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STABLE GEOMETRIC DIMENSION OF VECTOR BUNDLES OVER EVEN-DIMENSIONAL REAL PROJECTIVE SPACES

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ABSTRACT. In 1981, Davis, Gitler, and Mahowald determined the geometric dimension of stable vector bundles of order 2^e over RP^n if n is even and sufficiently large and $e \geq 75$. In this paper, we use the Bendersky-Davis computation of $v_1^{-1}\pi_*(SO(m))$ to show that the 1981 result extends to all $e \geq 5$ (still provided that n is sufficiently large). If $e \leq 4$, the result is often different due to anomalies in the formula for $v_1^{-1}\pi_*(SO(m))$ when $m \leq 8$, but we also determine the stable geometric dimension in these cases.

1. Statement of results

The geometric dimension $\mathrm{gd}(\theta)$ of a stable vector bundle θ over a space X is the smallest integer m such that θ is stably equivalent to an m-plane bundle. Equivalently, $\mathrm{gd}(\theta)$ is the smallest m such that the classifying map $X \xrightarrow{\theta} BO$ factors through $BO(m)$. The group $\widetilde{KO}(P^n)$ of equivalence classes of stable vector bundles over real projective space is a finite cyclic 2-group generated by the Hopf line bundle ξ_n. Many papers (e.g., [1], [22], [23], [24]) have been devoted to computing the geometric dimension of multiples $k\xi_n$ of the Hopf bundles, in part because certain cases are equivalent to determining whether P^n can be immersed in a certain Euclidean space (e.g., [10]).

In this paper, we prove the following theorem, which extends and completes a program initiated in [12]. It says that, for sufficiently large even n, the geometric dimension of a vector bundle over P^n depends only on its order in $\widetilde{KO}(P^n)$ and the mod 8 value of n.

Theorem 1.1. *Let* $\overline{n} = 2,\ 4,\ 6,$ *or* 8, *and* $e \geq 1$.

(1) *There is an integer* $\mathrm{sgd}(\overline{n}, e)$ *which equals the geometric dimension of all bundles of order* 2^e *in* $\widetilde{KO}(P^n)$ *for sufficiently large* n *satisfying* $n \equiv \overline{n}$ mod 8.
(2) *If* $e \geq 5$, *then*

$$\mathrm{sgd}(\overline{n}, e) = 2e + \delta(\overline{n}, e),$$

where $\delta(\overline{n}, e)$ *is defined by Table* 1.2.

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TABLE 1.2. Table of $\delta(\overline{n}, e)$

		e mod 4			
		0	1	2	3
	6, 8	0	2	2	1
$\overline{n}$	2, 4	0	0	−1	−2

(3) *For $e \le 4$, we have*

$$\operatorname{sgd}(\overline{n}, e) = \begin{cases} 5, & e = 1,\ \overline{n} = 2, 4, 6, 8, \\ 5, & e = 2,\ \overline{n} = 2, 4, \\ 6, & e = 3,\ \overline{n} = 2, 4, \\ 6, & e = 2,\ \overline{n} = 6, 8, \\ 7, & e = 3,\ \overline{n} = 6, 8, \\ 9, & e = 4,\ \overline{n} = 2, 4, 6, 8. \end{cases} \tag{1.3}$$

The notation $\operatorname{sgd}(\overline{n}, e)$ stands for "stable geometric dimension", which is stable as n increases within its mod 8 congruence class.

In [12], as corrected in [13], and [11] and [18], it was proved that $\operatorname{sgd}(\overline{n}, e) \ge 2e + \delta(\overline{n}, e)$ with equality if $e \ge 75$. As pointed out by the referee and described in the next two paragraphs, the elementary argument in the first paragraph of [12] that there is an sgd depending only on the mod 8 value of sufficiently large n seems to be incomplete.

For n even and k odd satisfying $n+8 < 2k-1$, there exists a map $P_{k+8}^{n+8} \xrightarrow{\phi} P_k^n$ which induces an isomorphism in $\widetilde{KO}(-)$. This map may be obtained as a compression of multiplication by 16, as we are in the stable range. Here P_k^n denotes the stunted projective space P^n/P^{k-1}. This map ϕ was first studied carefully in [21].

The argument in [12] concluded that the geometric dimension of vector bundles of order 2^e over P^n is a nonincreasing function of n for even $n \ge 4e+8$ in a fixed congruence class mod 8, and so must achieve a stable value. This argument seems to have made the tacit assumption that if $\overline{f} : P_k^n \to BO$ is a factorization of $f : P^n \to BO$, then $\operatorname{gd}(\overline{f}) = \operatorname{gd}(f)$. This is apparently not always necessarily true, for an obstruction to lifting $\overline{f}$ to $BO(m)$ might be in the indeterminacy for f. However, as discussed in the last paragraph of this section, all the obstructions used in [12] were of an elementary sort that are also nonzero for P^n. Thus the argument of [12], with the amendments in [13], [11], and [18], which said that, for sufficiently large even $n \equiv \overline{n} \mod 8$, the geometric dimension of any bundle of order 2^e over P^n is equal to or greater than $2e + \delta(\overline{n}, e)$, with equality if $e \ge 75$, is correct.

Stable maps $P^{n+8} \to P^n$ inducing a monomorphism in $KO^*(-)$ do not exist when n is odd, and so the situation for stable geometric dimension of bundles over odd-dimensional projective spaces is much more delicate, and will be discussed in a separate paper.

Our new approach makes heavy use of the computation of $v_1^{-1}\pi_*(SO(m))$ obtained in [2]. We begin by indicating the relationship between this computation and sgd.

Let $\nu(-)$ denote the exponent of 2 in an integer. For even n and $\nu(k) \ge n/2$, there are stable maps $\Sigma^{8k}P^n \to P^n$ inducing an isomorphism in $\widetilde{KO}(-)$. These can be obtained by combining James periodicity with the maps ϕ discussed above.

(See (1.6).) They are also called v_1-maps, e.g. in [19]. Bousfield ([6, p. 1251]) uses a v_1-map to define

$$v_1^{-1}\pi_i(Y;P^n) = \operatorname*{colim}_d [\Sigma^{8kd+i}P^n, Y]$$

for any space Y. When $i=0$, we will often write this as $v_1^{-1}[P^n,Y]$; note that this equals

$$\operatorname*{colim}_d [P^{n+8kd}_{1+8kd}, Y]. \tag{1.4}$$

When n is even, as it will be throughout this paper, $v_1^{-1}[P^n, BO] \approx \widetilde{KO}(P^n)$ is cyclic, as is $v_1^{-1}[P^n, BSO] \approx 2v_1^{-1}[P^n, BO]$. If h is a covariant functor on the homotopy category, and $i_{m*}: h(SO(m)) \to h(SO)$ is the stabilization, we define

$$\mathbf{s}h(SO(m)) = \operatorname{im}(i_{m*}) \approx h(SO(m))/\ker(i_{m*}).$$

This $\mathbf{s}$ denotes the "stable" part, under $SO(m) \to SO$. This is, of course, a different stabilization than the one involved in the definition of sgd. If G is a cyclic group, we abbreviate $\nu(|G|)$ as $\nu(G)$. The following elementary result is the key relationship between v_1-periodic homotopy groups and stable geometric dimension.

Proposition 1.5. *If $\nu(\mathbf{s}v_1^{-1}[P^n, BSO(m)]) \ge e$, then, for sufficiently large j, the geometric dimension of bundles of order 2^e over P^{n+8j} is equal to or less than m.*

Proof. Let $\nu(k) \ge n/2$, as above. Using (1.4) and the factorization

$$\begin{aligned} \Sigma^{8k(d+1)}P^n &\simeq P^{n+8k(d+1)}_{1+8k(d+1)} \xrightarrow{\phi} P^{n+8k(d+1)-8}_{1+8k(d+1)-8} \xrightarrow{\phi} \cdots \\ &\to P^{n+8kd+8}_{1+8kd+8} \xrightarrow{\phi} P^{n+8kd}_{1+8kd} \simeq \Sigma^{8kd}P^n, \end{aligned} \tag{1.6}$$

the hypothesis implies that for all sufficiently large j, $\nu(\mathbf{s}[P^{n+8j}_{1+8j}, BSO(m)]) \ge e$. This implies that the factorization of $2^{\varphi(n+8j)-e}\xi_{n+8j}$ through P^{n+8j}_{1+8j} lifts to $BSO(m)$. Here $\varphi(N)$ is the number of positive integers i satisfying $i \equiv 0,1,2,4$ mod 8 and $i \le N$, so that $2^{\varphi(N)}$ is the order of ξ_N in $\widetilde{KO}(P^N)$. Thus the bundle $2^{\varphi(n+8j)-e}\xi_{n+8j}$ on P^{n+8j} also lifts to $BSO(m)$. □

Although a nonlifting on P^{n+8j}_{1+8j} does not automatically imply a nonlifting on P^{n+8j}, and so the converse of 1.5 is not automatically true, we prove indirectly that the converse of 1.5 is true. This is done by calculating the explicit value of $\nu(\mathbf{s}v_1^{-1}[P^n, BSO(m)])$, and showing by other methods that whenever e is greater than this, the geometric dimension of bundles of order 2^e over P^{n+8j} is greater than m.

Next we describe how to compute $\mathbf{s}v_1^{-1}[P^n, BSO(m)]$. The situation when $n \equiv 6,8$ mod 8 is particularly simple. We will prove the following key result in Section 2.

Proposition 1.7. *If $n \equiv 6,8 \mod 8$ is sufficiently large, then*

$$\mathbf{s}v_1^{-1}[P^n, BSO(m)] \approx \mathbf{s}v_1^{-1}\pi_{-2}(SO(m)).$$

Note the simplification here—it essentially replaces the projective space by a sphere. The requirement that n be sufficiently large is not a problem for sgd, since sgd only cares about large values of n.

The groups $v_1^{-1}\pi_*(SO(m))$ were computed in [2], where the following result was proved.

Theorem 1.8. *If* $8i + d \geq 9$, *then*

$$\nu(\mathbf{s}v_1^{-1}\pi_{-2}(SO(8i+d))) = 4i + \begin{cases} -1, & d = -1, \\ 0, & d = 0,1,2,3, \\ 1, & d = 4,5, \\ 2, & d = 6. \end{cases}$$

The $\leq$-part of Theorem 1.1(2) when $\overline{n} = 6$ or 8 is an immediate consequence of 1.5, 1.7, and 1.8. Indeed, for $n \equiv 6, 8 \mod 8$ and $8i + d \geq 9$, the smallest d such that

$$\nu(\mathbf{s}v_1^{-1}[P^n, BSO(8i+d)]) \geq 4i + \langle 0,1,2,3\rangle$$

is $8i + \langle 0,4,6,7\rangle$.

Proof of Theorem 1.8. Because of the mammoth nature of [2], we guide the reader to the relevant results. Referring always to [2], the specific statements regarding $\nu(\mathbf{s}v_1^{-1}\pi_{-2}(SO(8i+d)))$ are in 1.2 for $d = \pm 1$, 3.10 for $d = 4 \pm 1$, and 3.13 for $d = 4 \pm 2$. Specific statements are not made for $d = 4$ or 8, but only with relation to the case $d - 1$. In 3.4(last case) (resp. 3.14(last case)), it is shown that the exponent when $d = 8$ (resp. $d = 4$) is 1 greater than when $d = 7$ (resp. $d = 3$).

There is a subtlety here for $SO(9)$ and $SO(10)$ which will be discussed at the beginning of Section 3. It is too technical to include in this introduction. □

When $\overline{n} = 2$ or 4, a similar program is followed, but we must define and compute a modified sort of v_1-periodic homotopy group. In Section 2, we will utilize the following definition and prove Theorem 1.10, which, with 1.5, implies the $\leq$-part of Theorem 1.1(2) when $\overline{n} = 2$ or 4 just as in the previous case.

Definition 1.9. Let $M^{n+1}(k) = S^n \cup_k e^{n+1}$ denote the usual Moore spectrum, and

$$N^{n+1}(k) = M^{n+1}(k) \cup_\eta e^{n+2} \cup_2 e^{n+3},$$

and define, for any space X and any integer i,

$$v_1^{-1}\pi_i'(X) = \operatorname*{colim}_{k,e}[N^{i+1+k2^{L_e}}(2^e), X].$$

The second part of this definition, analogous to the definition of $v_1^{-1}\pi_*(X)$ first given in [15], is made using v_1-maps $\Sigma^{2^{L_e}} N^i(2^e) \to N^i(2^e)$ and canonical maps $N^i(2^{e+1}) \to N^i(2^e)$, similarly to the situation for Moore spaces $M^i(2^e)$.

Theorem 1.10. (1) *If* $n \equiv 2, 4 \mod 8$ *is sufficiently large, then*

$$\mathbf{s}v_1^{-1}[P^n, BSO(m)] \approx \mathbf{s}v_1^{-1}\pi_{-2}'(SO(m)).$$

(2) *If* $8i + d \geq 9$, *then*

$$\nu(\mathbf{s}v_1^{-1}\pi_{-2}'(SO(8i+d))) = 4i + \begin{cases} 0, & d = 0,1, \\ 1, & d = 2, \\ 2, & d = 3, \\ 3, & d = 4,5,6,7. \end{cases}$$

The requirement that $e \geq 5$ in Theorem 1.1(2) is due to the condition $8i + d \geq 9$ in 1.8 and 1.10(2). In Section 3, we will prove the following result, which, with 1.5, 1.7, and 1.10(1), implies the $\leq$-part of (1.3).

Theorem 1.11. *For* $5 \le m \le 8$,

$$\nu(\mathbf{s}v_1^{-1}\pi_{-2}(SO(m))) = \begin{cases} 1, & m=5, \\ 2, & m=6, \\ 3, & m=7,8; \end{cases}$$

$$\nu(\mathbf{s}v_1^{-1}\pi'_{-2}(SO(m))) = \begin{cases} 2, & m=5, \\ 3, & m=6,7,8. \end{cases}$$

Also, $\mathbf{s}v_1^{-1}\pi_{-2}(SO(4))=0$ *and* $v_1^{-1}\pi'_{-2}(SO(4))=0$.

Remark 1.12. An alternative way of understanding why the results for $e \le 4$ sometimes differ from the pattern of those with $e > 4$ uses fiber-homotopy geometric dimension, as mentioned in [12, 1.2, 2.6]. For these bundles with small e, the fiber-homotopy geometric dimension can be strictly less than the (orthogonal) geometric dimension. As this viewpoint is only an alternative to the detailed argument of this paper, we shall not elaborate here.

Next we prove the $\ge$-part of 1.1. Adams proved in [1] that a vector bundle over RP^n which is not stably trivial and has $w_i = 0$ for $i \le 4$ must have geometric dimension ≥ 5. Thus $\operatorname{sgd}(\overline{n}, e) \ge 5$ for all even $\overline{n}$ if $e \ge 1$. He also proved in [1, Thm. 9] that the only bundles over P^n that could have geometric dimension 5 are those in $\mathbf{Z}/4 \subset \widetilde{KO}(P^n)$, i.e., those with $e \le 2$ in our notation. This establishes $\ge$ in the third case of (1.3).

The fourth and fifth cases of (1.3) obey the same formula as in Table 1.2, and so the argument of the last paragraph of this section for the $\ge$-part of 1.1(2) applies to them as well. Alternatively, Adams ([1]) proved the $\ge$-part of the fourth case of (1.3) by showing that if $n \equiv 6, 8 \bmod 8$, an element $\theta \in \widetilde{KO}(P^n)$ for which $\operatorname{gd}(\theta) \le 5$ must satisfy $2\theta = 0$. In [25], Lam and Randall extended Adams' representation-theoretic method to prove that if $n \equiv 6, 8 \bmod 8$ and $\operatorname{gd}(\theta) \le 6$, then $4\theta = 0$, establishing the $\ge$-part of the fifth case of (1.3).

In unpublished work performed in 1997-1998, Kee Lam, SiuPor Lam, and Randall proved the following result, using the same method as employed in [25]. We thank Kee Lam for pointing out this result, and allowing us to use it.

Proposition 1.13 ([20])**.** *If* $\operatorname{gd}(4k\xi_n) \le 8$, *then*

$$32k(k-1) \equiv 0 \mod |\widetilde{KO}(P^n)| \text{ or } 32(k-1)(k-2) \equiv 0 \mod |\widetilde{KO}(P^n)|.$$

This implies the $\ge$-part of the last case of (1.3), that a bundle of order 16 must have geometric dimension greater than 8.

The argument in [12, §2] seems to only establish the nonlifting on P_b^n, not on P^n. Again, we thank the referee for pointing out this flaw in our approach. However, in [11] and [18], all $\ge$-results asserted in 1.1(2), and the fourth and fifth cases of (1.3) were proved by other methods.

2. Proof of results about v_1-periodic homotopy groups

In this section we prove Proposition 1.7 and Theorem 1.10, which were shown in Section 1 to imply the $\le$-part of Theorem 1.1(2).

Let Φ denote the v_1-periodic spectrum functor described in [6, 7.2]. By [6, 7.2(i)], we have, if n is even,

$$\begin{aligned} v_1^{-1}[P^n, BSO(m)] &\approx [P^n, \Phi BSO(m)] \approx [P^n, \Phi SO(m)]_{-1} \\ &\approx v_1^{-1}\pi_{-1}(SO(m); P^n), \end{aligned} \tag{2.1}$$

or similarly with P^n replaced by another space with a v_1-map. We will use the four parts of (2.1) interchangeably.

Proof of Proposition 1.7. The proof utilizes the following result, which is part of [14, 4.2]. Here and throughout, $M^n(k) = S^{n-1} \cup_k e^n$ denotes a Moore spectrum.

Theorem 2.2 ([14])**.** *For $\epsilon = 0$ and 1, and L sufficiently large, there is a K_*-equivalence $M^{2^L}(2^{4k-\epsilon}) \to P^{2^L-2\epsilon}_{2^L+1-8k}$.*

We also note the following elementary result.

Proposition 2.3. *A K_*-equivalence $P^{n+8}_{b+8} \xrightarrow{\phi'} P^n_b$, with n even and b odd, induces an isomorphism*

$$v_1^{-1}\pi_*(Y; P^n_b) \xrightarrow{\phi'^*} v_1^{-1}\pi_*(Y; P^{n+8}_{b+8})$$

for any space Y.

Proof. A K_*-equivalence $\Sigma^{2^L} P^n_b \to P^n_b$ used in defining $v_1^{-1}\pi_*(Y; P^n_b)$ can be factored as

$$\Sigma^{2^L} P^n_b \to P^{n+8}_{b+8} \xrightarrow{\phi'} P^n_b;$$

thus ϕ'^* is injective. Similarly a K_*-equivalence $\Sigma^{2^L} P^{n+8}_{b+8} \to P^{n+8}_{b+8}$ used in defining $v_1^{-1}\pi_*(Y; P^{n+8}_{b+8})$ can be factored as

$$\Sigma^{2^L} P^{n+8}_{b+8} \xrightarrow{\Sigma^{2^L}\phi'} \Sigma^{2^L} P^n_b \to P^{n+8}_{b+8},$$

and so ϕ'^* is surjective. $\square$

Thus

$$v_1^{-1}\pi_*(Y; P^{8k-2\epsilon}) \approx v_1^{-1}\pi_*(Y; P^{-2\epsilon}_{1-8k}) \approx v_1^{-1}\pi_*(Y; M^0(2^{4k-\epsilon})). \tag{2.4}$$

Here we use that a K_*-equivalence induces an isomorphism in $[-, \Phi Y]$, since ΦY is K_*-local, and also use the fact ([19, 3.7]) that the maps of 2.2 asymptotically respect the v_1-maps of the two spaces.

With k sufficiently large, there is, by [15, 1.7], a natural split short exact sequence

$$0 \to v_1^{-1}\pi_{-1}(SO(m)) \to v_1^{-1}\pi_{-1}(SO(m); M^0(2^{4k-\epsilon})) \to v_1^{-1}\pi_{-2}(SO(m)) \to 0. \tag{2.5}$$

Recall that $\mathbf{s}v_1^{-1}\pi_i(Y; SO(m))$ equals the image of $v_1^{-1}\pi_i(Y; SO(m))$ in $v_1^{-1}\pi_i(Y; SO)$. Similarly to [5, 1.9], $v_1^{-1}\pi_{-1}(SO) = 0$. Thus (2.5) induces an isomorphism

$$\mathbf{s}v_1^{-1}\pi_{-1}(SO(m); M^0(2^{4k-\epsilon})) \to \mathbf{s}v_1^{-1}\pi_{-2}(SO(m)). \tag{2.6}$$

With (2.1) and (2.4), this yields the desired conclusion of Proposition 1.7. $\square$

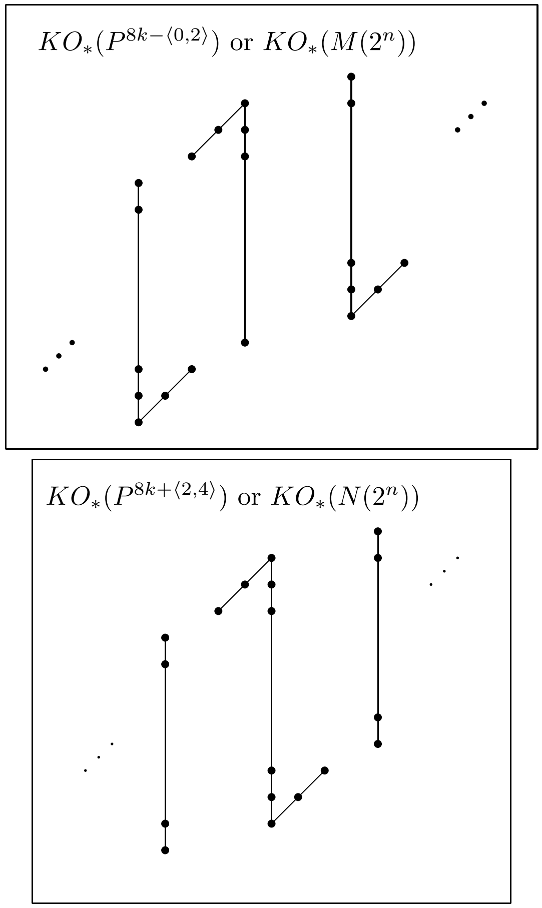

DIAGRAM 2.7.

Before we prove Theorem 1.10(1), we explain one way of seeing why the spectra $N^{n+1}(k)$ are necessary. This viewpoint for the relevance of $N^{n+1}(k)$ involves a comparison of charts of $KO_*(-)$ computed, for example, by the method of [12, p. 41] or [14, p. 133] as $v_1^{-1}ko_*(-)$. In Diagram 2.7, the top box is a chart of $KO_*(P^{8k-2\epsilon})$ with $\epsilon = 0$ or 1 and main groups of order $2^{4k-\epsilon}$, while the bottom box is $KO_*(P^{8k+2\delta})$ with $\delta = 1$ or 2 and larger (middle) groups of order $2^{4k+\delta+1}$. A chart for $KO_*(M(2^n))$ is given by the top box of Diagram 2.7 with main groups of order 2^n, while a chart for $KO_*(N(2^n))$ is given by the bottom box of the diagram with the larger groups of order 2^{n+2}. Here we have not listed a superscript for $M(-)$ or $N(-)$ since the effect of the superscript is just to translate the chart horizontally. These charts are not necessary for the proof; they merely form one way of understanding the need for resorting to $N^{n+1}(k)$. The charts for $M(2^n)$ match nicely with those of $P^{8k-\langle 0,2\rangle}$, but must be modified to those of $N(2^n)$ to match with $P^{8k+\langle 2,4\rangle}$.

Proposition 2.8. *For sufficiently large L, there exist K_*-equivalences*

$$\begin{array}{ccc} N^{2^{4k}L}(2^{4k}) & \xrightarrow{f_1} & P^2_{1-8k} \text{ and} \\ N^{2^{4k+1}L}(2^{4k+1}) & \xrightarrow{f_2} & P^4_{1-8k}. \end{array}$$

Proof. In [14, 4.2], a K_*-equivalence $M^0(2^{4k}) \xrightarrow{f} P^0_{1-8k}$ was constructed. Let $\mathcal{J} = v_1^{-1}J$ denote the periodic J-spectrum. The chart for $\mathcal{J}_*(P^2_{1-8k})$ in the range $-2 \le * \le 5$ is given in Diagram 2.9.

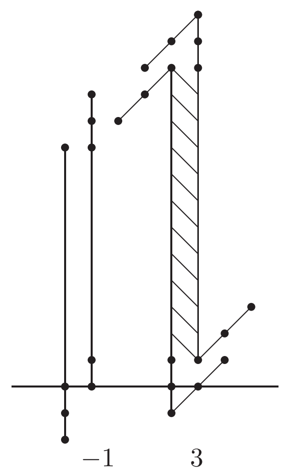

Diagram 2.9.

Let $M^1 = S^0 \cup_2 e^1$. Consider the composite

$$M^1 \xrightarrow{\tilde\eta} S^{-1} \xrightarrow{j} M^0(2^{4k}) \xrightarrow{f} P^0_{1-8k} \xrightarrow{i} P^2_{1-8k} \wedge \mathcal{J}, \tag{2.10}$$

with $\tilde\eta$ an extension of $\eta \in \pi_0(S^{-1})$. The cofiber of $j \circ \tilde\eta$ is $N^0(2^{4k})$. Note that $\mathcal{J}_*(P^{10}_{1-8k})$ has a chart like that of Diagram 2.9 with the same top and extending 4 units lower. The commutative diagram induced by the inclusion $P^2_{1-8k} \to P^{10}_{1-8k}$

$$\begin{array}{ccc} \mathcal{J}_{-1}(P^2_{1-8k}) & \xrightarrow{\tilde\eta^*} & [M^1, P^2_{1-8k} \wedge \mathcal{J}] \\ {\scriptstyle 2^4}\downarrow & & {\scriptstyle =}\downarrow \\ \mathcal{J}_{-1}(P^{10}_{1-8k}) & \xrightarrow{\tilde\eta^*} & [M^1, P^{10}_{1-8k} \wedge \mathcal{J}] \end{array}$$

implies that its top morphism is 0. Thus the composite (2.10) is trivial, and hence the extension $N^0(2^{4k}) \xrightarrow{\tilde f} P^2_{1-8k} \wedge \mathcal{J}$ of $i \circ f$ exists.

By [26], $P^2_{1-8k} \wedge \mathcal{J}$ is the telescope $v_1^{-1}P^2_{1-8k}$ over v_1-maps of $P^{2-2^{4k}L}_{1-8k-2^{4k}L}$ as $L \to \infty$. Thus the map $\tilde f$ factors through a map f_1 whose $(2^{4k}L)$-suspension is as in the statement of the proposition. This f_1 is a K_*-equivalence by the Five Lemma applied to

$$\begin{array}{ccccc} M^{2^{4k}L}(2^{4k}) & \longrightarrow & N^{2^{4k}L}(2^{4k}) & \longrightarrow & M^{2+2^{4k}L}(2) \\ \downarrow & & {\scriptstyle f_1}\downarrow & & \downarrow \\ P^0_{1-8k} & \longrightarrow & P^2_{1-8k} & \longrightarrow & M^2(2). \end{array}$$

In [14, 4.2], a K_*-equivalence $M^0(2^{4k+1}) \to P^0_{-8k-1}$ is constructed. As in the proof of the first part, this yields a K_*-equivalence $N^{2^{4k+1}L'}(2^{4k+1}) \xrightarrow{f_2'} P^2_{-8k-1}$. By [16, 3.1], there is a filtration-3 K_*-equivalence $P^2_{-8k-1} \xrightarrow{h_1} P^{-4}_{-8k-7}$. The filtration-4 K_*-equivalences ϕ mentioned in Section 1 yield a K_*-equivalence

$$P^{-4}_{-8k-7} \xrightarrow{h_2} P^{4-2^{4k+1}}_{1-8k-2^{4k+1}} \simeq \Sigma^{-2^{4k+1}} P^4_{1-8k}.$$

The 2^{4k+1}-fold suspension of $h_2 \circ h_1 \circ f_2'$ is our desired K_*-equivalence f_2. □

Thus for $\delta = 1, 2$, we have

$$\mathbf{s}v_1^{-1}[P^{8k+2\delta}, BSO(m)] \approx \mathbf{s}v_1^{-1}[N^0(2^{4k+\delta-1}), BSO(m)]). \tag{2.11}$$

Similarly to (2.5), for k sufficiently large, there is a split short exact sequence

(2.12)

$$0 \to v_1^{-1}\pi_{-1}(SO(m)) \to v_1^{-1}\pi_{-1}(SO(m); N^0(2^{4k+\delta-1})) \to v_1^{-1}\pi'_{-2}(SO(m)) \to 0,$$

which, similarly to (2.6), induces an isomorphism

$$\mathbf{s}v_1^{-1}\pi_{-1}(SO(m); N^0(2^{4k+\delta-1})) \approx \mathbf{s}v_1^{-1}\pi'_{-2}(SO(m)). \tag{2.13}$$

We will expand slightly upon the proof of (2.12) following Definition 2.14. Theorem 1.10(1) is an immediate consequence of (2.11), (2.1), and (2.13). □

We expand 1.9 to include another related spectrum.

Definition 2.14. Let $T^n = S^n \cup_\eta e^{n+2} \cup_2 e^{n+3}$.

The reason for the choice of names of the spectra T^n and $N^{n+1}(k)$ is "next letter of alphabet." The space T^n has appeared in other guises as variations on a sphere. In [7, 10.7], it was called C, and its K_*-localization was shown in [7, 10.6] to be the only other K_*-local spectrum to have the same $K_*(-)$-groups as S_K. The spectrum *bsp*, which was used in many papers of the second and third authors (e.g. [12, p. 41], [14, p. 127], [16, p. 41]) involving the J-spectrum, equals $T^0 \wedge bo$.

The split short exact sequence (2.12) is induced from cofiber sequences

$$T^{2^eL-2} \to N^{2^eL-1}(2^{4k+\delta-1}) \to S^{2^eL-1} \xrightarrow{2^{4k+\delta-1}}, \tag{2.15}$$

where k is large enough so that $SO(m)$ has H-space exponent $2^{4k+\delta-1}$, and e and L are large. This induces a split short exact sequence, for k sufficiently large,

$$0 \to \pi_{2^eL-1}(SO(m)) \to [N^{2^eL-1}(2^{4k+\delta-1}), SO(m)] \to [T^{2^eL-2}, SO(m)] \to 0,$$

and, similarly to [15, 2.6], there is a direct system of these split short exact sequences with respect to increasing 2^eL, the direct limit of which is (2.12). See also (2.17), which suggests that (2.12) can be obtained by applying $[-, \Phi SO(m)]$ to (2.15).

We will use the following spectral sequence to compute $v_1^{-1}\pi'_*(X)$, which was defined in 1.9.

Proposition 2.16. *If X is an odd sphere or simply-connected compact Lie group, there is a spectral sequence converging to $v_1^{-1}\pi'_*(X)$ with E_2-term*

$$\widetilde{E}_2^{s,t} \approx \operatorname{Ext}^s_{\mathcal{A}}(QK^1(X; \mathbf{Z}_2^\wedge)/\operatorname{im}(\psi^2), K^1(S^t; \mathbf{Z}_2^\wedge)).$$

Here $Q(-)$ denotes the indecomposables, and $\mathcal{A}$ the category of 2-adic stable Adams modules. Note that the E_2-term is isomorphic to that of the spectral sequence of [4] converging to $v_1^{-1}\pi_*(X)$. We will call it $\widetilde{E}_2(X)$ when it is the initial term of the spectral sequence converging to $v_1^{-1}\pi'_*(X)$.

Proof of Proposition 2.16. We begin by mimicking the proof of [6, 7.5]. With D denoting S-duality, there are isomorphisms

$$\begin{aligned} v_1^{-1}\pi'_*(X) &\approx \lim_k [N^1(2^k), \Phi(X)]_* \approx \operatorname*{colim}_k \pi_*(DN^1(2^k) \wedge \Phi(X)) \\ &\approx \pi_*(D(T^0) \wedge M^0(\mathbf{Z}/2^\infty) \wedge \Phi(X)) \\ &\approx [T^0, \Phi(X)]_*. \end{aligned} \tag{2.17}$$

Here we have used that applying $\wedge M^0(\mathbf{Z}/2^\infty)$ to the torsion spectra $S^0 \cup_2 e^1$ and $\Phi(X)$ leaves them unchanged.

By [6, 10.4],[1] there is a spectral sequence converging to $[T^0, \Phi(X)]_*$ with

$$E_2^{s,t} \approx \operatorname{Ext}^s_{\mathcal{A}}(K^*(\Phi(X); \mathbf{Z}_2^\wedge), K^*(T^t; \mathbf{Z}_2^\wedge)).$$

Bousfield ([8, 7.5,9.1]) has proved that for X as in this theorem, there is an isomorphism in $\mathcal{A}$

$$K^i(\Phi X) \approx \begin{cases} 0, & i = 0, \\ QK^1(X)/\operatorname{im}(\psi^2), & i = 1. \end{cases}$$

This is the 2-primary analogue of [6, 9.2]. The proof of the proposition is completed by noting that there is an isomorphism in $\mathcal{A}$, $K^*(S^t; \mathbf{Z}_2^\wedge) \approx K^*(T^t; \mathbf{Z}_2^\wedge)$. Indeed, the morphism in $K^*(-; \mathbf{Z}_2^\wedge)$ induced by the inclusion $S^t \hookrightarrow T^t$ is a monomorphism onto multiples of 2. $\square$

The next result gives the primed v_1-periodic homotopy groups of odd spheres. The conclusion is that the d_3-differentials between the eta-towers in the spectral sequence for $v_1^{-1}\pi'_*(S^{2n+1})$ are the opposite of the way they are in the spectral sequence for $v_1^{-1}\pi_*(S^{2n+1})$. Here n can be even or odd.

Theorem 2.18. *The spectral sequence of* 2.16 *converging to* $v_1^{-1}\pi'_*(S^{2n+1})$ *is as pictured in Diagram* 2.19. *Here* 8 *means* $\mathbf{Z}/8$, *while* C *is* $\mathbf{Z}/2^{\min(n,4+\nu(k+1))}$. *We do not picture many portions of eta-towers which are involved in nontrivial* d_3*'s. The dotted differential when* $n \equiv 1, 2$ *is nonzero unless* $\nu(k+1) + 4 > n$, *in which case* $d_3 = 0$ *and the extension in* $v_1^{-1}\pi'_{2n+8k+7}(S^{2n+1})$ *occurs. The action of* h_1 *on the generator of* C *in position* $(2n + 8k + 8, 1)$ *is nontrivial, but the class which it hits depends upon whether or not* $\nu(k + 1) + 4 > n$.

Proof. We begin by using a J-homology approach to determine $v_1^{-1}\pi'_*(S^{2n+1})$. These methods were developed in [26], and described quite thoroughly in [9, §3, §4, §5]. We assume that the reader has some familiarity with those methods. For a reader who has no such expertise, an alternate proof is given after this one.

Let $U^i = S^{i-3} \cup_2 e^{i-2} \cup_\eta e^i$. Note that T^i and U^{-i} are S-dual. The map $\Omega^{2n+1}S^{2n+1} \to QP^{2n}$ of [9, 3.3] induces an isomorphism in $v_1^{-1}\pi'_*(-)$. Thus

$$\begin{aligned} v_1^{-1}\pi'_i(S^{2n+1}) &\approx v_1^{-1}[T^i, \Sigma^\infty\Sigma^{2n+1}P^{2n}] \\ &\approx v_1^{-1}\pi_i(U^{2n+1} \wedge P^{2n}) \\ &\approx v_1^{-1}J_i(U^{2n+1} \wedge P^{2n}). \end{aligned}$$

Arguing similarly to [9, p. 1011], there is a short exact sequence of A_1-modules

$$0 \to H^*U^5 \to A_1//A_0 \to \mathbf{Z}_2 \to 0,$$

[1] Although many results of [6] only work when p is odd, this one also works when $p = 2$.

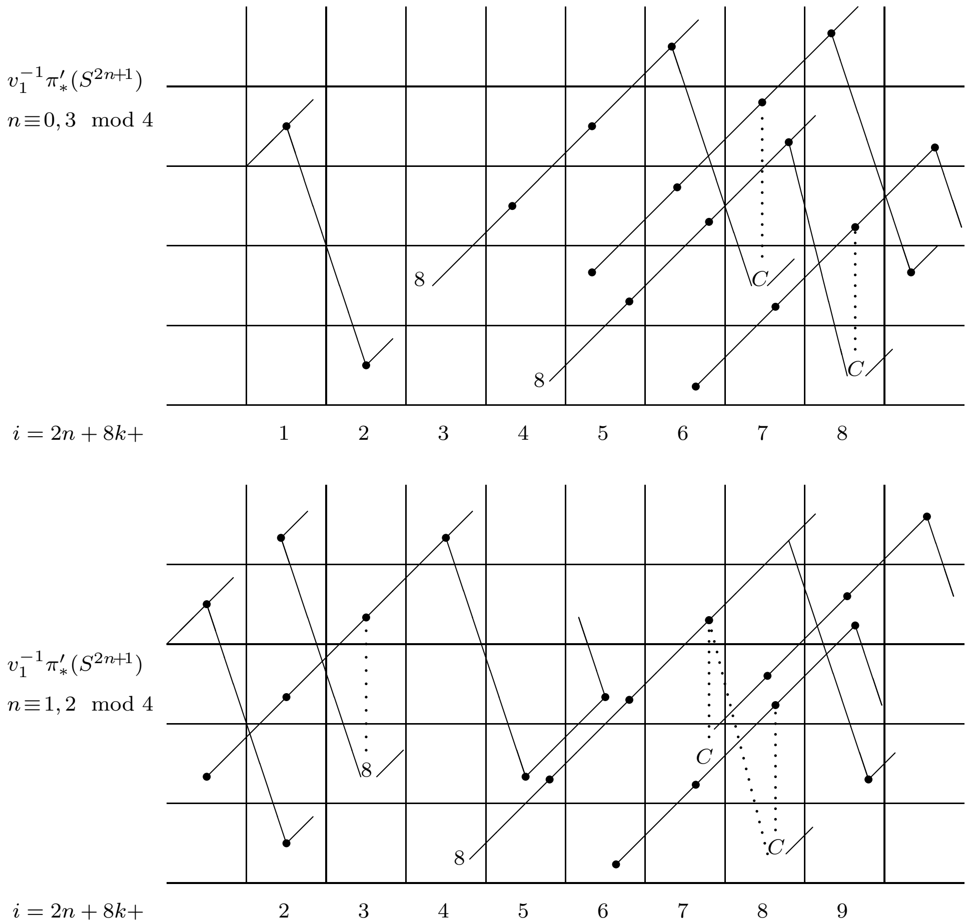

Diagram 2.19.

and hence isomorphisms

$$\mathrm{Ext}_{A_1}^{s,t}(H^*(U^5 \wedge X), \mathbf{Z}_2) \xrightarrow{\approx} \mathrm{Ext}_{A_1}^{s+1,t}(H^*X, \mathbf{Z}_2)$$

for $s > 1$ and any space X. Here A_1 is the subalgebra of the mod 2 Steenrod algebra generated by Sq^1 and Sq^2, which is relevant since the E_2-term of the Adams spectral sequence converging to $\pi_*(X \wedge bo)$ is $\mathrm{Ext}_{A_1}(H^*X, \mathbf{Z}_2)$. Inverting v_1, we conclude that

$$v_1^{-1}\pi_i(U^0 \wedge X \wedge bo) \approx v_1^{-1}\pi_{i+4}(X \wedge bo).$$

Thus, since $v_1^{-1}J$ is the fiber of $\psi^3 - 1 : v_1^{-1}bo \to v_1^{-1}bo$, the E_2-chart for

$$v_1^{-1}J_*(U^0 \wedge X)$$

is like that of $v_1^{-1}J_*(X)$ pushed back by 4, but the differentials between adjacent towers (corresponding to $\psi^3 - 1$) of $U^0 \wedge X$ are the same as those in X in the same dimension.

We obtain charts for $v_1^{-1}\pi'_*(S^{2n+1})$ as in Diagram 2.20. Here the differential between the second pair of towers in either box is $d_{\nu(4k+4)}$. The height (number of dots) of the towers in the left box is n. The height of the smaller (left) towers in the right box is $n-1$, while that of the larger towers is $n+1$.

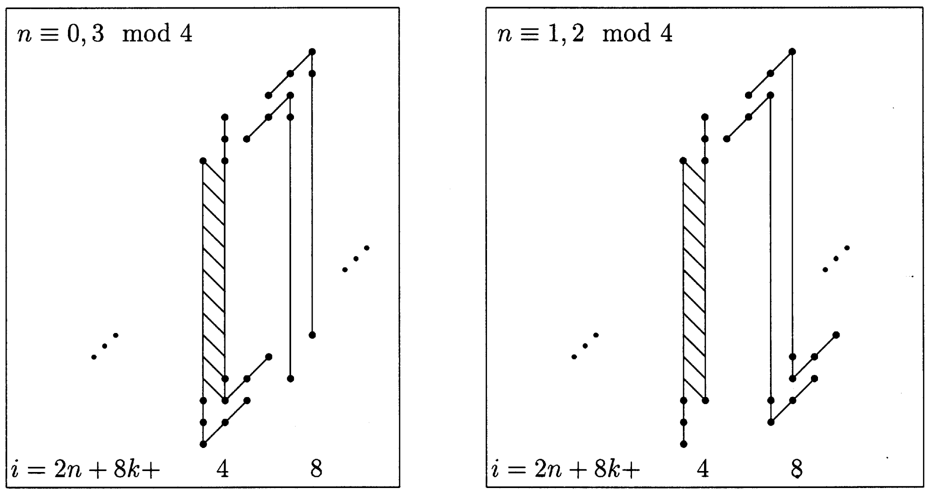

Diagram 2.20. $v_1^{-1}\pi'_i(S^{2n+1})$

By Proposition 2.16, the E_2-term in 2.19 is the same as that for $v_1^{-1}\pi_*(S^{2n+1})$ as given, for example, in [3, p. 488]. The d_3-differentials in 2.19 are the only way of inserting them to yield groups which agree with $v_1^{-1}\pi'_*(S^{2n+1})$ as given in 2.20. $\square$

Now we easily deduce the following key result. Note that our approach to the v_1-periodic homotopy groups of $SO(m)$ is via the spectral sequence for its simply-connected cover $\mathrm{Spin}(m)$, which satisfies $v_1^{-1}\pi_*(\mathrm{Spin}(m)) = v_1^{-1}\pi_*(SO(m))$.

Proposition 2.21. *The spectral sequence of* (2.16) *for* $v_1^{-1}\pi'_*(\mathrm{Spin}(m))$ *has* $\widetilde{E}_2$ *as given in* [2, 1.3, 3.4, 3.7, 3.12, 3.14] *but with* d_3*-differentials between eta-towers, the opposite of those given there.*

Proof. As described in [2, §5], an eta-tower is a family of $\mathbf{Z}_2$ elements related by $h_1 : \widetilde{E}_2^{s,t} \to \widetilde{E}_2^{s+1,t+2}$, beginning in filtration 1, 2, or 3. If x is an eta-tower, then there is an eta-tower with the same name appearing every 4 (horizontal) dimensions, and either all those whose dimension is congruent mod 8 to that of x support d_3-differentials hitting the others, or else all those congruent mod 8 to x are hit by d_3-differentials from the others. In [2], it was shown that all these d_3's in $\mathrm{Spin}(m)$ could be determined by naturality from those in the odd spheres. Since we saw in 2.18 that the d_3's in the spectral sequence for $v_1^{-1}\pi'_*(S^{2n+1})$ are opposite of those in the spectral sequence for $v_1^{-1}\pi_*(S^{2n+1})$, we can deduce that the same happens for $\mathrm{Spin}(m)$. $\square$

Now we give an alternate proof of Theorem 2.18 which does not involve J-chart technology. This argument can probably be used to prove Proposition 2.21 concurrently with 2.18.

Alternate proof of Theorem 2.18. Let t be odd, and let $M_t(\eta) = S^t \cup_\eta e^{t+2}$. The obvious cofibration induces a short exact sequence in $\mathcal{A}$

$$0 \to K^*(S^{t+2}) \to K^*(M_t(\eta)) \to K^*(S^t) \to 0,$$

and hence, for any $\mathcal{A}$-object N, an exact sequence

$$\operatorname{Ext}_{\mathcal{A}}^{s,t+2}(N) \to \operatorname{Ext}_{\mathcal{A}}^{s}(N, K^*M_t(\eta)) \to \operatorname{Ext}_{\mathcal{A}}^{s,t}(N) \xrightarrow{h_1} \operatorname{Ext}_{\mathcal{A}}^{s+1,t+2}(N).$$

If $\operatorname{Ext}_{\mathcal{A}}^{s,t}(N) \xrightarrow{h_1} \operatorname{Ext}_{\mathcal{A}}^{s+1,t+2}(N)$ is an isomorphism for $s > 2$, as is the case when $N = K^*(\Phi S^{2n+1})$ or $K^*(\Phi\,\mathrm{Spin}(n))$, then we deduce $\operatorname{Ext}_{\mathcal{A}}^{s}(N, K^*M_t(\eta)) = 0$ for $s > 2$.

Now we consider the cofiber sequence

$$S^{t+2} \xrightarrow{\alpha} M_t(\eta) \xrightarrow{i} T^t \xrightarrow{q} S^{t+3},$$

where α is a coextension of 2, i the inclusion, and q the collapse. It induces a short exact sequence in $\mathcal{A}$

$$0 \to K^*(T^t) \to K^*(M_t(\eta)) \to K^*(S^{t+2}) \to 0$$

and hence an exact sequence

$$\operatorname{Ext}_{\mathcal{A}}^{s}(N, K^*T^t) \to \operatorname{Ext}_{\mathcal{A}}^{s}(N, K^*M_t(\eta)) \to \operatorname{Ext}_{\mathcal{A}}^{s,t+2}(N) \xrightarrow{\delta} \operatorname{Ext}_{\mathcal{A}}^{s+1}(N, K^*T^t).$$

With N as above, since $\operatorname{Ext}_{\mathcal{A}}^{s}(N, K^*M_t(\eta)) = 0$ for $s > 2$, δ induces an isomorphism of eta-towers. Using that $K^*T^t \approx K^*S^t$ in $\mathcal{A}$ and h_1 is an isomorphism, we replace δ by the composite

$$\operatorname{Ext}_{\mathcal{A}}^{s,t+2}(N) \xrightarrow{\delta} \operatorname{Ext}_{\mathcal{A}}^{s+1,t}(N) \xrightarrow{h_1^{-1}} \operatorname{Ext}_{A}^{s,t-2}(N). \tag{2.22}$$

As noted in the proof of 2.21, names of eta-towers have period 4 in t. Thus (2.22) maps a set of eta towers to a set of eta towers with the same names. It can be shown, using the Small Complex of [2, §11], that (2.22) sends an eta tower to the one with the same name, at least if $N = K^*(\Phi S^{2n+1})$. Since the proof is somewhat involved and this is only an alternate proof, it is omitted here.

Finally we note that (2.22) commutes with d_3-differentials since it is induced by the map q. Thus the d_3-differential on eta towers in $\widetilde{E}_2^{s,t-2}$ of the spectral sequence for $v_1^{-1}\pi'_*(S^{2n+1})$ agree with d_3 on $E_2^{s,t+2}$ of the spectral sequence for $v_1^{-1}\pi_*(S^{2n+1})$, since they correspond under (2.22). The conclusion is that E_2 is the same for the two spectral sequences, but d_3 on eta-towers is opposite. For d_3 on the 1-line, more delicate analysis is required, which will be the focus of the next proposition. □

We close this section by proving Theorem 1.10(2), the determination of the required $\mathbf{s}v_1^{-1}\pi'_{-2}(SO(m))$. This is accomplished using the spectral sequence of 2.16, and follows from the following result.

Proposition 2.23. *Let $m \geq 11$. In the spectral sequence of* 2.16 *with* $X = \mathrm{Spin}(m)$,

- *If $4a \leq m \leq 4a+3$, then* $\nu(\mathbf{s}\widetilde{E}_2^{1,-1}) = 2a + \begin{cases} 0, & m \equiv 0,1,2 \mod 4, \\ 1, & m \equiv 3 \mod 4. \end{cases}$
- *There is a nontrivial extension* $(\cdot 2)$ *from* $\mathbf{s}\widetilde{E}_\infty^{1,-1}$ *to* $\widetilde{E}_\infty^{3,1}$ *if* $m \geq 11$.
- $d_3 : \widetilde{E}_3^{1,-1} \to \widetilde{E}_3^{4,1}$ *is nonzero if and only if* $m \equiv 0, \pm 1 \mod 8$.

The situation when $m = 9$ and 10 is slightly different, and will be described in Section 3.

Proof. We use the observation after 2.16 that $\widetilde{E}_2$ is isomorphic to the E_2-term of the spectral sequence converging to $v_1^{-1}\pi_*(X)$. From [2, 3.1], $\nu(\mathbf{s}E_2^{1,-1}(\mathrm{Spin}(2n{+}1))) = n$, while from [2, 3.3]

$$\nu(\mathbf{s}E_2^{1,-1}(\mathrm{Spin}(2n))) = \begin{cases} n-1, & \text{if } n \text{ odd,} \\ n, & \text{if } n \text{ even.} \end{cases}$$

The extension is into the class which would be labeled 1 in diagrams such as [2, 1.3]. This is the class corresponding to the element $x_1 \in K^1(\mathrm{Spin}(m))$. See, e.g., [2, 5.9, 5.19]. This class in position $(-3,4)$ is not depicted in [2, 1.3] because its entire eta-tower supports a nonzero d_3-differential, and such eta-towers are often omitted from the diagrams. But by 2.21, in the $\widetilde{E}_r(-)$-spectral sequence, the eta-tower labeled 1 passing through $(-3,4)$ is hit by d_3, and only in filtration ≥ 4. Thus this class x_1 lives in $\widetilde{E}_\infty^{3,1}(\mathrm{Spin}(m))$ as a candidate for an extension.

To see that this extension actually takes place, we note that in Spin, all the unstable classes are gone. We have

$$\widetilde{E}_\infty^{s,t}(\mathrm{Spin}) \approx \begin{cases} \mathbf{Z}/2^\infty, & s=1,\ t-s\equiv 2 \mod 4, \\ \mathbf{Z}_2, & 1\leq s\leq 3,\ t-2s\equiv 3 \mod 8, \\ 0, & \text{otherwise.} \end{cases}$$

The extensions must be nontrivial by a form of Bott periodicity. A similar situation is discussed in [5, 1.19]. The definition of $\mathbf{s}$ now implies that for all $m \geq 5$ the order of $\mathbf{s}v_1^{-1}\pi_{-2}(\mathrm{Spin}(m))$ must be twice as large as that of $\mathbf{s}\widetilde{E}_\infty^{1,-1}(\mathrm{Spin}(m))$. For $m \geq 11$, this will be achieved by an extension in the spectral sequence. For $5 \leq m \leq 10$, the way in which it is achieved will be discussed in Section 3.

In [2, §7], the authors determined d_3 from the 1-line of the spectral sequence converging to $v_1^{-1}\pi_*(\mathrm{Spin}(m))$ by noting that $d_3(x) = y$ iff $d_3(h_1x) = h_1y$. Since d_3 from the 2-line had already been computed, it sufficed to compute h_1x. Methods for computing h_1 from the 1-line were developed in [2, 7.2, 7.9]. The same methods work here in the spectral sequence converging to $v_1^{-1}\pi'_*(\mathrm{Spin}(m))$. The biggest difference is that, as shown in 2.21, the d_3's from the 2-line here are opposite from the way they were in [2].

We focus here on the cases where we must show $d_3 = 0$ on $\mathbf{s}\widetilde{E}_2^{1,-1}(\mathrm{Spin}(m))$. The nonzero d_3's when $m \equiv 0$ or ± 1 are implied by the $\geq$-part of 1.1(2), the proof of which (from [11]) was described in the last paragraph of Section 1. This says that if $\overline{n} = 2$ or 4 and $e \equiv 0,1 \bmod 4$, then $\mathrm{sgd}(\overline{n}, e) \geq 2e$, and hence by 1.5 and 1.10(1)

$$\nu(\mathbf{s}v_1^{-1}\pi'_{-2}(SO(2e-1)) \approx \nu(\mathbf{s}v_1^{-1}[P^n, BSO(2e-1)]) \leq e-1.$$

By the first two parts of this proposition (2.23), the only way for this group to be this small is by the claimed d_3-differential. Alternatively, the method used below to prove $d_3 = 0$ can also be used to obtain these nonzero d_3's.

We begin with the spectral sequence for $v_1^{-1}\pi'_*(\mathrm{Spin}(8i{+}3))$. The E_2-term equals that of [2, Diagram 3.7]. In Diagram 2.24, we present the relevant portion, with the d_3-differentials which apply to $v_1^{-1}\pi'_*(\mathrm{Spin}(8i+3))$.

The dual group $(\widetilde{E}_2^{2,1})^\#$ has basis $\{D, x_{4i-1}\} \cup B_C[2i, 4i]$, where

$$B_C[2i,4i] = \{x_j : 2i \leq j \leq 4i \text{ and } j - 2^{\nu(j)+1} < 2i\}.$$

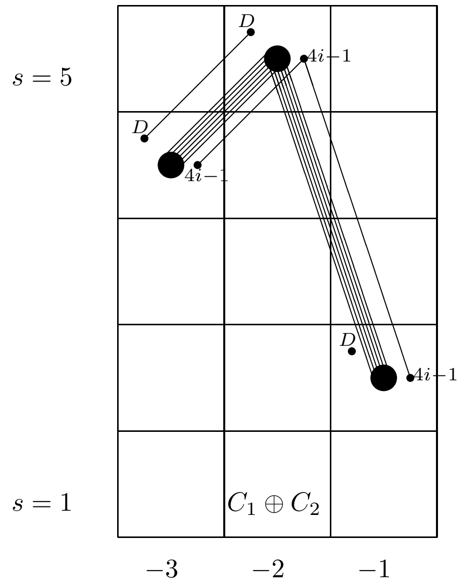

DIAGRAM 2.24. Part of the spectral sequence for $v_1^{-1}\pi'_*(\mathrm{Spin}(8i+3))$.

The set $B_C[2i,4i]$ has $[\log_2(16i/3)]+\delta_{\alpha(i),1}$ elements and is represented by the big $\bullet$ in Diagram 2.24. We use the same names for elements of the dual basis. By [2, 3.7] and Proposition 2.21, all basis elements of $\widetilde{E}_2^{2,1}$ except D support nonzero d_3 in the spectral sequence for $v_1^{-1}\pi'_*(\mathrm{Spin}(8i+3))$. By [2, 7.9], D is a summand of $h_1(g_1)$ in the case at hand; we will see why this is true in the next paragraph.

In order to show that $d_3(g_1)=0$, we must show that the basis elements of $\widetilde{E}_2^{2,1}$ other than D are not summands of $h_1(g_1)$. We adopt the dual point of view as explained in the proof of [2, 7.9]. In the notation of that proof, we are in the first case considered there—$4\ell+3=8k-1$ with $\nu >> n$. Since n which we have been using in this paper to denote the dimension of a projective space is not relevant to this proposition, we are free here to use n as it was used in the proof of [2, 7.9], namely $8i+3=2n+1$ so $n=4i+1$. The four relations described there which yield $(\widetilde{E}_2^{1,8k-1})^\#$ are

$$A_1 2^n\xi_1,\ A_2 2^n\xi_1 - 2^{n+1}\Delta,\ A_3 2^n\xi_1 - 2^n\Delta,\ u2^n\xi_1 + 2^\nu\Delta \tag{2.25}$$

with u odd.[2] In fact, A_1 is even by the discussion following [2, 8.1], and A_2 is even by [2, 3.2]. Hence in the $\mathbf{Z}/2^n \oplus \mathbf{Z}/2^n$ group presented by (2.25), it is only the last relation whose division by 2 lowers the order of the first (ξ_1) summand.

The fourth relation in (2.25) is due to $(\psi^3-3^{4k-1})(\Delta)$, the third to $\psi^2(\Delta)$, and the first two to ψ^2 and ψ^3-3^{4k-1} acting on various x_j. It was observed in the proof of [2, 7.9] that dividing the fourth relation by 2 corresponds to modding $(\widetilde{E}_2^{1,8k-1})^\#$ by $h_1^\#(D)$. Modding $(\widetilde{E}_2^{1,8k-1})^\#$ by $h_1^\#(b)$ for other elements b in the basis of $(\widetilde{E}_2^{2,8k+1})^\#$ corresponds to dividing other relations $\psi^2(\Delta)$, $\psi^2(x)$, or $(\psi^3-3^{4k-1})(x)$

[2]We use Δ to denote elements of $K^1(\mathrm{Spin}(8i+3))$ instead of the D that was used in [2] to avoid confusion with the element D of $(E_2^{2,8k-1})^\#$. Also note that k of [2, 7.9] is 0 here.

by 2. Since it is only dividing the fourth relation by 2 that lowers the order of the first summand, we deduce that the first component of $h_1^\#(\alpha_0 D + \sum \alpha_i x_i)$ in $(\widetilde{E}_2^{1,8k-1})^\#$ equals α_0 times the element of order 2, or dually that $h_1(g_1) = D$. This implies $d_3(g_1) = 0$ since $d_3(D) = 0$.

We prove now that $d_3 = 0 : \widetilde{E}_2^{1,-1}(\mathrm{Spin}(8i+2)) \to \widetilde{E}_2^{4,1}(\mathrm{Spin}(8i+2))$. By [2, 3.3], $\widetilde{E}_2^{1,-1}(\mathrm{Spin}(8i+1)) \to \widetilde{E}_2^{1,-1}(\mathrm{Spin}(8i+2))$ is bijective. By the proof of [2, 3.11], $\widetilde{E}_2^{4,1}(\mathrm{Spin}(8i+2)) \to \widetilde{E}_2^{4,1}(\mathrm{Spin}(8i+3))$ is injective.[3] By [2, 3.1], $\widetilde{E}_2^{1,-1}(\mathrm{Spin}(8i+1)) \approx \mathbf{Z}/2^{4i} \oplus \mathbf{Z}/2^{4i}$ and $\widetilde{E}_2^{1,-1}(\mathrm{Spin}(8i+3)) \approx \mathbf{Z}/2^{4i+1} \oplus \mathbf{Z}/2^{4i+1}$. Let $x \in \widetilde{E}_2^{1,-1}(\mathrm{Spin}(8i+2))$. Then $i_*(x) = 2y \in \widetilde{E}_2^{1,-1}(\mathrm{Spin}(8i+3))$. Hence $i_*(d_3(x))$ is divisible by 2, and hence is 0, since it lies in a $\mathbf{Z}_2$-vector space. The injectivity of i_* on $\widetilde{E}_2^{4,1}$ implies that $d_3(x) = 0$.

Next we consider $\mathrm{Spin}(8i+4)$. From [2, 6.1], we see that $\widetilde{E}_2^{4,1}(\mathrm{Spin}(8i+4))$ has basis dual to

$$\{x_{4i-1}, D_+\} \cup B_C[2i, 4i] \cup \{(D_+ - D_-)_s, (D_+ - D_-)_u\},$$

where the two classes $(D_+ - D_-)$ map nontrivially to $\widetilde{E}_2^{4,1}(S^{8i+3})$. By Diagram 2.19, d_3 acts injectively on $\widetilde{E}_2^{4,1}(S^{8i+3})$, and hence it does also on the classes $(D_+ - D_-)$. The element D_+ also supports a nonzero d_3 from $\widetilde{E}_3^{4,1}(\mathrm{Spin}(8i+4))$. This is true because of 2.21 and the fact that in [2, 3.7], the element D in position $(8k-3, 4)$ did not support a nonzero d_3 in the spectral sequence for $v_1^{-1}\pi_*(\mathrm{Spin}(8i+3))$. Thus the only elements that $d_3(g_1)$ might hit are dual to x_{4i-1} or $B_C[2i, 4i]$. By the argument used above in the case of $\mathrm{Spin}(8i+3)$, $h_1^\#$ does not send the corresponding elements of $\widetilde{E}_2^{2,1}(\mathrm{Spin}(8i+4))^\#$ to the element of order 2 in $\widetilde{E}_2^{1,-1}(\mathrm{Spin}(8i+4))^\#$ because dividing the corresponding relations by 2 will not lower the order of the first (ξ_1) summand. Thus $d_3(g_1) = 0$ on the stable summand of $\widetilde{E}_2^{1,-1}(\mathrm{Spin}(8i+4))$. That the same is true in $\mathrm{Spin}(8i+5)$ and $\mathrm{Spin}(8i+6)$ follows by naturality, since

$$\widetilde{E}_2^{1,-1}(\mathrm{Spin}(8i+4)) \to \widetilde{E}_2^{1,-1}(\mathrm{Spin}(8i+5)) \to \widetilde{E}_2^{1,-1}(\mathrm{Spin}(8i+6))$$

send the first summand bijectively. □

3. Proof of results for $SO(m)$ when $m \leq 10$

In this section, we prove Theorem 1.11, which we showed in Section 1 implies the $\leq$-part of Theorem 1.1(3). We will also pay some attention to Spin(9) and Spin(10), since, although their results fit into the pattern of the large $\mathrm{Spin}(m)$, the derivation of the result is somewhat unusual.

We begin by completing the proof of Theorem 1.8 when $m = 9$ and 10.

Proof of Theorem 1.8 *when* $m = 9$ *and* 10. The inclusions $SO(9) \to SO(10)$ and $\mathrm{Spin}(9) \to \mathrm{Spin}(10)$ induce isomorphisms in $v_1^{-1}\pi_{-2}(-)$ and $E_2^{1,-1}(-)$, referring to the spectral sequence used in [2]. There are isomorphisms

$$v_1^{-1}\pi_{-2}(SO(9)) \approx E_2^{1,-1}(\mathrm{Spin}(9)) \oplus \mathbf{Z}/2 \oplus \mathbf{Z}/2$$

and

$$E_2^{1,-1}(\mathrm{Spin}(9)) \approx \mathbf{Z}/8 \oplus \mathbf{Z}/32.$$

The Pontryagin dual of $E_2^{1,-1}(\mathrm{Spin}(9))$ was computed from $\widetilde{KO}(\mathrm{Spin}(9))$ in [4, 4.21] to have generators $g_1 = \xi_1 + 2uD_4$ and $g_2 = D_4$ of order 8 and 32, respectively.

[3] The proof there deals with $E_2^{4,8k+5}$ but also applies to $E_2^{4,8k+1}$.

Here u is odd. For $n > 4$, $E_2^{1,-1}(\mathrm{Spin}(2n+1))^{\#} \approx \mathbf{Z}/2^n \oplus \mathbf{Z}/2^n$, with generators ξ_1 and D_n. The morphism

$$E_2^{1,-1}(\mathrm{Spin}(2n+1))^{\#} \to E_2^{1,-1}(\mathrm{Spin}(2n-1))^{\#}$$

induced by inclusion sends ξ_1 to ξ_1 and $D_n \to 2D_{n-1}$. Thus, for n large,

$$E_2^{1,-1}(\mathrm{Spin}(2n+1))^{\#} \to E_2^{1,-1}(\mathrm{Spin}(9))^{\#}$$

sends ξ_1 to $g_1 - 2ug_2$, and so the dual morphism

$$E_2^{1,-1}(\mathrm{Spin}(9)) \to \mathbf{s}E_2^{1,-1}(\mathrm{Spin}(2n+1))$$

sends $\hat{g}_1$ to an element of order 8, and $\hat{g}_2$ to an element of order 16. Hence

$$\mathbf{s}v_1^{-1}\pi_{-2}(SO(9)) \approx \mathbf{Z}/16,$$

even though it is not the ξ_1-summand, but rather the D-summand, which yields the element of maximal order. □

Now we return to the proof of Theorem 1.11. By [2, 3.19] and the above proof of 1.8, for $m = 5$, 6, 7, 8, 9, 10, and 11, $E_2^{1,-1}(\mathrm{Spin}(m)) \approx \widetilde{E}_2^{1,-1}(\mathrm{Spin}(m))$ is given by

$$\begin{aligned}(3.1)\quad \mathbf{Z}_{16} \xrightarrow{\approx} \mathbf{Z}_{16} \xrightarrow{2g_1} \mathbf{Z}_8 \oplus \mathbf{Z}_8 \xrightarrow{i_7} \mathbf{Z}_8 \oplus \mathbf{Z}_8 \oplus \mathbf{Z}_8 &\xrightarrow{j_8} \mathbf{Z}_8 \oplus \mathbf{Z}_{32} \\ &\xrightarrow{\approx} \mathbf{Z}_8 \oplus \mathbf{Z}_{32} \xrightarrow{j_{10}} \mathbf{Z}_{32} \oplus \mathbf{Z}_{32}\end{aligned}$$

with i_7 inclusion into the first two summands, $j_8 i_7 = \begin{pmatrix} 1 & 4 \\ 0 & 8 \end{pmatrix}$, and $j_{10} = \begin{pmatrix} 4 & 2 \\ 0 & 2 \end{pmatrix}$. Thus, $\mathbf{s}E_2^{1,-1}(\mathrm{Spin}(m)) = E_2^{1,-1}(\mathrm{Spin}(m))/\ker(i_{m*})$ is given, for $5 \le m \le 11$, by

$$(3.2)\qquad \mathbf{Z}_4 \xrightarrow{\approx} \mathbf{Z}_4 \hookrightarrow \mathbf{Z}_8 \xrightarrow{\approx} \mathbf{Z}_8 \hookrightarrow \mathbf{Z}_{16} \xrightarrow{\approx} \mathbf{Z}_{16} \hookrightarrow \mathbf{Z}_{32}$$

with the $\mathbf{Z}_{16}$'s generated by g_2. Note that $4g$ in $E_2^{1,-1}(\mathrm{Spin}(5))$ is 0 in $\mathbf{s}E_2^{1,-1}(\mathrm{Spin}(5))$.

In the spectral sequence converging to $v_1^{-1}\pi_*(\mathrm{Spin}(m))$ when $m = 11$, the extension $(\cdot 2)$ from $\mathbf{s}E_\infty^{1,-1}$ to $E_\infty^{3,1}$ is trivial by [2, 3.8]. Hence, by the definition of $\mathbf{s}$, the extensions must also be trivial for $m < 11$. By [2, 1.4] and naturality, $d_3 = 0 : \mathbf{s}E_3^{1,-1}(\mathrm{Spin}(m)) \to E_4^{4,1}(\mathrm{Spin}(m))$ for $7 \le m \le 10$. When $m = 6$, $d_3 = 0$ by [2, 3.11] since there is nothing for d_3 to hit. When $m = 5$, this d_3 is nonzero, as can be seen by comparison with [17, 1.7], using that $\mathrm{Spin}(5) = Sp(2)$. Thus $\mathbf{s}v_1^{-1}\pi_{-2}(\mathrm{Spin}(m))$ is obtained from (3.2) by replacing the first $\mathbf{Z}_4$ by $\mathbf{Z}_2$, and so we obtain the first part of Theorem 1.11.

Since $\mathrm{Spin}(4) \approx S^3 \times S^3$, we deduce $4v_1^{-1}\pi_*(\mathrm{Spin}(4)) = 0$, and so $v_1^{-1}\pi_{-2}(\mathrm{Spin}(4)) \to v_1^{-1}\pi_{-2}(\mathrm{Spin}(5))$ cannot hit an element which stabilizes nontrivially, since 4 times such an element is nonzero in $v_1^{-1}\pi_{-2}(\mathrm{Spin}(5))$.

In the proof of 2.23, it was shown that in the spectral sequence $\widetilde{E}_r(\mathrm{Spin}(m))$, whose $\widetilde{E}_2^{1,-1}$ is given in (3.1), the order of $\mathbf{s}v_1^{-1}\pi_{-2}(\mathrm{Spin}(m))$ must be twice as large as that of $\mathbf{s}\widetilde{E}_\infty^{1,-1}(\mathrm{Spin}(m))$. This is obtained on the 1-line if $m = 5$, 6, 9, or 10, and by an extension to the 3-line otherwise.

We will show that d_3 from $\mathbf{s}\widetilde{E}_3^{1,-1}$ is injective when $m = 5$, 7, 8, and 9, but is 0 when $m = 6$ and 10. It is then immediate that $\mathbf{s}v_1^{-1}\pi'_{-2}(\mathrm{Spin}(m))$ is as in the second part of 1.11. Note that the generators of these homotopy groups, as compared with the $\widetilde{E}_2$-groups of (3.1), are, for $m = 5$, 6, 7, 8, 9, 10, and 11, given by $2g_1$, g_1, $2g_1$, $2g_1$, $2g_2$, g_2, and g_1.

First we show that d_3 is injective on $\mathbf{s}\widetilde{E}_3^{1,-1}(\mathrm{Spin}(m))$ when $m = 7$ and 9. This implies $d_3 \neq 0$ when $m = 8$, too. By the argument after [2, 7.2], $h_1 : \widetilde{E}_2^{1,-1}(\mathrm{Spin}(m)) \to \widetilde{E}_2^{2,1}(\mathrm{Spin}(m))$ is injective. Since d_3 on eta-towers of $\widetilde{E}_3(\mathrm{Spin}(m))$ is opposite to that on $E_3(\mathrm{Spin}(m))$, we deduce from [2, 1.3] that d_3 acts injectively on $\widetilde{E}^{2,1}(\mathrm{Spin}(m))$. Naturality of h_1 now implies that d_3 acts injectively on generators of $\widetilde{E}_3^{1,-1}(\mathrm{Spin}(m))$. Here $m = 7$ or 9.

Similarly to the proof in the previous section for $\mathrm{Spin}(8i+2)$ with $i > 1$, we deduce that $d_3 = 0$ from $\widetilde{E}_3^{1,-1}(\mathrm{Spin}(10))$. Indeed, $\widetilde{E}_2^{4,1}(\mathrm{Spin}(10)) \to \widetilde{E}_3^{4,1}(\mathrm{Spin}(11))$ is injective, but $\widetilde{E}_2^{1,-1}(\mathrm{Spin}(10)) \to \widetilde{E}_2^{1,-1}(\mathrm{Spin}(11))$ maps onto elements divisible by 2.

The groups $v_1^{-1}\pi'_*(\mathrm{Spin}(5)) = v_1^{-1}\pi'_*(Sp(2))$ can be obtained similarly to the J-chart determination of $v_1^{-1}\pi_*(Sp(2))$ in [17]. To obtain $v_1^{-1}\pi'_*(Sp(2))$, [17, Fig. 2.1] should be shifted by 4 dimensions, and d_1-differentials inserted from the new $8k+2$ to $8k+1$. But these differentials are not needed for our purposes. Since $v_1^{-1}\pi'_{-2}(S^3) = 0$ and $v_1^{-1}\pi'_{-2}(S^7) = \mathbf{Z}/8$, the exact sequence of the fibration $S^3 \to Sp(2) \to S^7$ implies that $v_1^{-1}\pi'_{-2}(Sp(2))$ is at most $\mathbf{Z}/8$. Thus the $\mathbf{Z}/16$ in $\widetilde{E}_2^{1,-1}(\mathrm{Spin}(5))$ must support a nonzero d_3. Finally d_3 is 0 on $\widetilde{E}_3^{1,-1}(\mathrm{Spin}(6))$ since its image in $\mathrm{Spin}(7)$ consists of multiples of 2, but the target classes $\widetilde{E}_3^{4,1}$ map injectively from $\mathrm{Spin}(6)$ to $\mathrm{Spin}(7)$ by [2, 6.1].

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