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ON THE CORRELATIONS OF DIRECTIONS IN THE EUCLIDEAN PLANE

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ABSTRACT. Let $\mathcal{R}^{(\nu)}_{(x,y),Q}$ denote the repartition of the ν-level correlation measure of the finite set of directions $P_{(x,y)}P$, where $P_{(x,y)}$ is the fixed point $(x,y) \in [0,1)^2$ and P is an integer lattice point in the square $[-Q,Q]^2$. We show that the average of the pair correlation repartition $\mathcal{R}^{(2)}_{(x,y),Q}$ over (x,y) in a fixed disc $\mathbb{D}_0$ converges as $Q \to \infty$. More precisely we prove, for every $\lambda \in \mathbb{R}_+$ and $0 < \delta < \frac{1}{10}$, the estimate

$$\frac{1}{\text{Area}(\mathbb{D}_0)} \iint\limits_{\mathbb{D}_0} \mathcal{R}^{(2)}_{(x,y),Q}(\lambda)\, dx\, dy = \frac{2\pi\lambda}{3} + O_{\mathbb{D}_0,\lambda,\delta}(Q^{-\frac{1}{10}+\delta}) \qquad \text{as } Q \to \infty.$$

We also prove that for each individual point $(x,y) \in [0,1)^2$, the 6-level correlation $\mathcal{R}^{(6)}_{(x,y),Q}(\lambda)$ diverges at any point $\lambda \in \mathbb{R}^5_+$ as $Q \to \infty$, and we give an explicit lower bound for the rate of divergence.

1. Introduction

In many problems one is led to consider in the Euclidean plane lines joining a fixed point P_0, which is not necessarily an integer lattice point, with a finite set of integer lattice points. A natural way of measuring the distribution of directions P_0P, $P \in \mathbb{Z}^2$, is via correlations and consecutive spacings. When the fixed point is the origin, the problem is related to the distribution of Farey fractions with multiplicities, each fraction $\frac{a}{q}$ in $\mathcal{F}_Q$ being counted $\left[\frac{Q}{q}\right]$ times. The consecutive h-level spacing measures of customary Farey fractions were computed for $h=1$ in [6] and for $h \geq 2$ in [1]. Limiting correlations of Farey fractions were shown to exist and computed recently in [5].

When the fixed point is not an integer lattice point, the problem of existence of limiting correlations/consecutive spacings is considerably more difficult. It is therefore natural to try to first prove some averaging results, letting the fixed point vary in a given region. In the first part of this paper we derive such a result for the limiting pair correlation measure. The limiting average pair correlation function is constant, as in the Poisson case. What is striking, however, is that this constant is not 1, as in the Poisson case, but $\frac{\pi}{3}$.

We now give a mathematical formulation of the problem. For each $Q \geq 1$, let $\square_Q$ denote the set of integer lattice points in the square $[-Q,Q]^2$, and set $N = N_Q = \#\square_Q = (2Q+1)^2$. Let $P_{(x,y)} = (x,y)$ be a fixed point and consider

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for every (large) integer Q the finite sequences $\big(\theta_P(x,y)\big)_{P\in\square_Q}$ of angles between the line $P_{(x,y)}P$ and the horizontal direction. The pair correlation of this finite sequence is defined as

$$\mathcal{R}^{(2)}_{(x,y),Q}(\lambda)=\frac{\#\left\{(P,P')\in\square_Q^2\,:\,P\neq P',\ \frac{N}{2\pi}\,|\theta_{P,P'}(x,y)|\leq\lambda\right\}}{N}\,,\qquad \lambda\in\mathbb{R}_+,$$

where $\theta_{P,P'}(x,y)$ denotes the measure of the angle $\angle PP_{(x,y)}P'$.

Throughout the paper we shall consider a fixed disc $\mathbb{D}_0$ of center $(x_0,y_0)\in[0,1)^2$ and radius r_0. We are interested in the asymptotic behavior of the average

$$R^{(2)}_{\mathbb{D}_0,Q}(\lambda)=\frac{1}{\pi r_0^2}\iint_{\mathbb{D}_0}\mathcal{R}^{(2)}_{(x,y),Q}(\lambda)\,dx\,dy \tag{1.1}$$

of $\mathcal{R}^{(2)}_{(x,y),Q}(\lambda)$ over $\mathbb{D}_0$, for fixed $\lambda>0$ and $Q\to\infty$.

The first three sections are concerned with the proof of the following result.

Theorem 1.1. *For every $\lambda>0$ and $\delta>0$*

$$R^{(2)}_{\mathbb{D}_0,Q}(\lambda)=\frac{2\pi\lambda}{3}+O_{\mathbb{D}_0,\lambda,\delta}(Q^{-\frac{1}{10}+\delta})\qquad \textit{as}\ \ Q\to\infty. \tag{1.2}$$

If one replaces $\mathbb{D}_0$ by a vertical or horizontal segment of length one, an identical asymptotic formula as in (1.2) turns out to be true. This can be proved by similar techniques as in this paper, or by using Erdös-Turán type discrepancy estimates. These results suggest that (1.2) may be true regardless of the shape of the range of the fixed point.

The behavior of higher level correlations appears to be different. In the last section we prove that the 6-level correlations diverge for every individual fixed point. When $\nu\geq 2$, the repartition of the ν-level correlation measure of the finite sequence $\big(\theta_P(x,y)\big)_{P\in\square_Q}$ is defined for each vector $\lambda=(\lambda_1,\dots,\lambda_{\nu-1})\in\mathbb{R}_+^{\nu-1}$ by

$$\mathcal{R}^{(\nu)}_{(x,y),Q}(\lambda)=\frac{\#\{(P_1,\dots,P_\nu)\in\square_Q^\nu : P_i\ \text{distinct},\ |\theta_{P_i,P_{i+1}}(x,y)|\leq\frac{2\pi\lambda_i}{N}\}}{N}\,. \tag{1.3}$$

For randomly chosen directions one would expect to obtain the Poissonian limit

$$\lim_{Q\to\infty}\mathcal{R}^{(\nu)}_{(x,y),Q}(\lambda)=\operatorname{Vol}\prod_{i=1}^{\nu-1}[-\lambda_i,\lambda_i]=2^{\nu-1}\lambda_1\dots\lambda_{\nu-1}. \tag{1.4}$$

It turns out however that (1.4) fails. More precisely, we will show that if $\nu\geq 6$, then for every point $(x,y)\in[0,1)^2$ and for every $(\lambda_1,\dots,\lambda_{\nu-1})\in\mathbb{R}_+^{\nu-1}$ we have $\lim\limits_{Q\to\infty}\mathcal{R}^{(\nu)}_{(x,y),Q}(\lambda)=\infty$. This is a consequence of

Theorem 1.2. *For every $(x,y)\in[0,1)^2$, every $\lambda=(\lambda_1,\dots,\lambda_5)\in\mathbb{R}_+^5$, and every $\delta>0$, for Q large enough in terms of x,y,λ and δ,*

$$\mathcal{R}^{(6)}_{(x,y),Q}(\lambda)>Q^{\frac14-\delta}. \tag{1.5}$$

As in Theorem 1.2 one can prove

Corollary 1.3. *The 6-level correlations of angles of directions $P_{(x,y)}P$, where P is a lattice point inside an expanding region $Q\Omega$, diverges as $Q\to\infty$ whenever Ω is a convex domain in $\mathbb{R}^2$ which contains the origin.*

The phenomenon is similar to the one encountered in the problem of the distribution of fractional parts of polynomials. There, one can handle the pair correlation problem generically (see [8], [3]). Moreover, in the case of the sequence $n^2\alpha \pmod 1$ one is able to solve the problem for all m-level correlations for a large class of irrational numbers α (see [9], [10]). However, as shown in [9], there are irrational numbers α for which the 5-level correlation of fractional parts of $n^2\alpha$, $1 \leq n \leq N$, diverges to infinity as $N \to \infty$. This occurs as a result of the presence of large clusters of such fractional parts. In the case of Theorem 1.2 above, large clusters of elements of the given sequence are responsible, too, for the divergence of the 6-level correlations, and hence of any other higher level correlations.

2. A first approximation for $R_Q^{(2)}(\lambda)$

For obvious practical reasons, from the beginning we try to replace $\theta_{P,P'}(x,y)$ by one of its trigonometric functions in the definition of $\mathcal{R}^{(2)}_{(x,y),Q}(\lambda)$. Suppose that two distinct points $P=(q,a)$, $P'=(q',a')\in\square_Q$, are such that $q,q'\geq 0$ and $\max\{a,a'\}>0>\min\{a,a'\}$. Then for sufficiently large Q (depending only on λ) we have

$$\min_{x,y\in[0,1]}|\theta_{P,P'}(x,y)|\geq \arcsin\frac{1}{\sqrt{Q^2+1}}>\frac{2\pi\lambda}{N}\ .$$

As a result, we may only consider in the definition of $\mathcal{R}^{(2)}_{(x,y),Q}$ points from the same quadrant. Thus if we set

$$\tilde{\square}_Q^2=\big\{(P,P')\in\square_Q^2\ :\ P\neq P' \text{ and } P,P' \text{ belong to the same quadrant}\big\}$$

and

$$\beta_{Q,\lambda}=\sin\frac{2\pi\lambda}{N}=\sin\frac{2\pi\lambda}{(2Q+1)^2}=\frac{\pi\lambda}{2Q^2}+O_\lambda\left(\frac{1}{Q^6}\right)\qquad\text{as } Q\to\infty, \tag{2.1}$$

then

$$\begin{aligned}\mathcal{R}^{(2)}_{(x,y),Q}(\lambda)&=\frac{\#\big\{(P,P')\in\tilde{\square}_Q^2:|\theta_{P,P'}(x,y)|\leq\frac{2\pi\lambda}{N}\big\}}{N}\\&=\frac{\#\big\{(P,P')\in\tilde{\square}_Q^2:|\sin\theta_{P,P'}(x,y)|\leq\beta_{Q,\lambda}\big\}}{N}\ .\end{aligned}\tag{2.2}$$

For $P=(q,a), P'=(q',a')$, $(x,y)\in\mathbb{R}^2$, we define

$$L_{P,P'}(x,y)=(a'-y)(q-x)-(a-y)(q'-x)=\begin{vmatrix}1&q&a\\1&q'&a'\\1&x&y\end{vmatrix}.$$

Then

$$|\sin\theta_{P,P'}(x,y)|=\frac{2\operatorname{Area}\triangle PP_{(x,y)}P'}{\|P_{(x,y)}P\|\cdot\|P_{(x,y)}P'\|}=\frac{|L_{P,P'}(x,y)|}{\|P_{(x,y)}P\|\cdot\|P_{(x,y)}P'\|}\ .$$

For each $P,P'\in\square_Q$, consider the weight

$$w_{P,P'}(Q,\lambda)=\operatorname{Area}\big\{(x,y)\in\mathbb{D}_0:|L_{P,P'}(x,y)|\leq\beta_{Q,\lambda}\|P_{(x,y)}P\|\cdot\|P_{(x,y)}P'\|\big\}.$$

From (2.2) and (1.1) we infer that

$$R_Q^{(2)}(\lambda)=R^{(2)}_{\mathbb{D}_0,Q}(\lambda)=\frac{1}{\pi r_0^2N}\sum_{(P,P')\in\tilde{\square}_Q^2}w_{P,P'}(Q,\lambda). \tag{2.3}$$

Denote

$$\gamma = \gamma_{P,P'}(Q) = \frac{\|OP\| \cdot \|OP'\|}{Q^2} = \frac{\sqrt{q^2+a^2}\,\sqrt{q'^2+a'^2}}{Q^2} ,$$

and define for every $\mu > 0$

$$\begin{aligned} & A_{P,P'}(Q,\mu) = \text{Area}\big\{ (x,y) \in \mathbb{D}_0 : |L_{P,P'}(x,y)| \le \mu \gamma_{P,P'}(Q) \big\}, \\ & G_Q(\mu) = \frac{1}{Q^2} \sum_{(P,P') \in \tilde{\square}_Q^2} A_{P,P'}(Q,\mu). \end{aligned} \tag{2.4}$$

In the remainder of this section we show that the asymptotic of $R_Q^{(2)}(\lambda)$ as $Q \to \infty$ is closely related to that of $G_Q\big(\frac{\pi\lambda}{2}\big)$.

For fixed P, P', denote by θ the angle between the line ℓ determined by P and P' and the horizontal direction. Also consider the lines $\ell_\pm$, parallel to ℓ and such that $\operatorname{dist}(\ell,\ell_\pm) = \frac{\mu\gamma\cos\theta}{|q'-q|}$. The equation of ℓ is given by

$$L_{P,P'}(x,y) = 0, \tag{ℓ}$$

while the equation of $\ell_\pm$ is given by

$$L_{P,P'}(x,y) = \pm \mu\gamma. \tag{$\ell_\pm$}$$

We see that

$$\operatorname{dist}(\ell_+,\ell_-) = \frac{2\mu\gamma}{|q'-q|} \cdot \cos\theta = \frac{2\mu\gamma}{\sqrt{(q'-q)^2+(a'-a)^2}} \le \frac{4\mu}{\sqrt{(q'-q)^2+(a'-a)^2}} .$$

The set whose area defines $A_{P,P'}(Q,\mu)$ is the intersection of the strip bounded by ℓ_+ and ℓ_- and the disc $\mathbb{D}_0$; thus

$$A_{P,P'}(Q,\mu) \le 2r_0 \operatorname{dist}(\ell_+,\ell_-) \le \frac{8\mu r_0}{\sqrt{(q'-q)^2+(a'-a)^2}} . \tag{2.5}$$

We also have

$$A_{P,P'}(Q,\mu) \ne 0 \quad \text{only if} \quad |a'q - aq'| \le 2\mu + |a'-a| + |q'-q|. \tag{2.6}$$

Lemma 2.1. *Let $\alpha \in (0,1]$. Let C be a compact set in $\mathbb{R}_+$. Then for all $\delta > 0$ and all $\mu \in C$*

$$\frac{1}{Q^2} \sum_{\substack{P \in \square_{Q^\alpha} \\ P' \in \square_Q \\ P \ne P'}} A_{P,P'}(Q,\mu) = O_{C,\mathbb{D}_0,\delta}(Q^{\alpha-1+\delta}).$$

Proof. The estimate (2.5) reads as $A_{P,P'}(Q,\mu) = O_{C,\mathbb{D}_0}\big(\frac{1}{\|PP'\|}\big)$. Combining it with (2.6) we see that it suffices to show that

$$A_Q := \sum_{\substack{P \in \square_{Q^\alpha},\, P' \in \square_Q \\ |a'q - aq'| \ll_{C,\delta} \|PP'\|}} \frac{1}{\|PP'\|} \ll_\delta Q^{\alpha+1+\delta}.$$

Taking $P'' = (q'', a'') = (q' - q, a' - a) \in \square_{2Q}$, we gather

$$\begin{aligned} A_Q &\leq \sum_{\substack{P \in \square_{Q^\alpha},\, O \neq P'' \in \square_{2Q} \\ |a'' q - a q''| \ll_C \|OP''\|}} \frac{1}{\|OP''\|} \\ &\leq \sum_{O \neq P'' \in \square_{2Q}} \frac{1}{\|OP''\|}\ \#\big\{(q,a) \in [-Q^\alpha, Q^\alpha]^2 : |a''q - aq''| \ll_C \|OP''\|\big\}. \end{aligned}$$

The two conditions above yield that (q, a) should belong to the intersection of the square $[-Q^\alpha, Q^\alpha]^2$ with a strip of width $\ll_C \frac{\|OP''\|}{\|OP''\|} = 1$ bounded by lines $y = \frac{a''}{q''}\,x \pm \lambda_C$. The number of integer lattice points inside this region is of order $O_C(Q^\alpha)$; thus

$$A_Q \ll_C Q^\alpha \sum_{O \neq P'' \in \square_{2Q}} \frac{1}{\|OP''\|} = Q^\alpha \sum_{0 < m^2 + n^2 \leq 4Q^2} \frac{1}{\sqrt{m^2 + n^2}}\ .$$

Since $r_2(k) = \{(m, n) \in \mathbb{Z}^2 : m^2 + n^2 = k\} = O_\delta(k^\delta)$, this gives

$$\begin{aligned} A_Q &\ll_C Q^\alpha \sum_{k=1}^{4Q^2} \sum_{m^2+n^2=k} \frac{1}{\sqrt{k}} = Q^\alpha \sum_{k=1}^{4Q^2} \frac{r_2(k)}{\sqrt{k}} \\ &\ll_\delta Q^\alpha \sum_{k=1}^{4Q^2} k^{\delta - \frac12} \ll_\delta Q^\alpha (Q^2)^{\delta + \frac12} = Q^{\alpha + 1 + 2\delta}, \end{aligned}$$

as desired. □

Lemma 2.2. *For every compact set $C \subset \mathbb{R}_+$ and every $\delta > 0$, there exist constants $M_1, M_2 > 0$ such that*

$$\begin{aligned} \frac{Q^2}{N}\ G_Q\bigg(\frac{\pi\lambda}{2} - M_1 Q^{-\frac13}\bigg) - M_2 Q^{-\frac13 + \delta} &\leq \pi r_0^2 R_Q^{(2)}(\lambda) \\ &\leq \frac{Q^2}{N}\ G_Q\bigg(\frac{\pi\lambda}{2} + M_1 Q^{-\frac13}\bigg) + M_2 Q^{-\frac13 + \delta}. \end{aligned}$$

Proof. The trivial estimate

$$\|P_{(x,y)}P\| = \|OP\| + O_{\mathbb{D}_0}(1)$$

and (2.1) yield, for all $P, P' \in \square_Q$, $\lambda \in C$, $(x, y) \in \mathbb{D}_0$, that

$$\begin{aligned} \beta_{Q,\lambda} \|P_{(x,y)}P\| \cdot \|P_{(x,y)}P'\| &= \beta_{Q,\lambda}\big(\|OP\| \cdot \|OP'\| + O_{\mathbb{D}_0}(Q)\big) \\ &= \bigg(\frac{\pi\lambda}{2Q^2} + O_C\Big(\frac{1}{Q^6}\Big)\bigg)\big(\|OP\| \cdot \|OP'\| + O_{\mathbb{D}_0}(Q)\big) \\ &= \frac{\pi\lambda \|OP\| \cdot \|OP'\|}{2Q^2} + O_{C,\mathbb{D}_0}\Big(\frac{1}{Q}\Big) \\ &= \frac{\pi\lambda}{2}\ \gamma_{P,P'}(Q) + O_{C,\mathbb{D}_0}\Big(\frac{1}{Q}\Big) \\ &= \gamma_{P,P'}(Q)\bigg(\frac{\pi\lambda}{2} + O_{C,\mathbb{D}_0}\Big(\frac{1}{Q\gamma_{P,P'}(Q)}\Big)\bigg). \end{aligned}$$

We first analyze the case $\min\{\|OP\|,\|OP'\|\} \geq Q^{\frac{2}{3}}$. Then $\gamma_{P,P'}(Q) \geq Q^{-\frac{2}{3}}$, and the relation above and the definitions of $w_{P,P'}$ and $A_{P,P'}$ yield $M_1 > 0$ such that

$$A_{P,P'}\bigg(Q, \frac{\pi\lambda}{2} - M_1 Q^{-\frac{1}{3}}\bigg) \leq w_{P,P'}(Q,\lambda) \leq A_{P,P'}\bigg(Q, \frac{\pi\lambda}{2} + M_1 Q^{-\frac{1}{3}}\bigg).$$

When $\min\{\|OP\|,\|OP'\|\} \leq Q^{\frac{2}{3}}$, we take $\alpha = \frac{2}{3}$ in Lemma 2.1. Since

$$\beta_{Q,\lambda}\|P_{(x,y)}P\| \cdot \|P_{(x,y)}P'\| \ll_C \pi\lambda\gamma_{P,P'}(Q) \qquad \text{as } Q \to \infty,$$

we get

$$\frac{1}{N} \sum_{\min\{\|OP\|,\|OP'\|\} \leq Q^{\frac{2}{3}}} w_{P,P'}(Q,\lambda) \ll_C \frac{1}{N} \sum_{\min\{\|OP\|,\|OP'\|\} \leq Q^{\frac{2}{3}}} A_{P,P'}(Q,\pi\lambda) \ll_{C,\delta} Q^{-\frac{1}{3}+\delta}.$$

□

3. A formula for $G_Q(\mu)$

An immediate consequence of (2.5) and (2.6) is that the contribution to $G_Q(\mu)$ of pairs of points $(P,P') \in \tilde{\square}_Q^2$ with $a' = a$ or with $q' = q$ is negligible. Indeed, we see from (2.6) that, when $a' = a \neq 0$, the term $A_{P,P'}(Q,\mu)$ is zero unless $|q'-q| \leq 2\mu + \frac{1}{|a|} \leq 2\mu + 1$; thus the total contribution of such points to $G_Q(\mu)$ is

$$\ll_C \frac{1}{Q^2} \sum_{\substack{|q| \leq Q \\ 0 < |q'-q| \leq 2\mu+1 \\ 0 < |a| \leq Q}} \frac{8\mu r_0}{|q'-q|} \ll_C \frac{1}{Q} \sum_{\substack{|q| \leq Q \\ 0 < |q'-q| < 2\mu+1}} \frac{1}{|q'-q|} \ll_C \frac{\log Q}{Q} \, .$$

The contribution of pairs of points $(P,P') \in \tilde{\square}_Q^2$ with $a' = a = 0$ to $G_Q(\mu)$ is

$$\ll_C \frac{1}{Q^2} \sum_{\substack{|q|,|q'| \leq Q \\ q' \neq q}} \frac{1}{|q'-q|} \ll \frac{\log Q}{Q} \, .$$

Similar estimates in the case $q' = q$ show that

$$G_Q(\mu) = \frac{1}{Q^2} \sum_{\substack{(P,P') \in \tilde{\square}_Q^2 \\ a' \neq a,\, q' \neq q}} A_{P,P'}(Q,\mu) \, + O_{C,\mathbb{D}_0}\bigg(\frac{\log Q}{Q}\bigg). \tag{3.1}$$

As a result, we shall subsequently assume that $a' \neq a$ and $q' \neq q$. We now set

$$\alpha = \frac{a'-a}{q'-q}\,, \qquad \beta = \frac{aq'-a'q}{q'-q}\,, \qquad \gamma_0 = \frac{\gamma}{|q'-q|} = \frac{\sqrt{q^2+a^2}\sqrt{q'^2+a'^2}}{Q^2|q'-q|}\,.$$

The remainder of this section is elementary and is concerned with putting $G_Q(\mu)$ in a tidy form, suitable for a precise estimation which will be completed in the next section.

Let C_0 denote the center of $\mathbb{D}_0$, let ℓ' be the line passing through C_0 and perpendicular to ℓ, and denote by A_+ and A_- the intersections of ℓ' with the circle

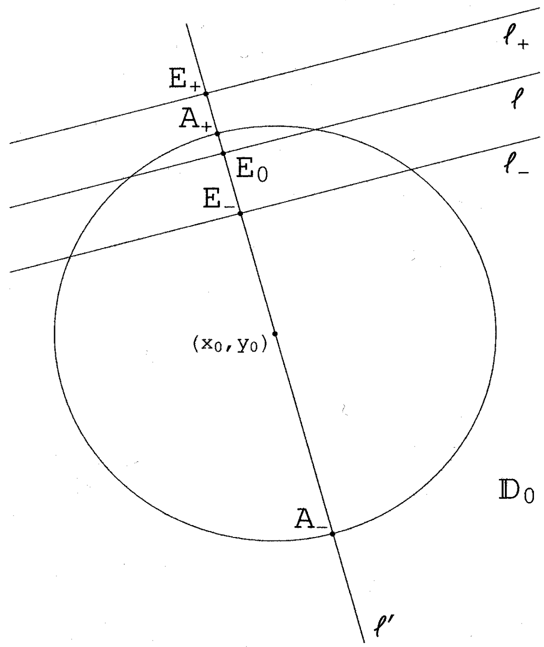

FIGURE 1. The intersection of the strip bounded by ℓ_+ and ℓ_- with $\mathbb{D}_0$.

$\partial\mathbb{D}_0$, by E_0 the intersection of ℓ' and ℓ, and by $E_\pm$ the intersection of ℓ' with $\ell_\pm$. Direct computation gives

$$\begin{aligned} x_{A_\pm} &= x_0 \mp \frac{\alpha r_0}{\sqrt{\alpha^2+1}} = x_0 \mp \frac{(a'-a)r_0}{\sqrt{(q'-q)^2+(a'-a)^2}}\,, \\ x_{E_\pm} &= \frac{\alpha y_0 + x_0 - \alpha\beta \mp \alpha\mu\gamma_0}{\alpha^2+1}\,, \\ \|E_-E_+\| &= \operatorname{dist}(\ell_+,\ell_-) = \frac{|x_{E_-}-x_{E_+}|}{|\sin\theta|} = \frac{2\mu\gamma_0}{\sqrt{\alpha^2+1}} = \frac{\mu\sqrt{q^2+a^2}\sqrt{q'^2+a'^2}}{Q^2\|PP'\|}\,. \end{aligned}$$

While ordering the points $x_{E_+} < x_{E_-}$ and $x_{A_+} < x_{A_-}$, the following situations may occur:

Case 1. $x_{E_+} < x_{A_+} < x_{A_-} < x_{E_-}$, that is,

$$\frac{\alpha y_0 + x_0 - \alpha\beta - \alpha\mu\gamma_0}{\alpha^2+1} < x_0 - \frac{\alpha r_0}{\sqrt{\alpha^2+1}} < x_0 + \frac{\alpha r_0}{\sqrt{\alpha^2+1}} < \frac{\alpha y_0 + x_0 - \alpha\beta + \alpha\mu\gamma_0}{\alpha^2+1}\,.$$

This gives $\mu\gamma_0 > r_0\sqrt{\alpha^2+1}$, hence

$$r_0\sqrt{(q'-q)^2+(a'-a)^2} < \frac{\mu\sqrt{q^2+a^2}\sqrt{q'^2+a'^2}}{Q^2} \le 2\mu.$$

Suppose first that $|a'-a| \le |q'-q|$. By (2.6) we know that for fixed (q,q'), the expression $D = aq' - a'q$ only takes values between $-2\mu - \frac{4\mu}{r_0}$ and $2\mu + \frac{4\mu}{r_0}$. Hence the number of solutions (a,a') of $aq' - a'q = D$ is of order $O_C(d)$, where d is the greatest common divisor of q and q'. But $d \leq \frac{2\mu}{r_0}$, hence this order is actually $O_C(1)$; thus the contribution to G_Q is

$$\ll_C \frac{1}{Q^2} \sum_{\substack{|q|,|q'|\leq Q \\ 0<|q'-q|\ll 1}} \sum_{\substack{|a|,|a'|\leq Q \\ A_{P,P'}\neq 0 \\ 0<|a'-a|\leq |q'-q|}} 1 \ll \frac{1}{Q^2} \sum_{\substack{|q|,|q'|\leq Q \\ 0<|q'-q|\ll 1}} 1 \ll \frac{1}{Q} .$$

The case $|q'-q| \leq |a'-a|$ is settled similarly by first summing over (a,a').

Case 2. $x_{A_+} < x_{E_+} < x_{A_-} < x_{E_-}$, that is,

$$|\mu\gamma_0 - r_0\sqrt{\alpha^2+1}| = r_0\sqrt{\alpha^2+1} - \mu\gamma_0 < y_0 - \alpha x_0 - \beta < \mu\gamma_0 + r_0\sqrt{\alpha^2+1} ,$$

or equivalently

$$\left| a'q - aq' + (q'-q)y_0 - (a'-a)x_0 - r_0\sqrt{(q'-q)^2+(a'-a)^2} \right| < \mu\gamma.$$

The change of variables $a'-a = a''$, $q'-q = q''$ gives

$$|a''q - aq'' - r_0\sqrt{q''^2+a''^2} + q''y_0 - a''x_0| < \mu\gamma \leq 2\mu.$$

So, keeping a'' and q'' fixed, the range of $a''q - aq''$ has cardinality $O_C(1)$. Now the equation $a''q - aq'' = K$ has either no solution (q,a) when $d = \gcd(a'',q'')$ does not divide K, or has $O\big(\frac{dQ}{q''}\big)$ solutions (q,a) when d divides K. Thus the contribution of terms $A_{P,P'}$ with $q''^2 + a''^2 = (q'-q)^2 + (a'-a)^2 > Q$ is

$$\ll_C \frac{1}{Q^2} \sum_{d=1}^{Q} \sum_{\substack{0<|q_0''|,|a_0''|\leq \left[\frac{Q}{d}\right] \\ \gcd(q_0'',a_0'')=1}} \frac{Q}{q_0''} \cdot \frac{1}{\sqrt{Q}} \leq \frac{1}{Q\sqrt{Q}} \sum_{d=1}^{Q} \frac{Q}{d} \sum_{0<|q_0''|\leq \left[\frac{Q}{d}\right]} \frac{1}{q_0''}$$

$$\ll \frac{\log^2 Q}{\sqrt{Q}} \ll_\delta Q^{-\frac{1}{2}+\delta}.$$

The contribution of terms $A_{P,P'}$ with $q''^2 + a''^2 = (q'-q)^2 + (a'-a)^2 \leq Q$ is

$$\ll_C \frac{1}{Q^2} \sum_{1\leq d\leq \sqrt{Q}} \sum_{0<|q_0''|,|a_0''|\ll \left[\frac{\sqrt{Q}}{d}\right]} \frac{Q}{q_0''} \cdot \frac{1}{dq_0''} \leq \frac{1}{Q} \sum_{d,q_0''=1}^{\infty} \frac{\sqrt{Q}}{d^2 q_0''} \ll Q^{-\frac{1}{2}} .$$

Case 3. $x_{E_+} < x_{A_+} < x_{E_-} < x_{A_-}$, that is,

$$-\mu\gamma_0 - r_0\sqrt{\alpha^2+1} < y_0 - \alpha x_0 - \beta < -|\mu\gamma_0 - r_0\sqrt{\alpha^2+1}|.$$

We infer as in Case 2 that the contribution of $A_{P,P'}$ is $O_\delta(Q^{-\frac{1}{2}+\delta})$ in this case too.

Case 4. $x_{A_+} < x_{E_+} < x_{E_-} < x_{A_-}$, that is,

$$\mu\gamma_0 - r_0\sqrt{\alpha^2+1} < y_0 - \alpha x_0 - \beta < -\mu\gamma_0 + r_0\sqrt{\alpha^2+1} ,$$

or equivalently

$$|L_{P,P'}(x_0,y_0)| < r_0\sqrt{(q'-q)^2+(a'-a)^2} - \mu\gamma = r_0\|PP'\| - \mu\gamma.$$

Denote $k = q'-q$ and $\ell = a'-a$. The interval $I_{k,\ell} = r_0\sqrt{k^2+\ell^2} - ky_0 + \ell x_0 + [-\mu\gamma, 0]$ has length $\mu\gamma \ll_C 1$. Hence we find that the contribution to G_Q of terms $A_{P,P'}$ for

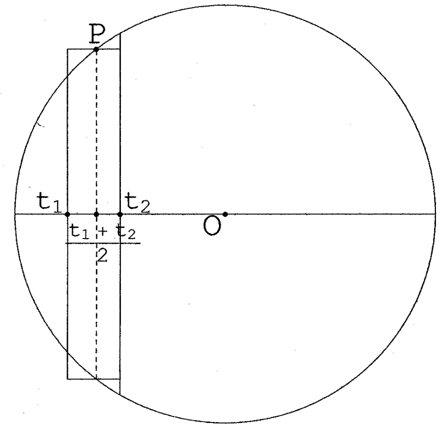

FIGURE 2.

which $r_0\|PP'\| - \mu\gamma < L_{P,P'}(x_0,y_0) < r_0\|PP'\|$ is

$$\ll_C \frac{1}{Q^2} \sum_{0<|k|,|\ell|\leq Q} \sum_{\substack{|q'|,|a'|\leq Q\\ -ka'-\ell q'\in I_{k,\ell}}} \frac{1}{\sqrt{k^2+\ell^2}} \ll \frac{1}{Q^2}\sum_{k,\ell=1}^{Q} \frac{Q\gcd(k,l)}{k}\cdot\frac{1}{\sqrt{k^2+\ell^2}}$$

$$\leq \frac{1}{Q}\sum_{d=1}^{Q}\frac{1}{d}\sum_{k_0,\ell_0=1}^{\left[\frac{Q}{d}\right]}\frac{1}{k_0\sqrt{k_0^2+\ell_0^2}} \ll \frac{1}{Q}\sum_{d=1}^{Q}\frac{1}{d}\sum_{k_0=1}^{\left[\frac{Q}{d}\right]}\frac{1}{k_0}\sum_{\ell_0=1}^{\left[\frac{Q}{d}\right]}\frac{1}{\ell_0}$$

$$\ll \frac{\log^3 Q}{Q}\,.$$

One similarly shows that the contribution of points P,P' for which $-r_0\|PP'\| + \mu\gamma < L_{P,P'}(x_0,y_0) < r_0\|PP'\|$ is of the same order. By (3.1) and the previous considerations we infer that

$$G_Q(\mu) = \frac{1}{Q^2}\sum_{\substack{(P,P')\in\tilde{\square}_Q^2\\ a'\neq a,\, q'\neq q\\ |L_{P,P'}(x_0,y_0)|<r_0\|PP'\|}} A_{P,P'}(Q,\mu) + O_{C,\mathbb{D}_0,\delta}(Q^{-\frac{1}{2}+\delta}). \tag{3.2}$$

The weights $A_{P,P'}$ are next approximated by elementary calculus.

Lemma 3.1. *The area of the region inside the circle of radius r_0 centered at the origin and inside the strip bounded by the vertical lines $y = t_1$ and $y = t_2$, $-r_0 < t_1 \leq t_2 < r_0$, is given, for small $t_2 - t_1$, by*

$$A_{r_0}(t_1,t_2) = 2(t_2-t_1)\sqrt{r_0^2 - \left(\frac{t_1+t_2}{2}\right)^2} + O_{r_0}\big((t_2-t_1)^{\frac{3}{2}}\big).$$

Proof. The error is seen to be given (see Figure 2) by

$$\int_{t_1}^{\frac{t_1+t_2}{2}} \left(\sqrt{r_0^2-\left(\frac{t_1+t_2}{2}\right)^2} - \sqrt{r_0^2-t^2}\right)dt + \int_{\frac{t_1+t_2}{2}}^{t_2}\left(\sqrt{r_0^2-t^2} - \sqrt{r_0^2-\left(\frac{t_1+t_2}{2}\right)^2}\right)dt.$$

It is $\ll (t_2-t_1)^{3/2}$ as a result of

$$\big|\sqrt{r_0^2-x^2}-\sqrt{r_0^2-y^2}\,\big| \leq \sqrt{|x^2-y^2|} \ll_{r_0} \sqrt{|x-y|}, \qquad x,y\in[-r_0,r_0].$$

□

We take

$$t_{E_\pm} = \frac{x_{E_\pm}-x_{E_0}}{\sin\theta} = \frac{\sqrt{1+\alpha^2}}{\alpha}\,(x_{E_\pm}-x_{E_0}) = \frac{L_{P,P'}(x_0,y_0)\mp\mu\gamma}{\|PP'\|}\ .$$

Note now that

$$t_{E_-}-t_{E_+} = \frac{2\mu\gamma}{\|PP'\|} \ll_C \frac{1}{\|PP'\|}\ , \tag{3.3}$$

denote $k=q'-q$, $\ell=a'-a$,

$$J_{k,\ell} = -ky_0+\ell x_0 + \big[-r_0\sqrt{k^2+\ell^2}, r_0\sqrt{k^2+\ell^2}\,\big], \tag{3.4}$$

and find that the contribution of the error provided by Lemma 3.1 in (3.2) is

$$\begin{aligned}
\frac{1}{Q^2}\sum_{\substack{(P,P')\in\tilde\square_Q^2\\ q'\neq q,\,a'\neq a\\ |L_{P,P'}(x_0,y_0)|<r_0\|PP'\|}} \frac{1}{\|PP'\|^{3/2}} &= \frac{1}{Q^2}\sum_{0<|k|,|\ell|\leq Q}\ \sum_{\substack{|q'|,|a'|\leq Q\\ ka'-\ell q'\in J_{k,\ell}}} \frac{1}{(k^2+\ell^2)^{3/4}}\\
&\ll \frac{1}{Q^2}\sum_{k,\ell=1}^{Q} \frac{Q\gcd(k,\ell)}{k}\cdot\sqrt{k^2+\ell^2}\cdot\frac{1}{(k^2+\ell^2)^{3/4}}\\
&\leq \frac{1}{Q^2}\sum_{d=1}^{Q}\sum_{k_0,\ell_0=1}^{\left[\frac{Q}{d}\right]} \frac{Qd}{dk_0}\cdot d(k_0^2+\ell_0^2)^{\frac12}\cdot\frac{1}{d^{3/2}(k_0^2+\ell_0^2)^{3/4}}\\
&= \frac{1}{Q}\sum_{d=1}^{Q}\frac{1}{d^{1/2}}\sum_{k_0,\ell_0=1}^{\left[\frac{Q}{d}\right]}\frac{1}{k_0(k_0^2+\ell_0^2)^{1/4}}\\
&\leq \frac{1}{Q}\sum_{d=1}^{Q}\frac{1}{d^{1/2}}\sum_{k_0=1}^{\left[\frac{Q}{d}\right]}\frac{1}{k_0}\sum_{\ell_0=1}^{\left[\frac{Q}{d}\right]}\frac{1}{\ell_0^{1/2}}\\
&\ll \frac{\log Q}{Q}\sum_{d=1}^{Q}\frac{1}{d^{1/2}}\cdot\frac{Q^{1/2}}{d^{1/2}}\\
&\ll \frac{\log^2 Q}{Q^{1/2}}\ .
\end{aligned}\tag{3.5}$$

By (3.2), Lemma 3.1, (3.3) and (3.5) we find that

$$G_Q(\mu) = \frac{1}{Q^2}\sum_{\substack{(P,P')\in\tilde\square_Q^2\\ q'\neq q,\,a'\neq a\\ |L_{P,P'}(x_0,y_0)|<r_0\|PP'\|}} B_{P,P'}(\mu)\ + O_{C,\mathbb{D}_0,\delta}(Q^{-\frac12+\delta}), \tag{3.6}$$

where $B_{P,P'}(Q,\mu)$ denotes the contribution of the main term in Lemma 3.1 to (3.2), that is,

$$\begin{aligned} B_{P,P'}(Q,\mu) &= 2(t_{E_-} - t_{E_+})\sqrt{r_0^2 - \left(\frac{t_{E_-}+t_{E_+}}{2}\right)^2} \\ &= 2\cdot\frac{2\mu\gamma}{\|PP'\|}\sqrt{r_0^2 - \frac{L_{P,P'}(x_0,y_0)^2}{\|PP'\|^2}} \\ &= \frac{4\mu\gamma\sqrt{r_0^2\|PP'\|^2 - L_{P,P'}(x_0,y_0)^2}}{\|PP'\|^2}\,. \end{aligned} \tag{3.7}$$

Finally we show that one can replace $\frac{\gamma}{\|PP'\|^2}$ by $\frac{qq'}{Q^2\max\{(q'-q)^2,(a'-a)^2\}}$ in (3.7) and (3.6). To see this, we assume without loss of generality that $|a'-a|\le|q'-q|$. Since $|L_{P,P'}(x_0,y_0)|<r_0\|PP'\|$, then $|a'q-aq'|\le\big(r_0+\sqrt{x_0^2+y_0^2}\,\big)\|PP'\|$ and

$$\begin{aligned} \left| q\sqrt{1+\frac{(a'-a)^2}{(q'-q)^2}} - q\sqrt{1+\frac{a^2}{q^2}}\,\right| &\le q\cdot\frac{\left|\frac{a'-a}{q'-q}-\frac{a}{q}\right|\left(\left|\frac{a'-a}{q'-q}\right|+\left|\frac{a}{q}\right|\right)}{\sqrt{1+\frac{(a'-a)^2}{(q'-q)^2}}+\sqrt{1+\frac{a^2}{q^2}}} \\ &\le q\cdot\frac{|a'q-aq'|}{q|q'-q|}\cdot\frac{1+\left|\frac{a}{q}\right|}{\sqrt{1+\frac{a^2}{q^2}}} \ll \frac{|a'q-aq'|}{|q'-q|} \ll \frac{|a'q-aq'|}{\|PP'\|} \ll_{\mathbb{D}_0} 1. \end{aligned}$$

This gives

$$\sqrt{q^2+a^2} = q\sqrt{1+\frac{(a'-a)^2}{(q'-q)^2}} + O_{\mathbb{D}_0}(1),$$

and similarly

$$\sqrt{q'^2+a'^2} = q'\sqrt{1+\frac{(a'-a)^2}{(q'-q)^2}} + O_{\mathbb{D}_0}(1).$$

Hence one can replace $B_{P,P'}(Q,\mu)$ in (3.6) by

$$W_{P,P'}(Q,\mu) = \frac{4\mu qq'\sqrt{r_0^2\|PP'\|^2 - L_{P,P'}(x_0,y_0)^2}}{Q^2\max\{(q'-q)^2,(a'-a)^2\}}\,,$$

at the cost of an error which is found to be, as in (3.5),

$$\begin{aligned} &\ll \frac{1}{Q^2}\sum_{\substack{(P,P')\in\tilde{\square}_Q^2\\ q'\ne q,\ a'\ne a\\ |L_{P,P'}(x_0,y_0)|<r_0\|PP'\|}} \frac{1}{Q\sqrt{(q'-q)^2+(a'-a)^2}} \\ &\qquad \ll \frac{1}{Q^3}\sum_{0<|k|,|\ell|\le Q}\sqrt{k^2+\ell^2}\cdot\frac{Q\gcd(k,\ell)}{k}\cdot\frac{1}{\sqrt{k^2+\ell^2}} \\ &\qquad \le \frac{1}{Q^3}\sum_{d=1}^{Q}\sum_{k_0,l_0=1}^{\left[\frac{Q}{d}\right]}\frac{Q}{k_0} \ll \frac{\log Q}{Q^2}\sum_{d=1}^{Q}\frac{Q}{d} \ll \frac{\log^2 Q}{Q}\,. \end{aligned}$$

In summary, we have shown for any μ in a fixed compact set $C \subset \mathbb{R}_+$ that

$$G_Q(\mu) = \frac{4\mu}{Q^4} \sum_{\substack{(P,P') \in \tilde{\square}_Q^2 \\ q' \neq q,\ a' \neq a \\ |L_{P,P'}(x_0,y_0)| < r_0 \|PP'\|}} \frac{qq' \sqrt{r_0^2 \|PP'\|^2 - L_{P,P'}(x_0,y_0)^2}}{\max\{(q'-q)^2, (a'-a)^2\}} + O_{C,\mathbb{D}_0,\delta}(Q^{-\frac{1}{2}+\delta}) \tag{3.8}$$

as $Q \to \infty$.

4. Estimating the sum S_Q

By reflecting $\mathbb{D}_0$ about the axes and about the line $y = x$, we see that it suffices to only estimate the contribution $A_Q(\mu)$ to $G_Q(\mu)$ of points $(P, P') \in (0, Q]^2$ with $0 < \alpha = \frac{a'-a}{q'-q} \leq 1$. We thus consider

$$A_Q(\mu) = 4\mu S_Q,$$

where

$$\begin{aligned} S_Q &= \frac{1}{Q^4} \sum_{\substack{0<q,q' \leq Q \\ 0<a,a' \leq Q \\ |L_{P,P'}(x_0,y_0)| < r_0 \|PP'\| \\ 0 < \frac{a'-a}{q'-q} \leq 1}} \frac{qq' \sqrt{r_0^2 \|PP'\|^2 - L_{P,P'}(x_0,y_0)^2}}{(q'-q)^2} \\ &= \frac{2}{Q^4} \sum_{\substack{0<a<a' \leq Q \\ 0<q<q' \leq Q \\ |L_{P,P'}(x_0,y_0)| < r_0 \|PP'\| \\ 0 < \frac{a'-a}{q'-q} \leq 1}} \frac{qq' \sqrt{r_0^2 \big((q'-q)^2 + (a'-a)^2\big) - L_{P,P'}(x_0,y_0)^2}}{(q'-q)^2} . \end{aligned}$$

Then we gather from (3.8) and the above formula for S_Q that

$$G_Q(\mu) = 8A_Q(\mu) + O_{C,\mathbb{D}_0,\delta}(Q^{-\frac{1}{10}+\delta}) = 32\mu S_Q + O_{C,\mathbb{D}_0,\delta}(Q^{-\frac{1}{10}+\delta}). \tag{4.1}$$

Changing q to $q' - q$ and a to $a' - a$, we may write

$$S_Q = \frac{2}{Q^4} \sum_{\substack{0<a<a' \leq Q \\ a \leq q < q' \leq Q \\ |y_0 q - x_0 a + aq' - a'q| < r_0 \sqrt{q^2+a^2}}} \frac{(q'-q)q' \sqrt{r_0^2(q^2+a^2) - (y_0 q - x_0 a + aq' - a'q)^2}}{q^2} .$$

Putting

$$D = aq' - a'q$$

and taking $J_{q,a}$ as in (3.4), that is,

$$J_{q,a} = -qy_0 + ax_0 + \big[-r_0 \sqrt{q^2+a^2}, r_0 \sqrt{q^2+a^2} \,\big],$$

we get

$$S_Q = \frac{2}{Q^4} \sum_{\substack{1 \leq a \leq q \leq Q \\ D \in J_{q,a}}} \sum_{\substack{q' \in [q,Q] \\ a' \in [a,Q] \\ aq' - a'q = D}} \frac{(q'-q)q' \sqrt{r_0^2(q^2+a^2) - (y_0 q - x_0 a + D)^2}}{q^2} . \tag{4.2}$$

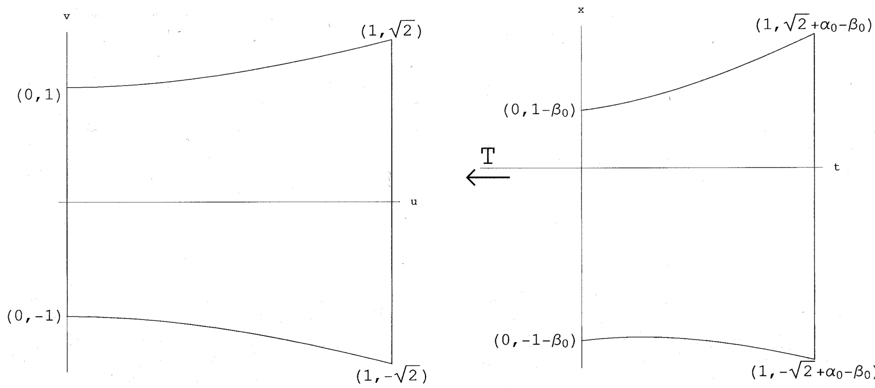

FIGURE 3. The regions $\mathcal{D}_{0,0}$ and $\mathcal{D}_{\alpha_0,\beta_0}$.

In this section we will prove

Proposition 4.1. $S_Q = \dfrac{\pi r_0^2}{6} + O_{\mathbb{D}_0,\delta}(Q^{-\frac{1}{10}+\delta})$ *for all* $\delta > 0$.

From this and (4.1) we infer

Corollary 4.2. $G_Q(\mu) = \dfrac{16\pi r_0^2 \mu}{3} + O_{C,\mathbb{D}_0,\delta}(Q^{-\frac{1}{10}+\delta})$.

Theorem 1.1 now follows combining Corollary 4.2 with Lemma 2.2.

We now start the proof of Proposition 4.1. We first lay out some notation and prove an elementary calculus lemma. Fix $\alpha_0, \beta_0 \in \mathbb{R}$ and consider the function

$$\Phi(t,x) = \Phi_{\alpha_0,\beta_0}(t,x) = 1 + t^2 - (\beta_0 - t\alpha_0 + x)^2$$

and the domain

$$\mathcal{D} = \mathcal{D}_{\alpha_0,\beta_0} = \{(t,x) : 0 \le t \le 1,\ \Phi(t,x) \ge 0\}.$$

Also consider the projection $\mathrm{pr}_2\,\mathcal{D}$ of $\mathcal{D}$ on the second coordinate, the x-section

$$I_x = \{t \in [0,1] : \Phi(t,x) \ge 0\}, \tag{4.3}$$

and the t-section

$$J_t = \{x \in \mathrm{pr}_2\,\mathcal{D} : \Phi(t,x) \ge 0\}. \tag{4.4}$$

Define the function $\psi = \psi_{\alpha_0,\beta_0} : \mathrm{pr}_2\,\mathcal{D} \to [0,\infty)$ by

$$\psi(x) = \int_{I_x} \sqrt{\Phi(t,x)}\ dt.$$

Lemma 4.3. *For every* α_0 *and* β_0

(i) $\displaystyle\int_{\mathrm{pr}_2\,\mathcal{D}_{\alpha_0,\beta_0}} |\psi'(x)|\,dx \le \sqrt{2} + \log(1+\sqrt{2})$.

(ii) $\displaystyle\int_{\mathrm{pr}_2\,\mathcal{D}_{\alpha_0,\beta_0}} \psi(x)\,dx = \iint_{\mathcal{D}_{\alpha_0,\beta_0}} \sqrt{\Phi(t,x)}\ dt\,dx = \tfrac{2\pi}{3}$.

Proof. (i) By the definition of Φ it is seen that I_x is the union of one or two intervals $[a(x), b(x)]$, where $a(x)$ and $b(x)$ are equal to 0, 1, or a root of $\Phi(t,x) = 0$. In all these cases

$$\Phi\big(a(x),x\big)a'(x) = \Phi\big(b(x),x\big)b'(x) = 0,$$

and as a result we get

$$\frac{d}{dx}\int\limits_{a(x)}^{b(x)} \sqrt{\Phi(t,x)}\,dt = \int\limits_{a(x)}^{b(x)} \frac{t\alpha_0-\beta_0-x}{\sqrt{\Phi(t,x)}}\,dt$$

and

$$\psi'(x) = \frac{d}{dx}\int\limits_{I_x} \sqrt{\Phi(t,x)}\,dt = \int\limits_{I_x} \frac{t\alpha_0-\beta_0-x}{\sqrt{\Phi(t,x)}}\,dt.$$

Using the triangle inequality and the change of variables $(u,v)=T(t,x)=(t,t\alpha_0-\beta_0-x)$ we obtain

$$\begin{aligned}\int\limits_{\mathrm{pr}_2\,\mathcal{D}_{\alpha_0,\beta_0}} |\psi'(x)|\,dx &= \int\limits_{\mathrm{pr}_2\,\mathcal{D}_{\alpha_0,\beta_0}} \left| \int\limits_{I_x} \frac{t\alpha_0-\beta_0-x}{\sqrt{\Phi(t,x)}}\,dt\right| dx \\ &\leq \int\limits_{\mathrm{pr}_2\,\mathcal{D}_{\alpha_0,\beta_0}} \int\limits_{I_x} \frac{|t\alpha_0-\beta_0-x|}{\sqrt{\Phi(t,x)}}\,dt\,dx = \iint\limits_{\mathcal{D}_{\alpha_0,\beta_0}} \frac{|t\alpha_0-\beta_0-x|}{\sqrt{1+t^2-(\beta_0-t\alpha_0+x)^2}}\,dt\,dx \\ &= \iint\limits_{\mathcal{D}_{0,0}} \frac{|v|}{\sqrt{1+u^2-v^2}}\,du\,dv = \sqrt{2}+\log(1+\sqrt{2}\,).\end{aligned}$$

(ii) The same change of variable as in (i) gives

$$\iint\limits_{\mathcal{D}_{\alpha_0,\beta_0}} \sqrt{\Phi(t,x)}\,dt\,dx = \iint\limits_{\mathcal{D}_{0,0}} \sqrt{1+u^2-v^2}\,du\,dv = \frac{2\pi}{3}\ .$$

□

We start to evaluate S_Q. If $D\in J_{q,a}$, then $D\in\Omega_q=\big(-(2+r_0\sqrt{2})q,(2+r_0\sqrt{2})q\big)$. The equality $aq'-a'q=D$ is equivalent to $a'=\frac{aq'-D}{q}$. Hence in the inner sum in (4.2) we should sum over $q'\in[q,Q]$ such that $aq'=D \pmod q$ and

$$a\leq \frac{aq'-D}{q}\leq Q,$$

or equivalently

$$\max\left\{q,q+\frac{D}{a}\right\}\leq q'\leq \min\left\{Q,\frac{qQ+D}{a}\right\}. \tag{4.5}$$

Next we show that the summation condition in the inner sum from (4.2) can be replaced by only $q'\in[q,Q]$ and $aq'=D \pmod q$. To see this, note first that for q' fixed, the relations $D=aq' \pmod q$ and $|D|\leq(r_0+\sqrt{2})q$ imply that D takes at most $O_{r_0}(1)$ values. So the total contribution to S_Q of terms with $0\leq q-a\leq 2+[r_0]$ is

$$\ll_{r_0} \frac{1}{Q^4}\sum_{q=1}^{Q}\sum_{q'=1}^{Q}\frac{(q'-q)q'}{q^2}\cdot q \ll \frac{1}{Q}\sum_{q=1}^{Q}\frac{1}{q}\ll \frac{\log Q}{Q}\ .$$

When $q-a\geq 3+[r_0]$, we get $-D\leq(3+[r_0])Q\leq(q-a)Q$; thus $Q\leq\frac{qQ+D}{a}$ and we can simply replace the second inequality in (4.5) by $q'\leq Q$. Suppose now that q' is between q and $q+\frac{D}{a}$. Owing again to $aq'=D \pmod q$, it follows that $aq'=D+kq$ for some integer k. But the range of q' is an interval of length $\frac{|D|}{a}$, hence the range of $k = \frac{aq'}{q} - \frac{D}{q}$ is an interval of length $\ll 1 + \frac{a}{q} \cdot \frac{|D|}{a} \leq 4 + [r_0]$. Thus k, and consequently q', take at most $O_{r_0}(1)$ values. Besides, in this case we have $0 \leq q' - q \leq \frac{|D|}{a}$. Hence the contribution to S_Q of terms with q' between q and $q + \frac{D}{a}$ is

$$\ll_{r_0} \frac{1}{Q^4} \sum_{1\leq a\leq q\leq Q} \sum_{D\in\Omega_q} \frac{\frac{|D|}{a} \cdot q}{q^2} \cdot q = \frac{1}{Q^4} \sum_{1\leq a\leq q\leq Q} \frac{1}{a} \sum_{D\in\Omega_q} |D|$$

$$\ll_{r_0} \frac{1}{Q^4} \sum_{a=1}^{Q} \frac{1}{a} \sum_{q=1}^{Q} q^2 \ll \frac{\log Q}{Q} \,.$$

Therefore we have shown that, up to an error term of order $\ll_{r_0} \frac{\log Q}{Q}$, one can replace the summation conditions in the inner sum from (4.2) by $q' \in [q, Q]$ and $aq' = D \pmod{q}$.

We write $x_0 = r_0\alpha_0$ and $y_0 = r_0\beta_0$. Take $\mathcal{D} = \mathcal{D}_{\alpha_0,\beta_0}$, $\Phi = \Phi_{\alpha_0,\beta_0}$ and $\psi = \psi_{\alpha_0,\beta_0}$, unless otherwise specified, and note that

$$r_0^2(q^2 + a^2) - (y_0 q - x_0 a + D)^2 = r_0^2 q^2 \Phi\left(\frac{a}{q}, \frac{D}{r_0 q}\right).$$

Then (4.1) and the previous considerations lead to

$$S_Q = \frac{2r_0}{Q^4} \sum_{1\leq a\leq q\leq Q} \sum_{D\in r_0 q J_{\frac{a}{q}}} \sum_{\substack{q'\in[q,Q] \\ aq'=D \pmod{q}}} \frac{(q'-q)q'}{q} \sqrt{\Phi\left(\frac{a}{q}, \frac{D}{r_0 q}\right)} + O_{r_0}\left(\frac{\log Q}{Q}\right)$$

$$= \frac{2r_0}{Q^4} \sum_{q=1}^{Q} \frac{1}{q} \sum_{D\in r_0 q \,\mathrm{pr}_2\, \mathcal{D}} \sum_{\substack{q'\in[q,Q] \\ a\in qI_{\frac{D}{r_0 q}} \\ aq'=D \pmod{q}}} (q'-q)q' \sqrt{\Phi\left(\frac{a}{q}, \frac{D}{r_0 q}\right)} + O_{r_0}\left(\frac{\log Q}{Q}\right),$$

where I_x and J_t are defined as in (4.3) and (4.4).

Take $d = \gcd(q, q')$ and write $q = dq_0$, $q' = dq_0'$. Then d divides D, so $D = dD_0$. The congruence $adq_0' = D \pmod{dq_0}$ is equivalent to $aq_0' = D_0 \pmod{q_0}$, and we may write the expression S_Q, up to an error term of order $O_{r_0}\big(\frac{\log Q}{Q}\big)$, as

$$\frac{2r_0}{Q^4} \sum_{d=1}^{Q} \sum_{q_0=1}^{\left[\frac{Q}{d}\right]} \frac{d}{q_0} \sum_{D_0\in r_0 q_0 \,\mathrm{pr}_2\, \mathcal{D}} \sum_{\substack{q_0'\in\left[q_0, \frac{Q}{d}\right] \\ \gcd(q_0', q_0)=1 \\ a\in dq_0 I_{\frac{D_0}{r_0 q_0}} \\ aq_0'=D_0 \pmod{q_0}}} (q_0'-q_0)q_0' \sqrt{\Phi\left(\frac{a}{dq_0}, \frac{D_0}{r_0 q_0}\right)}. \tag{4.6}$$

To estimate the inner sum above, we need some information about the distribution of solutions of the congruence $xy = h \pmod{q}$. We shall employ the following result, which is a consequence of Proposition A4 from the Appendix.

Proposition 4.4. *Assume that $q \geq 1$ and h are two given integers, $\mathcal{I}$ and $\mathcal{J}$ are intervals, and $f : \mathcal{I} \times \mathcal{J} \to \mathbb{R}$ is a C^1 function. Then for every integer $T > 1$ and* *every* $\delta > 0$

$$\sum_{\substack{a\in\mathcal{I},\, b\in\mathcal{J}\\ ab=h \pmod q\\ \gcd(b,q)=1}} f(a,b) = \frac{\varphi(q)}{q^2} \iint\limits_{\mathcal{I}\times\mathcal{J}} f(x,y)dxdy + \mathcal{E},$$

where

$$\mathcal{E} \ll_\delta \left(1+\frac{|\mathcal{I}|}{q}\right)\left(1+\frac{|\mathcal{J}|}{q}\right)Tq^{\frac{1}{2}+\delta}\gcd(h,q)^{\frac{1}{2}}\big(T\|f\|_\infty + q\|\nabla f\|_\infty\big) + \frac{|\mathcal{I}|\,|\mathcal{J}|\|\nabla f\|_\infty}{T}\ ,$$

and we denote $\|\cdot\|_\infty = \|\cdot\|_{\infty,\mathcal{I}\times\mathcal{J}}$.

We now return to the formula of S_Q given in (4.6) and first give an upper bound for the contribution to S_Q of quadruples (d, q_0, D_0, a) for which

$$(4.7)\qquad 0 \le r_0^2 d^2 q_0^2 \Phi\left(\frac{a}{dq_0}, \frac{D_0}{r_0 q_0}\right) = r_0^2(a^2 + d^2 q_0^2) - (dq_0 y_0 - a x_0 + dD_0)^2 \le L^2,$$

with $L = L_{q_0} > 1$ to be chosen later.

Lemma 4.5. *Let* $F(a) = ua^2 + va + w$ *with* $u \neq 0$. *Then for any* K *and* L

$$\big|\{a \in \mathbb{R} : K \le F(a) \le K + L^2\}\big| \le \frac{2|L|}{\sqrt{|u|}}\ .$$

Proof. Using

$$\{a : K \le F(a) \le K + L^2\} = \{a : -K - L^2 \le -F(a) \le -K\}$$

we see that it suffices to consider the case $u > 0$. In this case the statement follows from the fact that the double inequality $K \le F(t) \le K + L^2$ is equivalent to

$$\frac{1}{u}\left(K + \frac{v^2 - 4uw}{4u}\right) \le \left(a + \frac{v}{2u}\right)^2 \le \frac{L^2}{u} + \frac{1}{u}\left(K + \frac{v^2-4uw}{4u}\right),$$

and from the inequality

$$|\sqrt{x} - \sqrt{y}\,| \le \sqrt{|x-y|}\ .$$

$\square$

Suppose that (d, q_0, D_0) is fixed and consider the following two cases:

Case 1. $r_0 \neq x_0$.

By (4.7) and Lemma 4.5 the range of a is the union of at most two intervals of length $\frac{L_{q_0}}{r_0}$. Hence a can only assume $O(L_{q_0})$ values. But, for each a, q_0' belongs to $\big[q_0, \frac{Q}{d}\big]$ and is subject to the condition $q_0' a = D_0 \pmod{q_0}$. Hence q_0' takes $O\big(1+\frac{Q}{dq_0}\big) = O\big(\frac{Q}{dq_0}\big)$ values. Thus the contribution to S_Q of quadruples (d, q_0, D_0, a) which satisfy (4.7) is

$$(4.8)\qquad \ll \frac{1}{Q^4}\sum_{d=1}^{Q}\sum_{q_0=1}^{Q}\frac{d}{q_0}\sum_{|D_0|\ll q_0} L_{q_0}\cdot\frac{Q}{dq_0}\left(\frac{Q}{d}\right)^2\frac{L_{q_0}}{dq_0} \ll \frac{1}{Q}\sum_{q_0=1}^{Q}\frac{L_{q_0}^2}{q_0^2}\ .$$

Case 2. $r_0 = x_0$ thus $\alpha_0 = 1$.

In this case we collect directly from (3.7)

$$\frac{(dq_0y_0 + dD_0)^2 - r_0^2 d^2 q_0^2}{2r_0 d(D_0 + q_0 y_0)} \le a \le \frac{L_{q_0}^2 + (dq_0 y_0 + dD_0)^2 - r_0^2 d^2 q_0^2}{2r_0 d(D_0 + q_0 y_0)}\ .$$

Hence a can only assume $O\big(\frac{L_{q_0}^2}{dD_0}\big)$ values and we find, arguing as in Case 1, that the contribution to S_Q of quadruples (d, q_0, D_0, a) which satisfy (4.7) is

$$\begin{aligned}(4.9)\qquad &\ll \frac{1}{Q^4}\sum_{d=1}^{Q}\sum_{q_0=1}^{Q}\frac{d}{q_0}\sum_{|D_0|\ll q_0}\frac{L_{q_0}^2}{dD_0}\cdot\frac{Q}{dq_0}\left(\frac{Q}{d}\right)^2\frac{L_{q_0}}{dq_0} \ll \frac{1}{Q}\sum_{q_0=1}^{Q}\frac{\log q_0}{q_0^3}\cdot L_{q_0}^3 \\ &\ll_\delta \frac{1}{Q}\sum_{q_0=1}^{Q}\frac{L_{q_0}^3}{q_0^{3-\delta}}\ .\end{aligned}$$

Next we investigate the situation

$$r_0^2 d^2 q_0^2 \Phi\left(\frac{a}{dq_0}, \frac{D_0}{r_0 q_0}\right) = r_0^2(a^2 + d^2 q_0^2) - (dq_0 y_0 - a x_0 + dD_0)^2 \geq L^2.$$

Consider the range of q_0':

$$\mathcal{I}_{q_0,d} = \left[q_0, \frac{Q}{d}\right],$$

the range of a (which is the union of at most two intervals):

$$\mathcal{J}_{q_0,D_0,d,L} = \left\{y \in [0, dq_0] \,:\, \Phi\left(\frac{y}{dq_0}, \frac{D_0}{r_0 q_0}\right) \geq \frac{L^2}{r_0^2 d^2 q_0^2}\right\} \subseteq dq_0 I_{\frac{D_0}{r_0 q_0}}\ ,$$

and the functions:

$$\begin{aligned} G(x) &= G_{q_0}(x) = (x - q_0)x, \qquad x \in \mathcal{I}_{q_0,d}, \\ \Psi(y) &= \Psi_{q_0,D_0,d}(y) = \sqrt{\Phi\left(\frac{y}{dq_0}, \frac{D_0}{r_0 q_0}\right)}\ , \qquad y \in \mathcal{J}_{q_0,D_0,d,L} \subseteq [0, dq_0], \\ f(x,y) &= f_{q_0,D_0,d}(x,y) = G(x)\Psi(y), \qquad (x,y) \in \mathcal{I}_{q_0,d} \times \mathcal{J}_{q_0,D_0,d,L}. \end{aligned}$$

With this notation the following estimates hold on $\mathcal{I}_{q_0,d} \times \mathcal{J}_{q_0,D_0,d,L}$:

$$\begin{aligned} &\|G\|_\infty \ll \frac{Q^2}{d^2}\,, \qquad \|\Psi\|_\infty \ll 1, \quad \|f\|_\infty \leq \|G\|_\infty \|\Psi\|_\infty \ll \frac{Q^2}{d^2}\,, \\ &\|G'\|_\infty \ll \frac{Q}{d}\,, \qquad \|\Psi'\|_\infty = \sup_{y \in \mathcal{J}_{q_0,D_0,d,L}} \frac{\left|\frac{(1-\alpha_0^2)y}{d^2 q_0^2} + \frac{\alpha_0}{dq_0}\left(\frac{D_0}{r_0 q_0} + \beta_0\right)\right|}{\sqrt{\Phi\left(\frac{y}{dq_0}, \frac{D_0}{r_0 q_0}\right)}} \ll \frac{\frac{1}{dq_0}}{\frac{L_{q_0}}{dq_0}} = \frac{1}{L_{q_0}}\,, \\ &\|\nabla f\|_\infty \leq \|G\|_\infty \|\Psi'\|_\infty + \|G'\|_\infty \|\Psi\|_\infty \ll \frac{Q^2}{d^2}\cdot\frac{1}{L_{q_0}} + \frac{Q}{d} = \frac{Q^2}{dL_{q_0}}\left(\frac{1}{d} + \frac{L_{q_0}}{Q}\right) \\ &\qquad\qquad \leq \frac{Q^2}{dL_{q_0}}\left(\frac{1}{d} + \frac{q_0}{Q}\right) \ll \frac{Q^2}{d^2 L_{q_0}}\ . \end{aligned}$$

Applying Proposition 4.4 with $T=[q_0^{\frac{1}{5}}]$, $L=q_0^{\frac{9}{10}}$, we find that

$$\sum_{\substack{q_0'\in\left[q_0,\frac{Q}{d}\right]\\ \gcd(q_0',q_0)=1\\ a\in\mathcal{J}_{q_0,D_0,d,L}\\ aq_0'=D_0 \pmod{q_0}}} (q_0'-q_0)q_0\sqrt{\Phi\left(\frac{a}{dq_0},\frac{D_0}{r_0q_0}\right)} = \sum_{\substack{q_0'\in\left[q_0,\frac{Q}{d}\right]\\ \gcd(q_0',q_0)=1\\ a\in\mathcal{J}_{q_0,D_0,d,L}\\ aq_0'=D_0 \pmod{q_0}}} f_{q_0,D_0,d}(q_0',a)$$

$$= \frac{\varphi(q_0)}{q_0^2}\int_{q_0}^{\frac{Q}{d}} G_{q_0}(x)\,dx \int_{\mathcal{J}_{q_0,D_0,d,L}} \Psi_{q_0,D_0,d}(y)\,dy + \mathcal{E}_{q_0,D_0,d},$$

where

$$\begin{aligned}\mathcal{E}_{q_0,D_0,d} &\ll_\delta \frac{Q}{dq_0}\cdot dTq_0^{\frac{1}{2}+\delta}\gcd(D_0,q_0)^{\frac{1}{2}}\left(\frac{TQ^2}{d^2}+\frac{q_0Q^2}{d^2L^2}\right)+\frac{Q}{d}\cdot dq_0\cdot\frac{Q^2}{d^2L^2}\cdot\frac{1}{T}\\ &= \frac{Q^3}{d^2}\left(q_0^{-\frac{1}{10}+\delta}\gcd(D_0,q_0)^{\frac{1}{2}}+q_0^{-\frac{1}{5}+\delta}\gcd(D_0,q_0)^{\frac{1}{2}}+q_0^{-\frac{1}{10}}\right)\\ &\ll \frac{Q^3}{d^2}\cdot q_0^{-\frac{1}{10}+\delta}\gcd(D_0,q_0)^{\frac{1}{2}}.\end{aligned}$$

Using

$$\sum_{D=1}^{q_0}\gcd(D,q_0)^{\frac{1}{2}} \le \sum_{D=1}^{q_0}\gcd(D,q_0) = \sum_{d|q_0} d\varphi\left(\frac{q_0}{d}\right) \le q_0\tau(q_0) \ll_\delta q_0^{1+\delta},$$

we find that

$$\sum_{|D_0|\ll q_0}\mathcal{E}_{q_0,D_0,d} \ll_\delta \frac{Q^3}{d^2}\cdot q_0^{1-\frac{1}{10}+\delta}.$$

Thus the total contribution to S_Q of $\mathcal{E}_{q_0,D_0,d}$ is

$$\ll_\delta \frac{1}{Q}\sum_{d=1}^{Q}\sum_{q_0=1}^{Q}\frac{d}{q_0}\cdot\frac{1}{d^2}\cdot q_0^{1-\frac{1}{10}+\delta} \ll Q^{-\frac{1}{10}+2\delta}.$$

Moreover, the quantities in (4.8) and (4.9) are both $\ll_\delta Q^{-\frac{1}{10}+\delta}$. Thus we gather

$$S_Q = M_Q + O_\delta(Q^{-\frac{1}{10}+\delta}), \tag{4.10}$$

with

$$M_Q = \frac{2r_0}{Q^4}\sum_{d=1}^{Q}\sum_{q_0=1}^{\left[\frac{Q}{d}\right]}\frac{d}{q_0}\sum_{D_0\in r_0q_0\,\mathrm{pr}_2\,\mathcal{D}}\frac{\varphi(q_0)}{q_0^2}\int_{q_0}^{\frac{Q}{d}}G_{q_0}(x)\,dx\int_{\mathcal{J}_{q_0,D_0,d,q_0^{9/10}}}\Psi_{q_0,D_0,d}(y)\;dy.$$

Next we show that one can replace $\mathcal{J}_{q_0,D_0,d,q_0^{9/10}}$ by $dq_0 I_{\frac{D_0}{r_0q_0}}$ in the latter integral. Clearly $\mathcal{J}_{q_0,D_0,d,q_0^{9/10}} \subseteq dq_0 I_{\frac{D_0}{r_0q_0}}$ and by Lemma 4.5 we have

$$\left| dq_0 I_{\frac{D_0}{r_0q_0}} \setminus \mathcal{J}_{q_0,D_0,d,q_0^{9/10}}\right| \ll 2\sqrt{\frac{(q_0^{9/10})^2}{r_0^2-x_0^2}} \ll q_0^{\frac{9}{10}}.$$

Thus

$$0 \le \int\limits_{dq_0 I_{\frac{D_0}{r_0 q_0}} \setminus \mathcal{J}_{q_0,D_0,d,q_0^{9/10}}} \Psi_{q_0,D_0,d}(y)\,dy \le \left| dq_0 I_{\frac{D_0}{r_0 q_0}} \setminus \mathcal{J}_{q_0,D_0,d,q_0^{9/10}} \right| \cdot \frac{q_0^{9/10}}{dq_0} \ll \frac{q_0^{4/5}}{d} \ ,$$

and as a result the error in M_Q that results by replacing $\mathcal{J}_{q_0,D_0,d,L}$ by $dq_0 I_{\frac{D_0}{r_0 q_0}}$ is

$$\ll \frac{1}{Q^4} \sum_{d=1}^{Q} \sum_{q_0=1}^{\left[\frac{Q}{d}\right]} \frac{d\varphi(q_0)}{q_0^3} \cdot \frac{q_0^{4/5}}{d} \int\limits_{q_0}^{\frac{Q}{d}} G_{q_0}(x)\,dx.$$

On the other hand we find that

$$\int\limits_{q_0}^{\frac{Q}{d}} G_{q_0}(x)\,dx = \int\limits_{0}^{\frac{Q}{d}-q_0} x(x+q_0)\,dx = q_0^3\ g\Big(\frac{Q}{dq_0}\Big),$$

where

$$g(t) = \frac{(t-1)^3}{3} + \frac{(t-1)^2}{2}\ .$$

In particular, the integral of G_{q_0} on $\big[q_0, \frac{Q}{d}\big]$ is $\ll \frac{Q^3}{d^3}$, and we find that the total cost of replacing $\mathcal{J}_{q_0,D_0,d,q_0^{9/10}}$ by $dq_0 I_{\frac{D_0}{r_0 q_0}}$ in M_Q is

$$\ll \frac{1}{Q^4} \sum_{d=1}^{Q} \sum_{q_0=1}^{\left[\frac{Q}{d}\right]} \frac{\varphi(q_0)}{q_0^3} \cdot q_0^{\frac{4}{5}} \cdot \frac{Q^3}{d^3} \le \frac{1}{Q} \sum_{d=1}^{\infty} \frac{1}{d^3} \sum_{q_0=1}^{\infty} q_0^{-\frac{6}{5}} \ll \frac{1}{Q}\ .$$

Thus we infer that M_Q can be written, up to an error term of order $O\big(\frac{1}{Q}\big)$, as

$$\begin{aligned}
&\frac{2r_0}{Q^4} \sum_{d=1}^{Q} \sum_{q_0=1}^{\left[\frac{Q}{d}\right]} \frac{d}{q_0} \cdot \frac{\varphi(q_0)}{q_0^2} \cdot q_0^3\, g\Big(\frac{Q}{dq_0}\Big) \sum_{D_0 \in r_0 q_0\, \mathrm{pr}_2\, \mathcal{D}} \int\limits_{dq_0 I_{\frac{D_0}{r_0 q_0}}} \Psi_{q_0,D_0,d}(y)\,dy \\
&= \frac{2r_0}{Q^4} \sum_{d=1}^{Q} \sum_{q_0=1}^{\left[\frac{Q}{d}\right]} d\varphi(q_0)\, g\Big(\frac{Q}{dq_0}\Big) \sum_{D_0 \in r_0 q_0\, \mathrm{pr}_2\, \mathcal{D}} \int\limits_{dq_0 I_{\frac{D_0}{r_0 q_0}}} \sqrt{\Phi\Big(\frac{y}{dq_0}, \frac{D_0}{r_0 q_0}\Big)}\ dy \\
&= \frac{2r_0}{Q^4} \sum_{d=1}^{Q} \sum_{q_0=1}^{\left[\frac{Q}{d}\right]} d^2 q_0 \varphi(q_0)\, g\Big(\frac{Q}{dq_0}\Big) \sum_{D_0 \in r_0 q_0\, \mathrm{pr}_2\, \mathcal{D}} \int\limits_{I_{\frac{D_0}{r_0 q_0}}} \sqrt{\Phi\Big(t, \frac{D_0}{r_0 q_0}\Big)}\ dt \\
&= \frac{2r_0}{Q^4} \sum_{d=1}^{Q} \sum_{q_0=1}^{\left[\frac{Q}{d}\right]} d^2 q_0 \varphi(q_0)\, g\Big(\frac{Q}{dq_0}\Big) \sum_{D_0 \in r_0 q_0\, \mathrm{pr}_2\, \mathcal{D}} \psi\Big(\frac{D_0}{r_0 q_0}\Big).
\end{aligned} \tag{4.11}$$

By the Euler-MacLaurin summation formula and Lemma 4.3 the inner sum above is given by

$$\int\limits_{r_0 q_0 \operatorname{pr}_2 \mathcal{D}} \psi\Big(\frac{u}{r_0 q_0}\Big) du + O\Bigg(\sup_{x \in \operatorname{pr}_2 \mathcal{D}} |\psi(x)| + \int\limits_{r_0 q_0 \operatorname{pr}_2 \mathcal{D}} \bigg| \frac{1}{r_0 q_0} \cdot \psi'\Big(\frac{u}{r_0 q_0}\Big)\bigg| du \Bigg)$$
$$= r_0 q_0 \int\limits_{\operatorname{pr}_2 \mathcal{D}} \psi(v) dv + O(1) = \frac{2\pi r_0 q_0}{3} + O(1).$$

Inserting this back into (4.11) and putting $q = dq_0 \in [1, Q]$, we obtain

$$\begin{aligned} M_Q &= \frac{4\pi r_0^2}{3Q^4} \sum_{d=1}^{Q} \sum_{q_0=1}^{\left[\frac{Q}{d}\right]} \varphi(q_0)(dq_0)^2 g\Big(\frac{Q}{dq_0}\Big) + O\Big(\frac{\log Q}{Q}\Big) \\ &= \frac{4\pi r_0^2}{3Q^4} \sum_{q=1}^{Q} q^2 g\Big(\frac{Q}{q}\Big) \sum_{q_0 | q} \varphi(q_0) + O\Big(\frac{\log^2 Q}{Q}\Big) \\ &= \frac{4\pi r_0^2}{3Q^4} \sum_{q=1}^{Q} q^3 g\Big(\frac{Q}{q}\Big) + O\Big(\frac{\log^2 Q}{Q}\Big) \\ &= \frac{4\pi r_0^2}{3} \int_0^1 \Big(\frac{(1-x)^3}{3} + \frac{x(1-x)^2}{2} \Big) dx + O\Big(\frac{\log^2 Q}{Q}\Big) \\ &= \frac{\pi r_0^2}{6} + O\Big(\frac{\log^2 Q}{Q}\Big). \end{aligned} \tag{4.12}$$

Proposition 4.1 now follows from (4.10) and (4.12).

5. The divergence of the 6-level correlations

In this section we prove Theorem 1.2. We begin with a counting result.

Lemma 5.1. *Let a, b, d be positive integers with $\gcd(a, b, d) = 1$ and let d_1 denote the largest divisor of d which is relatively prime with b. Write $d_2 = \frac{d}{d_1}$. Then*

$$\#\{0 \le m \le 2d-1 \,:\, \gcd(a+bm, d) = 1\} = 2\varphi(d_1) d_2. \tag{5.1}$$

Proof. Using Möbius inversion we express the left-hand side of (5.1) as

$$\sum_{0 \le m \le 2d-1} \sum_{\substack{D|d \\ D|a+bm}} \mu(D) = \sum_{D|d} \mu(D)\ \#\{0 \le m \le 2d-1 \,:\, D|\, a+bm\}.$$

Note that if D does not divide d_1, then there is no m for which $D|a+bm$. Indeed, if for some m we have $D|a+bm$, then $a = Dk - bm$ for some $k \in \mathbb{Z}$. Hence a is divisible by $\gcd(D, b)$. Then $\gcd(D, b)$ divides $\gcd(a, b, d) = 1$; thus $\gcd(D, b) = 1$ and by the definition of d_1 it follows that D divides d_1. Therefore

$$\#\{0 \le m \le 2d-1 : \gcd(a+bm, d) = 1\} = \sum_{D|\, d_1} \mu(D)\ \#\{0 \le m \le 2d-1 : D|a+bm\}.$$

For $D|\, d_1$ we have $\gcd(D, b) = 1$ and there is a unique solution $m \pmod D$ to the congruence $a + bm = 0 \pmod D$. Thus there are exactly $\frac{2d}{D}$ values of m in

$\{0,1,\dots,2d-1\}$ for which $D|a+bm$. As a result we infer that

$$\begin{aligned}\#\{0 \le m \le 2d-1 : \gcd(a+bm,d)=1\} &= \sum_{D|\, d_1} \mu(D)\, \frac{2d}{D} = 2d \sum_{D|\, d_1} \frac{\mu(D)}{D} \\ &= 2d\, \frac{\varphi(d_1)}{d_1} = 2\varphi(d_1) d_2,\end{aligned}$$

which proves the lemma. □

For any positive integers a,b,q, consider the set

$$\mathcal{N}_{a,b,q} = \big\{(A,B) : 1 \le A,B \le 2q,\ \gcd(A,B)=1,\ q|Ab-Ba\big\}.$$

Lemma 5.2. *For q large and $1 \le a,b \le q$ such that $\gcd(a,b,q)=1$,*

$$\#\mathcal{N}_{a,b,q} \gg \frac{\varphi(q)}{\log q} \gg \frac{q}{\log q \log\log q}\ .$$

Remark. If in the definition of $\mathcal{N}_{a,b,q}$ we took the range of A and B to be $[1,q]$ instead of $[1,2q]$, the cardinality of $\mathcal{N}_{a,b,q}$ would be much smaller. For example, if $1 \le a \le q$, $\gcd(a,q)=1$, and $b=a$, then $q|A-B$, and in the range $1 \le A,B \le q$ this forces $A=B$. Then the only pair (A,B) with $\gcd(A,B)=1$ is $(1,1)$, so $\mathcal{N}_{a,b,q}$ will only contain one element. Lemma 5.2 shows a sudden increase in the cardinality of $\mathcal{N}_{a,b,q}$ when the range of A and B increases by a factor 2.

Proof of Lemma 5.2. To make a choice, assume in what follows next that $d = \gcd(a,q) \le \gcd(b,q)$. Then $\gcd(d,b)=1$ since $\gcd(q,a,b)=1$. It follows that for any solution (A,B) to the congruence $Ab = Ba \pmod q$, A has to be divisible by d. Write $q = dq_1$ and $a = da_1$, so $\gcd(a_1,q_1)=1$. Denote by $\bar{a}_1$ the multiplicative inverse of $a_1 \pmod{q_1}$ in the interval $[1,q_1]$.

Note that since q is divisible by the product $\gcd(a,q)\gcd(b,q)$, we have $d < \sqrt{q}$. Therefore $q_1 > \sqrt{q}$. So q_1 is large for large q, and by the Prime Number Theorem we know that

$$\#\{p \text{ prime} : q_1 < p \le 2q_1\} \sim \frac{q_1}{\log q_1}\ . \tag{5.2}$$

For any fixed prime p with $q_1 < p \le 2q_1$, we count the solutions $1 \le B \le 2q$ of the congruence

$$dpb = Ba \pmod q. \tag{5.3}$$

This is equivalent to

$$pb = Ba_1 \pmod{q_1}. \tag{5.4}$$

Since $\gcd(q_1,a_1)=1$, this congruence has a unique solution modulo q_1, namely $B = \overline{a}_1 pb \pmod{q_1}$. Denote by B_0 the solution to (5.4) which belongs to the interval $[1,q_1]$. Then (5.3) will have $2d$ solutions, given by

$$B = B_0 + q_1 m, \qquad 0 \le m \le 2d-1. \tag{5.5}$$

It remains to be seen how many of the numbers B from (5.5) are relatively prime with $A = dp$. Note first that at most two such numbers B are divisible by p. Assume that

$$p|B_0 + q_1 m_1, \quad p|B_0 + q_1 m_2, \quad \text{and} \quad p|B_0 + q_1 m_3, \tag{5.6}$$

with $0 \leq m_1 < m_2 < m_3 \leq 2d-1$. Since $p > q_1$, p does not divide q_1. Then (5.6) yields

$$p|m_2 - m_1 \qquad \text{and} \qquad p|m_3 - m_2. \tag{5.7}$$

Here at least one of the differences $m_2 - m_1$, $m_3 - m_2$ is less than d, and $d < \sqrt{q} < q_1 < p$. So it is impossible that both divisibilities in (5.7) hold true. Therefore at most two numbers from (5.5) are divisible by p. Note also, by the same reasoning, that for smaller values of d—more precisely for $d < \sqrt{\frac{q}{2}}$, that is, $d < \frac{q_1}{2}$—one has $2d < p$. Then one concludes that at most one number B from (5.5) can be a multiple of p.

We now count the numbers B from (5.5) which are relatively prime with d. We claim that $\gcd(B_0, q_1, d) = 1$. Indeed, let us assume this fails and choose a prime divisor p_1 of $\gcd(B_0, q_1, d)$. Since $p_1| d$, we have $p_1| a$ and $p_1| q$. Recall that B_0 satisfies (5.4), hence

$$pb = B_0 a_1 + q_1 k \tag{5.8}$$

for some $k \in \mathbb{Z}$. Here $p_1 | B_0$, $p_1 | q_1$, so p_1 must also divide the left side of (5.8). The inequalities $p_1 \leq B_0 \leq q_1 < p$ show that $p_1 \neq p$, so p divides b. But then $p_1| \gcd(a, b, q) = 1$, and we obtain a contradiction. This shows that $\gcd(B_0, q_1, d) = 1$. Then Lemma 5.1 is applicable to B_0, q_1, d. If d_1 denotes the largest divisor of d which is relatively prime with q_1, and $d_2 = \frac{d}{d_1}$, Lemma 5.1 provides

$$\#\{0 \leq m \leq 2d-1 : \gcd(B_0 + q_1 m, d) = 1\} = 2\varphi(d_1) d_2.$$

It follows in particular that there are always at least two numbers B as in (5.5) for which $\gcd(B, d) = 1$, and as soon as $d_2 \geq 2$ or $d_1 \geq 3$, there are at least four such numbers. Since at most two numbers B as in (5.5) are divisible by p, and for small values of d we know that at most one number B as in (5.5) is divisible by p, we conclude that in all cases

$$\#\{1 \leq B \leq 2d : \gcd(B, dp) = 1,\ dpb = Ba \pmod{q}\} \geq \varphi(d_1) d_2 \geq \varphi(d). \tag{5.9}$$

Combining (5.9) with (5.2) we infer that

$$\#\mathcal{N}_{a,b,q} \gg \frac{q_1 \varphi(d)}{\log q_1} = \frac{q\varphi(d)}{d \log q_1} \geq \frac{q}{\log q} \cdot \frac{\varphi(d)}{d}\ ,$$

and the lemma is completed using the inequalities

$$\frac{\varphi(d)}{d} \geq \frac{\varphi(q)}{q} \gg \frac{1}{\log \log q}\ .$$

□

Now let q be a large positive integer, let $a, b \in \{1, \dots, q\}$ such that $\gcd(a, b, q) = 1$, and let Q be a positive integer larger than q. In our applications Q will be at least of the order of magnitude of $q^{\frac{4}{3}}$. We will construct some sets of lattice points inside the square $[0, Q]^2$, indexed by the set $\mathcal{N}_{a,b,q}$. Precisely, we select a positive integer M, which will be chosen later in the proof of Theorem 1.2, and for each pair $(A, B) \in \mathcal{N}_{a,b,q}$ we construct a set $\mathcal{M}_{A,B} = \mathcal{M}_{A,B}(a, b, q, Q, M)$ as follows. Fix $(A, B) \in \mathcal{N}_{a,b,q}$. To make a choice assume that $B \leq A$. Let C be the integer defined by the equation

$$bA - aB = qC. \tag{5.10}$$

Denote by u the unique integer satisfying

$$u = -\overline{B}\,C \pmod{A}, \qquad 0 \le u \le A-1,$$

where $\overline{B}$ is the multiplicative inverse of B modulo A, and put $v = \frac{Bu+C}{A}$. Then v is an integer. Also, from the inequalities

$$-B \le -\frac{aB}{q} < \frac{bA-aB}{q} = C < \frac{bA}{q} \le A$$

it follows that $\frac{C}{A} \in (-\frac{B}{A}, 1) \subseteq (-1,1)$. Hence

$$-1 < \frac{C}{A} \le \frac{Bu+C}{A} \le \frac{Bu}{A} + 1 < B+1,$$

so that $v \in (-1, B+1)$. Thus $0 \le v \le B$, since v is an integer. Now let $s = \left[\frac{Q}{A}\right]$, and define $\mathcal{M}_{A,B}$ to be the set of lattice points given by

$$\mathcal{M}_{A,B} = \{(u+mA, v+mB) : s-M \le m \le s-1\}.$$

Note that the case $A=B$ can only occur when $A=B=1$, and in this situation we also get $C=u=v=0$, and so $\mathcal{M}_{1,1} = \{(m,m) : Q-M \le m \le Q-1\}$. We have constructed $\#\mathcal{N}_{a,b,q}$ sets of the form $\mathcal{M}_{A,B}$, each set $\mathcal{M}_{A,B}$ consisting of M lattice points. Note that u, v, s in the definition of $\mathcal{M}_{A,B}$ depend on the pair (A,B). In what follows we assume that M satisfies the inequality

$$M \le \left[\frac{Q}{4q}\right]. \tag{5.11}$$

Also define

$$\widetilde{\mathcal{M}}_{a,b,q} = \bigcup_{(A,B)\in\mathcal{N}_{a,b,q}} \mathcal{M}_{A,B}.$$

Some properties of these sets are collected in the following lemma.

Lemma 5.3. (i) $\operatorname{dist}\big([0,1]^2, \widetilde{\mathcal{M}}_{a,b,q}\big) \ge \frac{Q}{3}$.
(ii) $\widetilde{\mathcal{M}}_{a,b,q} \subseteq [0,Q]^2$.
(iii) *The sets $\mathcal{M}_{A,B}$ are disjoint.*

Proof. (i) Owing to (5.11) and to

$$[x] - \left[\frac{x}{2}\right] \ge \frac{x}{2} - 1,$$

we have for any $(A,B) \in \mathcal{N}_{a,b,q}$ and any point $(u+mA, u+mB) \in \mathcal{M}_{A,B}$ the inequalities

$$\begin{aligned} u+mA \ge mA \ge (s-M)A &\ge \left(\left[\frac{Q}{A}\right] - \left[\frac{Q}{4q}\right]\right)A \\ &\ge \left(\left[\frac{Q}{A}\right] - \left[\frac{Q}{2A}\right]\right)A \ge \frac{Q}{2} - A \ge \frac{Q}{2} - 2q. \end{aligned}$$

Recall that Q is much larger than q. It follows that the distance between any two points $P \in [0,1]^2$ and $P' \in \widetilde{\mathcal{M}}_{a,b,q}$ satisfies

$$\|PP'\| \ge \frac{Q}{2} - 2q - 1 \ge \frac{Q}{3}\ . \tag{5.12}$$

(ii) For any $(A,B) \in \mathcal{N}_{a,b,q}$ with, say, $A \ge B$, and any point $P = (u+mA, v+mB) \in \mathcal{M}_{A,B}$, one has $u+mA \le u+(s-1)A \le sA \le Q$. Also, $0 \le v+mB \le$

$v+(s-1)B \le sB = \left[\frac{Q}{A}\right]B \le Q$, since $B \le A$. Hence all points $P \in \widetilde{\mathcal{M}}_{a,b,q}$ lie inside the square $[0,Q]^2$.

(iii) Assume that there is a lattice point $P=(n_1,n_2)$ which belongs to two sets $\mathcal{M}_{A,B}$ and $\mathcal{M}_{A',B'}$ with $(A,B),(A',B') \in \mathcal{N}_{a,b,q}$ and $(A,B) \neq (A',B')$. Assume first that $B \le A$ and $B' \le A'$. Then

$$n_1 = u + mA, \qquad n_2 = v + mB \tag{5.13}$$

for some $m \in \{s-M,\dots,s-1\}$. Similarly

$$n_1 = u' + m'A', \qquad n_2 = v' + m'B'$$

for some $m' \in \{s'-M,\dots,s'-1\}$, with u,v,s,u',v',s' given by appropriate definitions.

We compute the ratio $\frac{qn_2-b}{qn_1-a}$ in two ways. First, by (5.13), (5.10) and the equality $v=\frac{Bu+C}{A}$ we have

$$\begin{aligned}\frac{qn_2-b}{qn_1-a} &= \frac{qv+qmB-b}{qu+qmA-a} = \frac{qBu+qC+AqmB-Ab}{A(qu+qmA-a)} \\ &= \frac{qBu-aB+AqmB}{A(qu+qmA-a)} = \frac{B}{A}\ .\end{aligned}$$

By a similar computation we also have

$$\frac{qn_2-b}{qn_1-a} = \frac{B'}{A'} = \frac{B}{A}\ .$$

Since A,B,A',B' are all positive and $\gcd(B,A)=\gcd(B',A')=1$, this forces $A=A'$ and $B=B'$.

In general the same argument gives $\min\{A,B\}=\min\{A',B'\}$ and $\max\{A,B\}=\max\{A',B'\}$, thus $(A',B') \in \{(A,B),(B,A)\}$. If $A'=B$ and $B'=A$, we get $\frac{A}{B}=\frac{B}{A}$, hence $A^2=B^2$ and $A=B=1$. $\square$

Now fix a point $P_{(x,y)} \in [0,1]^2$ and a pair $(A,B) \in \mathcal{N}_{a,b,q}$. Also, choose any two points $P,P' \in \mathcal{M}_{A,B}$, say $P=(u+mA,v+mB)$ and $P'=(u+m'A,v+m'B)$, with $m,m' \in \{s-M,\dots,s-1\}$. Consider the angle $\theta = \angle P'P_{(x,y)}P$. Then by (5.12) we gather

$$\begin{aligned}|\sin\theta| &= \frac{2\,\mathrm{Area}\,\triangle P'P_{(x,y)}P}{\|PP_{(x,y)}\|\,\|P'P_{(x,y)}\|} \le \frac{18\,\mathrm{Area}\,\triangle P'P_{(x,y)}P}{Q^2} \\ &= \frac{9|(u+m'A-x)(v+mB-y)-(u+mA-x)(v+m'B-y)|}{Q^2} \\ &= \frac{9|m'-m|\cdot|A(v-y)-B(u-x)|}{Q^2} \\ &\le \frac{9M|A(v-y)-B(u-x)|}{Q^2}\ .\end{aligned} \tag{5.14}$$

Next, using the equality $v=\frac{Bu+C}{A}$, we rewrite (5.14) as

$$|\sin\theta| \le \frac{9M|C+Bx-Ay|}{Q^2}\ . \tag{5.15}$$

Since $|C| \le \max\{A,B\} \le 2q$ and $0 \le x,y \le 1$, we see that $|C+Bx-Ay| \ll q$. If M satisfies (5.11), then $M|C+Bx-Ay| \ll Q$. As a consequence of (5.15) we also have

$$|\sin\theta| \ll \frac{1}{Q} .$$

From (5.15) we also infer that

$$|\theta| \leq \frac{\pi|\sin\theta|}{2} \ll \frac{M|C+Bx-Ay|}{Q^2} . \tag{5.16}$$

By (5.10) and (5.16) we derive that

$$|\theta| \ll \frac{M|Cq+Bqx-Aqy|}{qQ^2} = \frac{M|A(b-qy)+B(qx-a)|}{qQ^2} . \tag{5.17}$$

As a consequence of (5.17) and of the inequalities $1 \leq A, B \leq 2q$, we have

$$|\theta| \ll \frac{M(|b-qy|+|qx-a|)}{Q^2} , \tag{5.18}$$

uniformly for all pairs $(A,B) \in \mathcal{N}_{a,b,q}$ and all pairs of points $P, P' \in \mathcal{M}_{A,B}$.

Proof of Theorem 1.2. Fix $(x,y) \in [0,1]^2$, $\lambda = (\lambda_1, \ldots, \lambda_5) \in \mathbb{R}^5$ with $\lambda_1, \ldots, \lambda_5 > 0$, and $\delta > 0$. The case when both x and y are rational numbers is clear. In this case, if we fix an integer $m_0 \geq 1$ for which both $m_0 x$ and $m_0 y$ are integers, and consider the set of lattice points $\mathcal{A} = \{(\ell m_0 x, \ell m_0 y) : \ell = 1, 2, \ldots, \left[\frac{Q}{m_0}\right]\}$, then all these points lie on the same line that passes through $P_{(x,y)}$. Then all the 6-tuples of distinct elements from $\mathcal{A}$ will contribute to $\mathcal{R}^{(6)}_{(x,y),Q}(\lambda)$. Since $\#\mathcal{A} = \left[\frac{Q}{m_0}\right]$, it follows that

$$\mathcal{R}^{(6)}_{(x,y),Q}(\lambda) \gg \frac{1}{\#\square_Q} \cdot \frac{Q^6}{m_0^6} \gg \frac{Q^4}{m_0^6} .$$

Note that in this case the 3-level correlations already diverge as $Q \to \infty$.

Now consider the case when at least one of x, y is irrational. With x, y, λ and δ fixed, choose a large positive integer Q. Let $1 < T < Q$ be a parameter whose precise value will be chosen later and will be the integer part of a fractional power of Q. By Minkowski's convex body theorem (see Theorem 6.25 in [7] for the formulation used here), there exists an integer $1 \leq q \leq T$ for which

$$\langle qx \rangle \leq \frac{1}{\sqrt{T}} \qquad \text{and} \qquad \langle qy \rangle \leq \frac{1}{\sqrt{T}} , \tag{5.19}$$

where $\langle \cdot \rangle$ denotes here the distance to the closest integer. Let a and b denote the closest integers to qx and qy, respectively. Then $0 \leq a, b \leq q$, $\max\{a,b\} > 0$, and (5.19) gives

$$|qx-a| \leq \frac{1}{\sqrt{T}} \qquad \text{and} \qquad |qy-b| \leq \frac{1}{\sqrt{T}} . \tag{5.20}$$

Dividing if necessary a, b and q by $\gcd(a,b,q)$, we may assume in what follows that $\gcd(a,b,q) = 1$.

We will have $T \to \infty$ as $Q \to \infty$, and since at least one of x, y is irrational, this forces $q \to \infty$ as $Q \to \infty$. Then all our previous results, valid for large q, are applicable.

Let M be a positive integer satisfying (5.11), whose precise order of magnitude will be chosen later. Consider the disjoint subsets $\mathcal{M}_{A,B}$ of $\square_Q$, with $(A,B) \in$

$\mathcal{N}_{a,b,q}$. By (5.18) we know that for any $(A,B) \in \mathcal{N}_{a,b,q}$ and any $P, P' \in \mathcal{M}_{A,B}$, the measure of the angle $\angle PP_{(x,y)}P'$ satisfies

$$|\theta_{P,P'}| \ll \frac{M(|b-qy|+|qx-a|)}{Q^2} \ . \tag{5.21}$$

Plugging (5.20) in (5.21) we find that

$$|\theta_{P,P'}| \ll \frac{M}{Q^2\sqrt{T}} \ .$$

If we take the order of magnitude of M to be slightly smaller than that of both $\frac{Q}{q}$ and $\sqrt{T}$, for instance

$$M = \min\left\{ \left[\frac{Q}{4q}\right], \left[\frac{\sqrt{T}}{\log Q}\right] \right\}, \tag{5.22}$$

then M will satisfy (5.11) on the one hand, and on the other hand we will have

$$|\theta_{P,P'}| \ll \frac{1}{Q^2 \log Q} \ .$$

It follows that for Q large enough in terms of $\lambda_1, \dots, \lambda_5$, all the 6-tuples $(P_1, \dots, P_6)$ of distinct points from $\mathcal{M}_{A,B}$ will contribute to $\mathcal{R}^{(6)}_{(x,y),Q}(\lambda)$. Therefore, since $\#\mathcal{M}_{A,B} = M$ for each $(A,B) \in \mathcal{N}_{a,b,q}$, we derive that

$$\mathcal{R}^{(6)}_{(x,y),Q}(\lambda) \geq \frac{1}{N} \cdot \#\mathcal{N}_{a,b,q} \cdot M(M-1)\cdots(M-5).$$

Here $N = \#\square_Q = (2Q+1)^2$, and from (5.22) and Lemma 5.2 it follows that

$$\mathcal{R}^{(6)}_{(x,y),Q}(\lambda) \gg \frac{M^6 q}{Q^2 \log q \log\log q} \gg \min\left\{ \frac{Q^4}{q^5 \log^2 q} \, , \, \frac{T^3 q}{Q^2 \log^8 Q} \right\}. \tag{5.23}$$

We now choose $T = 1 + [Q^{\frac{3}{4}}]$. Then, no matter how small q might be, we have

$$\frac{T^3 q}{Q^2 \log^8 Q} \geq \frac{Q^{\frac{1}{4}}}{\log^8 Q} > Q^{\frac{1}{4}-\delta} \qquad \text{for large } Q. \tag{5.24}$$

Also, since $q \leq T$, it follows that

$$\frac{Q^4}{q^5 \log^2 q} \geq \frac{Q^4}{T^5 \log^2 T} > Q^{\frac{1}{4}-\delta} \qquad \text{for large } Q. \tag{5.25}$$

Now (1.5) follows from (5.23), (5.24) and (5.25). □

Appendix

For fixed integers $q \geq 2$, m, n, h and sets $I, I_1, I_2 \subset \mathbb{R}$, we consider the Kloosterman sum

$$K(m,n;q) = \sum_{\substack{x \pmod q \\ \gcd(x,q)=1}} e\left(\frac{mx + n\bar{x}}{q}\right),$$

the incomplete Kloosterman sum

$$S_I(m,n;q) = \sum_{\substack{x \in I \\ \gcd(x,q)=1}} e\left(\frac{mx + n\bar{x}}{q}\right),$$

and the set

$$N_{q,h}(I_1,I_2)=\big\{(x,y)\in I_1\times I_2:\gcd(x,q)=1,\ xy=h \pmod q\big\}.$$

Here $\bar{x}$ denotes the multiplicative inverse of $x \pmod q$.

The first two lemmas are probably well known (see for instance Lemmas 2.2 and 1.6 in [2]).

Lemma A1. *Suppose that f is a C^1 function on $[a,b]$. Then*

$$\sum_{\substack{a<k\le b\\ \gcd(k,q)=1}} f(k)=\frac{\varphi(q)}{q}\int_a^b f+O_\delta\Big(q^\delta(\|f\|_\infty+V_a^b f)\Big).$$

Lemma A2. *For any interval $I\subset[0,q)$ and any integer n*

$$|S_I(0,n;q)|\ll_\delta \gcd(n,q)^{\frac12}q^{\frac12+\delta}.$$

Proof. We write

$$\begin{aligned}
S_I(0,n;q)&=\sum_{\substack{x \pmod q\\ \gcd(x,q)=1}} e\bigg(\frac{n\bar{x}}{q}\bigg)\sum_{y\in I}\frac1q\sum_{k=1}^{q}e\bigg(\frac{k(y-x)}{q}\bigg)\\
&=\frac1q\sum_{k=1}^{q}\sum_{y\in I}e\bigg(\frac{ky}{q}\bigg)\sum_{\substack{x \pmod q\\ \gcd(x,q)=1}} e\bigg(\frac{-kx+n\bar{x}}{q}\bigg) \qquad \text{(A1)}\\
&=\frac1q\sum_{k=1}^{q}\sum_{y\in I}e\bigg(\frac{ky}{x}\bigg)K(-k,n;q).
\end{aligned}$$

The inner sum is a geometric progression and can be bounded as

$$\bigg|\sum_{y\in I}e\bigg(\frac{ky}{q}\bigg)\bigg|\le\min\bigg\{|I|,\frac{1}{2\|\frac kq\|}\bigg\}. \qquad \text{(A2)}$$

By (A1), (A2) and Weil-Salié estimates,

$$\begin{aligned}
|S_I(0,n;q)|&\le\frac1q\ |K(0,n;q)||I|+\frac1q\sum_{k=1}^{q-1}\frac{1}{2\|\frac kq\|}\ |K(-k,n;q)|\\
&\ll_\delta \frac{qq^{\frac12+\delta}\gcd(n,q)^{\frac12}}{q}+\frac{q^{\frac12+\delta}\gcd(n,q)^{\frac12}q\log q}{q}\ll_\delta \gcd(n,q)^{\frac12}q^{\frac12+2\delta}.
\end{aligned}$$

□

Proposition A3. *For any intervals $I_1,I_2\subset[0,q)$ and any integer h*

$$N_{q,h}(I_1,I_2)=\frac{\varphi(q)}{q^2}\ |I_1|\,|I_2|+O_\delta\big(q^{\frac12+\delta}\gcd(h,q)^{\frac12}\big).$$

Proof. We write

$$N_{q,h}(I_1,I_2)=\frac1q\sum_{\substack{x\in I_1,y\in I_2\\ \gcd(x,q)=1}}\sum_{k=0}^{q-1}e\bigg(\frac{k(y-h\bar{x})}{q}\bigg)=M+E,$$

where the main term can be expressed, as a result of Lemma A1, as

$$M = \frac{1}{q} \sum_{\substack{x\in I_1, y\in I_2 \\ \gcd(x,q)=1}} 1 = \frac{1}{q}\left(\frac{\varphi(q)}{q} \, |I_1| + O_\delta(q^\delta)\right)\big(|I_2| + O(1)\big) = \frac{\varphi(q)}{q^2} \, |I_1|\, |I_2| + O_\delta(q^\delta).$$

The error is given by

$$E = \frac{1}{q} \sum_{k=1}^{q-1} \sum_{y\in I_2} e\left(\frac{ky}{q}\right) \sum_{\substack{x\in I_1 \\ \gcd(x,q)=1}} e\left(-\frac{hk\bar{x}}{q}\right) = \frac{1}{q} \sum_{k=1}^{q-1} \sum_{y\in I_2} e\left(\frac{ky}{x}\right) S_{I_1}(0,-hk;q).$$

Owing to (A2), Lemma A2, and to the inequality $\gcd(kh,q) \le \gcd(h,q)\gcd(k,q)$, we have

$$|E| \ll_\delta \frac{\gcd(h,q)^{\frac12}}{q} \sum_{k=1}^{q-1} \frac{1}{2\left\|\frac{k}{q}\right\|} \; \gcd(k,q)^{\frac12} q^{\frac12+\delta} \le \gcd(h,q)^{\frac12} q^{\frac12+\delta} \sum_{k=1}^{\frac{q-1}{2}} \frac{1}{k} \; \gcd(k,q)^{\frac12}.$$

Writing $k = dm$, with $d = \gcd(k,m)$, we eventually get

$$|E| \ll_\delta \gcd(h,q)^{\frac12} q^{\frac12+\delta} \sum_{d|q} \frac{1}{d^{\frac12}} \sum_{m=1}^{\left[\frac{q}{d}\right]} \frac{1}{m} \ll_\delta \gcd(h,q)^{\frac12} q^{\frac12+2\delta}.$$

□

Proposition A4. *Assume that $q \ge 1$ and h are two given integers, $\mathcal{I}$ and $\mathcal{J}$ are intervals of length less than q, and $f : \mathcal{I}\times\mathcal{J} \to \mathbb{R}$ is a C^1 function. Then for any integer $T > 1$ and any $\delta > 0$*

$$\sum_{\substack{a\in\mathcal{I},\, b\in\mathcal{J} \\ ab=h \pmod q \\ \gcd(b,q)=1}} f(a,b) = \frac{\varphi(q)}{q^2} \iint_{\mathcal{I}\times\mathcal{J}} f(x,y)\,dxdy \; + \mathcal{E},$$

with

$$\mathcal{E} \ll_\delta T^2 \|f\|_\infty q^{\frac12+\delta} \gcd(h,q)^{\frac12} + T\|\nabla f\|_\infty q^{\frac32+\delta} \gcd(h,q)^{\frac12} + \frac{|\mathcal{I}|\,|\mathcal{J}|\,\|\nabla f\|_\infty}{T}.$$

The proof relies on Proposition A3 and is identical to that of Lemma 2.2 in [4].

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