amounts to replacing g_i by $p^r g_i/p^{si}$ and $\bar{f}$ by $p^s \bar{f}$. And if h is a root of $p^r \alpha(T/p^s)$ such that $h + \mathfrak{a}_x = p^m \bar{f}$, then $f = h/p^s$ is a root of $\alpha(T)$ such that $f + \mathfrak{a}_x = \bar{f}$. Under this assumption, we have a continuous map

$$c\colon U \to R^{n+1}, \quad c(y) = (g_0(y), \ldots, g_n(y)),$$

and Lemma 1.2.1 then gives a continuous map

$$\rho\colon V = (g_0(x), \ldots, g_n(x)) + (pR)^{n+1} \to R,$$

which maps $(a_0, \ldots, a_n)$ to the unique root $b \in \bar{f} + pR$ of $a_n T^n + \cdots + a_0$. The germ of the composite map

$$f\colon c^{-1}(V) \xrightarrow{c} V \xrightarrow{\rho} R$$

is the desired root of $\alpha(T)$. □

Proposition 1.2.3. *For all $q, n, v \geq 1$, there exist $m \geq n$ with the property that for all $j \geq 0$, there exists $k \geq j$ such that for all $0 < t \leq q$, the map induced from the canonical inclusion*

$$H_t(\mathrm{GL}_n(R, p^k R), \mathbb{Z}/p^v) \to H_t(\mathrm{GL}_m(R, p^j R), \mathbb{Z}/p^v)$$

is the zero homomorphism.

Proof. We show that for $t \geq 0$, there exists continuous homomorphisms

$$s_t \colon C_t(\mathrm{GL}_n(R[\tfrac{1}{p}]), \mathbb{Z}/p^v) \to C_{t+1}(\mathrm{GL}(R[\tfrac{1}{p}]), \mathbb{Z}/p^v),$$

defined in a neighborhood of zero, such that for $t \geq 1$, $ds_t + s_{t-1}d$ is the inclusion

$$\iota_t \colon C_t(\mathrm{GL}_n(R[\tfrac{1}{p}]), \mathbb{Z}/p^v) \to C_t(\mathrm{GL}(R[\tfrac{1}{p}]), \mathbb{Z}/p^v).$$

Here the chains are viewed as topological groups with a basis for the neighborhoods of zero given by the subgroups $C_t(\mathrm{GL}_n(R, p^jR), \mathbb{Z}/p^v)$, $j \geq 0$.

Granting this, we can find $m \geq n$ such that the s_t with $0 \leq t \leq q$ takes values in $C_{t+1}(\mathrm{GL}_m(R[\frac{1}{p}]), \mathbb{Z}/p^v)$. And the continuity of the s_t means that for all $j \geq 0$, there exists $k \geq j$ such that the restriction of the s_t to $C_t(\mathrm{GL}_n(R, p^kR), \mathbb{Z}/p^v)$ takes values in $C_{t+1}(\mathrm{GL}_m(R, p^jR), \mathbb{Z}/p^v)$. We thus obtain a null-homotopy, in homological degrees $1 \leq t \leq q$, of the canonical inclusion

$$C_*(\mathrm{GL}_n(R, p^kR), \mathbb{Z}/p^v) \to C_*(\mathrm{GL}_m(R, p^jR), \mathbb{Z}/p^v).$$

This proves the proposition.

We view $\mathrm{GL}_n(R[\frac{1}{p}])$ as a topological group with a basis for the neighborhoods of the unit given by the subgroups $\mathrm{GL}_n(R, p^iR)$, $i \geq 0$. Let $X^{\mathrm{cont}}_{n,t}$ be the t-fold product of copies of $\mathrm{GL}_n(R[\frac{1}{p}])$ considered as a topological space with the product topology, and let $\mathcal{O}^{\mathrm{cont}}_{n,t}$ be the ring of germs at the identity of continuous functions from $X^{\mathrm{cont}}_{n,t}$ to $R[\frac{1}{p}]$. Then the group $C_r(\mathrm{GL}(\mathcal{O}^{\mathrm{cont}}_{n,t}), \mathbb{Z}/p^v)$ is canonically isomorphic to the group of germs of continuous homomorphisms

$$C_t(\mathrm{GL}_n(R[\tfrac{1}{p}]), \mathbb{Z}/p^v) \to C_r(\mathrm{GL}(R[\tfrac{1}{p}]), \mathbb{Z}/p^v)$$

defined in a neighborhood of zero. Hence, to define the maps s_t above, we may instead define elements $\tilde{s}_t \in C_{t+1}(\mathrm{GL}(\mathcal{O}^{\mathrm{cont}}_{n,t}), \mathbb{Z}/p^v)$.

The requirement that $ds_t + s_{t-1}d = \iota_t$ is expressed in terms of the $\tilde{s}_t$ as follows. For varying t, the $X^{\mathrm{cont}}_{n,t}$ form a simplicial space with face maps

$$d_k(g_1, \ldots, g_t) = \begin{cases} (g_2, \ldots, g_t), & k = 0, \\ (g_1, \ldots, g_k g_{k+1}, \ldots, g_t), & 0 < k < t, \\ (g_1, \ldots, g_{t-1}), & k = t. \end{cases}$$

It follows that as t varies, the $C_r(\mathrm{GL}(\mathcal{O}^{\mathrm{cont}}_{n,t}), \mathbb{Z}/p^v)$ form a cosimplicial group and hence a cochain complex. Let δ denote the coboundary operator. Then the requirement is that

$$d\tilde{s}_t + \delta\tilde{s}_{t-1} = \tilde{\iota}_t.$$

Similarly, the equation $d\tilde{\iota}_t = \delta\tilde{\iota}_{t-1}$ expresses that ι_t is chain map.

Let $\tilde{s}_0 = 0$ and assume that $\tilde{s}_i$, $i < t$, has been constructed. The calculation

$$d(\tilde{\iota}_t - \delta\tilde{s}_{t-1}) = \delta d\tilde{s}_{t-1} + \delta\delta\tilde{s}_{t-2} - d\delta\tilde{s}_{t-1} = 0$$

shows that $\tilde{\iota}_t - \delta\tilde{s}_{t-1}$ is a cycle. But by Proposition 1.2.2, the pair $(\mathcal{O}^{\mathrm{cont}}_{n,t}, \mathfrak{a}^{\mathrm{cont}}_{n,t})$ is henselian, and [1] and [11, Corollary 1.6] then show that

$$H_t(\mathrm{GL}(\mathcal{O}^{\mathrm{cont}}_{n,t}, \mathfrak{a}^{\mathrm{cont}}_{n,t}), \mathbb{Z}/p^v) = 0.$$

Hence, there exists $\tilde{s}_t \in C_{t+1}(\mathrm{GL}(\mathcal{O}^{\mathrm{cont}}_{n,t}, \mathfrak{a}^{\mathrm{cont}}_{n,t}), \mathbb{Z}/p^v)$ such that $d\tilde{s}_t = \tilde{\iota}_t - \delta\tilde{s}_{t-1}$. This completes the proof. $\square$

Corollary 1.2.4. *For all $n, v \geq 1$ and $q \geq 0$, there exists $i \geq 1$ such that the map induced from the reduction*

$$H_q(\mathrm{GL}_n(R), \mathbb{Z}/p^v) \to H_q(\mathrm{GL}_n(R/p^iR), \mathbb{Z}/p^v)$$

is a monomorphism.

Proof. We consider the Hochschild-Serre spectral sequences

$$E^2_{s,t}(n,i) = H_s(\mathrm{GL}_n(R/p^iR), H_t(\mathrm{GL}_n(R, p^iR), \mathbb{Z}/p^v)) \Rightarrow H_{s+t}(\mathrm{GL}_n(R), \mathbb{Z}/p^v)$$

and the induced filtration $\{\mathrm{Fil}^i_s H_q(\mathrm{GL}_n(R), \mathbb{Z}/p^v)\}_{0 \leq s \leq q}$ of the abutment. The edge homomorphism is equal to the map of the statement, and hence this map has kernel $\mathrm{Fil}^i_{q-1} H_q(\mathrm{GL}_n(R), \mathbb{Z}/p^v)$ and image $E^\infty_{q,0}(n,i)$.

To prove the corollary, it suffices to show that the composite

$$H_q(\mathrm{GL}_n(R), \mathbb{Z}/p^v) \to H_q(\mathrm{GL}_n(R/p^iR), \mathbb{Z}/p^n) \to H_q(\mathrm{GL}(R/p^iR), \mathbb{Z}/p^v)$$

is a monomorphism. To this end, we show, inductively, that for all $n \geq 1$ and all $s < q$, there exists $i \geq 1$ such that the induced map

$$\mathrm{Fil}^i_s H_q(\mathrm{GL}_n(R), \mathbb{Z}/p^v) \to H_q(\mathrm{GL}(R/p^iR), \mathbb{Z}/p^v)$$

is zero. This is trivial for $s = -1$, and for $s = q-1$ is the statement we wish to prove. We assume the statement for $s-1$ and prove it for s. Given $n \geq 1$, we first use Proposition 1.2.3 to choose $m \geq n$ with the property that for all $j \geq 1$, there exists $k \geq j$ such that for all $0 < t \leq q$, the canonical map

$$H_t(\mathrm{GL}_n(R, p^kR), \mathbb{Z}/p^v) \to H_q(\mathrm{GL}_m(R, p^jR), \mathbb{Z}/p^v)$$

is zero. It follows that $E^r_{s,q-s}(n,k) \to E^r_{s,q-s}(m,j)$ is zero, which implies the factorization

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Fil}^k_{s-1} H_q(\mathrm{GL}_n(R)) \arrow[r] \arrow[d] & \mathrm{Fil}^k_s H_q(\mathrm{GL}_n(R)) \arrow[r] \arrow[d] \arrow[dl, dashed] & E^\infty_{s,q-s}(n,k) \arrow[r] \arrow[d, "0"] & 0 \\
0 \arrow[r] & \mathrm{Fil}^j_{s-1} H_q(\mathrm{GL}_m(R)) \arrow[r] & \mathrm{Fil}^j_s H_q(\mathrm{GL}_m(R)) \arrow[r] & E^\infty_{s,q-s}(m,j) \arrow[r] & 0
\end{tikzcd}$$

Hence, to finish the proof, it remains only to choose $j \geq 1$ such that

$$\mathrm{Fil}^j_{s-1} H_q(\mathrm{GL}_m(R), \mathbb{Z}/p^v) \to H_q(\mathrm{GL}(R/p^jR), \mathbb{Z}/p^v)$$

is zero. And this is the inductive hypothesis. □

Corollary 1.2.5. *For all $n, v \geq 1$ and $q \geq 0$, there exists $m \geq n$ with the property that for all $j \geq 1$, there exists $k \geq j$ such that the image of*

$$H_q(\mathrm{GL}_m(R), \mathbb{Z}/p^v) \to H_q(\mathrm{GL}_m(R/p^jR), \mathbb{Z}/p^v)$$

is contained in the image of

$$H_q(\mathrm{GL}_n(R/p^kR), \mathbb{Z}/p^v) \to H_q(\mathrm{GL}_m(R/p^jR), \mathbb{Z}/p^v).$$

Proof. We again consider the spectral sequences $E^r_{s,t}(n,i)$ from the proof of Corollary 1.2.4. By Proposition 1.2.3, we can choose, given $n \geq 1$ and $q \geq 0$, and $m \geq n$ such that for all $j \geq 1$, there exists $k \geq j$ such that for all $0 < t \leq q$, the map induced by the canonical inclusion

$$H_q(\mathrm{GL}_n(R, p^kR), \mathbb{Z}/p^v) \to H_q(\mathrm{GL}_m(R, p^jR), \mathbb{Z}/p^v)$$

is zero. It follows that in the diagram

$$\begin{tikzcd}
0 \arrow[r] & E^{r+1}_{q,0}(n,k) \arrow[r] \arrow[d] & E^{r}_{q,0}(n,k) \arrow[r, "d^r"] \arrow[d] \arrow[dl, dashed] & E^{r}_{q-r,r-1}(n,k) \arrow[d, "0"] \\
0 \arrow[r] & E^{r+1}_{q,0}(m,j) \arrow[r] & E^{r}_{q,0}(m,j) \arrow[r, "d^r"] & E^{r}_{q-r,r-1}(m,j)
\end{tikzcd}$$

the right-hand vertical map is zero, and this furnishes an induction argument which proves the corollary. □

1.3. To prove Theorem C it suffices by Panin [9, Corollary 5.8] to prove the following result. Let $\mathcal{O}$ be a local ring such that $(\mathcal{O}, p\mathcal{O})$ is a henselian pair and such that p is a non-zero-divisor.

Theorem 1.3.1. *For all $q \geq 0$ and all $v \geq 1$, the reduction induces an isomorphism of pro-abelian groups*

$$H_q(\mathrm{GL}(\mathcal{O}), \mathbb{Z}/p^v) \xrightarrow{\sim} \{H_q(\mathrm{GL}(\mathcal{O}/p^i\mathcal{O}), \mathbb{Z}/p^v)\}_{i\in\mathbb{N}}.$$

The same statement is true for the group SL.

Proof. Since $\mathcal{O}$ is a local ring, the map induced by the canonical inclusion

$$H_q(\mathrm{GL}_n(\mathcal{O}), \mathbb{Z}/p^v) \to H_q(\mathrm{GL}(\mathcal{O}), \mathbb{Z}/p^v)$$

is an isomorphism, for $n \geq 2q+1$, by Suslin [10]. The same is true for the rings $\mathcal{O}/p^i\mathcal{O}$, $i \geq 1$. For any $m \geq n$ and $k \geq j$, we have the diagram

$$\begin{tikzcd}
H_q(\mathrm{GL}_n(\mathcal{O}), \mathbb{Z}/p^v) \arrow[r] \arrow[d] & H_q(\mathrm{GL}_n(\mathcal{O}/p^k\mathcal{O}), \mathbb{Z}/p^v) \arrow[d] \\
H_q(\mathrm{GL}_m(\mathcal{O}), \mathbb{Z}/p^v) \arrow[r] & H_q(\mathrm{GL}_m(\mathcal{O}/p^j\mathcal{O}), \mathbb{Z}/p^v).
\end{tikzcd}$$

We assume that $n \geq 2q+1$ such that the groups are stable. In particular, the left-hand vertical map is an isomorphism. By Corollary 1.2.5, we can choose $m \geq n$ such that for all $j \geq 1$, there exists $k \geq j$ for which the image of the lower horizontal map contains, and hence equals, the image of the right-hand vertical map. By Corollary 1.2.4, we can choose $j \geq 1$ such that the lower horizontal map, and hence also the upper horizontal map, is a monomorphism. This completes the proof. □

Remark 1.3.2. The proof of Theorem C shows a little more than stated. Indeed, if $K(n)$ and $C(n)$ denote the kernel and cokernel, respectively, of the map

$$K_q(\mathcal{O}, \mathbb{Z}/p^v) \to K_q(\mathcal{O}/p^n\mathcal{O}, \mathbb{Z}/p^v),$$

then Theorem C states that given $n \geq 1$, there exists $m \geq n$ such that the induced maps $K(m) \to K(n)$ and $C(m) \to C(n)$ are zero. However, the proof shows that this $m \geq n$ depends on q and v only, and not on the ring $\mathcal{O}$. Hence, we may conclude that the map of sheaves of pro-abelian groups

$$i^*K_q(-, \mathbb{Z}/p^v) \to \{i_n^*K_q(-, \mathbb{Z}/p^v)^{\sim}\}_{n\in\mathbb{N}}$$

considered in the introduction is an isomorphism as stated.

2. Goodwillie calculus

2.1. We recall that if R is a ring and $I \subset R$ a two-sided nilpotent ideal, then by McCarthy [8], the cyclotomic trace induces an isomorphism

$$K_q(R, I, \mathbb{Z}/p^v) \xrightarrow{\sim} \mathrm{TC}_q(R, I; p, \mathbb{Z}/p^v).$$

In this section, we improve this result to an isomorphism of pro-abelian groups.

Theorem 2.1.1. *Let R be a (not necessarily commutative) ring and let $I \subset R$ be a two-sided nilpotent ideal. Then the canonical map*

$$\mathrm{TC}_q(R, I; p, \mathbb{Z}/p^v) \xrightarrow{\sim} \{\mathrm{TC}_q^m(R, I; p, \mathbb{Z}/p^v)\}_{m\in\mathbb{N}}$$

is an isomorphism of pro-abelian groups.

Remark 2.1.2. The proof below is similar to the proof of McCarthy's Theorem [8] and uses Goodwillie's calculus of functors [3, 4]. It would be desirable to have a more direct proof. However, this may not be so easy. For, in general, the map

$$\mathrm{TR}_q(R, I; p, \mathbb{Z}/p^v) \to \{\mathrm{TR}_q^m(R, I; p, \mathbb{Z}/p^v)\}_{m\in\mathbb{N}}$$

is *not* an isomorphism of pro-abelian groups. If, for example, $1 < n < p-1$ and $1 \le v < n$, one can identify the restriction

$$\mathrm{TR}_0(\mathbb{Z}, p^n\mathbb{Z}; p, \mathbb{Z}/p^v) \to \mathrm{TR}_0^m(\mathbb{Z}, p^n\mathbb{Z}; p, \mathbb{Z}/p^v)$$

with the canonical projection from a product indexed by $s \in \mathbb{N}_0$ of copies of $\mathbb{Z}/p^v$ to the corresponding product indexed by $0 \le s < n$.

The topological cyclic homology functor is defined also for simplicial rings. In effect, a simplicial ring determines a symmetric ring spectrum [2, §1]. We need two properties of this construction:

(i) if $A \to A'$ is a map of simplicial rings such that the underlying map of simplicial sets is a weak equivalence, then for all integers q, the induced map

$$\mathrm{TC}_q^m(A; p) \xrightarrow{\sim} \mathrm{TC}_q^m(A'; p)$$

is an isomorphism.

(ii) there is a natural spectral sequence

$$E^2_{s,t} = \pi_s([n] \mapsto \mathrm{TC}_t^m(A_n; p)) \Rightarrow \mathrm{TC}_{s+t}^m(A; p).$$

The same statements hold for $\mathrm{TC}(-; p)$ and for the corresponding theories with coefficients. The homotopy groups of a simplicial abelian group are canonically isomorphic to the homology of the associated chain complex [7, Theorem 22.1].

Lemma 2.1.3. *Suppose that Theorem* 2.1.1 *is true under the added hypothesis that the ideal I be square zero and that the projection $R \to R/I$ has a section. Then it is true in general.*

Proof. Let A be a simplicial ring, and let $I \subset A$ be a simplicial square zero ideal. We further assume that for all $n \ge 0$, the projection $A_n \to A_n/I_n$ has a section, but we do not assume that these sections respect the simplicial structure. Then, by assumption, the canonical map

$$\mathrm{TC}_q(A_n, I_n; p, \mathbb{Z}/p^v) \xrightarrow{\sim} \{\mathrm{TC}^m(A_n, I_n; p, \mathbb{Z}/p^v)\}_{m\in\mathbb{N}}$$

is an isomorphism of pro-abelian groups, for all $n \geq 0$ and for all integers q. It follows that

$$\pi_s([n] \mapsto \mathrm{TC}_t(A_n, I_n; p, \mathbb{Z}/p^v)) \xrightarrow{\sim} \{\pi_s([n] \mapsto \mathrm{TC}_t^m(A_n, I_n; p, \mathbb{Z}/p^v))\}_{m\in\mathbb{N}}$$

is an isomorphism of pro-abelian groups. The spectral sequence, which we recalled above, shows that for all integers q, the canonical map

$$\mathrm{TC}_q(A, I; p, \mathbb{Z}/p^v) \xrightarrow{\sim} \{\mathrm{TC}_q^m(A, I; p, \mathbb{Z}/p^v)\}_{m\in\mathbb{N}}$$

is an isomorphism of pro-abelian groups. Indeed, in each degree q, the filtration of these pro-abelian groups induced from the spectral sequence is finite.

We now suppose that R is a ring and that $I \subset R$ is a square zero ideal, but we no longer assume that the projection $R \to R/I$ has a section. Then we can find a simplicial ring A, such that for all $n \geq 0$, A_n is a free associative ring, and a map of simplicial rings $A \to R/I$ such that the underlying map of simplicial sets is a weak equivalence. Hence, if we form the pull-back

$$\begin{tikzcd} A' \arrow[r] \arrow[d, "\sim"] & A \arrow[d, "\sim"] \\ R \arrow[r] & R/I, \end{tikzcd}$$

then A is quotient of A' by a square zero ideal I', and for all $n \geq 0$, the projection $A'_n \to A_n$ has a section (since A_n is free). Therefore, the argument in the beginning of the proof shows that in the diagram

$$\begin{tikzcd} \mathrm{TC}_q(A', I'; p, \mathbb{Z}/p^v) \arrow[r, "\sim"] \arrow[d, "\sim"] & \{\mathrm{TC}_q^m(A', I'; p, \mathbb{Z}/p^v)\}_{m\in\mathbb{N}} \arrow[d, "\sim"] \\ \mathrm{TC}_q(R, I; p, \mathbb{Z}/p^v) \arrow[r] & \{\mathrm{TC}_q^m(R, I; p, \mathbb{Z}/p^v)\}_{m\in\mathbb{N}}, \end{tikzcd}$$

the top horizontal map is an isomorphism of pro-abelian groups. By property (i), the same holds for vertical maps. Hence also the lower horizontal map is an isomorphism of pro-abelian groups.

Finally, if R is a ring and $I \subset R$ an ideal, we have a map of cofibration sequences

$$\begin{tikzcd} \mathrm{TC}(R, I^n; p, \mathbb{Z}/p^v) \arrow[r] \arrow[d] & \mathrm{TC}(R, I; p, \mathbb{Z}/p^v) \arrow[r] \arrow[d] & \mathrm{TC}(R/I^n, I/I^n; p, \mathbb{Z}/p^v) \arrow[d] \\ \mathrm{TC}^m(R, I^n; p, \mathbb{Z}/p^v) \arrow[r] & \mathrm{TC}^m(R, I; p, \mathbb{Z}/p^v) \arrow[r] & \mathrm{TC}^m(R/I^n, I/I^n, \mathbb{Z}/p^v) \end{tikzcd}$$

and the statement now follows by easy induction on n. □

2.2. A tower of spectra is a diagram of spectra indexed by the natural numbers considered as a category with one morphism from $m+1$ to m, for all $m \in \mathbb{N}$. We say that a tower of spectra $\{X^m\}_{m\in\mathbb{N}}$ is *pro-n-connected* if the tower of homotopy groups $\{\pi_q X^m\}_{m\in\mathbb{N}}$ is pro-zero, for all $q \leq n$. A map $\{X^m\}_{m\in\mathbb{N}} \to \{Y^m\}_{m\in\mathbb{N}}$ of tower of spectra is *pro-n-connected* if the tower of homotopy fibers $\{F^m\}_{m\in\mathbb{N}}$ is pro-$(n-1)$-connected, or equivalently, if the induced map of homotopy groups

$$\{\pi_q X^m\}_{m\in\mathbb{N}} \to \{\pi_q Y^m\}_{m\in\mathbb{N}}$$

is an isomorphism of pro-abelian groups, for $q < n$, and an epimorphism of pro-abelian groups, for $q = n$. A map is a *pro-equivalence* if it is pro-n-connected, for all integers n.

Let S be a finite set, and let $\mathcal{P}(S)$ (resp. $\mathcal{P}_0(S)$) be the partially ordered set of subsets of S (resp. non-empty subsets of S). A functor $\mathcal{X}$ from $\mathcal{P}(S)$, viewed as a category under inclusion, to a category $\mathcal{C}$ is called an S-cube in $\mathcal{C}$. Following [4, 1.3], we say that an S-cube $\mathcal{X}$ of towers of spectra is *pro-n-cartesian* if the canonical map

$$\mathcal{X}(\emptyset) \to \operatorname{holim}(\mathcal{X}|_{\mathcal{P}_0(S)})$$

is pro-n-connected. It is *pro-cartesian* if it is pro-n-cartesian, for all integers n. Here the homotopy limit of a diagram of towers of spectra is formed level-wise.

We need the following generalization of [4, Theorem 5.3]. The reader is referred to *op.cit.* for the definitions and notation.

Theorem 2.2.1. *Let $\eta\colon \{F^m\}_{m\in\mathbb{N}} \to \{G^m\}_{m\in\mathbb{N}}$ be a natural transformation between towers of ρ-analytic functors from pointed spaces to spectra, and assume that for every map of pointed spaces $Y \to X$, the induced map of differentials*

$$\{(D_X F^m)(Y)\}_{m\in\mathbb{N}} \xrightarrow{\sim} \{(D_X G^m)(Y)\}_{m\in\mathbb{N}}$$

is a pro-equivalence. Then for every $(\rho+1)$-connected map $f\colon Y \to X$ of pointed spaces, the diagram

$$\begin{array}{ccc} \{F^i(Y)\}_{i\in I} & \xrightarrow{\eta} & \{G^j(Y)\}_{j\in J} \\ \downarrow{\scriptstyle F(f)} & & \downarrow{\scriptstyle G(f)} \\ \{F^i(X)\}_{i\in I} & \xrightarrow{\eta} & \{G^j(X)\}_{j\in J} \end{array}$$

is pro-cartesian.

Proof. The theorem is proved by repeating the proof of [4, Theorem 5.3]. One needs only check that the statements of *op.cit.*, Proposition 1.5 (iii) and Proposition 1.8 (iii) remain valid for cubes of towers of spectra. □

Proof of Theorem 2.1.1. According to Lemma 2.1.3 it suffices to show the statement when $I \subset R$ is a square zero ideal such that the projection $R \to R/I$ has a section. We prove a more general statement.

If X is a pointed simplicial set, let $\mathbb{Z}(X)$ be the simplicial abelian group which in simplicial degree n is the free abelian group generated by X_n modulo the subgroup generated by the base point. If M is an abelian group, let $M(X) = M \otimes \mathbb{Z}(X)$. If S is a (not necessarily commutative) ring and M is an S-bimodule, we let $S \ltimes M$ be the ring whose underlying abelian group is $S \times M$ with multiplication

$$(x, m) \cdot (x', m') = (xx', xm' + mx').$$

We show that for every (not necessarily commutative) ring S, every S-bimodule M, and every 0-connected map $Y \to X$ of based simplicial sets, the diagram

$$\begin{array}{ccc} \mathrm{TC}(S \ltimes M(Y); p, \mathbb{Z}/p^v) & \longrightarrow & \{\mathrm{TC}^m(S \ltimes M(Y); p, \mathbb{Z}/p^v)\}_{m\in\mathbb{N}} \\ \downarrow & & \downarrow \\ \mathrm{TC}(S \ltimes M(X); p, \mathbb{Z}/p^v) & \longrightarrow & \{\mathrm{TC}^m(S \ltimes M(X); p, \mathbb{Z}/p^v)\}_{m\in\mathbb{N}} \end{array}$$

is pro-cartesian. The original statement is the case where $S = R/I$, $M = I$, and where $Y \to X$ is the unique map from a two-point set to a one-point set.

We recall from [8, pp. 205–206] that the functors $\mathrm{TC}(S \ltimes M(-); p, \mathbb{Z}/p^v)$ and $\mathrm{TC}^m(S \ltimes M(-); p, \mathbb{Z}/p^m)$, $m \geq 1$, are (-1)-analytic. Hence

$$\mathrm{TC}(S \ltimes M(-); p, \mathbb{Z}/p^v) \to \{\mathrm{TC}^m(S \ltimes M(-); p, \mathbb{Z}/p^v)\}_{m\in\mathbb{N}}$$

is a natural transformation between towers of (-1)-analytic functors from pointed spaces to spectra, and we may employ Theorem 2.2.1. The statement follows given the following lemma. □

Lemma 2.2.2. *For every map of pointed spaces $Y \to X$, the natural map*

$$D_X\,\mathrm{TC}(S \ltimes M(-); p, \mathbb{Z}/p^v)(Y) \xrightarrow{\sim} \{D_X\,\mathrm{TC}^m(S \ltimes M(-); p, \mathbb{Z}/p^v)(Y)\}_{m\in\mathbb{N}}$$

is a pro-equivalence.

Proof. We prove that the canonical map

$$D_X\,\mathrm{TC}(S \ltimes M(-); p)(Y) \xrightarrow{\sim} \operatorname*{holim}_{m\in\mathbb{N}} D_X\,\mathrm{TC}^m(S \ltimes M(-); p)(Y)$$

is a weak equivalence, and that for all $v \geq 1$, the map of homotopy groups with $\mathbb{Z}/p^v$-coefficients induced from the canonical map

$$\operatorname*{holim}_{m\in\mathbb{N}} D_X\,\mathrm{TC}^m(S \ltimes M(-); p)(Y) \to \{D_X\,\mathrm{TC}^m(S \ltimes M(-); p)(Y)\}_{m\in\mathbb{N}}$$

is an isomorphism of pro-abelian groups. The first of these claims is an easy consequence of the analyticity of the functors $\mathrm{TC}^m(S \ltimes M(-); p)$. Indeed, it follows from [3, Proposition 1.15 (ii)] that

$$\tilde{T}^k_X\,\mathrm{TC}^m(S \ltimes M(-); p)(Y) \to D_X\,\mathrm{TC}^m(S \ltimes M(-); p)(Y)$$

is roughly $2k$-connected, i.e., $(2k - c)$-connected for some integer c independent of k. This remains true for the induced map of homotopy limits,

$$\operatorname*{holim}_{m\in\mathbb{N}} \tilde{T}^k_X\,\mathrm{TC}^m(S \ltimes M(-); p)(Y) \to \operatorname*{holim}_{m\in\mathbb{N}} D_X\,\mathrm{TC}^m(S \ltimes M(-); p)(Y).$$

Moreover, since homotopy limits commute (up to canonical isomorphism), the canonical map

$$\tilde{T}^k_X\,\mathrm{TC}(S \ltimes M(-); p)(Y) \xrightarrow{\sim} \operatorname*{holim}_{m\in\mathbb{N}} \tilde{T}^k_X\,\mathrm{TC}^m(S \ltimes M(-); p)(Y)$$

is an isomorphism. (Recall that $\tilde{T}^k_X F(Y)$ is the homotopy limit of a certain diagram naturally associated with F and $Y \to X$.) The composition of these maps is a roughly $2k$-connected map

$$\tilde{T}^k_X\,\mathrm{TC}(S \ltimes M(-); p)(Y) \to \operatorname*{holim}_{m\in\mathbb{N}} D_X\,\mathrm{TC}^m(S \ltimes M(-); p)(Y),$$

and the desired weak equivalence now follows by taking the homotopy colimit over k.

The proof of the second claim is more subtle and is based on a number of results from [5], which we now recall. First, recall from *op.cit.*, Proposition 2.1, that there is a natural decomposition

$$i_1\colon \bigvee_{a\in\mathbb{N}_0} \mathrm{TH}_a(S \ltimes M) \xrightarrow{\sim} \mathrm{TH}(S \ltimes M)$$

as $\mathbb{T}$-equivariant spectra, and hence, the induced map of $C_{p^{m-1}}$-fixed sets

$$i_m\colon \bigvee_{a\in\mathbb{N}_0} \mathrm{TH}_a(S\ltimes M)^{C_{p^{m-1}}} \xrightarrow{\sim} \mathrm{TR}^m(S\ltimes M;p)$$

is a weak equivalence. For the purpose of this proof, we will denote the left-hand side by $'\mathrm{TR}^m(S\ltimes M;p)$. It is proved in *op.cit.*, Lemma 2.2, that there are natural maps

$$'R, 'F\colon {}'\mathrm{TR}^m(S\ltimes M;p)\to {}'\mathrm{TR}^{m-1}(S\ltimes M;p)$$

such that

$$R\circ i_m = i_{m-1}\circ {}'R, \quad F\circ i_m = i_{m-1}\circ {}'F,$$

and hence i_m induces a natural weak equivalence

$$i_m\colon {}'\mathrm{TC}^m(S\ltimes M;p)\xrightarrow{\sim}\mathrm{TC}^m(S\ltimes M;p)$$

from the homotopy equalizer of $'R$ and $'F$ to the homotopy equalizer of R and F. Moreover, the map $'R$ takes the ath summand to the (a/p)th summand, and the map $'F$ takes the ath summand to the ath summand.

We next consider the natural map

$$p_m\colon {}'\mathrm{TR}^m(S\ltimes M;p)\to \prod_{s=0}^{m-1}\mathrm{TH}_1(S\ltimes M)^{C_{p^{m-1-s}}},$$

whose sth factor is the composite

$$\bigvee_{a\in\mathbb{N}_0}\mathrm{TH}_a(S\ltimes M)^{C_{p^{m-1}}}\to \mathrm{TH}_{p^s}(S\ltimes M)^{C_{p^{m-1}}}\xrightarrow{'R^s}\mathrm{TH}_1(S\ltimes M)^{C_{p^{m-1-s}}}.$$

Here the map on the left collapses the summands with a different from p^s to the basepoint. We write $''\mathrm{TR}^m(S\ltimes M;p)$ for the range of p_m and define

$$''R, ''F\colon {}''\mathrm{TR}^m(S\ltimes M;p)\to {}''\mathrm{TR}^{m-1}(S\ltimes M;p)$$

as follows. The map $''R$ takes the sth factor to the $(s-1)$st factor by the identity map, for $s\le 1$, and annihilates the factor $s=0$. The map $''F$ takes the sth factor to the sth factor by the natural inclusion map, for $s<m-1$, and annihilates the factor $s=m-1$. Clearly,

$$R'\circ p_m = p_{m-1}\circ {}'R, \quad F'\circ p_m = p_{m-1}\circ {}'F,$$

and hence we get a natural map

$$p_m\colon {}'\mathrm{TC}^m(S\ltimes M;p)\to {}''\mathrm{TC}^m(S\ltimes M;p)$$

from the homotopy equalizer of $'R$ and $'F$ to the homotopy equalizer of $''R$ and $''F$.

There is a natural map

$$j_m\colon \mathrm{TH}_1(S\ltimes M)^{C_{p^{m-1}}}\to {}''\mathrm{TR}^m(S\ltimes M;p)$$

whose sth factor is the iterated Frobenius

$$F^{m-1-s}\colon \mathrm{TH}_1(S\ltimes M)^{C_{p^{m-1}}}\to \mathrm{TH}_1(S\ltimes M)^{C_{p^s}}.$$

It is clear that $''R\circ j_m = {}''F\circ j_m$ such that we have an induced map

$$j_m\colon \mathrm{TH}_1(S\ltimes M)^{C_{p^{m-1}}}\xrightarrow{\sim} {}''\mathrm{TC}^m(S\ltimes M;p).$$

This map is a weak equivalence.

Finally, it follows from [5, Proposition 3.2, Lemma 3.3] that there is a natural weak equivalence

$$k_m\colon\ \mathrm{TH}(S,M)\vee\Sigma\,\mathrm{TH}(S,M)\xrightarrow{\sim}\mathrm{TH}_1(S\ltimes M)^{C_{p^{m-1}}},$$

where on the left $\mathrm{TH}(S,M)$ is the topological Hochschild homology of the ring A with coefficients in the bi-module M. Concluding, we have a natural chain of maps

$$\begin{aligned}\mathrm{TC}^m(S\ltimes M;p)\xleftarrow[\sim]{i_m}{}'\mathrm{TC}^m(S\ltimes M;p)\xrightarrow{p_m}{}''\mathrm{TC}^m(S\ltimes M;p)\\ \xleftarrow[\sim]{j_m}\mathrm{TH}_1(S\ltimes M)^{C_{p^{m-1}}}\xleftarrow[\sim]{k_m}\mathrm{TH}(S,M)\vee\Sigma\,\mathrm{TH}(S,M),\end{aligned}$$

and, as indicated, the maps i_m, j_m and k_m are weak equivalences.

The map p_m is usually not a weak equivalence, but [5, Lemma 2.2] shows that for a k-connected map $Y\to X$, the square

$$\begin{tikzcd}{}'\mathrm{TR}^m(S\ltimes M(Y);p)\arrow[r,"p_m"]\arrow[d] & {}''\mathrm{TR}^m(S\ltimes M(Y);p)\arrow[d]\\ {}'\mathrm{TR}^m(S\ltimes M(X);p)\arrow[r,"p_m"] & {}''\mathrm{TR}^m(S\ltimes M(X);p)\end{tikzcd}$$

is roughly $2k$-homotopy cartesian. Hence the same holds for the square

$$\begin{tikzcd}{}'\mathrm{TC}^m(S\ltimes M(Y);p)\arrow[r,"p_m"]\arrow[d] & {}''\mathrm{TC}^m(S\ltimes M(Y);p)\arrow[d]\\ {}'\mathrm{TC}^m(S\ltimes M(X);p)\arrow[r,"p_m"] & {}''\mathrm{TC}^m(S\ltimes M(X);p).\end{tikzcd}$$

But for any map $Y\to X$, the induced map $S^k_XY\to S^k_XX$ of k-fold fiberwise suspensions is at least $(k-1)$-connected, and therefore, the connectivity estimate above shows that p_m induces a roughly k-connected map

$$\tilde{T}^k_X({}'\mathrm{TC}^m(S\ltimes M(-);p))(Y)\to\tilde{T}^k_X({}''\mathrm{TC}^m(S\ltimes M(-);p))(Y).$$

Hence the induced map of homotopy colimits over k is a weak equivalence, that is,

$$D_X({}'\mathrm{TC}^m(S\ltimes M(-);p))(Y)\xrightarrow{\sim}D_X({}''\mathrm{TC}^m(S\ltimes M(-);p))(Y).$$

We have thus produced a natural chain of weak equivalences between the tower of spectra $\{D_X\,\mathrm{TC}^m(S\ltimes M(-);p)(Y)\}_{m\in\mathbb{N}}$ and the tower of spectra

$$\{\mathrm{TH}(S,M(Y))\vee\Sigma\,\mathrm{TH}(S,M(Y))\}_{m\in\mathbb{N}},$$

where the structure maps in the latter are given by $p\vee\mathrm{id}$. It follows that the map of homotopy groups with $\mathbb{Z}/p^v$-coefficients induced from the canonical map

$$\operatorname*{holim}_{m\in\mathbb{N}}D_X\,\mathrm{TC}^m(S\ltimes M(-);p)(Y)\to\{D_X\,\mathrm{TC}^m(S\ltimes M(-);p)(Y)\}_{m\in\mathbb{N}}$$

is an isomorphism of pro-abelian groups. □

3. Continuity for TC

3.1. It remains to prove that the analog of Theorem C is also valid for topological cyclic homology. We refer the reader to [2, §1] and [6, §1] for a discussion of topological cyclic homology. In fact, the analogous result is valid in a much greater generality.

Proposition 3.1.1. *Let R be a (not necessarily commutative) ring and assume that p is a non-zero-divisor in R. Then for all $m, v \geq 1$ and $q \geq 0$, the map induced from the reduction*

$$\mathrm{TR}^m_q(R; p, \mathbb{Z}/p^v) \xrightarrow{\sim} \{\mathrm{TR}^m_q(R/p^nR; p, \mathbb{Z}/p^v)\}_{n\in\mathbb{N}}$$

is an isomorphism of pro-abelian groups.

Proof. We first prove the case $m = 1$. The statement is that for all $v \geq 1$ and all integers q, the map of pro-abelian groups induced from the reduction

$$\pi_q(\mathrm{TH}(R), \mathbb{Z}/p^v) \to \{\pi_q(\mathrm{TH}(R/p^nR), \mathbb{Z}/p^v)\}_{n\in\mathbb{N}}$$

is an isomorphism. The topological Hochschild spectrum $\mathrm{TH}(R)$ is defined as the realization of a simplicial symmetric spectrum $\{[k] \mapsto \mathrm{TH}(R)_k\}$, so there is a natural first quadrant spectral sequence

$$E^2_{s,t} = \pi_s([k] \mapsto \pi_t(\mathrm{TH}(R)_k, \mathbb{Z}/p^v)) \Rightarrow \pi_{s+t}(\mathrm{TH}(R), \mathbb{Z}/p^v).$$

It will therefore suffice to show that for all $k, t \geq 0$ and all $v \geq 1$,

$$\pi_t(\mathrm{TH}(R)_k, \mathbb{Z}/p^v) \to \{\pi_t(\mathrm{TH}(R/p^nR)_k, \mathbb{Z}/p^v)\}_{n\in\mathbb{N}}$$

is an isomorphism of pro-abelian groups. Let $\mathrm{Ho}(\mathcal{S})$ be the homotopy category of spectra. This is a triangulated category with a compatible symmetric monoidal structure given by smash product of spectra. We identify a ring with its Eilenberg-MacLane spectrum. In the homotopy category, $\mathrm{TH}(R)_k$ represents the $(k+1)$-fold smash product of R. We claim that the map induced from the reduction

$$R \wedge M_{p^v} \to \{R/p^nR \wedge M_{p^v}\}_{n\in\mathbb{N}}$$

is an isomorphism in pro-$\mathrm{Ho}(\mathcal{S})$. Here M_{p^v} is a Moore spectrum for the group $\mathbb{Z}/p^v$. Indeed, one has, for every abelian group A, a natural triangle

$$A/p^v \to A \wedge M_{p^v} \to {}_{p^v}A[-1] \to A/p^v[-1],$$

and the claim then follows from the diagram

$$\begin{tikzcd}
R/p^vR \arrow[r] \arrow[d, equal] & R/p^{n+v} \wedge M_{p^v} \arrow[r] \arrow[d] \arrow[dl, dashed] & p^nR/p^{n+v}R[-1] \arrow[r] \arrow[d, "0"] & R/p^vR[-1] \arrow[d, equal] \\
R/p^vR \arrow[r] & R/p^n \wedge M_{p^v} \arrow[r] & p^{n-v}R/p^nR[-1] \arrow[r] & R/p^vR[-1].
\end{tikzcd}$$

It follows that the induced map of the $(k+1)$-fold smash products

$$R^{\wedge(k+1)} \wedge M_{p^v} \to \{(R/p^nR)^{\wedge(k+1)} \wedge M_{p^v}\}_{n\in\mathbb{N}}$$

is an isomorphism in pro-$\mathrm{Ho}(\mathcal{S})$. Hence, the induced map of homotopy groups,

$$\pi_q(\mathrm{TH}(R)_k, \mathbb{Z}/p^v) \to \{\pi_q(\mathrm{TH}(R/p^nR)_k, \mathbb{Z}/p^v)\}_{n\in\mathbb{N}},$$

is an isomorphism of pro-abelian groups as desired. This proves the statement for $m = 1$.

The proof of the general case is by induction on m based on the natural triangle

$$\mathbb{H}.(C_{p^{m-1}}, \mathrm{TH}(R)) \to \mathrm{TR}^m(R; p) \to \mathrm{TR}^{m-1}(R; p) \to \mathbb{H}.(C_{p^{m-1}}, \mathrm{TH}(R))[-1].$$

The homotopy groups with $\mathbb{Z}/p^v$-coefficients of the left-hand term are given by a natural spectral sequence

$$E^2_{s,t} = H_s(C_{p^{m-1}}, \pi_t(\mathrm{TH}(R) \wedge M_{p^v})) \Rightarrow \pi_{s+t}(\mathbb{H}.(C_{p^{m-1}}, \mathrm{TH}(R)) \wedge M_{p^v}).$$

Since $\pi_t(\mathrm{TH}(R) \wedge M_{p^v})$ is a trivial $C_{p^{m-1}}$-module, the case $m = 1$ shows that the map of E^2-terms induced from the reduction $R \to \{R/p^n\}_{n\in\mathbb{N}}$ is a pro-isomorphism in each bi-degree. This completes the proof. □

Addendum 3.1.2. *Let R be a ring and assume that p is a non-zero-divisor in R. Then for all $m, v \geq 1$ and $q \geq 0$, the map induced from the reduction*

$$\mathrm{TC}_q^m(R; p, \mathbb{Z}/p^v) \xrightarrow{\sim} \{\mathrm{TC}_q^m(R/p^n; p, \mathbb{Z}/p^v)\}_{n\in\mathbb{N}}$$

is an isomorphism of pro-abelian groups. □

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