ON THE COHEN-MACAUnty PROPERTY OF MULTIPLICATIVE INVARIANTS

MARTIN LORENZ

Abstract. We investigate the Cohen-Macaulay property for rings of invariants under multiplicative actions of a finite group \( G \). By definition, these are \( G \)-actions on Laurent polynomial algebras \( k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] \) that stabilize the multiplicative group consisting of all monomials in the variables \( x_i \). For the most part, we concentrate on the case where the base ring \( k \) is \( \mathbb{Z} \). Our main result states that if \( G \) acts non-trivially and the invariant ring \( \mathbb{Z}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]^G \) is Cohen-Macaulay, then the abelianized isotropy groups \( G_m^\text{ab} \) of all monomials \( m \) are generated by the bireflections in \( G_m \) and at least one \( G_m^\text{ab} \) is non-trivial. As an application, we prove the multiplicative version of Kemper’s 3-copies conjecture.

Introduction

This article is a sequel to [LPk]. Unlike in [LPk], however, our focus will be specifically on multiplicative invariants. In detail, let \( L \cong \mathbb{Z}^n \) denote a lattice on which a finite group \( G \) acts by automorphisms. The \( G \)-action on \( L \) extends uniquely to an action by \( k \)-algebra automorphisms on the group algebra \( k[L] \cong k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] \) over any commutative base ring \( k \). We are interested in the question of when the subalgebra \( k[L]^G \) consisting of all \( G \)-invariant elements of \( k[L] \) has the Cohen-Macaulay property. The reader is assumed to have some familiarity with Cohen-Macaulay rings; a good reference on this subject is [BH].

It is a standard fact that \( k[L] \) is Cohen-Macaulay precisely if \( k \) is. On the other hand, while \( k[L]^G \) can only be Cohen-Macaulay when \( k \) is so, the latter condition is far from sufficient, and rather stringent additional conditions on the action of \( G \) on \( L \) are required to ensure that \( k[L]^G \) is Cohen-Macaulay. Remarkably, the question of whether or not \( k[L]^G \) is Cohen-Macaulay, for any given base ring \( k \), depends only on the rational isomorphism class of the lattice \( L \), that is, the isomorphism class of \( L \otimes \mathbb{Q} \) as \( \mathbb{Q}[G] \)-module; see Proposition 3.4 below. This is in striking contrast with most other ring theoretic properties of \( k[L]^G \) (e.g., regularity, structure of the class group) which tend to be sensitive to the \( \mathbb{Z} \)-type of \( L \). For an overview, see [L1].

We will largely concentrate on the case where the base ring \( k \) is \( \mathbb{Z} \). This is justified in part by the fact that if \( \mathbb{Z}[L]^G \) is Cohen-Macaulay, then likewise is \( k[L]^G \).
Corollary. of Kemper’s 3-copies conjecture: 

for any Cohen-Macaulay base ring \( k \) (Lemma 3.2). Assuming \( \mathbb{Z}[L]^G \) to be Cohen-Macaulay, we aim to derive group theoretical consequences for the isotropy groups \( G_m = \{ g \in G \mid g(m) = m \} \) with \( m \in L \). An element \( g \in G \) will be called a \( k \)-reflection on \( L \) if the sublattice \( [g, L] = \{ g(m) - m \mid m \in L \} \) of \( L \) has rank at most \( k \) or, equivalently, if the \( g \)-fixed points of the \( \mathbb{Q} \)-space \( L \otimes \mathbb{Z} \mathbb{Q} \) have codimension at most \( k \). As usual, \( k \)-reflections with \( k = 1 \) and \( k = 2 \) will be referred to as reflections and bireflections. For any subgroup \( H \subseteq G \), we let \( H^{(2)} \) denote the subgroup generated by the elements of \( H \) that act as bireflections on \( L \). Our main result now reads as follows.

**Theorem.** Assume that \( \mathbb{Z}[L]^G \) is Cohen-Macaulay. Then \( G_m/G_m^{(2)} \) is a perfect group (i.e., equal to its commutator subgroup) for all \( m \in L \). If \( G \) acts non-trivially on \( L \), then some \( G_m \) is non-perfect.

It would be interesting to determine if the conclusion of the theorem can be strengthened to the effect that all isotropy groups \( G_m \) are in fact generated by bireflections on \( L \). I do not know if, for the latter to occur, it is sufficient that \( G \) is generated by bireflections. The corresponding fact for reflection groups is known to be true: if \( G \) is generated by reflections on \( L \) (or, equivalently, on \( L \otimes \mathbb{Z} \mathbb{Q} \)), then so are all isotropy groups \( G_m \); see [Sh Theorem 1.5] or [Bou Exercise 8(a) on p. 139].

There is essentially a complete classification of finite linear groups generated by bireflections. In arbitrary characteristic, this is due to Guralnick and Saxl [Gns]; for the case of characteristic zero, see Huffman and Wales [HuW]. Bireflection groups have been of interest in connection with the problem of determining all finite linear groups whose algebra of polynomial invariants is a complete intersection. Specifically, suppose that \( G \leq \text{GL}(V) \) for some finite-dimensional vector space \( V \) and let \( \mathcal{O}(V) = \mathfrak{S}(V^*) \) denote the algebra of polynomial functions on \( V \). It was shown by Kac and Watanabe [KW] and independently by Gordeev [G] that if the algebra \( \mathcal{O}(V)^G \) of all \( G \)-invariant polynomial functions is a complete intersection, then \( G \) is generated by bireflections on \( V \). The classification of all groups \( G \) so that \( \mathcal{O}(V)^G \) is a complete intersection has been achieved by Gordeev [G2] and by Nakajima [N].

The last assertion of the above Theorem implies in particular that if \( \mathbb{Z}[L]^G \) is Cohen-Macaulay and \( G \) acts non-trivially on \( L \), then some element of \( G \) acts as a non-trivial bireflection on \( L \). Hence we obtain the following multiplicative version of Kemper’s 3-copies conjecture:

**Corollary.** If \( G \) acts non-trivially on \( L \) and \( r \geq 3 \), then \( \mathbb{Z}[L^\text{Br}]^G \) is not Cohen-Macaulay.

The 3-copies conjecture was formulated by Kemper [K1] Vermutung 3.12 in the context of polynomial invariants. Using the above notation, the original conjecture states that if \( 1 \neq G \leq \text{GL}(V) \) and the characteristic of the base field of \( V \) divides the order of \( G \) (”modular case”), then the invariant algebra \( \mathcal{O}(V^\text{Br})^G \) will not be Cohen-Macaulay for any \( r \geq 3 \). This is still open. The main factors contributing to our success in the multiplicative case are the following:

- Multiplicative actions are permutation actions: \( G \) permutes the k-basis of \( k[L] \) consisting of all “monomials”, corresponding to the elements of the lattice \( L \). Consequently, the cohomology \( H^*(G, k[L]) \) is simply the direct sum of the various \( H^*(G_m, k) \) with \( m \) running over a transversal for the \( G \)-orbits in \( L \).
• Up to conjugacy, there are only finitely many finite subgroups of $GL_n(\mathbb{Z})$, and these groups are explicitly known for small $n$. A crucial observation for our purposes is the following: if $G$ is a non-trivial finite perfect subgroup of $GL_n(\mathbb{Z})$ such that no $1 \neq g \in G$ has eigenvalue 1, then $G$ is isomorphic to the binary icosahedral group and $n \geq 8$; see Lemma 2.3 below.

A brief outline of the contents of the this article is as follows. The short preliminary Section 1 is devoted to general actions of a finite group $G$ on a commutative ring $R$. This material relies rather heavily on [LPk]. We liberate a technical result from [LPk] from any a priori hypotheses on the characteristic; the new version (Proposition 1.4) states that if $R$ and $R^G$ are both Cohen-Macaulay and $H^i(G, R) = 0$ for $0 < i < k$, then $H^k(G, R)$ is detected by $k + 1$-reflections. Section 2 then specializes to the case of multiplicative actions. We assemble the main tools required for the proof of the Theorem, which is presented in Section 3. The article concludes with a brief discussion of possible avenues for further investigation and some examples.

1. Finite group actions on rings

1.1. In this section, $R$ will be a commutative ring on which a finite group $G$ acts by ring automorphisms $r \mapsto g(r)$ ($r \in R, g \in G$). The subring of $G$-invariant elements of $R$ will be denoted by $R^G$.

1.2. Generalized reflections. Following [GK], we will say an element $g \in G$ acts as a $k$-reflection on $R$ if

$$I_G(\mathfrak{p}) = \{ g \in G \mid g(r) - r \in \mathfrak{p} \forall r \in R \}$$

of some prime ideal $\mathfrak{p} \in \text{Spec } R$ with height $\mathfrak{p} \leq k$. The cases $k = 1$ and $k = 2$ will be referred to as reflections and bireflections, respectively. Define the ideal $I_R(g)$ of $R$ by

$$I_R(g) = \sum_{r \in R} (g(r) - r)R.$$

Evidently, $\mathfrak{p} \supseteq I_R(g)$ is equivalent to $g \in I_G(\mathfrak{p})$. Thus:

$g$ is a $k$-reflection on $R$ if and only if height $I_R(g) \leq k$.

For each subgroup $H \leq G$, we put

$$I_R(H) = \sum_{g \in H} I_R(g).$$

It suffices to let $g$ run over a set of generators of the group $H$ in this sum.

1.3. A height estimate. The cohomology $H^*(G, R) = \bigoplus_{n \geq 0} H^n(G, R)$ has a canonical $R^G$-module structure: for each $r \in R^G$, the map $\rho: R \to R$, $s \mapsto rs$, is $G$-equivariant and hence it induces a map on cohomology $\rho_*: H^*(G, R) \to H^*(G, R)$. The element $r$ acts on $H^*(G, R)$ via $\rho_*$. Let $\text{res}^G_H: H^*(G, R) \to H^*(H, R)$ denote the restriction map.

The following lemma extends [LPk] Proposition 1.4.

Lemma. For any $x \in H^*(G, R)$,

$$\text{height ann}_{R^G}(x) \geq \inf \{ \text{height } I_R(H) \mid H \leq G, \text{res}^G_H(x) \neq 0 \}.$$
Proof. Put $\mathfrak{X} = \{ \mathcal{H} \leq \mathcal{G} \mid \text{res}^G_{\mathcal{H}}(x) = 0 \}$. For each $\mathcal{H} \leq \mathcal{G}$, let $R^G_{\mathcal{H}}$ denote the image of the relative trace map $R^G \to R^G$, $r \mapsto \sum g(r)$, where $g$ runs over a transversal for the cosets $g\mathcal{H}$ of $\mathcal{H}$ in $\mathcal{G}$. By [LPK] Lemma 1.3,

$$R^G_{\mathcal{H}} \subseteq \text{ann}_{R^G}(x) \quad \text{for all } \mathcal{H} \in \mathfrak{X}.$$

To prove the lemma, we may assume that $\text{ann}_{R^G}(x)$ is a proper ideal of $R^G$; for, otherwise height $\text{ann}_{R^G}(x) = \infty$. Choose a prime ideal $\mathfrak{p}$ of $R^G$ with $\mathfrak{p} \supseteq \text{ann}_{R^G}(x)$ and height $\mathfrak{p} = \text{height} \text{ann}_{R^G}(x)$. If $\mathfrak{P}$ is a prime of $R$ that lies over $\mathfrak{p}$, then

$$R^G_{\mathcal{H}} \subseteq \mathfrak{P} \quad \text{for all } \mathcal{H} \in \mathfrak{X}$$

and height $\mathfrak{P} = \text{height} \mathfrak{p}$. By [LPK] Lemma 1.1], the above inclusion implies that

$$[I_G(\mathfrak{P}) : I_H(\mathfrak{P})] \in \mathfrak{P} \quad \text{for all } \mathcal{H} \in \mathfrak{X}.$$

Put $p = \text{char} R/\mathfrak{P}$ and let $\mathcal{P} \leq I_G(\mathfrak{P})$ be a Sylow $p$-subgroup of $I_G(\mathfrak{P})$ (so $\mathcal{P} = 1$ if $p = 0$). Then $I_R(\mathcal{P}) \subseteq \mathfrak{P}$ and $[I_G(\mathfrak{P}) : \mathcal{P}] \notin \mathfrak{P}$. Hence, $\mathcal{P} \notin \mathfrak{X}$ and height $I_R(\mathcal{P}) \leq \text{height} \mathfrak{P} = \text{height} \text{ann}_{R^G}(x)$. This proves the lemma.

We remark that the lemma and its proof carry over verbatim to the more general situation where $H^*(\mathcal{G}, R)$ is replaced by $H^*(\mathcal{G}, M)$, where $M$ is some module over the skew group ring of $\mathcal{G}$ over $R$; cf. [LPK]. However, we will not be concerned with this generalization here.

1.4. A necessary condition. In this section, we assume that $R$ is noetherian as an $R^G$-module. This assumption is satisfied whenever $R$ is an affine algebra over some noetherian subring $\mathbb{k} \subseteq R^G$; see [Bou2] Théorème 2 on p. 33). Put

$$(1.1) \quad \mathfrak{X}_k = \{ \mathcal{H} \leq \mathcal{G} \mid \text{height } I_R(\mathcal{H}) \leq k \}.$$

Note that each $\mathcal{H} \in \mathfrak{X}_k$ consists of $k$-reflections on $R$. The following proposition is a characteristic-free version of [LPK] Proposition 4.1.

Proposition. Assume that $R$ and $R^G$ are Cohen-Macaulay. If $H^i(\mathcal{G}, R) = 0$ for $0 < i < k$, then the restriction map

$$\text{res}^G_{\mathfrak{X}_{k+1}} : H^k(\mathcal{G}, R) \to \prod_{\mathcal{H} \in \mathfrak{X}_{k+1}} H^k(\mathcal{H}, R)$$

is injective.

Proof. We may assume that $H^k(\mathcal{G}, R) \neq 0$. Let $x \in H^k(\mathcal{G}, R)$ be non-zero and put $\mathfrak{a} = \text{ann}_{R^G}(x)$. By [LPK] Proposition 3.3], depth $\mathfrak{a} \leq k + 1$. Since $R^G$ is Cohen-Macaulay, depth $\mathfrak{a} = \text{height} \mathfrak{a}$. Thus, Lemma 1.3 implies that $k + 1 \geq \text{height } I_R(\mathcal{H})$ for some $\mathcal{H} \leq \mathcal{G}$ with $\text{res}^G_{\mathcal{H}}(x) \neq 0$. The Proposition follows.

Note that the vanishing hypothesis on $H^i(\mathcal{G}, R)$ is vacuous for $k = 1$. Thus, $H^1(\mathcal{G}, R)$ is detected by bireflections whenever $R$ and $R^G$ are both Cohen-Macaulay.

2. Multiplicative actions

2.1. For the remainder of this article, $L$ will denote a lattice on which the finite group $\mathcal{G}$ acts by automorphisms $m \mapsto g(m)$ ($m \in L, g \in \mathcal{G}$). The group algebra of $L$ over some commutative base ring $\mathbb{k}$ will be denoted by $\mathbb{k}[L]$. We will use additive notation in $L$. The $\mathbb{k}$-basis element of $\mathbb{k}[L]$ corresponding to the lattice element $m \in L$ will be written as

$$_{\mathfrak{X}^m}.$$
so \( x^0 = 1, x^{m+m'} = x^m x^{m'} \), and \( x^{-m} = (x^m)^{-1} \). The action of \( G \) on \( L \) extends uniquely to an action by \( k \)-algebra automorphisms on \( k[L] \):

\[
g(\sum_{m \in L} k_m x^m) = \sum_{m \in L} k_m x^{g(m)} .
\]

The invariant algebra \( k[L]^G \) is a free \( k \)-module: a \( k \)-basis is given by the \( G \)-orbit sums \( \sigma(m) = \sum_{\gamma \in G} x^{\gamma(m)} \), where \( G(m) \) denotes the \( G \)-orbit of \( m \in L \). Since all orbit sums are defined over \( \mathbb{Z} \), we have

\[
\]

2.2. Let \( H \) be a subgroup of \( G \). We compute the height of the ideal \( I_{k[L]}(H) \) from §4.1.2. Let

\[
L^H = \{ m \in L \mid g(m) = m \text{ for all } g \in H \}
\]
denote the lattice of \( H \)-invariants in \( L \) and define the sublattice \( [H, L] \) of \( L \) by

\[
[H, L] = \sum_{g \in H} [g, L] ,
\]

where \([g, L] = \{ g(m) - m \mid m \in L \} \). It suffices to let \( g \) run over a set of generators of the group \( H \) in the above formulas.

**Lemma.** With the above notation, \( k[L]/I_{k[L]}(H) \cong k[L]/[H, L] \) and

\[
\text{height } I_{k[L]}(H) = \text{rank}[H, L] = \text{rank } L - \text{rank } L^H .
\]

**Proof.** Since the ideal \( I_{k[L]}(H) \) is generated by the elements \( x^{\sigma(m)} - m - 1 \) with \( m \in L \) and \( g \in H \), the isomorphism \( k[L]/I_{k[L]}(H) \cong k[L]/[H, L] \) is clear.

To prove the equality \( \text{rank}[H, L] = \text{rank } L - \text{rank } L^H \), note that the rational group algebra \( Q[H] \) is the direct sum of the ideals \( Q(\sum_{g \in H} g) \) and \( \sum_{g \in H} Q(g-1) \). This implies \( L \otimes Q = (L^H \otimes Q) \oplus ([H, L] \otimes Q) \). Hence, \( \text{rank } L = \text{rank } L^H + \text{rank } [H, L] \).

To complete the proof, it suffices to show that

\[
\text{height } \mathfrak{P} = \text{rank}[H, L]
\]

holds for any minimal covering prime \( \mathfrak{P} \) of \( I_{k[L]}(H) \). Put \( A = L/[H, L] \) and \( \mathfrak{P} = \mathfrak{P}/I_{k[L]}(H) \), a minimal prime of \( k[L]/I_{k[L]}(H) = k[A] \). Further, put \( p = \mathfrak{P} \cap k = \mathfrak{P} \cap k \). Since the extension \( k \hookrightarrow k[A] = k[L]/I_{k[L]}(H) \) is free, \( p \) is a minimal prime of \( k \); see [Bou3] Cor. on p. AC VIII.15]. Hence, descending chains of primes in \( k[L] \) starting with \( \mathfrak{P} \) correspond in a 1-to-1 fashion to descending chains of primes of \( Q(k/p)[L] \) starting with the prime that is generated by \( \mathfrak{P} \). Thus, replacing \( k \) by \( Q(k/p) \), we may assume that \( k \) is a field. But then

\[
\text{height } \mathfrak{P} = \text{dim } k[L] - \text{dim } k[L]/\mathfrak{P} = \text{rank } L - \text{dim } k[L]/\mathfrak{P} .
\]

Let \( \mathfrak{P}_0 = \mathfrak{P} \cap k[A_0] \), where \( A_0 \) denotes the torsion subgroup of \( A \). Since \( \mathfrak{P} \) is minimal, we have \( \mathfrak{P} = \mathfrak{P}_0[k[A] \) and so \( k[L]/\mathfrak{P} \cong k_0[A/A_0] \), where \( k_0 = k[A_0]/\mathfrak{P}_0 \) is a field. Thus, \( \text{dim } k[L]/\mathfrak{P} = \text{rank } A/A_0 \). Finally, \( \text{rank } A/A_0 = \text{rank } A = \text{rank } L - \text{rank } [H, L] \), which completes the proof. □

Specializing the lemma to the case where \( H = \langle g \rangle \) for some \( g \in G \), we see that \( g \) acts as a \( k \)-reflection on \( k[L] \) if and only if \( g \) acts as a \( k \)-reflection on \( L \), that is,

\[
\text{rank } [g, L] \leq k .
\]
Moreover, the collection of subgroups $\mathcal{X}_k$ in equation (1.11) can now be written as
\[ \mathcal{X}_k = \{ \mathcal{H} \leq \mathcal{G} \mid \text{rank } L/L^H \leq k \} . \]

### 2.3. Fixed-point-free lattices for perfect groups

The $\mathcal{G}$-action on $L$ is called fixed-point-free if $g(m) \neq m$ holds for all $0 \neq m \in L$ and $1 \neq g \in \mathcal{G}$. Recall also that the group $\mathcal{G}$ is said to be perfect if $\mathcal{G}^{ab} = [\mathcal{G}, \mathcal{G}] = 1$.

#### Lemma

Assume that $\mathcal{G}$ is a non-trivial perfect group acting fixed-point-freely on the non-zero lattice $L$. Then $\mathcal{G}$ is isomorphic to the binary icosahedral group $2.A_5 \cong \text{SL}_2(\mathbb{F}_5)$ and rank $L$ is a multiple of $8$.

**Proof.** Put $V = L \otimes_{\mathbb{Z}} \mathbb{C}$, a non-zero fixed-point-free $\mathbb{C}[\mathcal{G}]$-module. By a well-known theorem of Zassenhaus (see [Wo, Theorem 6.2.1]), $\mathcal{G}$ is isomorphic to the binary icosahedral group $2.A_5$ and the irreducible constituents of $V$ are $2$-dimensional. The binary icosahedral group has two irreducible complex representations of degree $2$; they are Galois conjugates of each other and both have Frobenius-Schur indicator $-1$. We denote the corresponding $\mathbb{C}[\mathcal{G}]$-modules by $V_1$ and $V_2$. Both $V_1$ occur with the same multiplicity in $V$, since $V$ is defined over $\mathbb{Q}$. Thus, $V \cong (V_1 \oplus V_2)^m$ for some $m$ and rank $L = 4m$. We have to show that $m$ is even. Since both $V_1$ have indicator $−1$, it follows that $V_1 \oplus V_2$ is not defined over $\mathbb{R}$, whereas each $V_i^2$ is defined over $\mathbb{R}$; see [I, (9.21)]. Thus, $V_1 \oplus V_2$ represents an element $x$ of order $2$ in the cokernel of the scalar extension map $G_0(\mathbb{R}[\mathcal{G}]) \to G_0(\mathbb{C}[\mathcal{G}])$, and $mx = 0$. Therefore, $m$ must be even, as desired. □

We remark that the binary icosahedral group $2.A_5$ is isomorphic to the subgroup of the non-zero quaternions $\mathbb{H}^*$ that is generated by $(a+i+ja^*)/2$ and $(a+j+ka^*)/2$, where $a = (1 + \sqrt{5})/2$ and $a^* = (1 - \sqrt{5})/2$ and $\{1, i, j, k\}$ is the standard $\mathbb{R}$-basis of $\mathbb{H}$. Thus, letting $2.A_5$ act on $\mathbb{H}$ via left multiplication, $\mathbb{H}$ becomes a 2-dimensional fixed-point-free complex representation of $2.A_5$. It is easy to see that this representation can be realized over $K = \mathbb{Q}(i, \sqrt{5})$; so $\mathbb{H} = V \otimes_K \mathbb{C}$ with $\dim_{\mathbb{Q}} V = 2[K : \mathbb{Q}] = 8$. Any $2.A_5$-lattice for $V$ will be fixed-point-free and have rank $8$.

### 2.4. Isotropy groups

The isotropy group of an element $m \in L$ in $\mathcal{G}$ will be denoted by $\mathcal{G}_m$; so
\[ \mathcal{G}_m = \{ g \in \mathcal{G} \mid g(m) = m \} . \]

The $\mathcal{G}$-lattice $L$ is called faithful if $\text{Ker}_\mathcal{G}(L) = \bigcap_{m \in L} \mathcal{G}_m = 1$. The following lemma, at least part (a), is well known. We include the proof for the reader’s convenience.

#### Lemma

(a) The set of isotropy groups $\{ \mathcal{G}_m \mid m \in L \}$ is closed under conjugation and under taking intersections.

(b) Assume that the $\mathcal{G}$-lattice $L$ is faithful. If $\mathcal{G}_m$ ($m \in L$) is a minimal non-identity isotropy group, then $\mathcal{G}_m$ acts fixed-point-freely on $L/L^{\mathcal{G}_m} \neq 0$.

**Proof.** Consider the $\mathbb{Q}[\mathcal{G}]$-module $V = L \otimes_{\mathbb{Z}} \mathbb{Q}$. The collection of isotropy groups $\mathcal{G}_m$ remains unchanged when allowing $m \in V$. Moreover, for any subgroup $\mathcal{H} \leq \mathcal{G}$, $L/L^\mathcal{H}$ is an $\mathcal{H}$-lattice with $L/L^\mathcal{H} \otimes_{\mathbb{Z}} \mathbb{Q} \cong V/V^\mathcal{H}$.

(a) The first assertion is clear, since $\mathcal{G}_m = \mathcal{G}_{g(m)}$ holds for all $g \in \mathcal{G}, m \in V$. For the second assertion, let $M$ be a non-empty subset of $V$ and put $\mathcal{G}_M = \bigcap_{m \in M} \mathcal{G}_m$. We must show that $\mathcal{G}_M = \mathcal{G}_m$ for some $m \in V$. Put $W = V^{\mathcal{G}_M}$. If $g \in \mathcal{G} \setminus \mathcal{G}_M$, \[
then $W^g = \{ w \in W \mid g(w) = w \}$ is a proper subspace of $W$, since some element of $M$ does not belong to $W^g$. Any $m \in W \setminus \bigcup_{g \in G} W^g$ satisfies $G_m = G_M$.

(b) Let $\mathcal{H} = G_m$ be a minimal non-identity member of $\{ G_m \mid m \in V \}$. As $\mathbb{Q}[H]$-modules, we may identify $V$ and $V^H \oplus V/V^H$. If $0 \neq v \in V/V^H$, then $\mathcal{H}_v = H \cap G_v \subseteq H$. In view of (a), our minimality assumption on $\mathcal{H}$ forces $\mathcal{H}_v = 1$. Thus, $\mathcal{H}$ acts fixed-point-freely on $V/V^H$, and hence on $L/L^\mathcal{H}$.

**Proposition.** Assume that $L$ is a faithful $G$-lattice such that all minimal isotropy groups $1 \neq G_m$ ($m \in L$) are perfect. Then rank $L/L^\mathcal{H} \geq 8$ holds for every nonidentity subgroup $\mathcal{H} \leq G$.

In the notation of equation (2.2), the conclusion of the proposition can be stated as follows:

$$X_k = \{ 1 \} \text{ for all } k < 8.$$

**Proof of the Proposition.** Put $\bar{H} = \bigcap_{m \in L} G_m$. Then $\bar{H} \supseteq \mathcal{H}$ and $L^{\bar{H}} = L^\mathcal{H}$. Lemma 2.4(a) further implies that $\mathcal{H} \sim G_m$ for some $m$. Replacing $\mathcal{H}$ by $\bar{H}$, we may assume that $\mathcal{H}$ is a nonidentity isotropy group. If $\mathcal{H}$ is not minimal then replace $\mathcal{H}$ by a smaller nonidentity isotropy group; this does not increase the value of rank $L/L^\mathcal{H}$. Thus, we may assume that $\mathcal{H}$ is a minimal nonidentity isotropy group, and hence $\mathcal{H}$ is perfect. By Lemma 2.4(b), $\mathcal{H}$ acts fixed-point-freely on $L/L^\mathcal{H} \neq 0$ and Lemma 2.5 implies that rank $L/L^\mathcal{H} \geq 8$, proving the proposition. $\square$

### 2.5. Cohomology

Let $X$ denote any collection of subgroups of $G$ that is closed under conjugation and under taking subgroups. We will investigate injectivity of the restriction map

$$\text{res}_X^G : H^k(G, k[L]) \rightarrow \prod_{\mathcal{H} \in X} H^k(\mathcal{H}, k[L]).$$

This map was considered in Proposition 1.4 for $X = X_{k+1}$.

**Lemma.** The map $\text{res}_X^G : H^k(G, k[L]) \rightarrow \prod_{\mathcal{H} \in X} H^k(\mathcal{H}, k[L])$ is injective if and only if the restriction maps

$$H^k(G_m, k) \rightarrow \prod_{\mathcal{H} \in X} H^k(\mathcal{H}, k)$$

are injective for all $m \in L$.

**Proof.** As $k[G]$-module, $k[L]$ is a permutation module:

$$k[L] \cong \bigoplus_{m \in G \setminus L} k[G/G_m],$$

where $k[G/G_m] \cong k[G] \otimes_{k[G]} k$ and $G \setminus L$ is a transversal for the $G$-orbits in $L$. For each subgroup $\mathcal{H} \leq G$,

$$k[G/G_m]|_{\mathcal{H}} \cong \bigoplus_{g \in \mathcal{H} \setminus G/G_m} k[\mathcal{H}/gG_m \cap \mathcal{H}];$$

see [CR] 10.13). Therefore, $\text{res}_X^G$ is the direct sum of the restriction maps

$$H^k(G, k[G/G_m]) \rightarrow H^k(H, k[G/G_m]) = \bigoplus_{g \in H \setminus G/G_m} H^k(\mathcal{H}, k[H/gG_m \cap \mathcal{H}]).$$
By the Eckmann-Shapiro Lemma [BH III(5.2),(6.2)], \( H^k(\mathcal{G}, k[\mathcal{G}/\mathcal{G}_m]) \cong H^k(\mathcal{G}_m, k) \) and \( H^k(\mathcal{H}, k[\mathcal{H}/\mathcal{G}_m \cap \mathcal{H}]) \cong H^k(\mathcal{G}_m \cap \mathcal{H}, k) \). In terms of these isomorphisms, the above restriction map becomes
\[
\rho_{\mathcal{H}, m}: H^k(\mathcal{G}_m, k) \rightarrow \bigoplus_{g \in \mathcal{H} \setminus \mathcal{G}_m} H^k(\mathcal{G}_m \cap \mathcal{H}, k)
\]
where \([\cdot]\) denotes the cohomology class of a \( k \)-cocycle and \( g \) stands for a \( k \)-tuple of elements of \( \mathcal{G}_m \cap \mathcal{H} \). Therefore,
\[
\text{Ker } \rho_{\mathcal{H}, m} = \bigcap_{g \in \mathcal{H} \setminus \mathcal{G}_m} \text{Ker} \left( \text{res}_{\mathcal{G}_m \cap \mathcal{H}_g}^{\mathcal{G}_m}: H^k(\mathcal{G}_m, k) \rightarrow H^k(\mathcal{G}_m \cap \mathcal{H}_g, k) \right).
\]
Thus, \( \text{Ker } \text{res}_{\mathcal{X}}^{\mathcal{G}} \) is isomorphic to the direct sum of the kernels of the restriction maps
\[
H^k(\mathcal{G}_m, k) \rightarrow \prod_{\mathcal{H} \in \mathcal{X}} H^k(\mathcal{G}_m \cap \mathcal{H}_g, k)
\]
with \( m \in \mathcal{G} \setminus L \). Finally, by hypothesis on \( \mathcal{X} \), the groups \( \mathcal{G}_m \cap \mathcal{H}_g \) with \( \mathcal{H} \in \mathcal{X} \) are exactly the groups \( \mathcal{H} \in \mathcal{X} \) with \( \mathcal{H} \leq \mathcal{G}_m \). The lemma follows. \( \square \)

Corollary. Let \( \mathcal{K} = \mathbb{Z}/(\mathcal{G}) \) and \( k = 1 \). Then \( \text{res}_{\mathcal{X}}^{\mathcal{G}} \) injective if and only if all \( \mathcal{G}_m^{ab} \) \((m \in L)\) are generated by the images of the subgroups \( \mathcal{H} \leq \mathcal{G}_m \) with \( \mathcal{H} \in \mathcal{X} \).

Proof. By the lemma with \( k = 1 \), the hypothesis on the restriction map says that all restrictions
\[
H^1(\mathcal{G}_m, k) \rightarrow \prod_{\mathcal{H} \in \mathcal{X}} H^1(\mathcal{H}_g, k)
\]
are injective. Now, for each \( \mathcal{H} \leq \mathcal{G} \), \( H^1(\mathcal{H}_g, k) = \text{Hom}(\mathcal{H}_g^{ab}, k) \cong \mathcal{H}_g^{ab} \), where the last isomorphism holds by our choice of \( \mathcal{K} \). Therefore, injectivity of the above map is equivalent to \( \mathcal{G}_m^{ab} \) being generated by the images of all \( \mathcal{H} \leq \mathcal{G}_m \) with \( \mathcal{H} \in \mathcal{X} \). \( \square \)

3. The Cohen-Macaulay property

3.1. Continuing with the notation of [21] we now turn to the question of when the invariant algebra \( k[L]^{\mathcal{G}} \) is Cohen-Macaulay. Our principal tool will be Proposition [14]. We remark that the Cohen-Macaulay hypothesis of Proposition [14] simplifies slightly in the setting of multiplicative actions: it suffices to assume that \( k[L]^{\mathcal{G}} \) is Cohen-Macaulay. Indeed, in this case the base ring \( k \) is also Cohen-Macaulay, because \( k[L]^{\mathcal{G}} \) is free over \( k \), and then \( k[L] \) is Cohen-Macaulay as well; see [BH] Exercise 2.1.23 and Theorems 2.1.9, 2.1.3(b)].

3.2. Base rings. Our main interest is in the case where \( k = \mathbb{Z} \). As the following lemma shows, if \( \mathbb{Z}[L]^{\mathcal{G}} \) is Cohen-Macaulay, then so is \( k[L]^{\mathcal{G}} \) for any Cohen-Macaulay base ring \( k \).

Lemma. The following are equivalent:

(a) \( \mathbb{Z}[L]^{\mathcal{G}} \) is Cohen-Macaulay;
(b) \( k[L]^{\mathcal{G}} \) is Cohen-Macaulay whenever \( k \) is;
(c) \( k[L]^{\mathcal{G}} \) is Cohen-Macaulay for \( k = \mathbb{Z}/(\mathcal{G}) \);
(d) \( \mathbb{F}_p[L]^{\mathcal{G}} \) is Cohen-Macaulay for all primes \( p \) dividing \( |\mathcal{G}| \).
Proof. (a) \Rightarrow (b): Put \( S = \mathbb{k}[L]^{G} \) and consider the extension of rings \( \mathbb{k} \hookrightarrow S \).
This extension is free; see [2.1] By [3H] Exercise 2.1.23, \( S \) is Cohen-Macaulay if (and only if) \( \mathbb{k} \) is Cohen-Macaulay and, for all \( \mathfrak{p} \in \text{Spec} \, S \), the fibre \( S_{\mathfrak{p}}/pS_{\mathfrak{p}} \) is Cohen-Macaulay, where \( p = \mathfrak{p} \cap \mathbb{k} \). But \( S_{\mathfrak{p}}/pS_{\mathfrak{p}} \) is a localization of \( (S/pS)_{p} \neq \mathbb{Q}(\mathbb{k}/p)[L]^{G} \); see equation (2.4). Therefore, by [3H] Theorem 2.1.3(b)], it suffices to show that \( \mathbb{Q}(k/p)[L]^{G} \) is Cohen-Macaulay. In other words, we may assume that \( \mathbb{k} \) is a field. By [3H] Theorem 2.1.10, we may further assume that \( \mathbb{k} = \mathbb{Q} \) or \( \mathbb{k} = \mathbb{F}_{p} \). But equation (2.1) implies that \( \mathbb{Q}[L]^{G} = \mathbb{Z}[L]^{G}_{\mathbb{Z}} \) and \( \mathbb{F}_{p}[L]^{G} \cong \mathbb{Z}[L]^{G}/(p) \). Since \( \mathbb{Z}[L]^{G} \) is assumed Cohen-Macaulay, [3H] Theorem 2.1.3] implies that \( \mathbb{Q}[L]^{G} \) and \( \mathbb{F}_{p}[L]^{G} \) are Cohen-Macaulay, as desired.

(b) \Rightarrow (c) is clear.

(c) \Rightarrow (d): Write \( |G| = \prod \mathbb{Z}/p^{n} \). Then \( \mathbb{k}[L] \cong \prod \mathbb{Z}/(p^{n})[L]^{G} \) and \( \mathbb{Z}/(p^{n})[L]^{G} \) is a localization of \( k[L]^{G} \). Therefore, \( \mathbb{Z}/(p^{n})[L]^{G} \) is Cohen-Macaulay, by [3H] Theorem 2.1.3(b)]. If \( p_{n} \neq 0 \), then it follows from [3H] Theorem 2.1.3(a)] that \( \mathbb{Z}(p)[L]^{G} \) and \( \mathbb{F}_{p}[L]^{G} \cong \mathbb{Z}(p)[L]^{G}/(p) \) are Cohen-Macaulay.

(d) \Rightarrow (a): First, (d) implies that \( \mathbb{F}_{p}[L]^{G} \) is Cohen-Macaulay for all primes \( p \). For, if \( p \) does not divide \( |G| \), then \( \mathbb{F}[L]^{G} \) is always Cohen-Macaulay; see [3H] Corollary 6.4.6]. Now let \( \mathfrak{p} \) be a maximal ideal of \( \mathbb{Z}[L] \). Then \( \mathfrak{p} \cap \mathbb{Z} = (p) \) for some prime \( p \) and \( \mathbb{Z}[L]^{G}_{\mathfrak{p}}/(p) \) is a localization of \( \mathbb{Z}[L]^{G}/(p) = \mathbb{F}_{p}[L]^{G} \). Thus, \( \mathbb{Z}[L]^{G}_{\mathfrak{p}}/(p) \) is Cohen-Macaulay and [3H] Theorem 2.1.3(a)] further implies that \( \mathbb{Z}[L]^{G}_{\mathfrak{p}} \) is Cohen-Macaulay. Since, \( \mathfrak{p} \) was arbitrary, (a) is proved.

Since normal rings of (Krull) dimension at most 2 are Cohen-Macaulay, the implication (d) \Rightarrow (b) of the lemma shows that \( \mathbb{k}[L]^{G} \) is certainly Cohen-Macaulay whenever \( \mathbb{k} \) is Cohen-Macaulay and \( L \) has rank at most 2.

3.3. Proof of the Theorem. We are now ready to prove the Theorem stated in the Introduction. Recall that, for any subgroup \( H \leq G \), \( H(2) \) denotes the subgroup generated by the elements of \( H \) that act as bireflections on \( L \) or, equivalently, by the subgroups of \( H \) that belong to \( X_{2} \); see (2.2). Throughout, we assume that \( \mathbb{Z}[L]^{G} \) is Cohen-Macaulay.

We first show that \( G_{m}/G_{m}^{(2)} \) is a perfect group for all \( m \in L \). Put \( k = \mathbb{Z}/(|G|) \).
Then \( k[L]^{G} \) is Cohen-Macaulay, by Lemma 3.2]. Therefore, the restriction

\[
H^{1}(G, k[L]) \to \prod_{H \in X_{2}} H^{1}(H, k[L])
\]

is injective, by Proposition 1.4] see the remark in 3.1] Corollary 2.9 yields that all \( G^{ab} \) are generated by the images of the subgroups \( H \leq G_{m} \) with \( H \in X_{2} \). In other words, each \( G_{m}^{ab} \) is generated by the images of the bireflections in \( G_{m} \). Therefore, \( (G_{m}/G_{m}^{(2)})^{ab} = 1 \), as desired.

Now assume that \( G \) acts non-trivially on \( L \). Our goal is to show that some isotropy group \( G_{m} \) is non-perfect. Suppose otherwise. Replacing \( G \) by \( G/\text{Ker} G(L) \) we may assume that \( 1 \neq G \) acts faithfully on \( L \). Then \( X_{k} = \{1\} \) for all \( k < 8 \), by Proposition 2.4. It follows that

\[
k = \inf \{i > 0 \mid H^{i}(G, k[L]) \neq 0 \} \geq 7.
\]

Indeed, if \( k < 7 \), then Proposition 1.4] implies that \( 0 \neq H^{k}(G, k[L]) \) embeds into \( \prod_{H \in X_{k+1}} H^{k}(H, k[L]) \) which is trivial, because \( X_{k+1} = \{1\} \). By Lemma 2.5] with
Theorem 6.4.5: Let \( R \) be an integral extension of commutative rings having \( G \)-ring theoretic result of Hochster and Eagon [HE] (or see [BH, Theorem 6.4.5]): Let \( R \) be an integral extension of commutative rings having a Reynolds operator, that is, an \( S \)-linear map \( R \to S \) that restricts to the identity on \( S \). If \( R \) is Cohen-Macaulay, then \( S \) is Cohen-Macaulay as well.

To construct the requisite Reynolds operator, consider the truncation map

\[
\pi: k[L] \to k[L'], \quad \sum_{m \in L} k_m x^m \mapsto \sum_{m \in L'} k_m x^m.
\]

This is a Reynolds operator for the extension \( k[L] \supseteq k[L'] \) that satisfies \( \pi(g(f)) = g(\pi(f)) \) for all \( g \in G, f \in k[L] \). Therefore, \( \pi \) restricts to a Reynolds operator \( k[L]^G \to k[L']^G \) and the proposition follows.}

The proposition in particular allows us to reduce the general case of the Cohen-Macaulay problem for multiplicative invariants to the case of effective \( G \)-lattices. Recall that the \( G \)-lattice \( L \) is effective if \( L^G = 0 \). For any \( G \)-lattice \( L \), the quotient \( L/L^G \) is an effective \( G \)-lattice; this follows, for example, from the fact that \( L \) is rationally isomorphic to the \( G \)-lattice \( L^G \otimes L/L^G \).

**Corollary.** \( k[L]^G \) is Cohen-Macaulay if and only if this holds for \( k[L/L^G]^G \).

**Proof.** By the proposition, we may replace \( L \) by \( L' = L^G \oplus L/L^G \). But \( k[L']^G \cong k[L/L^G]^G \otimes k k[L]^G \), a Laurent polynomial algebra over \( k[L/L^G]^G \). Thus, by [BH, Theorems 2.1.3 and 2.1.9], \( k[L]^G \) is Cohen-Macaulay if and only if \( k[L/L^G]^G \) is Cohen-Macaulay. The corollary follows.

3.5. **Remarks and examples.**

3.5.1. **Abelian bireflection groups.** It is not hard to show that if \( G \) is a finite abelian group acting as a bireflection group on the lattice \( L \), then \( \mathbb{Z}[L]^G \) is Cohen-Macaulay. Using Corollary 3.3 and an induction on rank \( L \), the proof reduces to the verification...
that \( \mathbb{Z}[L]^{G} \) is Cohen-Macaulay for \( L = \mathbb{Z}^{n} \) and \( G = \text{diag}(\pm 1, \ldots, \pm 1) \cap \text{SL}_n(\mathbb{Z}) \). Direct computation shows that, for \( n \geq 2 \),

\[
\mathbb{Z}[L]^{G} = \mathbb{Z}[\xi_1, \ldots, \xi_n] \oplus \eta \mathbb{Z}[\xi_1, \ldots, \xi_n],
\]

where \( \xi_i = x^{c_i} + x^{-c_i} \) is the \( G \)-orbit sum of the standard basis element \( e_i \in \mathbb{Z}^n \) and \( \eta \) is the orbit sum of \( \sum e_i = (1, \ldots, 1) \).

It would be worthwhile to try and extend this result to larger classes of bireflection groups. The aforementioned classification of bireflection groups in [GuS] will presumably be helpful in this endeavor.

### 3.5.2. Subgroups of reflection groups

Assume that \( G \) acts as a reflection group on the lattice \( L \) and let \( H \) be a subgroup of \( G \) with \([G : H] = 2\). Then \( H \) acts as a bireflection group. (More generally, if \( G \) acts as a \( k \)-reflection group and \([G : H] = m\), then \( H \) acts as a \( km \)-reflection group; see [L]).) Presumably \( \mathbb{Z}[L]^{H} \) will always be Cohen-Macaulay, but I have no proof. For an explicit example, let \( G = S_n \) be the symmetric group on \( \{1, \ldots, n\} \) and let \( L = U_{n} \) be the standard permutation lattice for \( S_n \); so \( U_{n} = \bigoplus_{i=1}^{n} \mathbb{Z}e_i \) with \( s(e_i) = e_{s(i)} \) for \( s \in S_n \). Transpositions act as reflections on \( U_{n} \) and \( 3 \)-cycles as bireflections. Let \( A_n \leq S_n \) denote the alternating group.

To compute \( \mathbb{Z}[U_{n}]^{A_n} \), put \( x_i = x^{e_i} \in \mathbb{Z}[U_{n}] \). Then \( \mathbb{Z}[U_{n}] = \mathbb{Z}[x_1, \ldots, x_n][s_n^{-1}] \), where \( s_n = x\sum_{i} e_i = \prod_{i} x_i \) is the \( n \)-th elementary symmetric function, and \( S_n \) acts via \( s(x_i) = x_{s(i)} \) (\( s \in S_n \)). Therefore, \( \mathbb{Z}[U_{n}]^{A_n} = \mathbb{Z}[x_1, \ldots, x_n]^{A_n}[s_n^{-1}] \). The ring \( \mathbb{Z}[x_1, \ldots, x_n]^{A_n} \) of polynomial \( A_n \)-invariants has the following form; see [S] Theorem 1.3.5: \( \mathbb{Z}[x_1, \ldots, x_n]^{A_n} = \mathbb{Z}[s_1, \ldots, s_n] \oplus d\mathbb{Z}[s_1, \ldots, s_n] \), where \( s_i \) is the \( i \)-th elementary symmetric function and

\[
d = \frac{1}{2}(\Delta + \Delta_{+})
\]

with \( \Delta_{+} = \prod_{i<j}(x_i + x_j) \) and \( \Delta = \prod_{i<j}(x_i - x_j) \), the Vandermonde determinant. Thus,

\[
\mathbb{Z}[U_{n}]^{A_n} = \mathbb{Z}[s_1, \ldots, s_n-1, s_1^{\pm 1}] \oplus d\mathbb{Z}[s_1, \ldots, s_{n-1}, s_{n}^{\pm 1}].
\]

This is Cohen-Macaulay, being free over \( \mathbb{Z}[s_1, \ldots, s_{n-1}, s_{n}^{\pm 1}] \).

### 3.5.3. \( S_n \)-lattices

If \( L \) is a lattice for the symmetric group \( S_n \) such that \( \mathbb{Z}[L]^{S_n} \) is Cohen-Macaulay, then the Theorem implies that \( S_n \) acts as a bireflection group on \( L \), and hence on all simple constituents of \( L \otimes_{\mathbb{Z}} \mathbb{Q} \). The simple \( \mathbb{Q}[S_n] \)-modules are the Specht modules \( S_\lambda \) for partitions \( \lambda \) of \( n \). If \( n \geq 7 \), then the only partitions \( \lambda \) so that \( S_n \) acts as a bireflection group on \( S_\lambda \) are \((n),(1^n)\) and \((n-1,1)\); this follows from the lists in [Hu] and [W]. The corresponding Specht modules are trivial module, \( \mathbb{Q} \), the sign module \( \mathbb{Q}^{-} \), and the rational root module \( A_{n-1} \otimes_{\mathbb{Z}} \mathbb{Q} \), where \( A_{n-1} = \langle \sum_{i} z_i e_i \in U_n \mid \sum_{i} z_i = 0 \rangle \) and \( U_n \) is as in \{3.5.2\}. Thus, if \( n \geq 7 \) and \( \mathbb{Z}[L]^{S_n} \) is Cohen-Macaulay, then we must have

\[
L \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}^r \oplus (\mathbb{Q}^{-})^s \oplus (A_{n-1} \otimes_{\mathbb{Z}} \mathbb{Q})^t
\]

with \( s + t \leq 2 \). In most cases, \( \mathbb{Z}[L]^{S_n} \) is easily seen to be Cohen-Macaulay. Indeed, we may assume \( r = 0 \) by Corollary \{3.3\}. If \( s + t \leq 1 \), then \( S_n \) acts as a reflection group on \( L \) and so \( \mathbb{Z}[L]^{S_n} \) is Cohen-Macaulay by [L]. For \( t = 0 \) we may quote the last remark in \{3.2\}. This leaves the cases \( s = t = 1 \) and \( s = 0, t = 2 \) to consider.
If $s = t = 1$, then add a copy of $\mathbb{Q}$ so that $L$ is rationally isomorphic to $U_n \oplus \mathbb{Z}^-$. Using the notation of \[3.5.2\] and putting $t = x^{(0, u_n - 1)} \in \mathbb{Z}[U_n \oplus \mathbb{Z}^-]$ the invariants are

\[\mathbb{Z}[U_n \oplus \mathbb{Z}^-]^{S_n} = R \oplus R\varphi\]

with $R = \mathbb{Z}[s_1, \ldots, s_{n-1}, s_n^{\pm 1}, t + t^{-1}]$ and $\varphi = \frac{1}{2}(\Delta + \Delta)t + \frac{1}{2}(\Delta - \Delta)t^{-1}$.

If $s = 0$ and $t = 2$, then we may replace $L$ by the lattice $U_n^2 = U_n \oplus U_n$. By Lemma \[3.2\] $\mathbb{Z}[U_n^2]^{S_n}$ is Cohen-Macaulay precisely if $F_p[U_n^2]^{S_n}$ is Cohen-Macaulay for all primes $p \leq n$. As in \[3.5.2\], one sees that $F_p[U_n^2]^{S_n}$ is a localization of the algebra “vector invariants” $F_p[x_1, \ldots, x_n, y_1, \ldots, y_n]^{S_n}$. By \[K_2\] Corollary 3.5, this algebra is known to be Cohen-Macaulay for $n/2 < p \leq n$, but the primes $p \leq n/2$ apparently remain to be dealt with.

3.5.4. Ranks $\leq 4$. As was pointed out in \[3.2\] $\mathbb{Z}[L]^G$ is always Cohen-Macaulay when rank $L \leq 2$.

For $L = \mathbb{Z}^3$, there are 32 $\mathbb{Q}$-classes of finite subgroups $G \leq \text{GL}_3(\mathbb{Z})$. All $G$ are solvable; in fact, their orders divide 48. The Sylow 3-subgroup $H \leq G$, if non-trivial, is generated by a bireflection of order 3. Thus, $F_3[L]^H$ is Cohen-Macaulay, and hence so is $F_3[L]^G$. Therefore, by Lemma \[3.2\] $\mathbb{Z}[L]^G$ is Cohen-Macaulay if and only if $F_3[L]^G$ is Cohen-Macaulay, and for this to occur, $G$ must be generated by bireflections. It turns out that 3 of the 32 $\mathbb{Q}$-classes consist of non-bireflection groups; these classes are represented by the cyclic groups

\[\langle \begin{pmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{pmatrix} \rangle, \quad \langle \begin{pmatrix} -1 & 1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{pmatrix} \rangle, \quad \langle \begin{pmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{pmatrix} \rangle\]

of orders 2, 4 and 6 (the latter two classes each split into 2 $\mathbb{Z}$-classes). For the $\mathbb{Q}$-classes consisting of bireflection groups, Pathak \[PR\] has checked explicitly that $F_3[L]^G$ is indeed Cohen-Macaulay.

In rank 4, there are 227 $\mathbb{Q}$-classes of finite subgroups $G \leq \text{GL}_4(\mathbb{Z})$. All but 5 of them consist of solvable groups and 4 of the non-solvable classes are bireflection groups, the one exception being represented by $S_5$ acting on the signed root lattice $\mathbb{Z}^- \otimes_2 A_4$. Thus, if the group $G'G^{(2)}$ is perfect, then it is actually trivial, that is, $G$ is a bireflection group. It also turns out that, in this case, all isomorphism classes $G/m$ are bireflection groups. There are exactly 71 $\mathbb{Q}$-classes that do not consist of bireflection groups. By the foregoing, they lead to non-Cohen-Macaulay multiplicative invariant algebras. The $\mathbb{Q}$-classes consisting of bireflection groups have not been systematically investigated yet. The searches in rank 4 were done with \[GAP\].

Acknowledgments

Some of the research for this article was carried out during a workshop in Seattle (August 2003) funded by Leverhulme Research Interchange Grant F/00158/X and during the symposium “Ring Theory” in Warwick, UK (September 2003). The results described here were reported in the special session “Algebras and Their Representations” at the AMS-meeting in Chapel Hill (October 2003). Many thanks to Bob Guralnick for his helpful comments on an earlier version of this article.

References


COHEN-MACaulay Rings of Invariants


[N] H. Nakajima, Quotient singularities which are complete intersections, Manuscripta Math. 48 (1984), 163–187. MR0753729 (86h:14039)


Department of Mathematics, Temple University, Philadelphia, Pennsylvania 19122

E-mail address: lorenz@math.temple.edu