TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 358, Number 3, Pages 1347–1364
S 0002-9947(05)03753-0
Article electronically published on May 26, 2005

# QUIVERS WITH RELATIONS ARISING FROM CLUSTERS ($A_n$ CASE)

P. CALDERO, F. CHAPOTON, AND R. SCHIFFLER

ABSTRACT. Cluster algebras were introduced by S. Fomin and A. Zelevinsky in connection with dual canonical bases. Let $U$ be a cluster algebra of type $A_n$. We associate to each cluster $C$ of $U$ an abelian category $\mathcal{C}_C$ such that the indecomposable objects of $\mathcal{C}_C$ are in natural correspondence with the cluster variables of $U$ which are not in $C$. We give an algebraic realization and a geometric realization of $\mathcal{C}_C$. Then, we generalize the "denominator theorem" of Fomin and Zelevinsky to any cluster.

## 0. INTRODUCTION

Cluster algebras were introduced in the work of A. Berenstein, S. Fomin, and A. Zelevinsky [2, 5, 6, 7]. This theory appeared in the context of dual canonical basis and more particularly in the study of the Berenstein-Zelevinsky conjecture. Cluster algebras are now connected with many topics: double Bruhat cells, Poisson varieties, total positivity, Teichmüller spaces. The main results on cluster algebras are on one hand the classification of finite cluster algebras by root systems and on the other hand the realization of algebras of regular functions on double Bruhat cells in terms of cluster algebras.

Recall some facts about cluster algebras. Cluster algebras $U$ of rank $n$ form a class of algebras defined axiomatically in terms of a distinguished set of generators $\{u_1, \dots, u_n\}$. A cluster is a set of "cluster variables" $\{w_1, \dots, w_n\}$ obtained combinatorially from $\{u_1, \dots, u_n\}$. The so-called Laurent phenomenon asserts that each cluster variable is a Laurent polynomial in the set of variables given by a cluster. For each cluster $C$, one can define combinatorially an oriented quiver $Q_C$.

Suppose from now on that $U$ is a finite cluster algebra, *i.e.* there only exists a finite number of cluster variables. It is known that $U$ can be described by the data of a root system $X_n$. Moreover, there exists a cluster $\Sigma$ such that $Q_\Sigma$ is the alternating quiver on $X_n$. S. Fomin and A. Zelevinsky give in [6] a more precise description of the Laurent phenomenon: there exists a one-to-one correspondence $\alpha \mapsto w_\alpha$ between the set of almost positive roots of $X_n$, *i.e.* positive roots and simple negative roots, and the set of cluster variables, such that the denominator of $w_\alpha$ as a Laurent polynomial in $\Sigma$ is given by the decomposition of $\alpha$ in the basis of simple roots. Via Gabriel's Theorem, this property suggests a link between cluster algebras and representation theory of artinian rings. This relation was already investigated in [10] and is the core subject of the recent papers [3, 4].

Received by the editors February 25, 2004 and, in revised form, May 24, 2004.
2000 *Mathematics Subject Classification.* Primary 16G20, 16G70, 05E15, 20F55.

In this paper, we give conjectural relations $R_C$ on the quiver $Q_C$, such that for any cluster $C$, the denominators of the cluster variables as Laurent polynomial in $C$ are described by indecomposables of the category $\mathcal{C}_C$ of representations of $Q_C$ with relations $R_C$.

The main result of this article is the proof of this conjecture in the $A_n$ case.

Another important result of this paper is a geometric realization of the category $\mathcal{C}_C$ in the $A_n$ case. Recall that the algebra of regular functions on the 2-grassmannian of $\mathbb{C}^{n+3}$ is a finite cluster algebra $U$ of type $A_n$. Via this realization, the cluster variables are in natural bijection with the diagonals of a regular $(n+3)$ polygon. Moreover, a result of Fomin and Zelevinsky asserts that this bijection gives a one-to-one correspondence between the set of clusters of $U$ and the set of diagonal triangulations of the polygon. Theorem 4.4 gives a simple realization of the category $\mathcal{C}_C$ in terms of the diagonals of the $(n+3)$ polygon. There also exists a more canonical category associated to a finite cluster algebra and also studied in [3]. We give in the $A_n$ case a geometric realization of this category; see Theorem 5.2.

Let us mention that so-called cluster-tilted rings, introduced in [3, 4] as endomorphism rings of tilting objects in this canonical category, are conjectured there to be isomorphic to the path algebras of our quivers with relations. The viewpoint of these articles should provide a well-suited categorical background to generalize our constructions.

The authors thank Robert Marsh and Bernhard Keller for useful conversations. Our thanks also go to the LACIM at UQAM in Montreal, where this work was started, for its great hospitality.

## 1. Quivers of cluster type

Let $X_n$ be a simply-laced Dynkin diagram of rank $n$ and finite type. First we need to recall some material on clusters.

Each cluster $C$ of a cluster algebra of rank $n$ is associated with a sign-skew-symmetric square matrix $B_C$ whose rows and columns are indexed by the cluster variables of the cluster $C$. In the simply-laced case, coefficients of the matrices $B_C$ belong to the set $\{-1, 0, 1\}$. Hence it is easy and convenient to depict these matrices using oriented graphs (once a convention is chosen for the orientation). This oriented graph is called the quiver associated to the cluster $C$ and is denoted by $Q_C$. It is known that all triangles (and more generally cycles) in these quivers are oriented in a cyclic way [6, Proposition 9.7].

The mutation procedure of clusters contains in particular a mutation rule for the associated matrices, which can be translated as a mutation rule for the associated quivers. In the simply-laced case, the mutation rule can be further simplified. The result is as follows.

Let $C$ be a cluster in a cluster algebra of simply-laced type. The mutation $\mu_i(Q_C)$ of the quiver $Q_C$ at a vertex $i$ is described as follows. First, all arrows incident to $i$ in $Q_C$ are reversed in the mutated quiver. Then, for each pair of one incoming arrow $j \to i$ and one outgoing arrow $i \to k$ in $Q_C$, the arrow $j \to k$ is in the mutated quiver if and only if the arrow $k \to j$ is not in $Q_C$. The other arrows of $Q_C$ are kept unchanged in the mutated quiver.

By definition, a *shortest path* in the quiver $Q_C$ is an oriented path (with no repeated arrow) contained in an induced subgraph of $Q_C$ which is a cycle.

**Definition 1.** Let $Q_C$ be the quiver associated to the cluster $C$. For each arrow $i \to j$ in $Q_C$, a relation $\mathrm{Rel}_{i,j}$ is defined as follows. Consider the set of shortest paths from $j$ to $i$:

- If there are exactly two distinct paths $c$ and $c'$, then $c = c'$.
- If there exists only one path $c$, then $c = 0$.
- If there is no such path, there is no relation.

To each cluster $C$, one defines the (abelian) category $\mathrm{Mod}\,\mathsf{A}_C$ as the category of modules over the algebra $\mathsf{A}_C$ which is the quotient of the path algebra of the quiver $Q_C$ by the ideal generated by the relations $\mathrm{Rel}_{i,j}$ for all arrows $i \to j$ of $Q_C$.

*Remark.* It is clear in type $A$ that there is at most one shortest path for each arrow. One could show, using the geometric model of type $D$ cluster algebras, that there are at most two such paths in this case. In order for the conjecture to make sense also in the exceptional cases $E$, it remains to prove that this is also true in these cases. We will not consider this question here.

*Obvious remark.* There is a natural one-to-one correspondence between the vertices of the quiver $Q_C$ and the simple modules in $\mathrm{Mod}\,\mathsf{A}_C$. Therefore, the isomorphism class of the simple module associated to the vertex $i$ will be denoted $\alpha_i$.

Let $\mathrm{Ind}(Q_C)$ be the set of isomorphism classes of indecomposables of $\mathrm{Mod}\,\mathsf{A}_C$.

**Conjecture 1.1.** *Let $C = \{u_1, \dots, u_n\}$ be a cluster in a cluster algebra of simply-laced type and rank $n$. Let $V$ be the set of all cluster variables for this cluster algebra. There exists a bijection $b : \mathrm{Ind}(Q_C) \to V \backslash C$, $\alpha \mapsto w_\alpha$ such that $w_\alpha = \frac{P(u_1,\dots,u_n)}{\prod_i u_i^{n_i}}$, where $P$ is a polynomial prime to $u_i$ for all $i$ and where $n_i = n_i(\alpha)$ is the multiplicity of the simple module $\alpha_i$ in the module $\alpha$.*

*Remark* 1.2. Through Gabriel's celebrated theorem relating indecomposables and positive roots, this conjecture generalizes the theorem of Fomin and Zelevinsky ([6, Theorem 1.9]) which corresponds to the case of the alternating quiver.

The main aim of the present article is to prove Conjecture 1.1 in the case of cluster algebras of type $A_n$. This will be done using the geometric realization in terms of triangulations given by Fomin and Zelevinsky [7, §3.5].

## 2. Categories of diagonals

In this section, we introduce some terminology on triangulations and define a category $\mathcal{C}_T$ for each triangulation $T$ in a geometric way.

2.1. **Triangulations and diagonals.** Let us fix a non-negative integer $n$ and a triangulation $T$ of a regular polygon with $n+3$ vertices. The diagonals of this polygon will be called roots and will be designated by Greek letters. Let us call the roots belonging to $T$ *negative* and the other roots *positive*. Let $\Phi_+$ be the set of positive roots with respect to $T$. Let $I$ be the set of negative roots. By convention, the negative root corresponding to $i \in I$ will be called $-\alpha_i$. The *support* $\mathrm{Supp}\,\alpha \subseteq I$ of a positive root $\alpha$ is the set of negative roots which cross $\alpha$. Note that a positive root $\alpha$ is determined by its support. Indeed it is possible to recover the vertices of a positive diagonal from the sequence of crossed negative diagonals. A positive root $\alpha$ is related to a positive root $\alpha'$ by a *pivoting elementary move* if the associated diagonals share a vertex on the border (the pivot), the other vertices of $\alpha$ and $\alpha'$

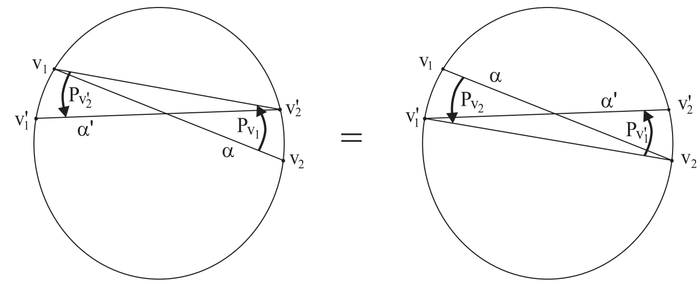


FIGURE 1. Mesh relation $P_{v'_2}P_{v_1} = P_{v'_1}P_{v_2}$

are the vertices of a border edge of the polygon and the rotation around the pivot is positive (for the trigonometric direction) from $\alpha$ to $\alpha'$. Let $P_v$ denote the pivoting elementary move with pivot $v$. A *pivoting path* from a positive root $\alpha$ to a positive root $\alpha'$ is a sequence of pivoting elementary moves starting at $\alpha$ and ending at $\alpha'$.

2.2. **Categories of diagonals.** One can define a combinatorial $\mathbb{C}$-linear additive category $\mathcal{C}_T$ as follows. The objects are positive integral linear combinations of positive roots. By additivity, it is enough to define morphisms between positive roots. The space of morphisms from a positive root $\alpha$ to a positive root $\alpha'$ is a quotient of the vector space over $\mathbb{C}$ spanned by pivoting paths from $\alpha$ to $\alpha'$.

The subspace which defines the quotient is spanned by the so-called *mesh relations* (see Figure 1). For any couple $\alpha, \alpha'$ of positive roots such that $\alpha$ is related to $\alpha'$ by two consecutive pivoting elementary moves with distinct pivots, we define the mesh relation $P_{v'_2}P_{v_1} = P_{v'_1}P_{v_2}$, where $v_1, v_2$ (respectively $v'_1, v'_2$) are the vertices of $\alpha$ (respectively $\alpha'$) such that $P_{v'_1}P_{v_2}(\alpha) = \alpha'$. That is, any two consecutive pivoting elementary moves using different pivots can in some sense be "exchanged".

In these relations, negative roots or border edges are allowed, with the following conventions:

(i) If one of the intermediate edges is a border edge, the corresponding term in the mesh relation is replaced by zero.
(ii) If one of the intermediate edges is a negative root, the corresponding term in the mesh relation is replaced by zero.

More generally, a mesh relation is an equality between two pivoting paths which differ only in two consecutive pivoting elementary moves by such a change.

We can now define the set of morphisms from a positive root $\alpha$ to a positive root $\alpha'$ to be the quotient of the vector space over $\mathbb{C}$ spanned by pivoting paths from $\alpha$ to $\alpha'$ by the subspace generated by mesh relations.

Therefore, the image of a pivoting path in the space of morphisms is either the zero morphism or only depends on the class of the pivoting path modulo the equivalence relation on the set of pivoting paths generated by the mesh relations with no vanishing terms.

The following lemma will be useful later.

**Lemma 2.1.** *The vector space* $\mathrm{Hom}_{\mathcal{C}_T}(\alpha, \alpha')$ *is not zero if and only if there exists* $i \in \mathrm{Supp}\,\alpha \cap \mathrm{Supp}\,\alpha'$ *such that the relative positions of* $\alpha$, $\alpha'$ *and* $-\alpha_i$ *are as in Figure* 2*. That is, let* $v_1, v_2$ *be the endpoints of* $-\alpha_i$ *and* $u_1, u_2$ *(respectively* $u'_1, u'_2$*) be the endpoints of* $\alpha$ *(respectively* $\alpha'$*). Then ordering the vertices of the polygon*

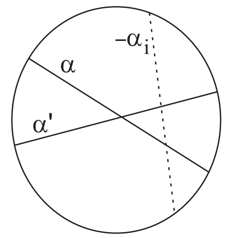


FIGURE 2. Relative position

*in the positive trigonometric direction starting at* $v_1$*, we have* $v_1 < u_1 \leq u_1' < v_2 < u_2 \leq u_2'$*. In this case,* $\mathrm{Hom}_{\mathcal{C}_T}(\alpha, \alpha')$ *is of dimension one.*

*Proof.* Suppose that $\mathrm{Hom}_{\mathcal{C}_T}(\alpha, \alpha')$ is not zero. Let $P \neq 0$ be a sequence $\alpha = \alpha^0 \xrightarrow{P_1} \alpha^1 \xrightarrow{P_2} \dots \xrightarrow{P_m} \alpha^m = \alpha'$, where the $\alpha^i$ are positive roots and the $P_i$ are pivoting elementary moves. One can map this sequence to a path in a grid using the following rules. Let $U$ and $R$ be the two vertices of $\alpha$. Then the grid path starts going up if the first pivot is $U$ and right if the first pivot is $R$. Then two consecutive steps in the grid path are drawn in the same direction (up or right) if and only if the corresponding elementary pivoting moves share the same pivot. The sequence of elementary pivoting moves can be recovered uniquely from the grid path. Now using the mesh relations, one can replace any pair of consecutive steps (up, right) in the grid path by a pair of consecutive steps (right, up) or the other way round. This gives another non-vanishing sequence of elementary pivoting moves from $\alpha$ to $\alpha'$. In this way, one can reach every grid path with the same numbers of up and right steps, *i.e.* ending at the same point of the grid. We may therefore suppose that the first $k$ moves $P_1, \dots, P_k$ have as pivot one of the vertices of $\alpha$ and the last $(m-k)$ moves $P_{k+1}, \dots, P_m$ have as pivot the common vertex of $\alpha^k$ and $\alpha'$. Denote by $V_1$ (respectively $V_2$) the set of vertices of $\alpha^1, \dots, \alpha^k$ (respectively $\alpha^{k+1}, \dots, \alpha^m$) other than the pivot of $P_1$ (respectively $P_{k+1}$). Since $P \neq 0$, this implies that $\alpha$ and $\alpha'$ intersect or share a vertex and that all diagonals with one vertex in $V_1$ and the other in $V_2$ are positive. Thus the vertices of $\alpha$ and $\alpha'$ form a quadrilateral in the polygon without any diagonals of the triangulation $T$ crossing it from $V_1$ to $V_2$. Because $T$ is a triangulation we must have a $-\alpha_i \in T$ crossing this quadrilateral in the other direction and we get the situation in the diagram. On the other hand, in the situation of the diagram it is clear that there is a non-zero morphism from $\alpha$ to $\alpha'$. Finally, the dimension of $\mathrm{Hom}_{\mathcal{C}_T}(\alpha, \alpha')$ is at most one, since any two non-zero pivoting paths from $\alpha$ to $\alpha'$ are in the same class. $\square$

## 3. QUIVERS AND TRIANGULATIONS

In this section, we define a quiver directly from a triangulation and introduce the category of modules on this quiver with some relations.

3.1. **Graphs and trees.** Let $T$ be a triangulation. Then one can define a planar tree $t_T$ as follows. Its vertices are the triangles of $T$ and its edges are between adjacent triangles (see the left part of Figure 3). Vertices of $t_T$ have valency 1, 2 or 3. It is clear that there is always at least one vertex of valency 1.

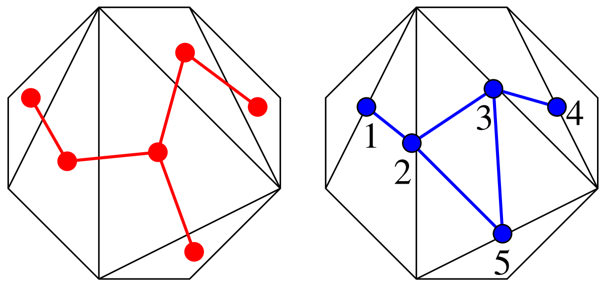


FIGURE 3. Tree $t_T$ and Graph $Q_T$ defined by a Triangulation $T$

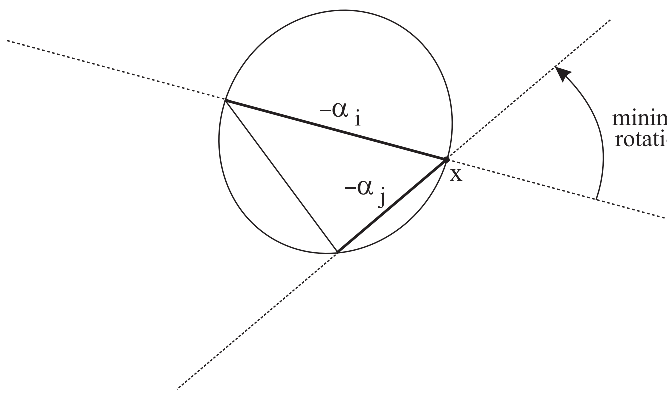


FIGURE 4. Definition of the arrows of $Q_T$

From $T$, one can also define a graph $Q_T$ as follows. The vertices of $Q_T$ are the inner edges of $T$ and are related by an edge if they bound part of the same triangle (see the right part of Figure 3).

In fact, it is possible to define the graph $Q_T$ starting from the planar tree $t_T$. Vertices of $Q_T$ are the edges of $t_T$. Two vertices of $Q_T$ are related by an edge in $Q_T$ if the corresponding edges of $t_T$ share a vertex in $t_T$. The equivalence with the previous definition is obvious.

A *leaf* is an edge $e$ of $t_T$ such that at least one of its vertices has valency 1. As there is always a vertex of valency 1 in $t_T$, there always exists a leaf.

Recall that the mutation of a triangulation at one of its diagonals is the unique triangulation which can be obtained by replacing this diagonal with another one.

3.2. **Quivers and triangulations.** Let $T$ be a triangulation. Let us define a quiver $Q_T$ with underlying graph the graph $Q_T$ defined in the previous section. Recall that its vertices are in bijection with $I$. Put a point at the middle of each negative root in $T$ and draw an edge between points in two negative roots $-\alpha_i, -\alpha_j$ which bound part of the same triangle. The orientation of the edge is defined as follows: denote by $x$ the common vertex of $-\alpha_i, -\alpha_j$, then $-\alpha_i \to -\alpha_j$ if the rotation with minimal angle around $x$ that sends the line through $-\alpha_i$ to the line through $-\alpha_j$ is in positive trigonometric direction (see Figure 4). From this description, it follows that all triangles in $Q_T$ are oriented.

**Lemma 3.1.** *The mutation of quivers, as defined in* §1, *corresponds to the mutation of triangulations described above.*

*Proof.* Left to the reader. □

One can define a $\mathbb{C}$-linear abelian category $\operatorname{Mod} Q_T$ as follows. This is the category of modules over the quiver $Q_T$ with the following relations, called *triangle relations*:

(1) In any triangle, the composition of two successive maps is zero.

These relations are exactly the relations prescribed by Definition 1.

The next two lemmas are steps for the proof of Lemma 3.4 which will be used later.

**Lemma 3.2.** *The support* $\operatorname{Supp}\alpha$ *of a positive root* $\alpha$ *is connected as a subset of the quiver* $Q_T$.

*Proof.* Let $-\alpha_i, -\alpha_j$ be two distinct diagonals in $\operatorname{Supp}\alpha$. We will show that there is an unoriented path from $i$ to $j$ in $Q_T \cap \operatorname{Supp}\alpha$. The diagonals $-\alpha_i$ and $-\alpha_j$ cut the polygon into three parts. Denote by $R_{ij}$ the part that contains both $-\alpha_i$ and $-\alpha_j$. We proceed by induction on $m$ the number of negative roots in $R_{ij}$. If $m = 1$, then $-\alpha_i = -\alpha_j$ and there is nothing to prove. Let us assume that $m > 1$. Let $\Delta$ be the unique triangle in $T$ that contains $-\alpha_i$ and lies in $R_{ij}$. Since $\alpha$ crosses both $-\alpha_i$ and $-\alpha_j$, it has to cross exactly one of the two sides different from $-\alpha_i$ in $\Delta$. This side cannot be a border edge of the polygon, hence it has to be a negative root, call it $-\alpha_k$. Thus there is an edge between $-\alpha_i$ and $-\alpha_k$ in $Q_T$ and $i, k \in \operatorname{Supp}\alpha$. We may suppose by induction that there is an unoriented path in $\operatorname{Supp}\alpha$ from $k$ to $j$, and we are done. □

**Lemma 3.3.** *Let* $\alpha, \alpha'$ *be positive roots, then* $\operatorname{Supp}\alpha \cap \operatorname{Supp}\alpha'$ *is connected.*

*Proof.* Suppose the contrary. Write $S = \operatorname{Supp}\alpha$ and $S' = \operatorname{Supp}\alpha'$ for short. Let $i, k \in S \cap S'$ be two vertices that belong to different connected components of $S \cap S'$. Since $S$ and $S'$ are connected (Lemma 3.2) we may choose two minimal paths $p : i = i_1, i_2, \ldots, i_p = k$ in $S$ and $p' : i = j_1, j_2, \ldots, j_q = k$ in $S'$. Let $m$ be the smallest integer such that $i_{m+1} \neq j_{m+1}$. In the triangulation $T$, each of the diagonals $-\alpha_{i_{m-1}}, -\alpha_{i_{m+1}}, -\alpha_{j_{m+1}}$ has a vertex in common with $-\alpha_{i_m}$. Since a positive root can only cross two sides of a triangle, we get that $-\alpha_{i_m}, -\alpha_{i_{m+1}}$ and $-\alpha_{j_{m+1}}$ form a triangle $\Delta$ in $T$. Moreover $i_{m+1} \in S \setminus S'$ and $j_{m+1} \in S' \setminus S$. Now cutting out the triangle $\Delta$ divides the polygon into three parts: $R_{i_m}, R_{i_{m+1}}$ and $R_{j_{m+1}}$ such that $R_l$ contains $-\alpha_l$, $l = i_m, i_{m+1}, j_{m+1}$. Clearly all $-\alpha_{i_l}$, $l \geq m+1$, lie in $R_{i_{m+1}}$ and all $-\alpha_{j_l}$, $l \geq m+1$, lie in $R_{j_{m+1}}$. But this contradicts the fact that $-\alpha_{j_q} = -\alpha_k = -\alpha_{i_p}$ and we have shown that $S \cap S'$ is connected. □

Let us introduce objects of $\operatorname{Mod} Q_T$ indexed by the positive roots. The module $(M^\alpha, f^\alpha) = (M_i^\alpha, f_{ij}^\alpha)$ is defined by

$$(2) \qquad M_i^\alpha = \begin{cases} \mathbb{C} & \text{if } i \in \operatorname{Supp}\alpha, \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \qquad f_{ij}^\alpha = \begin{cases} \mathrm{id}_{\mathbb{C}} & \text{if } M_i^\alpha = \mathbb{C} = M_j^\alpha, \\ 0 & \text{otherwise.} \end{cases}$$

This is indeed an object in $\operatorname{Mod} Q_T$ because a positive root $\alpha$ can only cross two sides of a triangle in $T$, which implies that in each triangle in $Q_T$ there is at most one arrow $i \to j$ such that $f_{ij}^\alpha \neq 0$ and hence the triangle relations (1) hold.

Then we have the following lemma.

**Lemma 3.4.** *The vector space* $\mathrm{Hom}_{\mathrm{Mod}\,Q_T}((M^\alpha, f^\alpha), (M^{\alpha'}, f^{\alpha'}))$ *is not zero if and only if the following conditions hold. Let* $S = \mathrm{Supp}\,\alpha$ *and* $S' = \mathrm{Supp}\,\alpha'$ *for short.*

(i) $S \cap S'$ *is not empty,*
(ii) *There is no arrow from* $S \setminus S'$ *to* $S \cap S'$ *in* $Q_T$*.*
(iii) *There is no arrow from* $S \cap S'$ *to* $S' \setminus S$ *in* $Q_T$*.*

*In this case,* $\mathrm{Hom}_{\mathrm{Mod}\,Q_T}((M^\alpha, f^\alpha), (M^{\alpha'}, f^{\alpha'}))$ *is of dimension one.*

*Proof.* Let $P$ be a non-zero element of $\mathrm{Hom}_{\mathrm{Mod}\,Q_T}((M^\alpha, f^\alpha), (M^{\alpha'}, f^{\alpha'}))$. Then condition (i) is clearly true. Let us show conditions (ii) and (iii). Suppose that condition (ii) is not true, thus there is an arrow $i \to j$ in $Q_T$ with $i \in S \setminus S'$, $j \in S \cap S'$ and such that the following diagram commutes:

$$\begin{tikzcd} M_i^\alpha \arrow[r, "\mathrm{id}_{\mathbb{C}}"] \arrow[d, "P_i"'] & M_j^\alpha \arrow[d, "P_j"] \\ 0 \arrow[r] & M_j^{\alpha'} \end{tikzcd}$$

Thus $P_i$ and $P_j$ are both zero. Now let $k$ be any vertex in $S \cap S'$, We will show that $P_k$ is zero. By Lemma 3.3, there is an unoriented path $k = k_0 — k_1 — \cdots — k_m = i$ in $Q_T$ such that each $k_i \in S \cap S'$. We proceed by induction on $m$. The case $m = 0$ is done above; suppose $m > 0$. By induction $P_{k_1}$ is zero and the commutativity of the diagram

$$\begin{tikzcd} M_k^\alpha \arrow[r, dash, "\mathrm{id}_{\mathbb{C}}"] \arrow[d, "P_k"'] & M_{k_1}^\alpha \arrow[d, "0"] \\ M_k^{\alpha'} \arrow[r, dash, "\mathrm{id}_{\mathbb{C}}"] & M_{k_1}^{\alpha'} \end{tikzcd}$$

implies that $P_k$ is zero for both possibilities of orientation of $k — k_1$ in $Q_T$. By contradiction, this shows (ii). Condition (iii) is proved by a similar argument.

In order to show the converse statement, let $\alpha, \alpha'$ be such that (i), (ii) and (iii) hold. Define $P \in \mathrm{Hom}_{\mathrm{Mod}\,Q_T}((M^\alpha, f^\alpha), (M^{\alpha'}, f^{\alpha'}))$ by $P_i = \mathrm{id}_{\mathbb{C}}$ whenever $i \in S \cap S'$ and $P_i = 0$ otherwise. Then (i) implies that $P$ is non-zero. We only have to check that $P$ is a morphism of quiver modules, *i.e.* that the diagram

$$\begin{tikzcd} M_i^\alpha \arrow[r, "f_{ij}^\alpha"] \arrow[d, "P_i"'] & M_j^\alpha \arrow[d, "P_j"] \\ M_i^{\alpha'} \arrow[r, "f_{ij}^{\alpha'}"] & M_j^{\alpha'} \end{tikzcd}$$

commutes for all $i \to j$ in $Q_T$. But this is true because of conditions (ii) and (iii). Finally, the dimension of $\mathrm{Hom}_{\mathrm{Mod}\,Q_T}((M^\alpha, f^\alpha), (M^{\alpha'}, f^{\alpha'}))$ is at most one, since all vector spaces $M_i^\alpha, M_i^{\alpha'}$ are of dimension zero or one and the intersection of the supports is connected (Lemma 3.3). $\square$

## 4. Equivalence of categories

In this section we define a functor and prove that it is an equivalence of categories.

4.1. **The $\Theta$ functor.** Let us define a $\mathbb{C}$-linear additive functor $\Theta$ from $\mathcal{C}_T$ to $\operatorname{Mod} Q_T$. On objects, it is sufficient by additivity to define $\Theta$ on positive roots. The image of the positive root $\alpha$ is taken to be the module $(M^\alpha, f^\alpha)$ defined by formula (2). Now we define the functor $\Theta$ on morphisms. By additivity, it is sufficient to define the functor on morphisms from a positive root to a positive root. Our strategy is to define first the functor on pivoting elementary moves, then check that the mesh relations hold. For any pivoting elementary move $P : \alpha \to \alpha'$, define the morphism $\Theta(P)$ from $(M^\alpha, f^\alpha)$ to $(M^{\alpha'}, f^{\alpha'})$ to be $\mathrm{id}_\mathbb{C}$ whenever possible and 0 otherwise. Let us now check that this is indeed a morphism in $\operatorname{Mod} Q_T$. For a given arrow $j \to i$ in $Q_T$, we have to check the commutativity of the following diagram:

$$\begin{tikzcd} M_j^\alpha \arrow[r, "f_{ji}^\alpha"] \arrow[d, "\Theta(P)_j"'] & M_i^\alpha \arrow[d, "\Theta(P)_i"] \\ M_j^{\alpha'} \arrow[r, "f_{ji}^{\alpha'}"] & M_i^{\alpha'} \end{tikzcd}$$

This is obvious if $M_j^\alpha = 0$ or $M_i^{\alpha'} = 0$ and also if both $M_i^\alpha$ and $M_j^{\alpha'}$ are 0. Suppose that $M_j^\alpha \neq 0$ and $M_i^{\alpha'} \neq 0$. If $M_i^\alpha \neq 0$ and $M_j^{\alpha'} \neq 0$, then all four maps are $\mathrm{id}_\mathbb{C}$ and the diagram commutes. The only remaining case is if exactly one of $M_i^\alpha$, $M_j^{\alpha'}$ is not zero. We will show that this cannot happen. Suppose that $M_j^{\alpha'} = 0$ and $M_i^\alpha \neq 0$, that is, $j, i \in \operatorname{Supp}\alpha,\ i \in \operatorname{Supp}\alpha'$ and $j \notin \operatorname{Supp}\alpha'$. Since $\alpha \to \alpha'$ is a pivoting elementary move we get that $-\alpha_j$ crosses $\alpha$, that $-\alpha_j$ and $\alpha'$ have a common point on the boundary of the polygon and that $-\alpha_i$ crosses $\alpha$ and $\alpha'$. This contradicts the orientation $j \to i$ in the quiver $Q_T$. The other case can be excluded by a similar argument. To show that the functor is well defined, it only remains to check the mesh relations. Let $\alpha \xrightarrow{P^1} \beta$, $\beta \xrightarrow{P^2} \gamma$, $\alpha \xrightarrow{P^3} \beta'$, $\beta' \xrightarrow{P^4} \gamma$ be pivoting elementary moves with $\alpha, \beta, \gamma$ positive roots and $\beta \neq \beta'$. Note that we can exclude the case where $\beta$ and $\beta'$ are both negative roots or border edges because in this case either $\alpha$ or $\gamma$ has to be negative too, since $T$ is a triangulation. Suppose first that $\beta'$ is positive. One has to check the commutativity of the diagram

$$\begin{tikzcd} M_i^\alpha \arrow[r, "\Theta(P^1)_i"] \arrow[d, "\Theta(P^3)_i"'] & M_i^\beta \arrow[d, "\Theta(P^2)_i"] \\ M_i^{\beta'} \arrow[r, "\Theta(P^4)_i"] & M_i^\gamma \end{tikzcd}$$

for all $i$. The only non-trivial case is when $i \in \operatorname{Supp}\alpha \cap \operatorname{Supp}\gamma$. In this case, we also have $i \in \operatorname{Supp}\beta \cap \operatorname{Supp}\beta'$ because any diagonal crossing both $\alpha$ and $\gamma$ must also cross $\beta$ and $\beta'$. Thus all maps are $\mathrm{id}_{\mathbb{C}}$ and the diagram commutes. Suppose now that $\beta'$ is negative or a border edge. We have to show that the composition $M_i^\alpha \xrightarrow{\Theta(P^1)_i} M_i^\beta \xrightarrow{\Theta(P^2)_i} M_i^\gamma$ is zero for all $i$. But in this case no negative root can cross both $\alpha$ and $\gamma$. So $\operatorname{Supp}\alpha \cap \operatorname{Supp}\gamma$ is empty, therefore the composition is zero. Hence the mesh relations hold, with the conventions made in its definition.

4.2. **Theorem of equivalence.** First we need a lemma.

**Lemma 4.1.** *Let $\alpha$ and $\alpha'$ be two positive roots. Then $\operatorname{Hom}_{\mathcal{C}_T}(\alpha, \alpha')$ is not zero if and only if $\operatorname{Hom}_{\operatorname{Mod} Q_T}((M^\alpha, f^\alpha), (M^{\alpha'}, f^{\alpha'}))$ is not zero.*

*Proof.* We have to show that the conditions in Lemma 2.1 and in Lemma 3.4 are equivalent.

Suppose $\alpha, \alpha'$ are as in Lemma 2.1. Then $i \in S \cap S'$ which implies (i). Suppose there is an arrow $j \to k$ in $Q_T$ such that $j \in S \setminus S'$ and $k \in S \cap S'$. Then $-\alpha_k$ crosses both $\alpha$ and $\alpha'$ while $-\alpha_j$ crosses only $\alpha$. Since $j \to k$, we know that $-\alpha_j$ and $-\alpha_k$ are related as in Figure 4. This is impossible because of the way that $-\alpha_i$ intersects $\alpha$ and $\alpha'$. We have shown that condition (ii) holds; condition (iii) can be shown similarly. This proves one direction of the lemma.

Suppose now that $\alpha, \alpha'$ satisfy the conditions (i), (ii) and (iii) of Lemma 3.4. By (i), there exists $-\alpha_i$ in $S \cap S'$. Let $v_1, v_2$ be the endpoints of $-\alpha_i$. Consider the two parts of the polygons $R_l$ and $R_r$ delimited by $-\alpha_i$. Each of them contains exactly one vertex of $\alpha$ and exactly one vertex of $\alpha'$. Consider the positive roots $\alpha, \alpha'$ as a path running from $R_r$ to $R_l$. The sequence of negative roots given by the successive intersections of the path $\alpha$ (respectively $\alpha'$) with elements of $T$ yields an ordering of $\operatorname{Supp}\alpha$ (respectively $\operatorname{Supp}\alpha'$). Let $S_l = \{-\alpha_i = -\alpha_{i_1}, -\alpha_{i_2}, \ldots, -\alpha_{i_p}\}$ (respectively $S'_l = \{-\alpha_i = -\alpha_{j_1}, -\alpha_{j_2}, \ldots, -\alpha_{j_q}\}$) be the set of negative roots in $R_l$ crossing $\alpha$ (respectively $\alpha'$) in that order. Let $m$ be the greatest integer such that $-\alpha_{i_m} = -\alpha_{j_m}$. We will distinguish four cases:

1. $m = p = q$, then on the boundary of $R_l$, going from an endpoint of $-\alpha_i$ in the positive direction, we meet $\alpha$ and $\alpha'$ at the same time.
2. $m = p < q$, then $-\alpha_{j_{m+1}}$ and $-\alpha_{j_m}$ bound part of the same triangle in $T$. The corresponding edge in $Q_T$ is oriented $j_{m+1} \to j_m$ by (iii). This implies that going from an endpoint of $-\alpha_i$ in the positive direction on the boundary of $R_l$, we meet $\alpha$ first and then $\alpha'$.
3. $m = q < p$, then $-\alpha_{i_{m+1}}$ and $-\alpha_{i_m}$ bound part of the same triangle in $T$. The corresponding edge in $Q_T$ is oriented $i_{m+1} \leftarrow i_m$ by (ii). This implies again that going from an endpoint of $-\alpha_i$ in the positive direction on the boundary of $R_l$, we meet $\alpha$ first and then $\alpha'$.
4. $m < p$ and $m < q$, then $-\alpha_{i_{m+1}}$, $-\alpha_{i_m}$ and $-\alpha_{j_{m+1}}$ are three different diagonals that bound part of the same triangle in $T$. The corresponding edges in $Q_T$ are oriented $i_{m+1} \leftarrow i_m$ by (ii) and $j_{m+1} \to j_m$ by (iii). This implies once more that going from an endpoint of $-\alpha_i$ in the positive direction on the boundary of $R_l$, we meet $\alpha$ first and then $\alpha'$.

By symmetry, we obtain the same results in the other part $R_r$. This implies that the relative positions of $\alpha, \alpha'$ and $\alpha_i$ are exactly as described in Lemma 2.1. □

**Proposition 4.2.** *The functor $\Theta$ is fully faithful.*

*Proof.* Using Lemma 4.1, it only remains to show that the image of a non-zero morphism is a non-zero morphism. It is sufficient to show this for all non-zero morphisms between positive roots. Let $P \in \operatorname{Hom}(\alpha, \alpha')$ be such a morphism. Then $P$ is given by a sequence of pivoting elementary moves $\alpha = \alpha^1 \to \ldots \to \alpha^m = \alpha'$. This sequence being a non-zero morphism implies that there exists a negative root $-\alpha_i$ crossing all the $\alpha^k$, $k = 1, \ldots, m$, by Lemma 2.1. By definition, $\Theta(P)_i$ is $\mathrm{id}_{\mathbb{C}}$, hence non-zero. □

*Remark* 4.3. If $i$ is a leaf (see §3.1) and $\alpha$ a positive root, then $i \in \operatorname{Supp} \alpha$ if and only if one endpoint of $\alpha$ is the vertex $x$ of the polygon that is cut off by $-\alpha_i$. Therefore there exists one positive root $\alpha^{Pr_i}$ such that the set of all positive roots with $i$ in their supports is equal to the set

$$\{\alpha^{Pr_i}, P_x(\alpha^{Pr_i}), P_x(P_x(\alpha^{Pr_i})), \ldots, P_x^{n-1}(\alpha^{Pr_i})\},$$

where $P_x$ is the pivoting elementary move with pivot $x$.

**Theorem 4.4.** *The functor $\Theta$ gives an equivalence of categories from $\mathcal{C}_T$ to $\operatorname{Mod} Q_T$.*

*Proof.* It only remains to show that the functor $\Theta$ is essentially surjective, *i.e.* that each indecomposable module in $\operatorname{Mod} Q_T$ is the image of a positive root under $\Theta$. In fact, we will characterize the indecomposable modules of $\operatorname{Mod} Q_T$ with the help of the Auslander-Reiten theory, and this will enable us to conclude by proving that there are $\frac{n(n+1)}{2}$ indecomposable $Q_T$-modules.

In the following, we refer to [8]; see also [1] for definitions, notation and results in representation theory of finite-dimensional algebras. In this proof, we use the following notations in $\operatorname{Mod} Q_T$: $P_i$ (respectively $I_i$) is the $i-$th projective (respectively injective) indecomposable module, with the convention that $P_\emptyset = I_\emptyset = 0$. Hence $(P_i)_l = \mathbb{C}$ if there is an oriented path in $Q_T$ modulo the triangle relations from $i$ to $l$ and $(P_i)_l = 0$ otherwise. Similarly, $(I_i)_l = \mathbb{C}$ if there is an oriented path in $Q_T$ modulo the triangle relations from $l$ to $i$ and $(I_i)_l = 0$ otherwise. The maps of $P_i$ and of $I_i$ are $\mathrm{id}_{\mathbb{C}}$ whenever possible and zero otherwise. In particular, these modules are multiplicity free. Fix a triangulation $T$. In the sequel, we set $Q = Q_T$ when no confusion occurs and we denote by $Q_0$ its set of vertices. Given a subset of vertices $S$ of $Q_0$, a *full subquiver* of $Q$ with vertices $S$ will be the set $S$ together with the set of all arrows (with relations) of $Q$ joining vertices of $S$. We say that a $Q_T$-module $M$ is *of type $A$* if the full subquiver of $Q_T$ on the support of $M$ is of type $A_k$ for some $k \in \mathbb{N}$.

**Lemma 4.5.** *Let $M$ be an indecomposable $Q$-module of type $A$ and let $N$ be any indecomposable $Q$-module. If $\operatorname{Hom}_Q(N, M)$ or $\operatorname{Hom}_Q(M, N)$ contains an irreducible morphism, then $N$ is of type $A$.*

*Proof.* The proof is based on the construction of irreducible morphisms via the Nakayama functor [8, §4.4]. The dual functor gives an (anti)-equivalence between $\operatorname{Mod} Q$ and $\operatorname{Mod} Q^{opp} = \operatorname{Mod} Q_{T^*}$, where $T^*$ is the triangulation obtained from $T$ by a reflection w.r.t. a line containing the center of the regular polygon. Using this (contravariant) functor, we can easily reduce the proof to the case where $\operatorname{Hom}_Q(N, M)$ contains an irreducible.

Suppose that the support of $M$ is given by the set $Q_0' := \{1, \ldots, m\}$. Let $Q'$ be the full subquiver of $Q$ given by $\operatorname{Supp} M$. By assumption, $Q'$ is of type $A_m$ with extremal vertices 1 and $m$ and we can suppose that the edges link $i$ with $i \pm 1$. Remark that, as $M$ is indecomposable of type $A$, $M$ is multiplicity free.

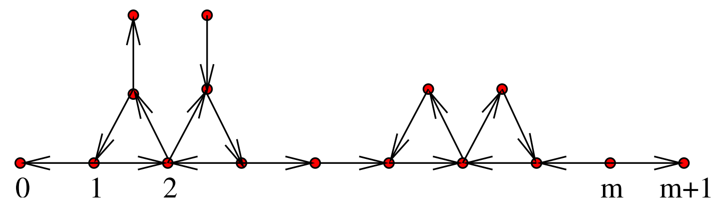


FIGURE 5. Quiver $Q$ and subquiver $Q'$ of type $A$

In order to go further, we need a more precise description of the quiver $Q'$ inside $Q$. In the sequel, we freely use §3.1. The reader may like to follow the argument on the example provided by Figure 5. Suppose that the module $M$ is not simple, *i.e.* $m \neq 1$. Now we define the vertices 0 and $m+1$ of Figure 5, which may or may not exist. There exists at most one vertex 0 in $Q_0 \backslash Q'_0$ such that $1 \to 0$ and such that there exist no other edges between $Q'_0$ and 0. In the same way, there exists at most one vertex $m+1$ in $Q_0 \backslash Q'_0$ such that $m \to m+1$ and such that there exist no other edges between $Q'_0$ and $m+1$. Note also that for all $k$ in $Q'_0$, there exist at most two vertices $i(k)^{\pm}$ such that $i(k)^{\pm} \notin Q'_0$ and $i(k)^{\pm} \to k$ is an arrow of $Q$. We can define $i(k)^+$, resp. $i(k)^-$, to be the vertex such that $k+1 \to i(k)^+$, resp. $k-1 \to i(k)^-$. By convention, if the vertices $i(k)^+$, $i(k)^-$, 0, $m+1$ do not exist, we define the corresponding symbol to be the empty set.

Let $S_0$, resp. $S_{m+1}$, be the support of the injective module associated to 0, resp. $m+1$, in the full subquiver of $Q$ with set of vertices $Q \backslash 1$, resp. $Q \backslash m$. It is clear from the tree structure of $Q$ (see §3.1) that the set $Q''_0 := S_0 \cup Q'_0 \cup S_{m+1}$ is the set of vertices of a full subquiver $Q''$ of $Q$ of type $A$.

For each source $k$ or for $k = 1, m$, let $S_k^{\pm}$ be the support of the injective module associated to $i(k)^{\pm}$ in the full subquiver of $Q$ with set of vertices $Q_0 \backslash Q'_0$.

Note that if $M$ is projective, then the module $N$ is a direct summand of the radical of $M$ and the lemma is true in this case. Suppose now that $M$ is not projective. Then, we can calculate the Auslander-Reiten translate $\tau M$ via the Nakayama functor. We have a minimal projective presentation of $M$: $P^1 \to P^0 \to M \to 0$ by setting $P^0 := \bigoplus_{i \in R} P_i$, where $i$ runs over the set $R$ of sources of $Q'$, and $P^1 := \bigoplus_{j \in R'} P_j$, where $j$ runs over the set $R'$ given by the union of $\{0, m+1\}$ with the sinks of $Q'_0 \backslash \{1, m\}$.

By [8, Remark 3.1], with the help of the Nakayama functor, we obtain a minimal injective representation of the Auslander-Reiten translate $\tau M$ of $M$: $0 \to \tau M \to I^1 \to I^0$, with $I^0 := \bigoplus_{i \in R} I_i$, and $I^1 := \bigoplus_{j \in R'} I_j$. For $k$ in $R'$, let $k^+$, resp. $k^-$, be the source of $Q'$ succeeding, resp. preceding, $k$, with the convention that $k^+ = m$, resp. $k^- = 1$, if there is no such source.

Then, for each sink $k$ of $Q'_0$, the support of $I_k$ is given by $\mathrm{Supp}(I_k) = S^-_{k^-} \cup S^+_{k^+} \cup (\mathrm{Supp}(I_k) \cap Q'_0)$. The support of $I_0$, resp. $I_{m+1}$, is $S^+_l \cup (\mathrm{Supp}(I_0) \cap Q''_0)$, resp. $S^-_g \cup (\mathrm{Supp}(I_{m+1}) \cap Q''_0)$, where $l$, resp. $g$, is the lowest, resp. greatest, source of $Q'_0$. The support of $I_k$, $k \in R$ contains $S_k^{\pm}$.

This implies that the support of $\tau M$ is a subset of $Q''_0$.

Now, let $X$ be the middle term in the Auslander-Reiten sequence $0 \to \tau M \to X \to M \to 0$. We have $\operatorname{Supp} X \subset \operatorname{Supp} \tau M \cup \operatorname{Supp} M \subset Q_0''$. Hence, $X$ is of type $A$. The Auslander-Reiten Theorem asserts that the module $N$ of the lemma is a direct summand of $X$. So, we obtain the lemma in this case. The case where $M$ is simple is very similar and left to the reader. □

We can now prove the theorem. Let $M$ be an indecomposable $Q$-module of type $A$. By the lemma, the component of $M$ in the Auslander-Reiten quiver contains only modules of type $A$, therefore can only be finite. Hence, by [8, Proposition 6.3], every indecomposable module is of type $A$ and in particular is multiplicity free. So, there exists a one-to-one correspondence between indecomposable $Q$-modules and full subquivers of $Q$ of type $A$. Let $i$ be a leaf of $Q$ and let $j$ be any vertex of $Q$. By §3.1, there exists a unique full subquiver of type $A$ of $Q$ whose extreme vertices are $i$ and $j$. This implies by induction that the number of such subquivers is $\frac{n(n+1)}{2}$. Hence, there are $\frac{n(n+1)}{2}$ indecomposable $Q$-modules as required. □

**Corollary 4.6.** *The category $\mathcal{C}_T$ is abelian.*

**Corollary 4.7.** *There exists a bijection $\varphi$ between $\operatorname{Ind}(Q_T)$ and the diagonals of the polygon not in $T$. Moreover, for $M$ in $\operatorname{Ind}(Q_T)$ and any vertex $i$ of $Q_T$, the multiplicity of the simple module $S_i$ in the module $M$ is $1$ if $\varphi(M)$ crosses the $i$-th diagonal of $T$ and $0$ if not. In particular, for two isoclasses $M$, $M'$ in $\operatorname{Ind}(Q_T)$, we have $M = M'$ if and only if $n_i(M) = n_i(M')$ for all $i$.*

## 5. The orbit category

This section is not used in the sequel. Here we give a description of the category $\mathcal{C}_T$, using the equivalence of categories proved above. Then, we prove that the orbit category introduced in [3] has a nice geometric realization in the $A_n$ case. Let $r^+$, resp. $r^-$, be the elementary rotation of the polygon in the positive, resp. negative, direction.

**Theorem 5.1.** *Let $T$ be a triangulation of the $n+3$ polygon, and let $\mathcal{C}_T$ be the corresponding category, then:*

(i) *The irreducible morphisms of $\mathcal{C}_T$ are direct sums of the generating morphisms given by pivoting elementary moves.*
(ii) *The mesh relations of $\mathcal{C}_T$ are the mesh relations* [1] *of the Auslander-Reiten quiver of $\mathcal{C}_T$.*
(iii) *The Auslander-Reiten translate is given on diagonals by $r^-$.*
(iv) *The projective indecomposable objects of $\mathcal{C}_T$ are diagonals in $r^+(T)$.*
(v) *The injective indecomposable objects of $\mathcal{C}_T$ are diagonals in $r^-(T)$.*

*Proof.* (i) and then (ii) are clear by construction of the category $\mathcal{C}_T$. By (i) and (ii), extremal terms of an almost split sequence are given by the diagonals $\alpha$ and $\alpha'$ of Figure 2. This proves (iii). (iv) and (v) follow from (iii). □

Assertions (iv) and (v) of Theorem 5.1 suggest an interpretation of the diagonals of the triangulation $T$ in terms of a category. Indeed, we will consider those diagonals, at least in the hereditary case, as shifts of the projectives in the derived category $\mathcal{DC}_T$.

In order to simplify the construction, suppose that $T$ is a triangulation corresponding to the unioriented quiver $A_n$ with simple projective $P_1$. The category $\mathcal{C}_T \simeq \operatorname{Mod} Q_T$ is hereditary, so the indecomposable objects of the derived category $\mathcal{D}\operatorname{Mod} Q_T$ are the shifts $M[m]$, $m \in \mathbb{Z}$, of the indecomposables $M$ of $\operatorname{Mod} Q_T$. Let $F$ be the functor of $\mathcal{D}\operatorname{Mod} Q_T$ given by $M \mapsto \tau^{-1}M[1]$, where $\tau$ is the Auslander-Reiten translate in the derived category. We define the orbit category $\overline{\mathcal{D}\operatorname{Mod} Q_T}$ whose objects are objects of $\mathcal{D}\operatorname{Mod} Q_T$ and morphisms are given by

$$\operatorname{Hom}_{\overline{\mathcal{D}\operatorname{Mod} Q_T}}(M,N) := \bigoplus_{i\in\mathbb{Z}} \operatorname{Hom}_{\mathcal{D}\operatorname{Mod} Q_T}(M, F^i(N)), \qquad M,N \in \mathcal{D}\operatorname{Mod} Q_T.$$

The set $\operatorname{Ind}(Q_T) \cup \{P_i[1],\, 1 \leq i \leq n\}$ is the set of indecomposable objects of $\overline{\mathcal{D}\operatorname{Mod} Q_T}$ up to isomorphism. Note that the category $\overline{\mathcal{D}\operatorname{Mod} Q_T}$ is not abelian in general. By a result of Bernhard Keller [9], it is a triangulated category.

We can also construct the total category $\mathcal{C}$ generated by all the diagonals of the $(n+3)$ polygon. The construction is analogue to the construction of $\mathcal{C}_T$: indecomposable objects are positive roots and simple negative ones. The homomorphisms and the mesh relations are defined as in §2.2 without the point (ii) in the convention made there.

**Theorem 5.2.** *Let $T$ be a triangulation corresponding to an orientation of the quiver $A_n$. Then, the categories $\mathcal{C}$ and $\overline{\mathcal{D}\operatorname{Mod} Q_T}$ are equivalent.*

Here we give a sketch of the proof. The derived category of representations of an oriented quiver of type A is well known. In particular, it does not depend on the orientation of the quiver. Hence, we can suppose that $T$ corresponds to the unioriented quiver as above. The indecomposable objects of $\mathcal{D}\operatorname{Mod} Q_T$ can be indexed by $\kappa$: $\mathbb{Z}\times\{1,\dots,n\} \to \operatorname{Ind}\mathcal{D}\operatorname{Mod} Q_T$, by the rule $\kappa(1,i) = P_i$, $\kappa(i+1,j) = \tau^{-1}(\kappa(i,j))$, $\kappa(i+j, n+1-j) = \kappa(i,j)[1]$. This implies that the indecomposable objects of $\overline{\mathcal{D}\operatorname{Mod} Q_T}$ can be indexed by $\mathbb{Z}\times\{1,\dots,n\}/(i,j) \equiv (i+j+1, n-j+1)$.

For all $(i,j)$ in $\mathbb{Z}\times\{1,\dots,n\}$, we define the quadrilateral $R_{(i,j)}$ by its vertices $(i,j)$, $(i,n)$, $(i+j-1,1)$, $(i+j-1, n-j+1)$. Let $M$ be the indecomposable object in $\overline{\mathcal{D}\operatorname{Mod} Q_T}$ indexed by $(i,j)$, and let $N$ be any indecomposable object. Then, $\operatorname{Hom}_{\overline{\mathcal{D}\operatorname{Mod} Q_T}}(M,N) \neq 0$ if and only if $N$ is indexed by a point inside $R_{(i,j)}$. In this case, it is $\mathbb{C}$ as a space and the composition of morphisms is given by the multiplication. Now, let us index the vertices of the $(n+3)$ polygon by the group $\mathbb{Z}/(n+3)\mathbb{Z}$ and let $[i,j]$ be the diagonal from $i$ to $j$, $j-i \neq 1, 0, -1$. By the description above, the additive functor defined by $[i,j] \mapsto \kappa(i, j-i-1)$ gives an equivalence of categories.

*Remark* 5.3. Using the equivalence above, it is easy to see that given two diagonals $\alpha$ and $\alpha'$, the group $\operatorname{Ext}^1(\alpha,\alpha')$ is non-zero if and only if $\alpha$ and $\alpha'$ cross. Hence, a triangulation of the polygon correspond to a maximal set of pairwise extension free diagonals.

*Remark* 5.4. The orbit category $\overline{\mathcal{D}\operatorname{Mod} Q_T}$ was introduced in [3] for all simply-laced root systems. Its construction was given to us by Bernhard Keller.

An example of the Auslander-Reiten quiver is provided in Figure 6, for the quiver with relation shown in the right part of Figure 3.

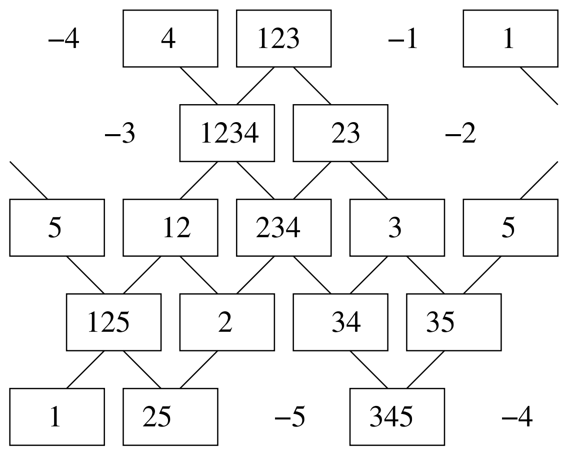


Figure 6. Example of Auslander-Reiten quiver

## 6. Denominators of Laurent polynomials

In this section, we will prove Conjecture 1.1 for cluster algebras of type $A$. The existence of the bijection will be an easy consequence of §4 and older results of Fomin and Zelevinsky. Most of this section is concerned with the calculation of the exponents in the denominators of the Laurent polynomials, in order to prove the equation of the conjecture.

Throughout this section we will use the following setup. Let $\Phi_{\geq -1}$ be the set of almost positive roots and let $\tau_+$ and $\tau_-$ be the involutions on $\Phi_{\geq -1}$ defined by Fomin and Zelevinsky in [7]. For any $\alpha \in \Phi_{\geq -1}$, let $w_\alpha$ be the cluster variable corresponding to $\alpha$ by the bijection of Fomin and Zelevinsky [6, Theorem 1.9]. Let $C = \{u_1, \ldots, u_n\}$ be a cluster and let $\beta_1, \ldots, \beta_n$ be the almost positive roots such that $w_{\beta_i} = u_i$. Recall the following properties of $\tau_\pm$:

**Proposition 6.1.** 1. *Every $\langle \tau_-, \tau_+ \rangle$–orbit in $\Phi_{\geq -1}$ contains a negative simple root.*

2. *There is a unique function $(\,||\,) : \Phi_{\geq -1} \times \Phi_{\geq -1} \to \mathbb{Z}_{\geq 0}$ such that*
   (i) $(-\alpha_i \mid\mid \beta) = \max(n_i(\beta), 0)$,
   (ii) $(\tau_\pm \alpha \mid\mid \tau_\pm \beta) = (\alpha \mid\mid \beta)$.

*Furthermore $(\,||\,)$ is symmetric for simply-laced root systems.*

3. *The set $\tau_\pm(C) = \{w_{\tau_\pm(\beta_i)} \mid i = 1, \ldots, n\}$ is a cluster.*

*Proof.* 1. is shown in [7, Theorem 2.6], 2. in [7, section 3.1] and 3. in [7, Proposition 3.5]. □

By the Laurent phenomenon [5], we can write, for any almost positive root $\alpha$,

$$w_\alpha = \frac{R_{\alpha,C}}{\prod_{i=1}^n u_i^{[\alpha,\beta_i,C]}}, \tag{3}$$

where $R_{\alpha,C}$ is a polynomial in the variables $u_1, \ldots, u_n$ such that none of the $u_i$ divides $R_{\alpha,C}$, and $[\alpha, \beta_i, C] \in \mathbb{Z}$. The following lemma is crucial.

**Lemma 6.2.** *For any pair of almost positive roots $\alpha, \beta_i$ and any pair of clusters $C, C'$ such that $u_i = w_{\beta_i} \in C \cap C'$, we have*

$$[\alpha, \beta_i, C] = [\alpha, \beta_i, C'].$$

*Proof.* It is sufficient to prove assertions (a) and (b) below.

(a) All clusters containing the given cluster variable $u_i$ are connected in the mutation graph.
(b) In mutations which do not exchange $u_i$, the exponent of $u_i$ in the denominator of $w_\alpha$ is unchanged.

Assertion (a) can either be seen as a classical statement on the link of a simplex in a simplicial sphere, or can be checked directly using the recursive properties of clusters. First the adjacency graph of clusters containing a fixed cluster variable is mapped to an isomorphic graph by the action of $\tau_+$ and $\tau_-$ because this action respects the compatibility function. Thus one can suppose that the fixed cluster variable corresponds to a negative simple root. But then the graph of clusters containing this negative simple root is isomorphic to a product of whole cluster adjacency graphs for smaller sub-root systems. Therefore it is connected as a product of connected graphs.

Let us show (b) now. Consider the mutation $C \to C'$ that exchanges the cluster variables $u_j \in C$ and $u_j' \in C'$. The exchange relation gives $u_j = \frac{M_1+M_2}{u_j'}$, where $M_1$ and $M_2$ are monomials without common divisors in the variables $C \setminus \{u_j\}$. From equation (3), we obtain by substitution

$$w_\alpha = \frac{R_{\alpha,C}(u_1, \ldots, u_{j-1}, \frac{M_1+M_2}{u_j'}, u_{j+1}, \ldots, u_n)}{\prod_{l\neq j} u_l^{[\alpha,\beta_l,C]}(\frac{M_1+M_2}{u_j'})^{[\alpha,\beta_j,C]}}.$$

By the Laurent phenomenon, we know that $w_\alpha$ is a Laurent polynomial in the cluster variables $u_1, \ldots, u_j', \ldots, u_n$. We want to prove that the exponent of $u_i$ in the denominator is still $[\alpha, \beta_i, C]$. Clearly, by properties of $M_1$ and $M_2$, this is true if and only if the Laurent polynomial $R_{\alpha,C}(u_1, \ldots, \frac{M_1+M_2}{u_j'}, \ldots, u_n)$ is not zero after evaluation at $u_i = 0$. To conclude, remark that, by properties of the monomials $M_1$ and $M_2$ stated above, the value of this Laurent polynomial at $u_i = 0$ is obtained by an invertible substitution from the value of $R_{\alpha,C}$ at $u_i = 0$, which is known not to be zero. □

Thus $[\alpha, \beta, C]$ will be denoted simply $[\alpha, \beta]$ from now on.

The following lemma is proved for all simply-laced root-systems.

**Lemma 6.3.** *Let $\alpha$,$\beta$ be two almost positive roots. Then*

$$[\alpha, \beta] = [\tau_\pm \alpha, \tau_\pm \beta]. \tag{4}$$

*Proof.* Let us consider a sequence of adjacent clusters

$$C_0 \leftrightarrow C_1 \leftrightarrow \cdots \leftrightarrow C_N, \tag{5}$$

where $\alpha \in C_0$ and $\beta \in C_N$. The exchange relations depend only on the matrices $B(C_0), \ldots, B(C_N)$ associated to these clusters. As the action of $\tau$ respects the compatibility relation, one gets another chain of adjacent clusters

$$\tau(C_0) \leftrightarrow \tau(C_1) \leftrightarrow \cdots \leftrightarrow \tau(C_N), \tag{6}$$

where $\tau(\alpha) \in \tau(C_0)$ and $\tau(\beta) \in \tau(C_N)$. By Lemma 4.8 in [6], one has

$$B_{\tau\gamma',\tau\gamma''}(\tau(C)) = -B_{\gamma',\gamma''}(C), \tag{7}$$

for any cluster $C$ and roots $\gamma', \gamma''$ in it. This minus sign does not change the exchange relations. From this one deduces that the expression of the cluster variable

$u_\alpha$ in the variables of the cluster $C_N$ is the same as the expression of the cluster variable $u_{\tau(\alpha)}$ in the variables of the cluster $\tau(C_N)$. This proves the lemma. □

**Lemma 6.4.** *Let $-\alpha_i$ be a simple negative root and let $\alpha$ be an almost positive root. Then*

$$[\alpha, -\alpha_i] = n_i(\alpha).$$

*Proof.* The quantity $[\alpha, -\alpha_i]$ is computed using the expression of the cluster variables $\alpha$ in the cluster made of negative roots. Then the bijection of Fomin and Zelevinsky between cluster variables and roots ([6, Theorem 1.9]) implies that this is $(\alpha \mid\mid -\alpha_i)$. By Proposition 6.1.2.(i) and symmetry of $(\mid\mid)$ in the simply-laced cases, the conclusion follows. □

**Proposition 6.5.** *Let $\alpha$,$\beta$ be two distinct almost positive roots. Then*

$$[\alpha, \beta] = (\alpha \mid\mid \beta).$$

*Proof.* Define a function $b : \Phi_{\geq -1} \times \Phi_{\geq -1} \to \mathbb{Z}_{\geq 0}$ by

$$b(\alpha, \beta) = \begin{cases} [\alpha, \beta] & \text{if } \alpha \neq \beta, \\ 0 & \text{if } \alpha = \beta. \end{cases}$$

This function is well defined by Lemma 6.2. Moreover

(1) $b(-\alpha_i, \beta) = \max(n_i(\beta), 0)$ (by Lemma 6.4),
(2) $b(\tau_\pm \alpha, \tau_\pm \beta) = b(\alpha, \beta)$ (by Lemma 6.3).

By Proposition 6.1.2, the function $(\mid\mid) : \Phi_{\geq -1} \times \Phi_{\geq -1} \to \mathbb{Z}_{\geq 0}$ is the unique function having the properties (1) and (2), thus $b(\alpha, \beta) = (\alpha \mid\mid \beta)$. Therefore $[\alpha, \beta] = (\alpha \mid\mid \beta)$ if $\alpha \neq \beta$. □

The following theorem establishes the Conjecture 1.1 for the type $A_n$.

**Theorem 6.6.** *Let $C = \{u_1, \dots, u_n\}$ be a cluster of a cluster algebra of type $A_n$ and let $V$ be the set of all cluster variables of the algebra. Let $Q_C$ be the quiver with relations associated to $C$ and let $\operatorname{Ind}(Q_C)$ be the set of isoclasses of indecomposable modules. Then there is a bijection*

$$\operatorname{Ind}(Q_C) \to V \setminus C, \ \alpha \mapsto w_\alpha,$$

*such that*

$$w_\alpha = \frac{P(u_1, \dots, u_n)}{\prod_{i=1}^n u_i^{n_i(\alpha)}},$$

*where $P$ is a polynomial such that none of the $u_i$ divides $P$ $(i = 1, \dots, n)$ and $n_i(\alpha)$ is the multiplicity of the simple module $\alpha_i$ in the module $\alpha$.*

*Proof.* Let $T_C = \{\beta_1, \dots, \beta_n\}$ be the triangulation of the $(n+3)$ polygon corresponding to the cluster $C$ and let $D$ be the set of diagonals of the polygon; thus $T_C \subset D$. Let $T_0 = \{-\alpha_1, \dots, -\alpha_n\}$ be the “snake triangulation” [6, 12.2]. $Q_{T_0}$ is the alternating quiver of type $A_n$ and the diagonals $-\alpha_i \in T_0$ are the negative simple roots. Fomin and Zelevinsky have shown that there is a bijection $\alpha \mapsto w_\alpha$ between the set of almost positive roots $\Phi_{\geq -1}$ and the set of cluster variables $V$. In type $A$, they identified $\Phi_{\geq -1}$ with $D$ and proved that for any cluster $C$ there is a bijection $\alpha \mapsto w_\alpha$ between $D \setminus T_C$ and $V \setminus C$. In §4, we have shown the bijection $M_\alpha \mapsto \alpha$ between $\operatorname{Ind}(Q_C)$ and $D \setminus T_C$. This establishes a bijection $\operatorname{Ind}(Q_C) \to V \setminus C$. This bijection sends the simple module in $\operatorname{Ind}(Q_C)$ at the vertex $i \in Q_C$ to the variable

$w_\beta$, where $\beta$ is the unique diagonal in $D \setminus T_C$ that crosses $\beta_i$ and does not cross any diagonal in $T_C \setminus \{\beta_i\}$.

Let $M_\alpha$ be an element of $\mathrm{Ind}(Q_C)$ with $\alpha$ the corresponding diagonal in $D \setminus T_C$. By Lemma 6.2, we have $w_\alpha = \frac{P(u_1,\dots,u_n)}{\prod_{i=1}^n u_i^{[\alpha,\beta_i]}}$. We have to show that $[\alpha, \beta_i] = n_i(\alpha)$ for all $i = 1, \dots, n$. Note that $\alpha \neq \beta_i$ since $\alpha \notin T_C$, hence using Proposition 6.5, we get $[\alpha, \beta_i] = (\alpha \mid\mid \beta_i)$ and by [6, 12.2] this is equal to 1 if the diagonals $\alpha$ and $\beta_i$ are crossing, and zero otherwise. Thus

$$[\alpha, \beta_i] = \left\{ \begin{array}{ll} 1 & \text{if } i \in \operatorname{Supp} \alpha \text{ in the sense of §4} \\ 0 & \text{otherwise} \end{array} \right\} = n_i(\alpha).$$

□

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Institut Camille Jordan, Université Claude Bernard Lyon 1, 69622 Villeurbanne Cedex, France

Institut Camille Jordan, Université Claude Bernard Lyon 1, 69622 Villeurbanne Cedex, France

School of Mathematics and Statistics, Carleton University, Ottawa, Ontario, Canada K1S 5B6