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INTEGRAL GEOMETRY AND THE GAUSS-BONNET THEOREM IN CONSTANT CURVATURE SPACES

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ABSTRACT. We give an integral-geometric proof of the Gauss-Bonnet theorem for hypersurfaces in constant curvature spaces. As a tool, we obtain variation formulas in integral geometry with interest in its own.

1. INTRODUCTION

Let $M^n(k)$ be the n-dimensional simply connected Riemannian manifold of constant sectional curvature k. When the dimension n is odd, the *total curvature* of a compact immersed hypersurface S (i.e., the integral of its *extrinsic curvature* K) equals

$$\int_S K(x)\mathrm{d}x = \frac{\mathrm{vol}(\mathbb{S}^{n-1})}{2}\chi(S) - \sum_i c_i k^i \int_S K_{n-2i-1}(x)\mathrm{d}x \tag{1}$$

where χ is the Euler characteristic, the constants c_i depend only on the dimensions, and K_i is the i-th mean curvature of S. This is a consequence of the Gauss-Bonnet-Chern theorem for abstract Riemannian manifolds (cf. [1]). When the dimension n is even, a similar formula holds for embedded hypersurfaces. Indeed, let $Q \subset M^n(k)$ be a compact domain with smooth boundary $S = \partial Q$. If n is even and V denotes the volume of Q, then

$$\int_S K(x)\mathrm{d}x = \mathrm{vol}(\mathbb{S}^{n-1})\chi(Q) - \sum_i c_i k^i \int_S K_{n-2i-1}(x)\mathrm{d}x - ck^{n/2}V. \tag{2}$$

This formula can be obtained by applying the intrinsic Gauss-Bonnet-Chern theorem for manifolds with boundary (cf. [2]) to the domain Q.

In euclidean space these formulae were first obtained by Hopf in an *extrinsic* way (cf. [4, 5]). Also in the spherical space, a very simple extrinsic proof was given by Teufel in [10] by means of integral geometry. Furthermore, that work provided a geometric interpretation of the linear combinations of the mean curvature integrals that appear in (1) and (2), which play an analogous role to the defect of non-euclidean triangles. In fact, these *defects* of hypersurfaces were shown to be the measure (with multiplicity) of the set of codimension 2 great spheres intersecting the hypersurface. Note that this generalizes the well-known identification of defect and area of spherical triangles. Unfortunately, the method used there could not

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be generalized to hyperbolic space. The present paper extends that results to the hyperbolic geometry. Moreover, our approach provides a new proof also in the spherical case.

Let $\mathcal{L}_{n-2}$ be the space of $(n-2)$-dimensional totally geodesic submanifolds of $M^n(k)$. This is a homogeneous space of the isometry group of $M^n(k)$ with a unique (up to a constant factor) measure $\mathrm{d}L$ invariant under this group. The main result is the following theorem, from which we easily deduce formulas (1) and (2).

Theorem 1. *If $Q \subset M^n(k)$ is a compact domain with smooth boundary, then*

$$\int_{\partial Q} K(x)\mathrm{d}x = \mathrm{vol}(\mathbb{S}^{n-1})\chi(Q) - k\frac{2(n-1)}{\mathrm{vol}(\mathbb{S}^{n-2})}\int_{\mathcal{L}_{n-2}} \chi(L\cap Q)\ \mathrm{d}L. \tag{3}$$

If S is a compact immersed hypersurface and n is odd, then

$$\int_S K(x)\mathrm{d}x = \frac{\mathrm{vol}(\mathbb{S}^{n-1})}{2}\chi(S) - k\frac{n-1}{\mathrm{vol}(\mathbb{S}^{n-2})}\int_{\mathcal{L}_{n-2}} \chi(L\cap S)\ \mathrm{d}L. \tag{4}$$

In other words, the *defect* of a hypersurface in a non-euclidean space is the measure of codimension 2 planes meeting it, counted with multiplicity. Note that equations (1) and (2) do not allow such a geometric interpretation.

The idea of the proof is to study the variation properties of the integrals involved. To be precise, in the case of embedding we prove the following formula for a smooth deformation Q_t of Q:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{L}_{n-2}} \chi(L\cap \partial Q_t)\ \mathrm{d}L = -\frac{\mathrm{vol}(\mathbb{S}^{n-2})}{2}\int_{\partial Q} \phi(x)K_{n-2}(x)\mathrm{d}x$$

where $\phi(x)$ is the normal part of the deformation. From [8] one knows that the variation of the total curvature is

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\partial Q} K(x)\mathrm{d}x = k(n-1)\int_{\partial Q} \phi(x)K_{n-2}(x)\mathrm{d}x,$$

and we have that both sides of (3) have the same variation properties. Then, one deforms Q to be contained in a small ball of $M^n(k)$, so that Q becomes 'almost' euclidean. Finally, one applies the Gauss-Bonnet theorem for euclidean hypersurfaces to get Theorem 1. The scheme for even dimensional immersed hypersurfaces is the same. By induction, and by using the classical reproductibility formulas of integral geometry, we get (2) and (1) from (3) and (4).

2. Definitions and preliminaries

We use the following model for $M^n(k)$ when $k \neq 0$. Endow $\mathbb{R}^{n+1}$ with the (pseudo-)metric

$$\langle x, y\rangle = \mathrm{sign}(k)\, x_0y_0 + x_1y_1 + \cdots + x_ny_n.$$

Then $M^n(k)$ is the connected component of $\{x \in \mathbb{R}^{n+1} \,|\, \langle x, x\rangle = k\}$ containing $(1/\sqrt{|k|}, 0, \ldots, 0)$. The isometry group $G(k)$ is the linear subgroup preserving $\langle\ ,\ \rangle$ and $M^n(k)$. An orthonormal frame defined on a manifold U will be a smooth mapping $g : U \to M^n(k)$ together with a smooth collection $g_i : U \to \mathbb{R}^{n+1}$ ($i = 1, \ldots, n$) such that $\langle g_i, g_j\rangle = \delta_{ij}$ and $\langle g, g_i\rangle = 0$. Then we consider in U the differential forms $\omega_i(\cdot) = \langle \mathrm{d}g(\cdot), g_i\rangle$, and $\omega_{ij}(\cdot) = \langle \mathrm{d}g_i(\cdot), g_j\rangle$ for $i, j = 1, \ldots, n$. When $U \subset M^n(k)$ and $g = id$, we just have the usual dual forms ω_i, and ω_{ij} are the connection forms of $M^n(k)$. In euclidean space $M^n(0)$, one considers orthonormal frames and defines ω_i and ω_{ij} analogously.

The intersection of a linear $(r+1)$-dimensional subspace of $\mathbb{R}^{n+1}$ with $M^n(k)$ is an r-dimensional totally geodesic submanifold that we call an *r-plane*. The space $\mathcal{L}_r$ of all r-planes is homogeneous under $G(k)$. When $k>0$, it is the Grassmann manifold $G(n+1,r+1)$ and when $k<0$ it is an open subset of $G(n+1,r+1)$. In euclidean space $M^n(0)$, the space $\mathcal{L}_r$ of r-dimensional affine subspaces is also homogeneous below the rigid motion group. In any case $\mathcal{L}_r$ admits a measure (unique up to a constant factor) invariant under $G(k)$. Given a moving frame defined on an open set $U\subset\mathcal{L}_r$ so that $g(L_r)\in L_r$, and $\langle g_1(L_r),\dots,g_r(L_r)\rangle = T_{g(L_r)}L_r$, this measure is given by (cf. [9, p. 305])

$$\mathrm{d}L_r = \left|\bigwedge \omega_h \bigwedge \omega_{ij}\right|, \qquad 1\le i\le r<j,h\le n. \tag{5}$$

One takes the absolute value to integrate densities instead of differential forms. Note that $\mathcal{L}_r$ need not be orientable.

The classical formulas of integral geometry are devoted to integrate geometric magnitudes of intersections of such planes with submanifolds of $M^n(k)$. For instance, if S is a compact q-dimensional submanifold (possibly with boundary), and $n-r\le q\le n$, then (cf. [9, p. 309])

$$\int_{\mathcal{L}_r} \mathrm{vol}_{r+q-n}(L_r\cap S)\mathrm{d}L_r = \frac{O_n\cdots O_{n-r}O_{r+q-n}}{O_r\cdots O_1 O_q}\cdot \mathrm{vol}_q(S)$$

where $O_i=\mathrm{vol}(\mathbb{S}^i)$. This is called the reproductive property of the volume. There is an analogous property for the mean curvature integrals of a hypersurface S. Indeed, denote

$$M_i(S)=\int_S K_i(x)\mathrm{d}x$$

where the i-th mean curvature K_i is the mean value of all the products of i principal curvatures of S (according to some orientation if i is odd), and $\mathrm{d}x$ is the volume element on S. One has (cf. [9, p. 248])

$$\int_{\mathcal{L}_r} M_i^{(r)}(S\cap L_r)\mathrm{d}L_r = 2\frac{O_{n-2}\cdots O_{n-r}O_{n-i}}{O_{r-2}\cdots O_0 O_{r-i}}M_i(S) \tag{6}$$

where $M_i^{(r)}$ denotes the i-th mean curvature integral considered in some r-plane. Besides volumes and curvature integrals, the Euler characteristic of intersections is also usually considered.

Definition 2.1. Let Q be a compact domain in $M^n(k)$. For $r=1,\dots,n-1$ set

$$W_r(Q)=\frac{(n-r)\cdot O_{r-1}\cdots O_0}{n\cdot O_{n-2}\cdots O_{n-r-1}}\int_{\mathcal{L}_r}\chi(L_r\cap Q)\ \mathrm{d}L_r.$$

Furthermore, set

$$W_0(Q)=V(Q) \qquad \text{and} \qquad W_n(Q)=\frac{O_{n-1}}{n}\chi(Q).$$

The constants are chosen in such a way that when $k=0$ and Q is convex, $W_r(Q)$ are the so-called Quermassintegrale (or mean cross-sectional measures). In euclidean space, one has the Cauchy formula when ∂Q is smooth: $W_r(Q)= nM_{r-1}(\partial Q)$. In non-euclidean geometry, $W_r(Q)$ was expressed by Santaló in [9] as a linear combination of several mean curvature integrals on ∂Q. We will obtain this result in a completely new way (cf. Corollary 8).

For odd dimensional planes, the previous definition extends to immersions.

Definition 2.2. Let $i : S \to M^n(k)$ be a compact immersed hypersurface. When r is odd we define

$$W_r(S) = \frac{1}{2}\frac{(n-r)O_{r-1}\cdots O_0}{n\cdot O_{n-2}\cdots O_{n-r-1}}\int_{\mathcal{L}_r}\chi(i^{-1}(L_r))\mathrm{d}L_r.$$

Note that if $S = \partial Q$, then $W_r(S) = W_r(Q)$ since $\chi(L_r \cap S) = 2\chi(L_r \cap Q)$.

3. Variation formulas

Our first step is to study the variation of $W_r(Q)$ (resp. $W_r(S)$) when Q (resp. S) is perturbed. We will need the following formula of M. Morse (cf. [7]) which is a generalization of the Poincaré-Hopf index theorem. This formula is also used in [3] to prove the Gauss-Bonnet theorem in euclidean space.

Theorem 2 (Morse [7]). *Let X be a smooth field in a compact manifold N with boundary ∂N. Assume that X has no zero in ∂N and it coincides with the inner normal at isolated points of ∂N. Then the sum $IndX$ of the indices of X at singular points of N is*

$$IndX = \chi(N) - Ind\_\partial X$$

where $Ind\_\partial X$ is the sum of the indices of the projection of X at ∂N in the singular points where X is inward (normal).

3.1. **Embedding case.** Let Q be a compact n-dimensional manifold with boundary, and let $\varphi : Q \times I \longrightarrow M^n(k)$ be a smooth mapping such that for every $t \in I = (-\epsilon, \epsilon)$, the restriction $\varphi_t = \varphi(\cdot, t)$ is an embedding. We denote $Q_t = \varphi_t(Q)$ and call them a *deformation* of Q_0. It is clear that φ_t is an embedding of $S = \partial Q$ and that the image $S_t = \varphi_t(S) = \partial Q_t$.

In the following proposition and in the subsequent, the normal curvature of an oriented hypersurface S in the direction of some linear subspace $l \subset T_xS$ stands for the determinant of the second fundamental form of S restricted to l.

Proposition 3. *If Q_t is a deformation of a domain Q_0, then for almost every r-plane L_r, and $0 < s < \epsilon$,*

$$\chi(L_r \cap Q_s) - \chi(L_r \cap Q_0) = -\sum \mathrm{sign}\langle\frac{\partial\varphi}{\partial t}, \mathrm{n}\rangle \mathrm{sign}K(L_r)$$

where the sum runs over the points $\varphi_t(x)$ where L_r is tangent to S_t with $t \in (0, s)$, and $K(L_r)$ is the normal curvature of S_t in the direction $T_{\varphi(x,t)}L_r$ with respect to the inner normal n.

Proof. Consider the mapping $Q \times I \longrightarrow M^n(k) \times I$ defined by $(p, t) \mapsto (\varphi(p, t), t)$. By hypothesis, the image is a domain M of $M^n(k) \times I$. Reduce for the moment I to $(0, s)$. Then $\partial M \subset M^n(k) \times I$ is a smooth hypersurface. For almost every L_r we can assume $L_r \times I$ to be transverse to this hypersurface. Thus, $N = M \cap (L_r \times I)$ is a domain of $L_r \times I$ with smooth boundary (cf. Figure 1). Consider the unit vertical field ∂t and its orthogonal projection onto ∂N,

$$X = \partial t - \langle \partial t, \mathrm{n}' \rangle \mathrm{n}'$$

where n' is the inner unit normal to ∂N. Let us place on a singular point $y = \varphi_t(p) \in \partial N$ of X. Next we compute the index ι of X in y. Take (local) normal coordinates in $L_r \times I$ around y, and let $\mathrm{d}X : T_y\partial N \to T_y\partial N$ be the derivative of X written in local coordinates. We will show that generically X is not degenerate

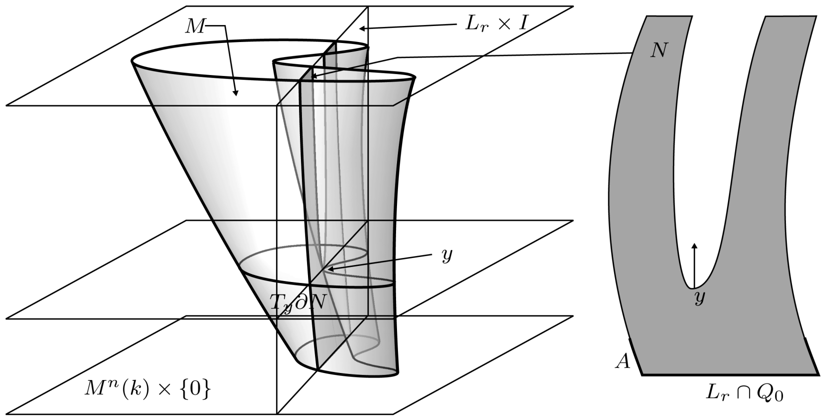

FIGURE 1.

so that its index is ± 1 according to the sign of the determinant of $\mathrm{d}X$ (cf. [6, p. 37]). Since we took normal coordinates, we have $\mathrm{d}X(Y) = \nabla_Y X$, where ∇ is the Levi-Civita connection of $L \times I$. For $Z \in T_y N$,

$$
\begin{aligned}
(7)\quad \mathrm{d}X(Z) = \nabla_Z X &= \nabla_Z \partial t - \nabla_Z \langle \partial t, \mathrm{n}' \rangle \mathrm{n}' \\
&= -Z(\langle \partial t, \mathrm{n}' \rangle)\mathrm{n}' - \langle \partial t, \mathrm{n}' \rangle \nabla_Z \mathrm{n}' = -\langle \partial t, \mathrm{n}' \rangle \nabla_Z \mathrm{n}'
\end{aligned}
$$

and by hypothesis $\partial t = \pm \mathrm{n}'$. Thus, the determinant of $\mathrm{d}X$ is, up to the sign, the (extrinsic) curvature K' of ∂N in y as a hypersurface of $L_r \times I$ and with respect to n',

$$\det \mathrm{d}X = \langle \partial t, \mathrm{n}' \rangle^r K'.$$

Since n' points inwards to M, by Meusnier's theorem, K' is a positive multiple of the normal curvature of ∂M in the direction $T_y \partial N$ with respect to the inner normal to ∂M. By the same reason, this normal curvature is a positive multiple of $K(L_r)$, the normal curvature of $S_t \equiv (M^n(k) \times \{t\}) \cap \partial M$ with respect to n in the direction $T_p L_r$. Thus

$$\det \mathrm{d}X = \lambda \langle \partial t, \mathrm{n}' \rangle^r K(L_r)$$

with $\lambda > 0$. In particular, y is a degenerate singular point if and only if S_t has normal curvature 0 at x in the direction of L_r. This can happen only for a null measure set of L_r. To see this, it is enough to apply the Sard theorem to the mapping $\{(p,l) | l \leq T_p S\} \times I \to \mathcal{L}_r$ defined by $((p,l),t) \mapsto \exp_{\varphi(p,t)} l$. As seen in [11], its critical values are precisely the r-planes which are tangent to some S_t in such a way that $K(L_r) = 0$. Therefore, for almost every L_r we can assume all the singular points y of X to be non-degenerate, and thus isolated. Hence their index ι is ± 1 according to the sign of the determinant of $\mathrm{d}X$ (cf. [6]); i.e.

$$(8)\qquad \iota(y) = \operatorname{sign} \det(\mathrm{d}X_y) = \langle \partial t, \mathrm{n}' \rangle^r \operatorname{sign} K(L_r).$$

We now want to relate $\chi(L_r \cap Q_s) - \chi(L_r \cap Q_0)$ to the index sum of X. First we extend I to $[0, s]$ and we slightly modify N in such a way that the new boundary ∂N in $M^n(k) \times I$ is smooth. This modification can be made outside the region $B = N \cap (L_r \times [\delta, s - \delta])$, with $\delta > 0$ small enough. Moreover, we can assume the new ∂N to be orthogonal to ∂t only at isolated points. Consider the open set $A = \partial N \cap (L_r \times [0, \delta))$ in ∂N. The field X, orthogonal projection of ∂t on ∂N, points outwards to A at ∂A. By Theorem 2 we have that $\chi(A)$ is the index sum of X in the singular points contained in A. Applying Theorem 2 again gives

$$\chi(N)=\sum_{C^+}\iota=\sum_{C^+\cap A}\iota+\sum_{C^+\setminus A}\iota=\chi(A)+\sum_{C^+\setminus A}\iota$$

where C^+ is the set of singular points of X where ∂t is interior. Then

$$\chi(L_r\cap Q_0)=\chi(N)-\sum_{C^+\cap B}\iota.$$

Analogously one sees

$$\chi(L_r\cap Q_s)=\chi(N)-\sum_{C^-\cap B}\iota'$$

where C^- contains those singularities of $-X$ where $-\partial t$ is interior, and ι' is the index of $-X$. Since $\iota'=(-1)^r\iota$,

$$\chi(L_r\cap Q_s)-\chi(L_r\cap Q_0)=-\sum_{C^-\cap B}\iota'+\sum_{C^+\cap B}\iota=\sum_{C}\langle\partial t,\mathrm{n}'\rangle^{r+1}\iota$$

where $C=(C^+\cup C^-)\cap B$ and n' is the inner normal to ∂N. We finish the proof by substituting (8) in the latter equation, and by noting that ∂t is interior to M if and only if $\partial\varphi/\partial t$ is exterior to Q_t. □

Now we can prove the following variation formula for domains with smooth boundary in $M^n(k)$.

Theorem 4. *For a deformation of domains Q_t,*

$$\left.\frac{\mathrm{d}}{\mathrm{dt}}\right|_{t=0}\int_{\mathcal{L}_r}\chi(L_r\cap Q_t)\mathrm{d}L_r=-\mathrm{vol}(G(r,n-1))\int_{S_0}\phi(x)K_r(x)\mathrm{d}x$$

where $\phi(x)=\langle\partial\varphi/\partial t,\mathrm{n}\rangle$ and n is the inner unit normal.

Proof. By the previous proposition, for almost all L_r, we have

$$\chi(L_r\cap Q_t)-\chi(L_r\cap Q_0)=-\sum\mathrm{sign}\phi\ \mathrm{sign}K(L_r)$$

where the sum runs over the tangencies of L_r with the hypersurfaces S_t. Integrating with respect to L_r gives

$$\int_{\mathcal{L}_r}(\chi(L_r\cap Q_t)-\chi(L_r\cap Q_0))\mathrm{d}L_r=-\int_{\mathcal{L}_r}\left(\sum\mathrm{sign}\phi\ \mathrm{sign}K(L_r)\right)\mathrm{d}L_r.$$

Consider $G_r(S)=\{(p,l)|l\leq T_pS\ \ \dim l=r\}$, and

$$\begin{array}{rcl}\gamma:G_r(S)\times(-\epsilon,\epsilon)&\longrightarrow&\mathcal{L}_r\\ (\ (p,l),\ t\ \ )&\longmapsto&\exp_{\varphi_t(p)}l.\end{array}$$

Since

$$\gamma^*(\mathrm{d}L_r)=\iota_{\partial t}(\gamma^*(\mathrm{d}L_r))\mathrm{dt}=\gamma_t^*(\iota_{\mathrm{d}\gamma\partial t}\mathrm{d}L_r)\mathrm{d}t,$$

where $\gamma_t=\gamma(\cdot,t)$, the co-area formula gives

$$\begin{aligned}\int_{\mathcal{L}_r}\left(\sum\mathrm{sign}\phi\ \mathrm{sign}K(L_r)\right)\mathrm{d}L_r&=\int_0^t\int_{G_r(S)}\mathrm{sign}\phi\ \mathrm{sign}K(L_r)\gamma^*(\mathrm{d}L_r)\\ &\int_0^t\int_{G_r(S)}\mathrm{sign}\phi\ \mathrm{sign}K(L_r)\gamma_t^*(\iota_{\mathrm{d}\gamma\partial t}\mathrm{d}L_r)\mathrm{d}t.\end{aligned}$$

Of course, we can restrict the integration to regular points since $\gamma^*(\mathrm{d}L_r)$ vanishes at critical points. Taking derivatives we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0}\int_0^t\int_{G_r(S)}\mathrm{sign}\phi\ \mathrm{sign}K(L_r)\gamma_t^*(\iota_{\mathrm{d}\gamma\partial t}\mathrm{d}L_r)\mathrm{d}t$$
$$=\int_{G_r(S)}\mathrm{sign}\phi\ \mathrm{sign}K(L_r)\gamma_0^*(\iota_{\mathrm{d}\gamma\partial t}\mathrm{d}L_r).$$

Now take a (local) orthonormal frame $g; g_1,\ldots,g_n$ defined on $G_r(S)\times(-\epsilon,\epsilon)$ so that $g((p,l),t)=\varphi(p,t)$, $\gamma=\langle g,g_1,\ldots,g_r\rangle\cap M^n(k)$, and $g_n=\mathrm{n}$. Since we are restricted to regular points of γ, we can consider this frame as defined on an open set of $\mathcal{L}_r$. On the other hand, consider the curve $L_r(t)$ given by the parallel translation of L_r along the geodesic given by n. Let T be the tangent vector of $L_r(t)$. Then

$$\omega_i(T)=\langle\mathrm{d}g(T),g_i\rangle=\langle\frac{\mathrm{d}}{\mathrm{d}t}g(L_r(t)),g_i\rangle=0\quad r<i<n,$$
$$\omega_{ij}(T)=\langle\mathrm{d}g_i(T),g_j\rangle=\langle\frac{\mathrm{d}}{\mathrm{d}t}g_i(L_r(t)),g_j\rangle=0\qquad 1\le i\le r<j\le n,$$

since $L_r(t)$ are contained in $\langle g,g_1,\ldots,g_r,g_n\rangle$. Now by (5) we see

$$\mathrm{d}L_r=|\omega_n|\cdot\iota_T\mathrm{d}L_r,$$

and

$$\iota_{\mathrm{d}\gamma\partial t}\mathrm{d}L_r=|\omega_n(\mathrm{d}\gamma\partial t)|\iota_{\mathrm{T}}\mathrm{d}L_r+|\omega_n|\iota_{\mathrm{d}\gamma\partial t}\iota_T\mathrm{d}L_r.$$

But

$$\omega_n(\mathrm{d}\gamma\partial t)=\langle\mathrm{d}g\mathrm{d}\gamma\partial t,g_n\rangle=\langle\frac{\partial\varphi}{\partial t},\mathrm{n}\rangle=\phi,$$
$$\gamma_0^*(\omega_n)(v)=\langle\mathrm{d}g\mathrm{d}\gamma_0(v),g_n\rangle=0\quad\forall v\in T_{p,l}G_r(S).$$

Thus,

$$\gamma_0^*(\iota_{\mathrm{d}\gamma\partial t}\mathrm{d}L_r)=|\phi|\gamma_0^*(\iota_{\mathrm{T}}\mathrm{d}L_r)$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0}\int_{\mathcal{L}_r}\chi(L_r\cap Q_t)\mathrm{d}L_r=-\int_{G_r(S)}\phi\ \mathrm{sign}K(L_r)\gamma_0^*\iota_T\mathrm{d}L_r.$$

Finally, from [11, Lemma 1], $\gamma_0^*(\iota_T\mathrm{d}L_r)=|K(l)|\mathrm{d}l\mathrm{d}x$, and since the mean value of the normal curvatures is the mean curvature (cf. [11, Prop. 1]), we get

$$-\int_{G_r(S)}\phi\ K(L_r)\mathrm{d}l\mathrm{d}x=-\mathrm{vol}(G(r,n-1))\int_{S_0}\phi\ K_r(x)\mathrm{d}x.$$

□

3.2. **Immersion case.** We can extend these results to immersions if we restrict the parity of some dimensions. Thus, suppose $i: S\times I\longrightarrow M^n(k)$ is a smooth mapping such that for each $t\in I=(-\epsilon,\epsilon)$, the restriction $i_t=i(\cdot,t)$ is an immersion (not necessarily injective) of a compact hypersurface S. We will say that we have a *deformation* of the immersion i_0. In this setting we have variation formulas for $W_r(S)$ with odd r (cf. Definition 2.2). Using essentially the same ideas as in the embedded case one can prove the following proposition.

Proposition 5. *For a deformation of immersions i_t, and for almost every odd-dimensional r-plane L_r,*

$$\chi(i_s^{-1}(L_r)) - \chi(i_0^{-1}(L_r)) = -2\sum \mathrm{sign} K(L_r)$$

where the sum runs over the contact points $i_t(p)$ of L_r with the hypersurfaces S_t with $t \in (0,s)$, and $K(L_r)$ is the normal curvature of S_t in the direction $T_{i(p,t)}L_r$ with respect to the unit normal n *that makes $\langle \partial i/\partial t, \mathrm{n}\rangle > 0$.*

From this, a proof analogous to that of Theorem 4 gives the following:

Theorem 6. *For a deformation i_t of immersions of S, if r is odd,*

$$\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \int_{\mathcal{L}_r} \frac{\chi(i_t^{-1}(L_r))}{2} \mathrm{d}L_r = -\mathrm{vol}(G(r,n-1)) \int_{S_0} \langle \partial i/\partial t, \mathrm{n}\rangle K_r(x) \mathrm{d}x$$

where K_r is the mean curvature with respect to a unit normal n.

When the S_t are embedded this coincides with Theorem 4.

4. The Gauss-Bonnet theorem

Next we prove our main result, and we use it to obtain some known integral-geometric formulas, as well as to prove the Gauss-Bonnet theorem in constant curvature spaces.

To get Theorem 1, the idea is to shrink any hypersurface to almost collapse it to a point, and integrate the variation of the integrals during this deformation.

Theorem 1. *Let S be a hypersurface of $M^n(k)$ bounding a compact domain Q. Then,*

$$\int_S K(x)\mathrm{d}x = O_{n-1}\chi(Q) - k\frac{2(n-1)}{O_{n-2}} \int_{\mathcal{L}_{n-2}} \chi(L_{n-2} \cap Q)\mathrm{d}L_{n-2}.$$

If $i : S \to M^n(k)$ is a compact immersed hypersurface with odd n, then

$$\int_S K(x)\mathrm{d}x = \frac{O_{n-1}\chi(S)}{2} - k\frac{n-1}{O_{n-2}} \int_{\mathcal{L}_{n-2}} \chi(i^{-1}(L_{n-2}))\mathrm{d}L_{n-2}.$$

Proof. Let us restrict to the first case. Take some smooth deformation of Q to get Q' contained in a ball of arbitrarily small radius. By Theorem 4,

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathcal{L}_{n-2}} \chi(L_{n-2} \cap Q_t)\mathrm{d}L_r = -\frac{O_{n-2}}{2} \int_{\partial Q_t} \langle \partial\varphi/\partial t, \mathrm{n}\rangle K_{n-2}(x)\mathrm{d}x.$$

On the other hand, it was proven by Reilly in [8] that

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{S_t} K(x)\mathrm{d}x = k(n-1) \int_{S_t} \langle \partial\varphi/\partial t, \mathrm{n}\rangle K_{n-2}(x)\mathrm{d}x.$$

Therefore,

$$k\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathcal{L}_{n-2}} \chi(L_{n-2} \cap Q_t)\mathrm{d}L_{n-2} = -\frac{O_{n-2}}{2(n-1)} \frac{\mathrm{d}}{\mathrm{d}t} \int_{S_t} K(x)\mathrm{d}x,$$

and integrating with respect to t, we have

$$-k\int_{\mathcal{L}_{n-2}}(\chi(L_{n-2}\cap Q)-\chi(L_{n-2}\cap Q'))\mathrm{d}L_{n-2} = \frac{O_{n-2}}{2(n-1)}(\int_{\partial Q}K(x)\mathrm{d}x-\int_{\partial Q'}K(x)\mathrm{d}x).$$

Since the metric of a small ball in $M^n(k)$ is almost euclidean, and the curvature depends continuously on the metric, by the Gauss-Bonnet formula for hypersurfaces in euclidean space we have that the total curvature of $\partial Q'$ is as close to $O_{n-1}\chi(Q)$ as we want. On the other hand, $\chi(L_{n-2}\cap Q')\neq 0$ only for an arbitrarily small subset of $\mathcal{L}_{n-2}$. □

Moreover, Theorem 1 can be moved to lower order curvatures using the reproductive property (6).

Proposition 7. *If $Q\subset M^n(k)$ is a compact domain with smooth boundary S, then*

$$M_r(\partial Q)=n\left(W_{r+1}(Q)-k\frac{r}{n-r+1}W_{r-1}(Q)\right).$$

If S is a compact immersed hypersurface and r is odd, then

$$M_r(S)=n\left(W_{r+1}(S)-k\frac{r}{n-r+1}W_{r-1}(S)\right).$$

Proof. For almost every geodesic r-plane L_r, the intersection $Q\cap L_r$ has smooth boundary. By the previous theorem we have

$$M_{r-1}(S\cap L_r)=O_{r-1}\chi(L_r\cap Q)-\frac{r(r-1)}{2}W_{r-2}(Q\cap L_r).$$

Integrating with respect to L_r gives

$$\begin{aligned}(9)\qquad \int_{\mathcal{L}_r}M_{r-1}(S\cap L_r)\mathrm{d}L_r &= O_{r-1}\int_{\mathcal{L}_r}\chi(L_r\cap Q)\mathrm{d}L_r \\ -\frac{r(r-1)}{2}\int_{\mathcal{L}_r}W_{r-2}(Q\cap L_r)\mathrm{d}L_r &= O_{r-1}\int_{\mathcal{L}_r}\chi(L_r\cap Q)\mathrm{d}L_r \\ &\quad -\frac{(r-1)\cdot O_0}{O_{r-2}}\int_{\mathcal{L}_r}\int_{\mathcal{L}_{[r](r-2)}}\chi(L_{r-2}\cap Q)\mathrm{d}L_{[r](r-2)}\mathrm{d}L_r\end{aligned}$$

where $\mathcal{L}_{[r](r-2)}$ is the space of $(r-2)$-planes contained in L_r and $\mathrm{d}L_{[r](r-2)}$ is the corresponding measure. But one has (cf. [9, p. 207]) that

$$\mathrm{d}L_{[r](r-2)}\mathrm{d}L_r=\mathrm{d}L_{r[r-2]}\mathrm{d}L_{r-2}$$

where $\mathrm{d}L_{r[r-2]}$ is the natural measure in the space of r-planes containing L_{r-2}. Thus,

$$\int_{\mathcal{L}_r}\int_{\mathcal{L}_{[r](r-2)}}\chi(L_{r-2}\cap Q)\mathrm{d}L_{[r](r-2)}\mathrm{d}L_r = \frac{O_{n-r+1}O_{n-r}}{O_1O_0}\int_{\mathcal{L}_{r-2}}\chi(L_{r-2}\cap Q)\mathrm{d}L_{r-2}.$$

On the other hand, by the reproductive property of the mean curvature integrals (6),

$$\int_{\mathcal{L}_r} M_{r-1}(S \cap L_r)\mathrm{d}L_r = \frac{O_{n-2}\cdots O_{n-r}O_{n-r+1}}{O_{r-2}\cdots O_0 O_1} M_{r-1}(S).$$

Substituting the two latter equations in (9) one gets the desired formula. The proof in the immersed case is analogous. □

From here we can express W_r as a linear combination of several M_i.

Corollary 8. *If $Q \subset M^n(k)$ is a domain with C^2 boundary, then for $r = 2l$*

$$W_r(Q) = \frac{1}{n}\sum_{i=0}^{l-1} k^i \frac{(r-1)!!(n-r)!!}{(r-2i-1)!!(n-r+2i)!!} M_{r-2i-1}(\partial Q) + k^l \frac{(r-1)!!(n-r)!!}{n!!}\mathrm{vol}(Q), \tag{10}$$

and for $r = 2l + 1$,

$$W_r(Q) = \frac{1}{n}\sum_{i=0}^{l} k^i \frac{(r-1)!!(n-r)!!}{(r-2i-1)!!(n-r+2i)!!} M_{r-2i-1}(\partial Q). \tag{11}$$

Proof. Use the recurrence

$$W_{r+1}(Q) = \frac{1}{n} M_r(\partial Q) + k\frac{r}{n-r+1} W_{r-1}(Q)$$

and finish with

$$W_1(Q) = \frac{1}{n} M_0(\partial Q) \qquad W_0(Q) = V.$$

□

Remark. Formula (11) holds also for immersed hypersurfaces.

In the particular case $r = n$ we have the Gauss-Bonnet formulae (1) and (2) for hypersurfaces in constant curvature manifolds. Indeed, recall that $W_n(Q) = O_{n-1}\chi(Q)/n$, and $W_n(S) = O_{n-1}\chi(S)/2n$ when n is odd. In the remaining cases ($r < n$), these formulae were obtained by Santaló (cf. [9, p. 310]) in a completely different way. Note that the simple relationship between W_i and M_j shown in Proposition 7 remains hidden in Corollary 8.

Another consequence of Theorem 1 is the following fact, which was well known for $k = 0$, but is apparently new for $k < 0$.

Corollary 9. *If $k \le 0$ and $Q \subset Q' \subset M^n(k)$ are compact convex domains, then $M_i(Q) \le M_i(Q')$.*

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