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ISOVARIANT BORSUK-ULAM RESULTS FOR PSEUDOFREE CIRCLE ACTIONS AND THEIR CONVERSE

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Abstract. In this paper we shall study the existence of an S^1-isovariant map from a rational homology sphere M with pseudofree action to a representation sphere SW. We first show some isovariant Borsuk-Ulam type results. Next we shall consider the converse of those results and show that there exists an S^1-isovariant map from M to SW under suitable conditions.

Introduction

The original Borsuk-Ulam theorem proved by [1] states that if there is a continuous map $f : S^m \to S^n$ so that $f(-x) = -f(x)$ for all $x \in S^m$, then $m \leq n$ holds (and as easily seen, the converse is true as well). For over seventy years this theorem was generalized in various directions and applied in several fields of mathematics (cf. [13], [18], [19]). In the viewpoint of the equivariant topology, Borsuk-Ulam type theorems are deeply concerned with the problem of whether or not a G-equivariant map exists between given G-spaces. Recently one of such Borsuk-Ulam type results played a significant role in a partial solution of the 11/8-conjecture [10].

On the other hand, in the equivariant topology, there are various theories and results in the isovariant setting, e.g., (isovariant) surgery theory on stratified sets [3], [7], isovariant homotopy theory [8], the isovariant s-cobordism theorem [3], [12, Theorem 4.42], etc.

In this paper we shall study Borsuk-Ulam type results and their converse in the isovariant setting. A G-equivariant map $f : X \to Y$ is called *G-isovariant* if f preserves the isotropy groups, i.e., $G_x = G_{f(x)}$ for all $x \in X$. (Throughout this paper all maps are assumed to be continuous.) Let $\mathrm{Iso}(X)$ denote the set of isotropy groups of a G-space X. In [20] and [15], isovariant variants of the Borsuk-Ulam theorem were studied, which provide nonexistence results of isovariant maps between representation spheres. We shall consider a similar problem in the following situation: G is the circle group S^1, the target space is a representation sphere, and the source space is a rational homology sphere M with effective smooth S^1-action of which singular set $M_s := \bigcup_{H \neq 1} M^H$ is empty or of $\dim M_s/S^1 = 0$. We call such an action the *discrete singularity action* or the *DS-action* for short. A rational homology sphere means a smooth closed manifold of which homology groups with rational coefficients are isomorphic to those of S^m, $m = \dim M$. As is easily seen,

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discrete singularity S^1-actions on a connected closed orientable manifold M are divided into three types:

Type F: M_s is empty, namely, S^1 acts freely on M.

Type SF: M_s is nonempty and consists of finitely many isolated fixed points, namely, S^1 acts semifreely on M with isolated fixed points.

Type PF: M_s is nonempty and consists of finitely many exceptional orbits (i.e., the isotropy groups of nonfree orbits are nontrivial finite groups), which is just the pseudofree action in the sense of Montgomery and Yang [14] or Petrie [17].

Note that there is no mixed type of SF and PF, in fact $\dim M$ is even in type SF and odd in type PF, since the slice representation of a fixed point or a singular orbit has even dimension. In section 1 we shall show the following isovariant Borsuk-Ulam type results.

Theorem A. *Let M be an S^1-rational homology sphere of type F, SF or PF, and let SW be the representation sphere of an orthogonal S^1-representation W. If an S^1-isovariant map $f : M \to SW$ exists, the following inequalities hold: In type F,*

(F): $\dim M + 1 \leq \dim SW - \dim SW^H$ *for* $1 \neq H \leq S^1$.

In type SF,

(SF): $\dim M \leq \dim SW - \dim SW^H$ *for* $1 \neq H \leq S^1$.

In type PF,

(PF1): $\dim M - 1 \leq \dim SW - \dim SW^H$ *for* $H \neq 1$ *such that* $H \leq C$ *for some* $C \in \mathrm{Iso}(M)$, *and*

(PF2): $\dim M + 1 \leq \dim SW - \dim SW^H$ *for* $H \neq 1$ *such that* $H \not\leq C$ *for any* $C \in \mathrm{Iso}(M)$.

Here if SW^H is empty, then we set $\dim SW^H = -1$ by convention. In sections 2–4, we shall investigate the converse of Theorem A using equivariant obstruction theory. Our results on the converse are summarized as follows:

Theorem B. *Let M and SW be as above. Suppose that* $\mathrm{Iso}(M) \subset \mathrm{Iso}(SW)$. *In each type, if the corresponding inequalities hold, then there exists an S^1-isovariant map from M to SW.*

Remark. The condition $\mathrm{Iso}(M) \subset \mathrm{Iso}(SW)$ is obviously necessary.

In section 2 we shall prove Theorem B in types F and SF by showing that the obstruction groups vanish. In the case of type PF, the obstruction groups do not necessarily vanish, hence we need to detect the obstruction class itself. To do that, in section 3, we introduce the *multidegree* of certain S^1-maps and show some Hopf type results. In section 4 we shall prove Theorem B in type PF in the following way. Let N_i be closed S^1-tubular neighborhoods of the exceptional orbits. Set $N = \coprod_i N_i$ and $X = M \setminus (\mathrm{int}\, N)$. We first construct isovariant maps $f_i : N_i \to SW$ and next discuss the extendability of $\coprod_i f_{i\,|\partial N_i} : \partial X \to SW$ to an isovariant map from M using the multidegree. The essential point is that the obstruction vanishes for suitable choices of f_i.

In section 5, we shall illustrate some examples, in particular, we shall give an answer to (an analogue of) the question posed in [16]. Let T_n, $n \in \mathbb{Z}$, denote the irreducible complex representation (space) of S^1 given by $\rho_n : S^1 \to SU(1)$ $(= S^1)$, $z \mapsto z^n$.

Example. Let p, q, r be pairwise coprime integers greater than 1. There exists an S^1-isovariant map

$$f : S(T_p \oplus T_q \oplus T_r) \to S(T_1 \oplus T_{pq} \oplus T_{qr} \oplus T_{rp}).$$

1. Some isovariant Borsuk-Ulam type results

We begin by recalling well-known Borsuk-Ulam type results. Let G be a compact Lie group, and let SV and SW be representation spheres with free G-action. A Borsuk-Ulam type theorem states that if there is a G-map from SV to SW, then the inequality $\dim SV \leq \dim SW$ holds (see for example [6]). This result is generalized by many authors, for example, from [4, Theorem 6.4] the following generalization can be derived.

Proposition 1.1. *Let X and Y be finite-dimensional S^1-CW complexes with finitely many isotropy groups and without S^1-fixed points (but not necessarily free). Suppose $H_*(X;\mathbb{Q}) \cong H_*(S^m;\mathbb{Q})$ and $H_*(Y;\mathbb{Q}) \cong H_*(S^n;\mathbb{Q})$. If there is an S^1-map from X to Y, then $m \leq n$ holds.*

Isovariant variants of the Borsuk-Ulam theorem were first studied in [20] (see also [15]). One of them is stated as follows:

Proposition 1.2. *Suppose that G is solvable. If there is a G-isovariant map from SV to SW, then the inequality*

$$\dim SV - \dim SV^G \leq \dim SW - \dim SW^G$$

holds.

These Borsuk-Ulam type theorems provide some dimensional conditions for the existence of an isovariant map. We first consider an S^1-rational homology sphere M of type F, namely S^1 acts freely on M. Let $f : M \to SW$ be an S^1-isovariant map. By isovariance, f maps M into $SW \setminus SW^H$ for $1 \neq H \leq S^1$. Since $SW \setminus SW^H$ is S^1-homotopy equivalent to $S(W^H)^\perp$, where $S(W^H)^\perp$ denotes the unit sphere of the subrepresentation defined as the orthogonal complement $(W^H)^\perp$ of W^H in W (i.e., $W = W^H \oplus (W^H)^\perp$), there is an S^1-map $\bar{f} : M \to S(W^H)^\perp$. By Proposition 1.1 one can see

$$\dim M \leq \dim S(W^H)^\perp = \dim SW - \dim SW^H - 1.$$

Thus we obtain:

Proposition 1.3. *Let M be an S^1-rational homology sphere of type F. If there is an S^1-isovariant map $f : M \to SW$, then*

(F): $\dim M + 1 \leq \dim SW - \dim SW^H$ *for* $1 \neq H \leq S^1$.

We next consider the case of type SF. By definition S^1 acts semifreely on M with isolated fixed points. Let $x \in M^{S^1}$ and let $f : M \to SW$ be an S^1-isovariant map. By the slice theorem (see [2], [11]), there is a (small) invariant neighborhood N of x which is S^1-diffeomorphic to some representation disk DV. Restricting f to $\partial N \cong SV$, we have an S^1-isovariant map $\bar{f} : SV \to SW$. Since S^1 acts freely on SV, it follows from Proposition 1.2 that $\dim SV + 1 \leq \dim SW - \dim SW^H$ for $1 \neq H \leq S^1$. Thus we obtain the following proposition.

Proposition 1.4. *Let M be of type SF. If there is an S^1-isovariant map $f : M \to SW$, then*

(SF): $\dim M \leq \dim SW - \dim SW^H$ *for* $1 \neq H \leq S^1$.

Remark. In this proposition, M does not need to be a rational homology sphere.

Remark. Inequalities (F) and (SF) are equivalent to the following conditions, respectively.

(F′): $\dim M + 1 \leq \dim SW - \dim SW_s$.

(SF′): $\dim M \leq \dim SW - \dim SW_s$.

Here SW_s denotes the singular set of SW, i.e., $SW_s = \bigcup_{H \neq 1} SW^H$, and $\dim SW_s = \max\{\dim SW^H | H \neq 1\}$.

In the case of type PF, we can see the following.

Proposition 1.5. *Let M be an S^1-rational homology sphere of type PF. If there is an S^1-isovariant map $f : M \to SW$, then*

(PF1): $\dim M - 1 \leq \dim SW - \dim SW^H$ *for* $H \neq 1$ *such that* $H \leq C$ *for some* $C \in \mathrm{Iso}(M)$, *and*

(PF2): $\dim M + 1 \leq \dim SW - \dim SW^H$ *for* $H \neq 1$ *such that* $H \not\leq C$ *for any* $C \in \mathrm{Iso}(M)$.

Proof. Let H be a nontrivial finite subgroup of S^1 such that $H \leq C$ for some $C \in \mathrm{Iso}(M)$. Take an exceptional orbit isomorphic to S/C in M. By the slice theorem its (closed) S^1-tubular neighborhood N has the form of $S^1 \times_C DU$ for some C-representation U. Since S^1 acts freely on the outside of (1-dimensional) exceptional orbits, C acts freely on SU, and $\dim SU = \dim M - 2$. Restricting f to SU, we have a C-isovariant map $\bar{f} : SU \to SW$, and by restricting the action, $\bar{f}$ is regarded as an H-isovariant map. It follows from Proposition 1.2 that

$$\dim SU + 1 \leq \dim SW - \dim SW^H,$$

which shows (PF1): $\dim M - 1 \leq \dim SW - \dim SW^H$. Let H be a nontrivial finite subgroup of S^1 such that $H \not\leq C$ for any $C \in \mathrm{Iso}(M)$. Then, one can see that f maps M into $SW \setminus SW^H$. By the same argument as in Proposition 1.3 we obtain (PF2): $\dim M + 1 \leq \dim SW - \dim SW^H$. □

Remark. In the proof of (PF1), M does not need to be a rational homology sphere.

Thus the proof of Theorem A is complete.

2. Existence of isovariant maps in types F and SF

In this section we shall discuss the converse of Theorem A in the cases of types F and SF. The main tool is equivariant obstruction theory as described in [5, II 3]. For convenience we recall necessary notations and facts. Let (X, A) be a relative G-CW complex such that G acts freely on $X \setminus A$. Denote by X_n the n-skeleton of (X, A) and by $C_*(X, A)$ $(= H_*(X_*, X_{*-1}; \mathbb{Z}))$ the cellular chain complex. The G-action on X induces a G-action on $C_*(X, A)$ and the identity component G_0 acts trivially on $C_*(X, A)$. Therefore $C_*(X, A)$ inherits a (left) $\mathbb{Z}(G/G_0)$-module structure. Taking a $\mathbb{Z}(G/G_0)$-module π as coefficients, we define the (equivariant) cochain complex $C^*_G(X, A; \pi)$ to be $\mathrm{Hom}_{\mathbb{Z}(G/G_0)}(C_*(X, A), \pi)$. We denote by $\mathfrak{H}^*_G(X, A; \pi)$ its cohomology group. Note [5, p.112] that there is an isomorphism

$$\mathfrak{H}^*_G(X, A; \pi) \cong H^*(X/G, A/G; \pi).$$

(Here $H^*(X/G, A/G; \pi)$ is a cohomology group with local coefficients in general.) Let Y be a path-connected n-simple G-space, $n \geq 1$. The homotopy group $\pi_n(Y)$ $(= [S^n, Y])$ inherits a $\mathbb{Z}(G/G_0)$-module structure. Obstruction theory asserts that the obstruction to an extension of a G-map $f : A \to Y$ to $F : X \to Y$ lies in $\mathfrak{H}^*_G(X, A; \pi_{*-1}(Y))$, and the obstruction to the existence of an equivariant homotopy between extensions F and F' of f lies in $\mathfrak{H}^*_G(X, A; \pi_*(Y))$.

Let SW be an S^1-representation sphere with effective action, i.e., W is a faithful representation. This is also equivalent to being $SW \setminus SW_s \neq \emptyset$. Set $Y = SW \setminus SW_s$, and let $\mathcal{A}$ be the set of isotropy groups H of SW with $\dim SW^H = \dim SW_s$.

Lemma 2.1. *Let W be a faithful S^1-representation. Set $k = \dim SW - \dim SW_s$.*

(1) *Y is $(k-2)$-connected and $(k-1)$-simple.*

(2) *There is an isomorphism*

$$\Phi : H_{k-1}(Y) \to \bigoplus_{H \in \mathcal{A}} H_{k-1}(S(W^H)^\perp) \cong \bigoplus_{H \in \mathcal{A}} \mathbb{Z}$$

by the following composite of isomorphisms:

$$H_{k-1}(Y) \xrightarrow{j_*} \bigoplus_{H \in \mathcal{A}} H_{k-1}(SW \setminus SW^H) \xleftarrow{\bigoplus i_{H*}} \bigoplus_{H \in \mathcal{A}} H_{k-1}(S(W^H)^\perp),$$

where i_H, j are the inclusions.

(3) *The Hurewicz homomorphism $h : \pi_{k-1}(Y) \to H_{k-1}(Y)$ is an isomorphism.*

Proof. (1): Note $k \geq 2$. The $(k-2)$-connectivity follows by a general position argument. If $k > 2$, then Y is 1-connected, hence $(k-1)$-simple. If $k = 2$, then $\pi_1(Y)$ is abelian as seen below, so Y is 1-simple.

(2): Set $\mathcal{B} := \mathrm{Iso}(SW) \setminus \{1\}$. Note $Y = \bigcap_{H \in \mathcal{B}} (SW \setminus SW^H)$. Note also that

$$\dim(SW^H \cap SW^K) = \dim SW^{HK} \leq \dim SW_s - 2,$$

provided $H \neq K$ $(H, K \in \mathcal{B})$, in fact, since $\dim SW^H - \dim SW^K$ is even whenever $H \leq K$, it suffices to show $\dim SW^{HK} < \dim SW_s$. If $H \notin \mathcal{A}$ or $K \notin \mathcal{A}$, this is obvious. In the case where $H \in \mathcal{A}$ and $K \in \mathcal{A}$, if $\dim SW^{HK} = \dim SW_s$, then one can see that $SW^H = SW^{HK} = SW^K$, and hence $H = K$, which is a contradiction.

Using the Mayer-Vietoris exact sequence inductively, one can see that

$$H_{k-1}(Y) \xrightarrow{j_*} \bigoplus_{H \in \mathcal{B}} H_{k-1}(SW \setminus SW^H)$$

is an isomorphism. Since the inclusion $i_H : S(W^H)^\perp \to SW \setminus SW^H$ is a homotopy equivalence and $\dim S(W^H)^\perp = \dim SW - \dim SW^H - 1$, we have that $H_{k-1}(SW \setminus SW^H) \cong \mathbb{Z}$ when $\dim SW^H = \dim SW_s$, and $H_{k-1}(SW \setminus SW^H) = 0$ when $\dim SW^H < \dim SW_s$.

(3): If $k > 2$, this follows from the Hurewicz theorem. In case of $k = 2$ it suffices to show that $\pi_1(Y)$ is abelian. Set $Y' = SW \setminus \bigcup_{H \in \mathcal{A}} SW^H$. Since $\dim SW - \dim SW^K \geq k + 2$ for $1 \neq K \in \mathrm{Iso}(SW) \setminus \mathcal{A}$, the inclusion $Y \subset Y'$ induces an isomorphism $\pi_1(Y) \cong \pi_1(Y')$. Moreover Y' is a deformation retract of $Z := W \setminus \bigcup_{H \in \mathcal{A}} W^H$. Hence it suffices to see that $\pi_1(Z)$ is abelian. By assumption, $(W^H)^\perp$ is 2-dimensional, and hence irreducible. Moreover $(W^H)^\perp \neq (W^{H'})^\perp$ for $H \neq H'$, hence W is decomposed as $W = \bigoplus_{H \in \mathcal{A}} (W^H)^\perp \oplus W'$ for some W', and hence one can see that Z is homeomorphic to $\prod_{H \in \mathcal{A}} ((W^H)^\perp \setminus \{0\}) \times W'$. Thus we have $\pi_1(Z) \cong \bigoplus_{H \in \mathcal{A}} \mathbb{Z}$. $\square$

We first show Theorem B in the case of type F.

Proposition 2.2. *Let M be of type F and SW an S^1-representation sphere. Suppose that*

(F): $\dim M + 1 \leq \dim SW - \dim SW^H$ *for* $1 \neq H \leq S^1$.

Then there exists an S^1-isovariant map from M to SW.

Proof. Set $k = \dim SW - \dim SW_s$ as before. Condition (F) implies that

$$\dim M + 1 \leq \dim SW - \dim SW_s$$

and so $k \geq 2$ and W is faithful. It suffices to show the existence of an S^1-map from M to $Y = SW \setminus SW_s$. Since Y is $(k-2)$-connected and $(k-1)$-simple by Lemma 2.1 and $\dim M/S^1 \leq k-2$, it follows that the obstruction groups $\mathfrak{H}^*_{S^1}(M; \pi_{*-1}(Y)) \cong H^*(M/S^1; \pi_{*-1}(Y))$ vanish, hence there exists an S^1-isovariant map. □

Remark. In this proposition, M does not need to be a rational homology sphere, however (F) is not deduced from the existence of an isovariant map in general, for example, the projection $p : S^1 \times M \to S^1$ is an S^1-isovariant map for an arbitrary manifold M.

A similar argument is valid in semifree case. We show:

Proposition 2.3. *Let M be of type SF and SW an S^1-representation sphere with S^1-fixed points. Suppose that*

(SF): $\dim M \leq \dim SW - \dim SW^H$ *for* $1 \neq H \leq S^1$.

Then there exists an S^1-isovariant map from M to SW.

Proof. By definition M_s consists of isolated fixed points. Take a (small) invariant disk neighborhood N_i at each fixed point x_i, which is S^1-diffeomorphic to some representation disk, say DU_i. Since S^1 acts semifreely on M, every SU_i has a free action. Take $y \in SW^{S^1}$ and let B be an invariant disk neighborhood of y which is S^1-diffeomorphic to some DV. We claim that there exists an S^1-isovariant map $f_i : SU_i \to SV$. Noting that $\dim SU_i = \dim M - 1$ and that $\dim SV - \dim SV_s = \dim SW - \dim SW_s$, we have $\dim SU_i + 1 \leq \dim SV - \dim SV_s$ by (SF). Applying Proposition 2.2 to this case, we obtain an S^1-isovariant map $f_i : SU_i \to SV$. By the radial extension, f_i can be extended to an isovariant map $\tilde{f}_i : DU_i \to DV$, so we obtain an isovariant map $g_i : N_i \to B$ equivalent to $\tilde{f} : DU_i \to DV$.

The next step is to extend $g := \coprod_i g_i : \coprod_i N_i \to B \subset SW$ to an isovariant map from M to SW. To do this it suffices to see the existence of an S^1-map $F : X := M \setminus (\coprod_i \operatorname{int} N_i) \to Y = SW \setminus SW_s$ extending $g_{|\partial N} : \partial X \to Y$. The obstruction lies in $\mathfrak{H}^*_{S^1}(X, \partial X; \pi_{*-1}(Y)) \cong H^*(X/S^1, \partial X/S^1; \pi_{*-1}(Y))$. Since $\dim X/S^1 = \dim M/S^1 \leq k-1$ by condition (SF), the obstruction groups vanish, and hence there exists the required S^1-map F. □

Remark. In this proposition, indeed, M does not need to be a rational homology sphere.

3. The multidegree and Hopf type results

In the case of type PF, the obstruction groups do not necessarily vanish, so we need to detect the obstruction class itself. For that purpose we shall introduce the

multidegree and show a variant of the equivariant Hopf theorem. (See [5, II 4], [9] for the equivariant Hopf theorem.)

Throughout this section, M is an orientable, connected, closed S^1-manifold of type PF and we assume that $\mathrm{Iso}(M) \subset \mathrm{Iso}(SW)$. Set $k = \dim SW - \dim SW_s$. We also assume that $\dim M - 1 = k$. Let S be an exceptional orbit of M isomorphic to S^1/C, $C \neq 1$, S^1. By the slice theorem, S admits a (closed) S^1-tubular neighborhood N which is S^1-diffeomorphic to $S^1 \times_C DU$, where U is a k-dimensional C-representation so that C acts freely on $U \setminus \{0\}$. We fix an orientation of $M \setminus \mathrm{int}\, N$ and consider the orientation of ∂N (homologically) induced as the boundary of $M \setminus \mathrm{int}\, N$. Taking the standard orientation of S^1 ($\subset \mathbb{C}$), we orient SU such that $\partial N \cong S^1 \times_C SU$ is an orientation-preserving S^1-diffeomorphism. For simplicity we often identify N [resp. ∂N] with $S^1 \times_C DU$ [resp. $S^1 \times_C SU$] hereafter.

Let $f : S^{k-1} \to Y$ be a (nonequivariant) map. Then f induces the homomorphism $f_* : H_{k-1}(S^{k-1}) \to H_{k-1}(Y)$. Composing the isomorphism Φ in Lemma 2.1 with f_*, we have a homomorphism

$$\Phi \circ f_* : H_{k-1}(S^{k-1}) \to \bigoplus_{H \in \mathcal{A}} H_{k-1}(S(W^H)^\perp) = \bigoplus_{H \in \mathcal{A}} \mathbb{Z}.$$

(We fix orientations of $S(W^H)^\perp$, $H \in \mathcal{A}$, and identify $\bigoplus_{H \in \mathcal{A}} H_{k-1}(S(W^H)^\perp)$ with $\bigoplus_{H \in \mathcal{A}} \mathbb{Z}$.)

We define $\text{m-deg}\, f$, called the *multidegree* of f, by setting

$$\text{m-deg}\, f = \Phi \circ f_*([S^{k-1}]) \in \bigoplus_{H \in \mathcal{A}} \mathbb{Z},$$

where $[S^{k-1}]$ is the fundamental class of S^{k-1}.

Let $f : \partial N \to Y$ be an S^1-map. Restricting f to SU, we have a C-map $\bar{f} := \mathrm{Res}\, f : SU \to Y$. By assumption, $\dim SU = k - 1$.

Definition. We define the *multidegree* $\text{m-Deg}\, f$ of an S^1-map $f : \partial N \to Y$ by setting

$$\text{m-Deg}\, f := \text{m-deg}\, \bar{f} = \Phi \circ \bar{f}_*([SU]) \in \bigoplus_{H \in \mathcal{A}} \mathbb{Z}.$$

Since N is orientable, $\text{m-Deg}\, f$ does not depend on the choice of U and hence m-Deg is an S^1-homotopy invariant for S^1-maps. We fix an S^1-map $f_0 : \partial N \to Y$. One of purposes in this section is to show the following Hopf type result.

Theorem 3.1. *With the notation above,*

(1) $\text{m-Deg} : [\partial N, Y]_{S^1} \to \bigoplus_{H \in \mathcal{A}} \mathbb{Z}$ *is injective.*
(2) *The image of* $D := \text{m-Deg} - \text{m-Deg}\, f_0$ *coincides with* $\bigoplus_{H \in \mathcal{A}} |C|\mathbb{Z}$.

To show this we need some lemmas.

Lemma 3.2. *The homotopy group* $\pi_{k-1}(Y)$ *is isomorphic to* $\bigoplus_{H \in \mathcal{A}} \mathbb{Z}$, *and* C *acts trivially on* $\pi_{k-1}(Y)$.

Proof. The first half follows from Lemma 2.1. The C-action on $\pi_{k-1}(Y)$ is coming from the S^1-action, which induces the trivial action on the homotopy group. □

Lemma 3.3. *The restriction* $\mathrm{Res} : [\partial N, Y]_{S^1} \to [SU, Y]_C$ *is a bijection.*

Proof. See [5, I (4.7)]. □

Lemma 3.4. *The assignment $[f] \mapsto \gamma_C(f, f_0)$, where $\gamma_C(f, f_0)$ is the obstruction to the existence of a C-homotopy between f and f_0, gives a bijection:*

$$\rho_C(f_0) : [SU, Y]_C \to \mathfrak{H}_C^{k-1}(SU; \pi_{k-1}(Y)).$$

Proof. See [5, II (3.17)]. □

Remark. The same result holds in nonequivariant case, i.e., $\rho(f_0) : [SU, Y] \to H^{k-1}(SU; \pi_{k-1}(Y))$ is a bijection.

The bijection in Lemma 3.4 depends on the choice of f_0. We call f_0 the *reference* map.

Lemma 3.5. *The forgetful maps $\varepsilon : \mathfrak{H}_C^{k-1}(SU; \pi_{k-1}(Y)) \to H^{k-1}(SU; \pi_{k-1}(Y))$ and $\iota : [SU, Y]_C \to [SU, Y]$ are injective.*

Proof. Set $\pi = \pi_{k-1}(Y)$, $C_C^* = \mathrm{Hom}_{\mathbb{Z}C}(C_*(SU), \pi)$ and $C^* = \mathrm{Hom}_{\mathbb{Z}}(C_*(SU), \pi)$. By definition ε is induced by forgetting the C-action. Let $\bar{\tau} : C^{k-1} \to C_C^{k-1}$ be the homomorphism defined by setting $\bar{\tau}(f)(c) = \sum_{g \in C} gf(g^{-1}c)$, $f \in C^{k-1}$ and $c \in C_{k-1}$, which induces the norm homomorphism $\tau : H^*(SU; \pi) \to \mathfrak{H}_C^*(SU; \pi)$ [5, p. 123]. It is easily seen that $\tau\varepsilon$ is multiplication by $|C|$. Since $\mathfrak{H}_C^{k-1}(SU; \pi) \cong \pi$ is torsion free, it follows that ε is injective. Moreover there is the following commutative diagram:

$$\begin{array}{ccc} [SU, Y]_C & \xrightarrow{\iota} & [SU, Y] \\ {\scriptstyle \rho_C(f_0)}\downarrow & & \downarrow{\scriptstyle \rho(f_0)} \\ \mathfrak{H}_C^{k-1}(SU; \pi) & \xrightarrow{\varepsilon} & H^{k-1}(SU; \pi). \end{array}$$

Since $\rho_C(f_0)$ and ε are injective, it follows that ι is injective. □

Lemma 3.6. *The norm homomorphism $\tau : H^{k-1}(SU; \pi) \to \mathfrak{H}_C^{k-1}(SU; \pi)$ is an isomorphism.*

Proof. Let $C_{k-1} = C_{k-1}(SU)$. Since C_{k-1} is a free $\mathbb{Z}C$-module, it follows [5, II (4.5)] that $\bar{\tau} : \mathrm{Hom}_{\mathbb{Z}}(C_{k-1}, \pi) \to \mathrm{Hom}_{\mathbb{Z}C}(C_{k-1}, \pi)$ is surjective, and hence so is τ. Since $H^{k-1}(SU; \pi)$ and $\mathfrak{H}_C^{k-1}(SU; \pi)$ are free abelian groups with the same rank, it follows that τ is an isomorphism. □

Proof of Theorem 3.1. (1) One can see that m-Deg coincides with the following composite map:

$$[\partial N, Y]_{S^1} \xrightarrow{\mathrm{Res}} [SU, Y]_C \xrightarrow{\iota} [SU, Y] = \pi_{k-1}(Y) \underset{\cong}{\xrightarrow{h}} H_{k-1}(Y) = \bigoplus_{H \in \mathcal{A}} \mathbb{Z},$$

which is injective by Lemmas 3.3 and 3.5.

(2) To show this we first give a cohomological description of the multidegree. Let $f : \partial N \to Y$ be an S^1-map, and set $\bar{f} = \mathrm{Res}\, f : SU \to Y$. Using the universal

coefficient theorem, we have the following commutative diagram:

$$\begin{array}{ccc}
H^{k-1}(Y;\pi) & \xrightarrow{\bar{f}^*} & H^{k-1}(SU;\pi) \\
{\scriptstyle\kappa}\downarrow\cong & & {\scriptstyle\kappa}\downarrow\cong \\
\mathrm{Hom}_{\mathbb{Z}}(H_{k-1}(Y),\pi) & \xrightarrow{\mathrm{Hom}_{\mathbb{Z}}(\bar{f}_*)} & \mathrm{Hom}_{\mathbb{Z}}(H_{k-1}(SU),\pi) \\
{\scriptstyle h^*}\downarrow\cong & & {\scriptstyle h^*}\downarrow\cong \\
\mathrm{Hom}_{\mathbb{Z}}(\pi,\pi) & \xrightarrow{\mathrm{Hom}_{\mathbb{Z}}(\bar{f}_*)} & \mathrm{Hom}_{\mathbb{Z}}(\pi_{k-1}(SU),\pi),
\end{array}$$

where h denotes the Hurewicz homomorphism. Set

$$\iota(Y) = (h^*\kappa)^{-1}(id_\pi) \in H^{k-1}(Y,\pi).$$

Chasing the diagram, one can see that $\langle \bar{f}^*\iota(Y), [SU]\rangle = \bar{f}_*([SU]) \in \pi$, where $[SU]$ is the fundamental class in $H_{k-1}(SU;\pi) = H_{k-1}(SU)\otimes\pi$, and hence we obtain

$$\text{m-Deg}\, f = \Phi \circ h(\langle \bar{f}^*\iota(Y), [SU]\rangle). \tag{3.1}$$

It is well known (cf. [5, II (3.19)]) that

$$\gamma(\bar{f},\bar{f}_0) = \bar{f}^*\iota(Y) - \bar{f}_0^*\iota(Y) \in H^{k-1}(Y;\pi),$$

hence we obtain

$$\text{m-Deg}\, f - \text{m-Deg}\, f_0 = \Phi \circ h(\langle \gamma(\bar{f},\bar{f}_0), [SU]\rangle). \tag{3.2}$$

Note that the image of ε coincides with $|C|H^{k-1}(SU;\pi)$, since $\tau\varepsilon$ is multiplication by $|C|$ and τ is an isomorphism by Lemma 3.6. Thus we obtain

$$\gamma(\bar{f},\bar{f}_0) = \varepsilon(\gamma_C(\bar{f},\bar{f}_0)) \in |C|H^{k-1}(SU;\pi), \tag{3.3}$$

and hence

$$D(f) = \text{m-Deg}\, f - \text{m-Deg}\, f_0 \in \bigoplus_{H\in\mathcal{A}} |C|\mathbb{Z}.$$

By Lemmas 3.3 and 3.4, every element of $\mathfrak{H}_C^{k-1}(SU;\pi)$ is realized as $\gamma_C(\bar{f},\bar{f}_0)$ for some S^1-map $f : \partial N \to Y$, thus we see that the image of D coincides with $\bigoplus_{H\in\mathcal{A}} |C|\mathbb{Z}$. □

In the rest of this section, using the multidegree, we shall discuss the extendability of $f : \partial N \to Y$ to an isovariant map from N to SW. Take a point $y_0 \in SW$ with isotropy group C, and let V be the slice representation of the orbit through y_0, namely, the closed S^1-tubular neighborhood of the orbit is S^1-diffeomorphic to $S^1 \times_C DV$ and y_0 corresponds to $[1,0] \in S^1 \times_C DV$.

Lemma 3.7. *There exists a C-isovariant map $\bar{g} : SU \to SV$. Moreover any C-isovariant map $\bar{g} : SU \to SV$ can be extended to some S^1-isovariant map $g : N \to SW$.*

Proof. Note that the isotropy group at each point of $S^1 \times_C DV$ is a subgroup of C. One can see that, for any $1 \neq K \leq C$,

$$\begin{aligned}
\dim S^1 \times_C DU - 1 &\leq \dim SW - \dim SW^K \ (\text{by } \dim M - 1 = k) \\
&= \dim S^1 \times_C DV - \dim(S^1 \times_C DV)^K \\
&= \dim DV - \dim DV^K.
\end{aligned}$$

This implies that $\dim SU + 1 \leq \dim SV - \dim SV^K$ for any $1 \neq K \leq C$, and hence we obtain

$$\dim SU + 1 \leq \dim SV - \dim SV_s.$$

Then the obstruction groups $\mathfrak{H}^*_C(SU; \pi_{*-1}(SV \setminus SV_s))$ vanish, because $SV \setminus SV_s$ is $(l-2)$-connected and $(l-1)$-simple, $l = \dim SV - \dim SV_s$ (which is proved by a similar argument as in Lemma 2.1), so there exists a C-map from SU to $SV \setminus SV_s$. Since C acts freely on SU and $SV \setminus SV_s$, composing the inclusion $SV \setminus SV_s \subset SV$, we obtain a C-isovariant map $\bar{g} : SU \to SV$.

By the radial extension of $\bar{g}$, we obtain a C-isovariant map $\tilde{g} : DU \to DV$, which induces an S^1-isovariant map $S^1 \times_C \tilde{g} : S^1 \times_C DU \to S^1 \times_C DV$. □

Remark. Under condition (PF1), this lemma still holds without the assumption $\dim M - 1 = k$.

We denote by $d_H(f)$ or $d_H(\bar{f})$ the H-component of $\text{m-Deg}\, f = \text{m-deg}\, \bar{f}$.

Lemma 3.8. *If an S^1-map $f : \partial N \to Y$ can be extended to an S^1-isovariant map $\tilde{f} : N \to SW$, then the H-component $d_H(f) = 0$ for every $H \in \mathcal{A}$ with $H \not\leq C$.*

Proof. Note that $\tilde{f}(N) \subset SW \setminus SW^H$, in fact, if $\tilde{f}(x) \in SW^H$ for some $x \in N$, then $H \leq G_{\tilde{f}(x)} = G_x \leq C$, which is a contradiction. Hence a C-map $\bar{f} = \text{Res}\, f : SU \to Y$ has the extension $\tilde{f}_{|DU} : DU \to SW \setminus SW^H$ ($\simeq S(W^H)^\perp$). This implies $d_H(f) = 0$. □

Conversely the extendability of f to an isovariant map is detected by the multi-degree. We have:

Proposition 3.9. *An S^1-map $f : \partial N \to Y$ can be extended to an S^1-isovariant map from N to SW if and only if $d_H(f) = 0$ for every $H \in \mathcal{A}$ with $H \not\leq C$. Moreover, for such an S^1-map f and for any $a \in \bigoplus_{H \in \mathcal{A}_C} |C|\mathbb{Z} \subset \bigoplus_{H \in \mathcal{A}} |C|\mathbb{Z}$, where $\mathcal{A}_C := \{H \in \mathcal{A} | H \leq C\}$, there exists an S^1-map $f' : \partial N \to Y$ such that f' can be extended to an isovariant map and* $\text{m-Deg}\, f' = \text{m-Deg}\, f + a$.

Proof. The "only if" part has been already shown in Lemma 3.8. We show the converse. With the same notation as before, by Lemma 3.7 there is a C-isovariant map

$$\bar{g} : SU \to Y' := SV \setminus SV_s \subset Y,$$

and such a map is always extended to an S^1-isovariant map $g : \partial N \to SW$. It suffices to show that, for any S^1-isovariant map $f : \partial N \to SW$ so that $d_H(f) = 0$ for every $H \in \mathcal{A}$ with $H \not\leq C$, $\bar{f} := \text{Res}\, f : SU \to Y$ is C-homotopic to some C-map $\bar{g}$ as above.

Let $H \in \mathcal{A}_C$ and let B be an H-invariant neighborhood H-diffeomorphic to $DT_{y_0}(SW)$ a disk of the tangent space at y_0. Note $T_{y_0}(SW) \cong \text{int}\, D(W - \mathbb{R})$ as H-representations. Choosing a small slice DV, one may suppose $DV \subset \text{int}\, B$ and $D(V^H)^\perp \subset \text{int}\, (B^H)^\perp$. Noting $\dim S(V^H)^\perp = \dim \partial(B^H)^\perp = k - 1$, one can see that $[S(V^H)^\perp] = [\partial(B_H)^\perp]$ in $H_{k-1}(B \setminus B^H)$. Moreover, since the inclusion $\partial B \setminus \partial B^H \subset SW \setminus SW^H$ is a homotopy equivalence, it follows that $[\partial(B^H)^\perp] = [S(W^H)^\perp] \in H^{k-1}(SW \setminus SW^H)$. Thus the inclusion $SV \setminus SV^H \subset SW \setminus SW^H$ induces an isomorphism $H_{k-1}(SV \setminus SV^H) \cong H_{k-1}(SW \setminus SW^H)$ and $[S(V^H)^\perp]$

maps to $[S(W^H)^\perp]$ for any $H \in \mathcal{A}_C$. Consequently we obtain that, for every C-map $\bar{g} : SU \to Y'$ $(\subset Y)$ and for every $H \in \mathcal{A}_C$,

$$d_H(\bar{g} : SU \to Y') = d_H(\bar{g} : SU \to Y).$$

From this, fixing $\bar{g}_0 : SU \to SV$ as in Lemma 3.7, we have the following commutative diagram:

$$\begin{array}{ccc} [SU, Y']_C & \xrightarrow[\cong]{\text{m-deg} - \text{m-deg}\, \bar{g}_0} & \bigoplus_{H \in \mathcal{A}_C} |C|\mathbb{Z} \\ {\scriptstyle i_*}\big\downarrow & & \big\downarrow{\scriptstyle j} \\ [SU, Y]_C & \xrightarrow[\text{m-deg} - \text{m-deg}\, \bar{g}_0]{\cong} & \bigoplus_{H \in \mathcal{A}} |C|\mathbb{Z}, \end{array}$$

where $i : Y' \to Y$ and j are the natural inclusions. Since $d_H(\bar{f}) = 0$ for any $H \in \mathcal{A}$ with $H \not\leq C$, it follows that $[\bar{f}] \in [SU, Y]_C$ is in the image i_*, which completes the proof of the first half. The last half follows from Theorem 3.1 and the first half. □

4. Existence of isovariant maps in type PF

Let M be of type PF and SW a representation sphere. Suppose $\mathrm{Iso}(M) \subset \mathrm{Iso}(SW)$ and conditions (PF1) and (PF2). Let S_i, $i = 1, \ldots, t$, be the exceptional orbits with isotropy group C_i of M. Let N_i be a closed S^1-tubular neighborhood of S_i with the slice representation U_i. By Lemma 3.7 and its remark there exists an S^1-isovariant $g_i : N_i \to SW$. Set $N = \coprod_i N_i$ and $X = M \setminus \operatorname{int} N$. Let $k = \dim SW - \dim SW_s$.

We shall discuss the extendability of an isovariant map $g := \coprod_i g_i$ to an isovariant map from the whole space M. Set $f_i = g_{i|\partial N_i} : \partial N_i \to Y := SW \setminus SW_s$ and $f = \coprod_i f_i : \partial X \to Y$. It is sufficient to see the existence of an S^1-map $F : X \to Y$ extending f. Conditions (PF1) and (PF2) imply that $\dim M - 1 \leq k$, so we divide the problem into two cases:

(1) $\dim M - 1 < k$,
(2) $\dim M - 1 = k$.

In case (1) the proof is analogous to that of free or semifree case. The obstruction lies in $\mathfrak{H}^*_{S^1}(X, \partial X; \pi_{*-1}(Y))$, but this group vanishes since Y is $(k-2)$-connected and $\dim X/S^1 \leq k - 1$ by assumption (1).

Next we consider the problem in case (2). The obstruction class $\gamma_{S^1}(f)$ to the existence of an S^1-map $X \to Y$ extending f lies in $\mathfrak{H}^k_{S^1}(X, \partial X; \pi) \cong \pi$, $\pi = \pi_{k-1}(Y)$. Fix an S^1-map $F_0 : X \to Y$ (which is not necessarily extending f). Note that such F_0 always exists, in fact, since Y is $(k-2)$-connected and $\dim X/S^1 = k$ by assumption (2), the obstruction lies in $\mathfrak{H}^k_{S^1}(X; \pi_{k-1}(Y))$, however, since X has a (nonempty) boundary, this group also vanishes and hence there exists an S^1-map F_0. Set $f_0 = F_0|_{\partial N} : \partial N \to Y$ and $f_{0,i} = F_0|_{\partial N_i} : \partial N_i \to Y$ as reference maps.

Lemma 4.1. *Let $\delta : \mathfrak{H}^{k-1}_{S^1}(\partial X; \pi) \to \mathfrak{H}^k_{S^1}(X, \partial X; \pi)$ be the connecting homomorphism. Then $\delta\gamma_{S^1}(f, f_0) = \gamma_{S^1}(f)$.*

Proof. Note [5, II (3.14)] that $\delta\gamma_{S^1}(f, f_0) = \gamma_{S^1}(f) - \gamma_{S^1}(f_0)$. Since f_0 has the extension F_0, it follows that $\gamma_{S^1}(f_0) = 0$. □

Let $p : X \to X/S^1$ and $\hat{p} = \coprod p_i : \partial X = \coprod \partial N_i \to \partial X/S^1 = \coprod \partial N_i/S^1$ be the projections. By [5, II (3.4)], these projections induce the natural isomorphisms

$$p^* : H^k(X/S^1, \partial X/S^1; \pi) \to \mathfrak{H}^{k-1}_{S^1}(X, \partial X; \pi),$$
$$\hat{p}^* : H^k(\partial X/S^1; \pi) \to \mathfrak{H}^{k-1}_{S^1}(\partial X; \pi).$$

We define isomorphisms $E : \mathfrak{H}^{k-1}_{S^1}(X, \partial X; \pi) \to \pi$ and $\bar{E} : \mathfrak{H}^{k-1}_{S^1}(\partial X; \pi) \to \bigoplus_i \pi$ by $E = ev \circ p^{*-1}$ and $\bar{E} = ev \circ \hat{p}^{*-1}$, respectively. Here ev denotes the evaluation map by the fundamental class.

Proposition 4.2. *Identifying π with $\bigoplus_{H \in \mathcal{A}} \mathbb{Z}$ via $\Phi \circ h$, we have:*

(1) $\bar{E}(\gamma_{S^1}(f, f_0)) = ((\text{m-Deg}\, f_i - \text{m-Deg}\, f_{0,i})/|C_i|)_i$,
(2) $E(\gamma_{S^1}(f)) = \sum_i (\text{m-Deg}\, f_i - \text{m-Deg}\, f_{0,i})/|C_i|$.

Proof. Let $r = \coprod_i r_i : \coprod_i SU_i \to \partial X = \coprod_i \partial N_i$ be the inclusion, which induces a diffeomorphism $\bar{r} = \coprod_i \bar{r}_i : \coprod_i SU_i/C_i \to \partial X/S^1 = \coprod_i \partial N_i/S^1$. Let $\bar{p}_i : SU_i \to SU_i/C_i$ be the projection. Consider the following commutative diagram:

$$\begin{array}{ccccc}
\mathfrak{H}^k_{S^1}(X, \partial X; \pi) & \xleftarrow{p^*} & H^k(X/S^1, X/S^1; \pi) & \xrightarrow{ev} & \pi \\
\uparrow{\scriptstyle \delta} & & \uparrow{\scriptstyle \delta} & & \uparrow{\scriptstyle \sigma} \\
\mathfrak{H}^{k-1}_{S^1}(\partial X; \pi) & \xleftarrow{\hat{p}^*} & H^{k-1}(\partial X/S^1; \pi) & \xrightarrow{ev} & \bigoplus_i \pi \\
\downarrow{\scriptstyle r^* = \oplus r_i^*} & & \downarrow{\scriptstyle \bar{r}^*} & & \| \\
\bigoplus_i \mathfrak{H}^{k-1}_{C_i}(SU_i; \pi) & \xleftarrow{\oplus \bar{p}_i^*} & \bigoplus_i H^{k-1}(SU_i/C_i; \pi) & \xrightarrow{ev} & \bigoplus_i \pi \\
\downarrow{\scriptstyle \oplus \varepsilon} & & \downarrow{\scriptstyle \oplus \bar{p}_i^*} & & \downarrow{\scriptstyle \oplus |C_i|} \\
\bigoplus_i H^{k-1}(SU_i; \pi) & = & \bigoplus_i H^{k-1}(SU_i; \pi) & \xrightarrow{ev} & \bigoplus_i \pi.
\end{array}$$

Here ε is the forgetful map and σ is defined by setting $\sigma(a_i)_i = \sum_i a_i$. Since $\bar{r}_i^* : \mathfrak{H}^{k-1}_{S^1}(\partial N_i; \pi) \to \mathfrak{H}^{k-1}_{C_i}(SU_i; \pi)$ maps $\gamma_{S^1}(f_i, f_{0,i})$ to $\gamma_{C_i}(\bar{f}_i, \bar{f}_{0,i})$, it follows from formulae (3.2) and (3.3) in section 3 that

$$ev \circ \varepsilon \circ r_i^*(\gamma_{S^1}(f_i, f_{0,i})) = \text{m-Deg}\, f_i - \text{m-Deg}\, f_{0,i}.$$

Hence, chasing the diagram, we conclude the first assertion. By Lemma 4.1, we see that

$$E(\gamma_{S^1}(f)) = E(\delta \gamma_{S^1}(f, f_0)) = \sigma \bar{E}(\gamma_{S^1}(f, f_0)).$$

Hence the second assertion is obtained from (1). □

Corollary 4.3. *An S^1-map $f : \partial X \to Y$ is extended to an S^1-map from X if and only if $\sum_i (\text{m-Deg}\, f_i - \text{m-Deg}\, f_{0,i})/|C_i| = 0$.*

Proof. Since E is an isomorphism, it follows from Proposition 4.2 that $\gamma_{S^1}(f) = 0$ if and only if $\sum_i (\text{m-Deg}\, f_i - \text{m-Deg}\, f_{0,i})/|C_i| = 0$. □

We now prove the remaining case of Theorem B.

Proposition 4.4. *Let M be of type PF and SW a representation sphere. Suppose $\text{Iso}(M) \subset \text{Iso}(SW)$ and $\dim M - 1 = \dim SW - \dim SW_s$ $(= k)$. If inequalities* (PF1) *and* (PF2) *are fulfilled, then there exists an S^1-isovariant map from M to SW.*

Proof. It suffices to see that one can choose S^1-maps $f_i : \partial N_i \to Y$ such that every f_i is extended to an S^1-isovariant map from N_i to SW and such that $\gamma_{S^1}(f) = 0$ ($f = \coprod_i f_i$). By Lemma 3.7 there are S^1-isovariant maps $g_i : \partial N_i \to SW$ extended to isovariant maps from N_i. Consider the obstruction class $\gamma_{S^1}(g)$, $g = \coprod_i g_i$, which is determined by $\sum_i (\text{m-Deg}\, g_i - \text{m-Deg}\, f_{0,i})/|C_i|$. Suppose that some H-component of $\sum_i (\text{m-Deg}\, g_i - \text{m-Deg}\, f_{0,i})/|C_i|$ is nonzero. We note that, for any $H \in \mathcal{A}$, there exists some $C_j \in \text{Iso}(M)$ so that $H \leq C_j$. In fact, if $H \not\leq C_i$ for every C_i, then it follows from (PF2) that $\dim M + 1 \leq k$, which contradicts the assumption $\dim M - 1 = k$. Therefore, using Proposition 3.9, one can replace g_j with other isovariant map $f_j : \partial N_j \to Y$ so that

$$\sum_{i \neq j} (d_H(g_i) - d_H(f_{0,i}))/|C_i| + (d_H(f_j) - d_H(f_{0,j}))/|C_j| = 0$$

and $d_K(f_j) = d_K(g_j)$ for $K \neq H$. Continuing this procedure, we obtain the desired map f. □

Remark. In this proposition, it is sufficient that M is an orientable closed manifold, however (PF2) is not necessarily deduced from the existence of an isovariant map. (See Example 5.2 in the next section.)

5. Some examples

Let T_n, $n \in \mathbb{Z}$, denote the irreducible (complex) representation of S^1 given by $\rho_n : S^1 \to SU(1)$ $(= S^1)$, $z \mapsto z^n$. In the following, p, q, r are pairwise coprime integers greater than 1. As a simple example of Theorem A, we have:

Example 5.1. There is no S^1-isovariant map from $S(T_p \oplus T_1)$ to $S(T_p \oplus T_q)$.

Proof. Condition (PF2) is not fulfilled (although (PF1) is fulfilled). □

Remark. Obviously there exists an S^1-map $f : S(T_p \oplus T_1) \to S(T_p \oplus T_q)$, for example, $f(z, w) = (z, w^q)/\|(z, w^q)\|$ is S^1-equivariant.

Next we illustrate an example in the case when M is not a rational homology sphere. Set $M = ST_p \times S(T_1 \oplus \mathbb{R})$ and $SW = S(T_p \oplus T_q)$. Note that M is an S^1-manifold of type PF with two exceptional orbits whose isotropy groups are the same subgroup $\mathbb{Z}_p$.

Example 5.2. There exists an S^1-isovariant map $f : M \to SW$.

Proof. One can directly construct an S^1-isovariant map. Define $f : M \to SW$ by setting

$$f(z, (w, t)) = (z, tw^q)/\|(z, tw^q)\|.$$

It is easily verified that f is S^1-isovariant. □

Remark. In this example, condition (PF2) is not fulfilled, however one can see that the reference map f_0 can be taken so that $d_{\mathbb{Z}_q}(f_{0,1}) + d_{\mathbb{Z}_q}(f_{0,2}) = 0$. Hence the proof of Proposition 4.4 still works.

The following two examples give an answer to (an analogue of) the question posed in [16].

Example 5.3. There exists an S^1-isovariant map

$$f : S(T_p \oplus T_q \oplus T_r) \to S(T_1 \oplus T_{pq} \oplus T_{qr} \oplus T_{rp}).$$

Proof. Set $V = T_p \oplus T_q \oplus T_r$ and $W = T_1 \oplus T_{pq} \oplus T_{qr} \oplus T_{rp}$. Note that $\mathrm{Iso}(SV) \subset \mathrm{Iso}(SW)$. Moreover, it is easily seen that SV is of type PF and that both (PF1) and (PF2) are fulfilled. □

Let $C = \mathbb{Z}_n$ be the cyclic group of order n. Let g be a generator of C. The irreducible orthogonal C-representations R_i are given by the following: $R_0 = \mathbb{R}$ with trivial action, and R_i is the 2-dimensional C-representation on which g acts by rotation $2\pi i/n$ for each $0 < i < n/2$. When n is even, $R_{n/2}$ is the 1-dimensional C-representation on which g acts antipodally. We have:

Example 5.4. There exists a $\mathbb{Z}_{pqr}$-isovariant map

$$f : S(R_p \oplus R_q \oplus R_r) \to S(R_1 \oplus R_{pq} \oplus R_{qr} \oplus R_{rp}),$$

provided that none of p, q, r are equal to 2. Otherwise there is no C-isovariant map.

Proof. If none of p, q, r are equal to 2, then each R_i is obtained by restricting the S^1-representation T_i to the $\mathbb{Z}_{pqr}$-action, and hence restricting an S^1-isovariant map as in Example 5.3, we obtain the required map. Suppose that one of p, q, r is equal to 2, say $p = 2$. Note that $\dim R_{qr} = 1$. If there is a $\mathbb{Z}_{pqr}$-isovariant map

$$f : SV := S(R_2 \oplus R_q \oplus R_r) \to SW := S(R_1 \oplus R_{2q} \oplus R_{qr} \oplus R_{2r}),$$

it follows from Proposition 1.2 that

$$4 = \dim SV - \dim SV^{\mathbb{Z}_2} \leq \dim SW - \dim SW^{\mathbb{Z}_2} = 3$$

by regarding f as a $\mathbb{Z}_2$-isovariant map. This is a contradiction. □

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