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THE GEOMETRY OF SYMPLECTIC PAIRS

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ABSTRACT. We study the geometry of manifolds carrying symplectic pairs consisting of two closed 2-forms of constant ranks, whose kernel foliations are complementary. Using a variation of the construction of Boothby and Wang we build contact-symplectic and contact pairs from symplectic pairs.

1. INTRODUCTION

A symplectic pair on a smooth manifold M is a pair of non-trivial closed two-forms ω_1, ω_2 of constant and complementary ranks, for which ω_1 restricts as a symplectic form to the leaves of the kernel foliation of ω_2, and vice versa. This definition is analogous to that of contact pairs and of contact-symplectic pairs introduced by the first author [1, 2]. In this paper we exhibit several constructions of symplectic pairs on closed manifolds, and use them to show that even in dimension four there is a surprisingly rich supply of examples, with very different geometric features. One reason why this is suprising is that it seems that not many explicit examples are known of closed four-manifolds which admit pairs of complementary two-dimensional foliations. It was only recently that Morita and the second author constructed certain interesting examples within the class of foliated bundles; cf. [17]. Another reason why the plethora of symplectic pairs (ω_1, ω_2) on closed four-manifolds is surprising is that they give rise to symplectic forms $\omega_\pm = \omega_1 \pm \omega_2$ compatible with different orientations of M. It is known that manifolds which are symplectic for both choices of orientation, or just have non-trivial Seiberg–Witten invariants for both orientations, are rather special; see for example [15].

On a four-manifold a symplectic pair (ω_1, ω_2) can be equivalently defined by two symplectic forms $\omega_\pm$ with the properties

$$\omega_+ \wedge \omega_- = 0 \quad \text{and} \quad \omega_+ \wedge \omega_+ = -\omega_- \wedge \omega_- .$$

If we were to change the sign in the last condition to $\omega_+ \wedge \omega_+ = \omega_- \wedge \omega_-$, we would obtain the definition of a conformal symplectic couple in the sense of Geiges [9], who gave a complete classification of the diffeomorphism types of four-manifolds which admit such couples. We will see that symplectic pairs, whose definition is very similar, are much more common than conformal symplectic couples.

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While we believe that this paper shows symplectic pairs to be interesting geometric objects in their own right, the original motivation for this work came from two other sources. First, symplectic pairs appear naturally in the study of Riemannian metrics for which all products of harmonic forms are harmonic (see [16]) and in the investigation of the group cohomology of symplectomorphism groups; see [17]. Second, symplectic pairs can be used to construct new examples of contact-symplectic and of contact pairs in the sense of [1, 2, 4]. In Section 2 we show how a variation of the classical Boothby–Wang construction [6] allows one to construct contact-symplectic pairs from symplectic pairs for which ω_1 represents an integral cohomology class in M, and contact pairs from contact-symplectic pairs in which the leafwise symplectic form represents an integral cohomology class. In particular, if we have a symplectic pair for which both ω_i represent integral classes, then the fiber product of the corresponding Boothby–Wang fibrations yields a contact pair on a principal T^2-bundle over M.

In Section 3 we give several constructions of symplectic pairs. Considering their Boothby–Wang fibrations one obtains many new examples of contact-symplectic and of contact pairs which go beyond the examples exhibited in [1, 2, 4]. In Section 4 we study Riemannian metrics compatible with a symplectic pair, and in Section 5 we give some further applications of our constructions.

In a sequel to this paper written jointly with Ghiggini [3], we formulate and prove the appropriate adaptation of Moser's theorem [20] for symplectic pairs.

2. The Boothby–Wang construction

Let (M, ω) be a closed symplectic manifold. After a small perturbation and multiplication by a constant we may assume that ω represents an integral class in $H^2(M; \mathbb{R})$. Let $\pi \colon E \to M$ be the principal S^1-bundle with Euler class $[\omega]$. There is a connection 1-form α on this circle bundle with curvature ω, i.e. we have $d\alpha = \pi^*\omega$. As ω is assumed to be symplectic on M, it follows that α is a contact form on the total space E. This is the construction of Boothby and Wang [6] associating so-called regular contact forms to integral symplectic forms.

Now if ω is an arbitrary closed 2-form representing an integral cohomology class, we can again find a connection 1-form α with curvature ω, because every closed 2-form representing an integral class is the curvature of some connection. If ω has constant rank $2k$, then it follows that α has constant class $2k+1$, meaning $\alpha \wedge (d\alpha)^k \neq 0$, and $(d\alpha)^{k+1} \equiv 0$. Recall the definition of a contact-symplectic pair:

Definition 1 ([1, 2])**.** A contact-symplectic pair on a manifold N consists of a 1-form α of constant class $2k+1$ and a closed 2-form β of constant rank $2l$ such that the kernel foliations of $\alpha \wedge (d\alpha)^k$ and of β are complementary, α restricts as a contact form to the leaves of the kernel foliation of β, and β restricts as a symplectic form to the leaves of the kernel foliation of $\alpha \wedge (d\alpha)^k$.

Note that the kernel distributions are integrable because the forms β and $\alpha \wedge (d\alpha)^k$ are closed. The assumption that the kernel foliations are complementary implies that the dimension of N must be $2k+2l+1$.

Our discussion above immediately yields the following:

Theorem 2. *Let M be a closed manifold with a symplectic pair (ω_1, ω_2). If $[\omega_1] \in H^2(M; \mathbb{R})$ is an integral cohomology class, then the total space of the circle bundle $\pi \colon E \to M$ with Euler class $[\omega_1]$ carries a natural contact-symplectic pair.*

Indeed, if α is a connection form with curvature ω_1 and $\beta = \pi^*\omega_2$, then all the required properties are satisfied.

Note that if $\mathcal{F}_i$ denotes the kernel foliation of ω_i, then in the above construction the classical Boothby–Wang construction is performed leafwise over the leaves of $\mathcal{F}_2$, to which ω_1 restricts as an integral symplectic form. On E the kernel foliation of $\beta = \pi^*\omega_2$ consists of the circle bundles over the leaves of $\mathcal{F}_2$, whereas the kernel foliation of $\alpha \wedge (d\alpha)^k$ is complementary and obtained by lifting the leaves of $\mathcal{F}_1$ to the horizontal subspaces for the connection α.

Next recall the definition of a contact pair:

Definition 3 ([1, 4])**.** A contact pair on a manifold N consists of a pair of 1-forms (α, γ) of constant class $2k+1$ and $2l+1$ respectively, such that the kernel foliations of $\alpha \wedge (d\alpha)^k$ and of $\gamma \wedge (d\gamma)^l$ are complementary, α restricts as a contact form to the leaves of the kernel foliation of $\gamma \wedge (d\gamma)^l$, and γ restricts as a contact form to the leaves of the kernel foliation of $\alpha \wedge (d\alpha)^k$.

Again the kernel distributions are integrable because the defining forms are closed. The assumption that the kernel foliations are complementary implies that the dimension of N must be $2k + 2l + 2$.

Our discussion above yields the following:

Theorem 4. *Let M be a closed manifold with a contact-symplectic pair (α, β). If $[\beta] \in H^2(M; \mathbb{R})$ is an integral cohomology class, then the total space of the circle bundle $\pi\colon E \to M$ with Euler class $[\beta]$ carries a natural contact pair.*

Indeed, if γ is a connection form with curvature β and we identify α with its pullback under π, then all the required properties are satisfied.

Finally, combining Theorems 2 and 4, we obtain:

Corollary 5. *If a closed manifold M has a symplectic pair (ω_1, ω_2) such that both $[\omega_i] \in H^2(M; \mathbb{R})$ are integral, then the fiber product of the two circle bundles with Euler classes equal to $[\omega_1]$ and $[\omega_2]$ respectively carries a natural contact pair.*

An important difference between this leafwise Boothby–Wang construction and the classical one is that we cannot perturb the defining forms in a symplectic pair so as to make them rational (and integral after multiplication with a constant), because one cannot control the rank under such perturbations. Therefore, we will check in each example of a symplectic pair we construct in the next section, whether the defining forms represent integral cohomology classes.

3. Constructions of symplectic pairs

The most obvious examples of symplectic pairs are of course products of symplectic manifolds with the induced split symplectic forms. In this case one can obviously choose the forms ω_i to represent integral classes.

We now discuss non-trival sources of examples.

3.1. **Flat bundles with symplectic total holonomy.** Let (B, ω_B) and (F, ω_F) be closed symplectic manifolds, and let $\rho\colon \pi_1(B) \to \mathrm{Symp}(F, \omega_F)$ be a representation of the fundamental group of B in the group of symplectomorphisms of (F, ω_F). The suspension of ρ defines a horizontal foliation on the fiber bundle $\pi\colon M \to B$

with fiber F and total space

$$M = (\tilde{B} \times F)/\pi_1(B),$$

where $\pi_1(B)$ acts on $\tilde{B}$ by covering transformations and on F via ρ. As the image of ρ preserves the symplectic form ω_F, the pullback of this form to the product $\tilde{B} \times F$ descends to M as a closed form of constant rank, whose kernel foliation is exactly the horizontal foliation complementary to the fibers. Pulling back ω_B to the total space M we obtain another closed form of constant rank, which is a defining form for the vertical foliation whose leaves are the fibers of the fibration. As the two foliations are complementary by construction, the forms ω_F and $\pi^*\omega_B$ form a symplectic pair on M.

Note that if we choose ω_B to be integral, then so is its pullback. For ω_F checking integrality is more subtle. In particular, it turns out that starting with an integral form on F, though necessary, is not usually sufficient.

A special case of the above construction is given by taking a single symplectic diffeomorphism $\varphi \in \mathrm{Symp}(F, \omega_F)$, and forming the product of its mapping torus M_φ with S^1. If φ is isotopic to the identity through symplectomorphisms φ_t, with $\varphi_1 = \varphi$ and $\varphi_0 = Id_F$, then M_φ is diffeomorphic to $F \times S^1$ by a diffeomorphism encoding the isotopy. It was proved in Lemma 8 of [17] that under this diffeomorphism, the cohomology class $[\omega_F] \in H^2(M_\varphi)$ corresponds to

$$[\omega_F] + Flux(\varphi_t) \otimes \nu \in H^2(F) \oplus (H^1(F) \otimes H^1(S^1)),$$

where ν is the fundamental cohomology class of S^1. Thus, the cohomology class of ω_F on such a symplectic mapping torus is integral if and only if $Flux(\varphi_t)$ is an integral class in $H^1(F)$.

While symplectic mapping tori have a rather simple topology, determined completely by φ, there are more complicated flat bundles with symplectic total holonomy which exhibit more complex topology. For example, in the simplest possible case, where B and F are both 2-dimensional, Kotschick and Morita [17] proved the following:

Theorem 6 ([17])**.** *For every $g \geq 3$ there exist foliated oriented surface bundles $\pi\colon M \rightarrow B$ over closed oriented surfaces B with fibers F of genus g, which have non-zero signature and whose total holonomy group is contained in the symplectomorphism group* $\mathrm{Symp}(F, \omega_F)$ *with respect to a prescribed symplectic form ω_F on F. In fact, one can restrict the holonomy to be in* $\mathrm{Symp}(F; D^2)$*, the group of compactly supported symplectomorphisms of $F \setminus D^2$.*

The first part is Theorem 1 in [17], whereas the addendum restricting to symplectomorphisms relative to an embedded disk follows from the proof of Theorem 3 in [17]. This addendum is useful for the construction of further symplectic pairs, see Subsection 3.2 below, because it implies that the 4-manifold M in the statement of the theorem contains a product neighbourhood $D^2 \times B$ to which the symplectic pairs restrict in the obvious way, so that the two foliations are given by $D^2 \times \{\star\}$ and by $\{\star\} \times B$. In particular, the horizontal foliation has an open set of closed leaves.

Remark 7. If the base manifold B is not just symplectic, but has a symplectic pair, then any flat bundle over B with symplectic total holonomy inherits something we may naturally call a symplectic triple. From this one can combine several different symplectic pairs. The same remark applies if the total holonomy preserves a symplectic pair on the fiber F.

Foliated bundles can also be used to construct contact-symplectic and contact pairs directly. For example, if B carries a contact or symplectic structure and the image of a homomorphism $\rho\colon \pi_1(B) \to \mathrm{Diff}(F)$ preserves a contact form on F, then $M = (\tilde{B} \times F)/\pi_1(B)$ obtained by suspending ρ inherits a contact or contact-symplectic pair.

3.2. **The Gompf sum for symplectic pairs.** Gompf [11] has shown that symplectic manifolds with closed symplectic submanifolds of codimension 2 admit certain cut-and-paste constructions which build new symplectic manifolds out of old ones. Suppose that (M_1, ω_1) and (M_2, ω_2) are closed symplectic manifolds of dimension $2n$ admitting symplectic submanifolds $\Sigma_i \subset M_i$ of codimension 2 with trivial normal bundles, and such that (Σ_1, ω_1) and (Σ_2, ω_2) are symplectomorphic. Then by the symplectic tubular neighbourhood theorem they have symplectomorphic neighbourhoods. In this situation $M_1 \setminus \Sigma_1$ and $M_2 \setminus \Sigma_2$ can be glued together symplectically along punctured tubular neighbourhoods of the Σ_i. The gluing map turns a punctured normal disk inside out symplectically.

This construction sometimes works for manifolds with symplectic pairs if one of the foliations has codimension 2 and has an open set of compact leaves. Let $(M_1, \omega_1, \omega_2)$ and (M_2, η_1, η_2) be closed manifolds of dimension $2n$ with symplectic pairs for which $rk(\omega_1) = rk(\eta_1) = 2n - 2$. Suppose that the kernel foliations $\mathcal{F}_1$ of ω_2 and $\mathcal{F}_2$ of η_2 each have closed leaves Σ_1 and Σ_2 respectively, such that our symplectic pairs admit product structures in open neighbourhoods of the Σ_i. This means that we assume that Σ_1 has an open neighbourhood $U_1 \subset M_1$ which is diffeomorphic to $\Sigma_1 \times D^2$ in such a way that

$$\omega_1|_{U_1} = \pi_1^*(\omega_1|_{\Sigma_1}) \quad \text{and} \quad \omega_2|_{U_1} = \pi_2^*(\omega_2|_{D^2}),$$

where the π_i are the projections to the factors; and similarly for $\Sigma_2 \subset M_2$. Then we may assume without further loss of generality that $\omega_2|_{D^2}$ and $\eta_2|_{D^2}$ coincide with the standard area form $dx \wedge dy$ on the disk. Suppose further that there is a symplectomorphism

$$f\colon (\Sigma_1, \omega_1) \longrightarrow (\Sigma_2, \eta_1) \ .$$

Then the Gompf sum $M_1 \natural_f M_2$ of the M_i along the submanifolds Σ_i carries a natural symplectic pair.

As in Gompf's original construction [11], the assumptions are particularly easy to verify when the M_i are 4-dimensional. In this case ω_1 and η_1 are volume forms on the Σ_i and, by Moser's theorem [20], a symplectomorphism f as above exists as soon as Σ_1 and Σ_2 have the same genus, and

$$\int_{\Sigma_1} \omega_1 = \int_{\Sigma_2} \eta_1 \ .$$

We can use the flat bundles in Theorem 6 as building blocks for the Gompf sum, because, by construction, their horizontal foliations have product structures on an open set. For the vertical foliations we trivially have product structures around every fiber. Performing the Gompf sum of symplectic pairs by matching fibers with fibers or sections[1] with sections does not lead to any new examples. However,

[1] = closed leaves of the horizontal foliations.

taking a flat bundle over a surface of genus g, and another one with fibers of genus g, we can, after scaling one of the 2-forms involved by a constant, perform the sum of symplectic pairs matching a fiber in one fibration with the section in the other fibration. This gives new examples of manifolds admitting symplectic pairs which are not surface bundles over surfaces.

3.3. **Four-dimensional Thurston geometries.** A geometry in the sense of Thurston consists of a model space X which is a simply connected complete Riemannian manifold, together with a group G of effective isometries acting transitively and admitting a discrete subgroup Γ for which the quotient space X/Γ is a compact smooth manifold. Such compact quotients are said to admit a Thurston geometry of type (X, G).

The four-dimensional Thurston geometries have been classified by Filipkiewicz (unpublished). We refer the reader to Wall's papers [27, 28] for an account of this classification. We now want to show that for some of these geometries there are natural G-invariant symplectic pairs on the model spaces, which then descend to all compact quotients. As the isometries we consider preserve a symplectic structure, they are orientation-preserving.

Example 8. Consider the model spaces $S^2 \times \mathbb{R}^2$, $S^2 \times \mathbb{H}^2$, $\mathbb{R}^2 \times \mathbb{H}^2$ with the product metrics obtained from the standard constant curvature metrics on the factors. In this case any isometry preserves the local product structure and its factors. In the maximal group of orientation-preserving isometries those which preserve a pair of given orientations on the factors form a subgroup G of index 2. The volume forms of the metrics on the factors then form a G-invariant symplectic pair on X.

Example 9. The discussion in the previous example applies to the model spaces $S^2 \times S^2$ and $\mathbb{H}^2 \times \mathbb{H}^2$, except that these also admit isometries interchanging the two factors.

Example 10. The model space $\mathbb{R}^4$ with its standard flat metric has as its compact quotients the flat Riemannian 4-manifolds M. If such a manifold is orientable, then $b_1(M) > 0$. It is known that $b_1(M) \leq 4$, with equality if and only if M is diffeomorphic to T^4, and that $b_1(M) \neq 3$. Moreover, if $b_1(M) = 1$, then the vanishing of the Euler characteristic shows that $b_2(M) = 0$, so that M cannot be symplectic. Thus the only interesting case is when $b_1(M) = 2$. The classification of flat 4-manifolds in [13, 26] shows that in the case $b_1(M) = 2$ they are all quotients of $\mathbb{R}^4$ by isometry groups preserving a product structure $\mathbb{R}^2 \times \mathbb{R}^2$, and acting on each factor preserving its orientation. In a different guise, this statement appears in the classification of compact complex surfaces, where these particular flat Riemannian manifolds appear as so-called hyperelliptic[2] sufaces; see [5], p. 148. They are in fact quotients of products of elliptic curves by free diagonal actions of finite groups of holomorphic automorphisms. Thus they carry natural symplectic pairs.

Example 11. Consider the model space $X = Sol^3 \times \mathbb{R}$ with its maximally symmetric product metric. Then the maximal connected isometry group G_0 is also $Sol^3 \times \mathbb{R}$, acting on itself by left multiplication; cf. [25], pp. 518, 519. This Lie group admits a parallelization by left-invariant one-forms $\alpha_1, \ldots, \alpha_4$ with $d\alpha_1 = \alpha_1 \wedge \alpha_4$,

[2]These surfaces are sometimes called bielliptic, because they have two different elliptic fibrations.

$d\alpha_3 = \alpha_4 \wedge \alpha_3$, $d\alpha_2 = d\alpha_4 = 0$. It follows that $\omega_1 = \alpha_1 \wedge \alpha_3$ and $\omega_2 = \alpha_2 \wedge \alpha_4$ form a left-invariant symplectic pair.

Example 12. Consider the model space $X = Nil^4$ with its maximally symmetric metric. Then again the maximal connected isometry group G_0 coincides with X, acting on itself by left multiplication; cf. [25], p. 518. This Lie group admits a parallelization by left-invariant one-forms $\alpha_1, \ldots, \alpha_4$ with $d\alpha_2 = \alpha_1 \wedge \alpha_4$, $d\alpha_3 = \alpha_2 \wedge \alpha_4$, $d\alpha_1 = d\alpha_4 = 0$. It follows that $\omega_1 = \alpha_1 \wedge \alpha_2$ and $\omega_2 = \alpha_3 \wedge \alpha_4$ form a left-invariant symplectic pair.

Example 13. Consider the model space $X = Nil^3 \times \mathbb{R}$ with its maximally symmetric product metric. This Lie group admits a parallelization by left-invariant one-forms $\alpha_1, \ldots, \alpha_4$ with $d\alpha_3 = \alpha_1 \wedge \alpha_2$ and $d\alpha_1 = d\alpha_2 = d\alpha_4 = 0$. It follows that $\omega_1 = \alpha_1 \wedge \alpha_3$ and $\omega_2 = \alpha_2 \wedge \alpha_4$ form a left-invariant symplectic pair. In this case the maximal connected group of isometries is larger than $Nil^3 \times \mathbb{R}$, because it contains the rotations in the plane spanned by α_1 and α_2. But these rotations do not preserve the symplectic pair.

It turns out that the remaining Thurston geometries do not support any symplectic pairs:

Theorem 14. *The model spaces S^4, $\mathbb{C}P^2$, $\mathbb{H}^4$, $\mathbb{C}H^2$, $\widetilde{PSL_2}(\mathbb{R}) \times \mathbb{R}$, $\mathbb{H}^3 \times \mathbb{R}$, $S^3 \times \mathbb{R}$, Sol_0^4, Sol_1^4 and $Sol_{m,n}$ with $m \neq n$ with their standard metrics do not admit any transitive groups of isometries containing cocompact lattices which also preserve a symplectic pair.*

We have formulated the theorem in such a way that it covers non-maximal geometries in the sense of [27, 28], i.e. we rule out symplectic pairs invariant under transitive subgroups which need not be the maximal isometry groups.

Proof. We proceed case by case. The four-sphere admits no symplectic structure, and so is ruled out. Any compact quotient of $S^3 \times \mathbb{R}$ is finitely covered by $S^3 \times S^1$, and so admits no symplectic structure.

The complex projective plane does admit a symplectic structure, but its tangent bundle has no decomposition into a direct sum of two oriented plane bundles. (This is equivalent to the well-known fact that $\mathbb{C}P^2$ endowed with the non-complex orientation admits no almost-complex structure.) Thus $\mathbb{C}P^2$ is also ruled out. Concerning its non-compact dual $\mathbb{C}H^2$, Wall [28] proved that the isotropy subgroup of any transitive isometry group admitting a cocompact lattice contains $U(2)$. As this does not preserve any splitting of $\mathbb{R}^4$ into a direct sum of proper subspaces, $\mathbb{C}H^2$ cannot carry any invariant symplectic pair.

This last argument also applies to the geometries $\mathbb{H}^4$ and $\mathbb{H}^3 \times \mathbb{R}$. In these cases the isotropy subgroup of any transitive isometry group admitting a cocompact lattice contains $SO(4)$, respectively $SO(3)$. These groups do not preserve any splitting of $\mathbb{R}^4$ into a direct sum of 2-dimensional subspaces.

Finally, for the Lie group geometries $\widetilde{PSL_2}(\mathbb{R}) \times \mathbb{R}$, Sol_0^4, Sol_1^4 and $Sol_{m,n}$ with $m \neq n$, any transitive isometry group must contain the Lie group itself, acting by left multiplication. However, in these cases it is easy to check using the structure constants in [27] that there are no left-invariant symplectic forms; cf. [12]. $\square$

4. Compatible metrics

In this section we clarify the metric properties of symplectic pairs. As a first step, we have the following:

Proposition 15. *Let M be a manifold endowed with two smooth complementary foliations $\mathcal{F}$ and $\mathcal{G}$ which admit closed defining forms. Then there are Riemannian metrics g on M for which $\mathcal{F}$ and $\mathcal{G}$ are orthogonal and have minimal leaves.*

Proof. This is a consequence of the minimality criterion of Rummler and Sullivan; see [10], pages 371, 372. Given an arbitrary foliation $\mathcal{F}$ with leaves of dimension d and a form of degree d which is relatively closed for $\mathcal{F}$ and restricts as a volume form to the leaves of $\mathcal{F}$, one can construct metrics g making the leaves of $\mathcal{F}$ minimal, and such that the given d-form is the volume form of the restricted metric. These metrics g can be chosen to make the kernel of the d-form orthogonal to $\mathcal{F}$, and the restriction to this orthogonal complement is arbitrary.

Suppose that $\mathcal{F} = \mathrm{Ker}(\alpha)$ and $\mathcal{G} = \mathrm{Ker}(\beta)$, with α and β closed and of degrees equal to the codimensions of $\mathcal{F}$ and $\mathcal{G}$, respectively. As $\mathcal{F}$ and $\mathcal{G}$ are assumed to be complementary, α is a leafwise volume form on $\mathcal{G}$ and β is a leafwise volume form on $\mathcal{F}$. Define a metric g by requiring $T\mathcal{F}$ and $T\mathcal{G}$ to be orthogonal, and choosing g along $\mathcal{F}$ so that β is the Riemannian volume form of $g|T\mathcal{F}$, and choosing g along $\mathcal{G}$ so that α is the Riemannian volume form of $g|T\mathcal{G}$. These requirements clearly underdetermine the metric, and any such metric has all the desired properties. □

Corollary 16. *A manifold endowed with a symplectic, contact-symplectic or contact pair admits metrics for which the characteristic foliations are orthogonal with minimal leaves.*

In many of the examples constructed above there are metrics which in addition to making the foliations orthogonal with minimal leaves have further good properties. For example, the flat bundles always have metrics for which the vertical foliation is Riemannian and the horizontal foliation has totally geodesic leaves. The following theorem shows that a general symplectic pair does not admit any metric with properties more restrictive than the ones specified in Corollary 16.

Theorem 17. *There are symplectic pairs on closed four-manifolds for which both foliations are not geodesible and not Riemannian.*

Proof. Consider foliated surface bundles M over surfaces with symplectic total holonomy. The normal bundle of the horizontal foliation is the tangent bundle along the fibers, and its first Pontryagin number is three times the signature $\sigma(M)$, because the Pontryagin number of the tangent bundle of the horizontal foliation vanishes. If the signature is non-zero, then Pasternack's refinement [21] of the Bott vanishing theorem for Riemannian foliations implies that the horizontal foliation is not Riemannian. To see this, recall that for Riemannian foliations Pasternack shows that the Pontryagin numbers of the normal bundle vanish in degrees strictly larger than the codimension of the foliation, which improves the range of vanishing in Bott's theorem by a factor of two. In our situation this means that the first Pontryagin number of the normal bundle, which is in degree 4, vanishes, as the codimension equals 2.

Now take two such foliated bundles, M_1 and M_2. By Theorem 6 we can choose both of them with non-zero signature, such that the base genus of M_2 equals the fiber genus of M_1, and such that the horizontal foliation in M_2 has an open set of compact leaves with trivial normal bundle. Let M be the Gompf sum $M_1 \natural_f M_2$, where a section in M_2 is identified with a fiber in M_1, as discussed in Subsection 3.2 above. This sum M carries an induced symplectic pair, and it is clear that on M the first Pontryagin number of both $T\mathcal{F}$ and $T\mathcal{G}$ is non-zero, because we have chosen both M_i to have non-zero signature. Note that each of these bundles is the normal bundle for the complementary foliation. Thus neither of the two foliations can be Riemannian, by Pasternack's theorem [21].

Suppose now that in M one of the foliations, say $\mathcal{F}$, is geodesible. If a metric making $\mathcal{F}$ totally geodesic also makes it orthogonal to $\mathcal{G}$, then the duality theorem for totally geodesic and bundle-like foliations implies that $\mathcal{G}$ is Riemannian; see [10], p. 190. This is a contradiction.

Next assume that we can choose a metric g for which $\mathcal{F}$ is totally geodesic, without assuming that its orthogonal complement is $\mathcal{G}$. Cairns and Ghys [7] have shown that for any two-dimensional geodesible foliation on a 4-manifold we may choose g to make the leaves both totally geodesic and of constant Gaussian curvature. As $\mathcal{F}$ has closed leaves of genus ≥ 2, the constant curvature is negative. Another result of [7] then tells us that the g-orthogonal complement $T\mathcal{F}^{\perp}$ is integrable, and defines a foliation $\mathcal{H}$ (which may be different from $\mathcal{G}$). By the duality theorem, g is bundle-like for $\mathcal{H}$. But $\mathcal{H}$ has normal bundle $T\mathcal{F}$, which has non-zero first Pontryagin number, and so we again have a contradiction with Pasternack's theorem. $\square$

There are special cases of symplectic pairs for which it is possible to find a metric which makes the two foliations orthogonal and totally geodesic, for example the Thurston geometries which are products of two-dimensional geometries. When performing a Boothby–Wang construction on such an example one can choose a submersion metric on the total space which also has the property that the foliations of the contact-symplectic pair are orthogonal and totally geodesic. This will be used in Subsection 5.2 below.

5. Some applications

5.1. **Torus-bundles over the torus.** We now want to prove the following:

Theorem 18. *Every oriented T^2-bundle over T^2 admits a symplectic pair (ω_1, ω_2) for which the cohomology classes of the ω_i are integral.*

This can be seen as generalizing a result of Geiges [8], who proved that these manifolds admit symplectic structures. His proof, like ours, depends in the classification of T^2-bundles over T^2 due to Sakamoto and Fukuhara [22], and on the fact that all these manifolds carry compatible Thurston geometries; cf. [25].

Proof. The classification of orientable T^2-bundles over T^2 is summarized in the table in the Appendix. We will proceed case by case and use the information given in the table. In case (a), for the four-torus, the claim is trivial.

Case (b) consists of manifolds with Thurston geometry $Nil^3 \times \mathbb{R}$. As the first Betti number equals 3, these manifolds are nilmanifolds (rather than infranil manifolds), i.e. they are quotients of our Lie group by lattices in the group itself acting by left translations; cf. [14], p. 170. We saw in Example 13 that there is a left-invariant symplectic pair on the group. Thus this descends to all manifolds under discussion here.

In case (c) we have the flat orientable four-manifolds with $b_1 = 2$. These have symplectic pairs by Example 10.

Case (d) consists of manifolds with Thurston geometry Nil^4. As their first Betti number equals 2, these manifolds are nilmanifolds (rather than infranil manifolds); cf. [14], p. 170. We saw in Example 12 that there is a left-invariant symplectic pair on the group. Thus this descends to all manifolds under discussion in this case.

Cases (e) and (f) consist of infranil manifolds for the group $Nil^3 \times \mathbb{R}$. As we do not have a symplectic pair on the model space invariant under the full group of orientation-preserving isometries, we argue instead as in [8]. Geiges [8] showed that identifying the model space with $\mathbb{R}^4$ with coordinates (x, y, z, t), the two-forms $dy \wedge dt$ and $dx \wedge dz - xdx \wedge dy$ are invariant under the lattices arising as fundamental groups in this case. Clearly they are closed of constant rank equal to 2, and their wedge product is a volume form. Thus they give rise to a symplectic pair.

Cases (g) and (h) consist of manifolds with Thurston geometry $Sol^3 \times \mathbb{R}$. It was shown in [8] that identifying the model space with $\mathbb{R}^4$ with coordinates (x, y, z, t), the two-forms $dx \wedge dy$ and $dz \wedge dt$ are invariant under the lattices arising as fundamental groups. They are closed of constant rank equal to 2, and their wedge product is a volume form. Thus they give rise to a symplectic pair.

It remains to address the integrality of the cohomology classes of the forms involved. This can trivially be arranged in the case of T^4. For the nilmanifolds of $Nil^3 \times \mathbb{R}$ the integrality of the cohomology classes for the symplectic pair we have exhibited can be checked by direct calculation, or using [8]. For the remaining cases, (c)–(h), we give a uniform argument as follows. All T^2-bundles over T^2 have vanishing Euler characteristic and signature. Therefore, if $b_1 = 2$, we conclude that $b_2 = 2$, and the intersection form is indefinite. Thus, $H^2(M; \mathbb{R})$ equipped with the cup product form is hyperbolic, and the classes of square 0 make up the light cone. For a symplectic pair the classes $[\omega_i]$ have square zero and $[\omega_1] \cdot [\omega_2] \neq 0$, thus they span the light cone. It follows that after constant rescaling these classes are integral. □

5.2. **Irreducible quotients of the polydisk.** For our final application we return to the Thurston geometry with model space $\mathbb{H}^2 \times \mathbb{H}^2$, which we discussed briefly in Example 9. The connected component of the identity in the isometry group is $PSL_2(\mathbb{R}) \times PSL_2(\mathbb{R})$, acting on the model space preserving the symplectic pair formed by the volume forms ω_1 and ω_2 of the hyperbolic metrics on the factors. Note that the product metric on $\mathbb{H}^2 \times \mathbb{H}^2$ is Kähler for both choices of orientation, with Kähler forms $\omega_1 \pm \omega_2$.

It is well known that there are irreducible cocompact lattices $\Gamma \subset PSL_2(\mathbb{R}) \times PSL_2(\mathbb{R})$, where by irreducibility we mean that Γ is not commensurate to a product of lattices in $PSL_2(\mathbb{R})$. While the existence of irreducible lattices can be deduced from a general theorem due to Borel, there are actually explicit constructions in this case due to Kuga (cf. [5]) and Shavel [23] using the theory of quadratic forms. The quotients $(\mathbb{H}^2 \times \mathbb{H}^2)/\Gamma$ are compact complex-algebraic surfaces of general type with Kähler class $\omega_1 \pm \omega_2$ (up to scale, the sign depending on the choice of orientations). It follows that the ω_i represent integral classes in cohomology.

These irreducible quotients of the polydisk have already been used to exhibit various interesting phenomena in both differential and algebraic geometry; cf. [7, 24]. Here we shall add one more, in the form of the following result.

Proposition 19. *There exist closed* 5*-manifolds* M *with two complementary foliations and a Riemannian metric for which the foliations are orthogonal and totally geodesic, and such that* M *does not admit any finite cover by a product of manifolds of strictly smaller dimension.*

Proof. Let X be the quotient of the polydisk by a torsion-free irreducible cocompact lattice Γ. This carries two complementary foliations which are orthogonal and totally geodesic with respect to the metric induced from the product metric on the universal covering.

As the cohomology class of the form ω_1 is integral, we can perform the leafwise Boothby–Wang construction of Section 2 to obtain a closed manifold M, which is the total space of the corresponding circle bundle over X. On M we obtain a contact-symplectic pair, and a metric for which the two foliations are orthogonal and totally geodesic. In fact, the Riemannian universal covering of M is isometric to the direct product $\mathbb{H}^2 \times \widetilde{PSL_2}(\mathbb{R})$, where we think of $\widetilde{PSL_2}(\mathbb{R})$ as the universal covering of the unit tangent bundle of $\mathbb{H}^2$.

It remains to prove that M does not have any finite covering which splits as a direct product of two manifolds of positive dimension. Now it is known that X has vanishing first Betti number (see [23]) and therefore M also has vanishing first Betti number by the Gysin sequence of the circle fibration. It is easy to see that the same conclusion must hold for any finite covering of M. Thus, no such covering can split off a circle, and if it is homotopy equivalent to a product of a 2-manifold and a 3-manifold, then these factors must be real homology spheres. By the classification of surfaces the 2-dimensional factor is then S^2, contradicting the fact that M and all its finite coverings are aspherical. $\square$

Remark 20. Note that we have excluded all splittings of finite coverings of M, without assuming that they are induced by the foliations.

Remark 21. Proposition 19 answers a question of Matveev [19], related to his work in [18]. He noted that in dimensions 2 and 3 every closed Riemannian manifold with a local product structure given by a pair of orthogonal totally geodesic foliations admits a finite covering which is a genuine product (not necessarily induced by the foliations). In dimension 4 this result is false because of the existence of irreducible quotients of the polydisk, and similar examples also exist in dimensions ≥ 6.

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Appendix: Orientable T^2-bundles over T^2

Table 1 summarizes the classification of orientable T^2-bundles over T^2 due to Sakamoto and Fukuhara [22], and the information about their Thurston geometries due to Ue [25]; compare also [8].

The given matrices describe the monodromy corresponding to the two generators of $\pi_1(T^2) = \mathbb{Z}^2$, and the pairs of integers (m, n) represent the Euler class.

Table 1.

	b_1	Monodromy & Euler class	Geometry
(a)	4	$\{I, I, (0,0)\} = T^4$	$\mathbb{R}^4$
(b)	3	$\{I, I, (m,n)\}, (m,n) \neq (0,0)$	$Nil^3 \times \mathbb{R}$
(c)	2	$\left\{\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, I, (0,0)\right\}$ $\left\{\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, I, (-1,0)\right\}$ $\left\{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, I, (0,0)\right\}$ $\left\{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, I, (-1,0)\right\}$ $\left\{\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}, I, (0,0)\right\}$ $\{-I, I, (0,0)\}$ $\{-I, I, (-1,0)\}$	$\mathbb{R}^4$
(d)	2	$\left\{\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}, I, (m,n)\right\}, \lambda \neq 0, n \neq 0$	Nil^4
(e)	2	$\left\{\begin{pmatrix} -1 & \lambda \\ 0 & -1 \end{pmatrix}, I, (m,n)\right\}, \lambda \neq 0$	$Nil^3 \times \mathbb{R}$
(f)	2	$\left\{\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}, -I, (m,n)\right\}, \lambda \neq 0$	$Nil^3 \times \mathbb{R}$
(g)	2	$\{C, I, (m,n)\}, \lvert trC\rvert \geq 3, C \in SL_2(\mathbb{Z})$	$Sol^3 \times \mathbb{R}$
(h)	2	$\{C, -I, (m,n)\}, trC \geq 3, C \in SL_2(\mathbb{Z})$	$Sol^3 \times \mathbb{R}$

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