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STABLE MAPPING CLASS GROUPS OF 4-MANIFOLDS WITH BOUNDARY

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Abstract. We give a complete algebraic description of the mapping class groups of compact simply connected 4-manifolds with boundary up to connected sum with copies of $S^2 \times S^2$.

1. Introduction

It has been shown in [15] that two orientation preserving homeomorphisms of a *closed* simply connected 4-manifold are homotopic if and only if the induced automorphisms on the second homology group coincide with each other (see also [4, §5]). Using this, Quinn [15] has shown that two homeomorphisms inducing the same automorphism on the second homology group are actually topologically isotopic (see also [13]). Quinn has also shown that two diffeomorphisms inducing the same automorphism are smoothly isotopic after the connected sum with the identity diffeomorphism of some copies of $S^2 \times S^2$. In this paper we give a similar smooth stable isotopy criterion for the case where the simply connected 4-manifold has nonempty connected boundary.

Let M be a compact 1-connected 4-manifold with nonempty connected boundary. In this paper we first give an algebraic criterion for two diffeomorphisms of M to be stably isotopic (see §2), where two diffeomorphisms h_0 and h_1 of M which are the identity on the boundary are *stably isotopic* relative to boundary if $h_0 \sharp k(\mathrm{id})$ and $h_1 \sharp k(\mathrm{id})$ are smoothly isotopic relative to boundary as diffeomorphisms of $M \sharp k(S^2 \times S^2)$ for some $k \geq 0$. We will see that certain homomorphisms

$$H_2(M, \partial M; \mathbf{Z}) \to H_2(M; \mathbf{Z}),$$

called variation maps, associated with such diffeomorphisms play an essential role.

Then in §3, we introduce the notion of the stable mapping class group of M and give its complete algebraic description using the result of §2. The algabraic objects that play an important role here are homomorphisms, called variational homomorphisms, $H_2(M, \partial M; \mathbf{Z}) \to H_2(M; \mathbf{Z})$ which are defined to be abstract homomorphisms satisfying certain algebraic conditions. We will see that the set of variational homomorphisms form a group with respect to a certain multiplication.

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The main content of §3 is a construction of a diffeomorphism which realizes a given variational homomorphism.

In §4, we study the relationship between the group of variational homomorphisms and the group of isometries of $H_2(M)$ endowed with the intersection form, which will be used in §5.

In §5, we consider the stable mapping class group of boundary-free diffeomorphims of 4-manifolds whose boundaries are certain spherical 3-manifolds.

Throughout the paper, we work in the smooth category unless otherwise specified. All the homology and cohomology groups are with integer coefficients unless otherwise indicated. We use the symbol "$\cong$" to denote a diffeomorphism between smooth manifolds or an appropriate isomorphism between algebraic objects.

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2. Isotopy of 1-connected 4-manifolds with boundary

Let M be a compact 1-connected 4-manifold with nonempty connected boundary. We first define the variation map associated with a diffeomorphism of M, which will play an important role throughout the paper.

Definition 2.1. Let $h : M \to M$ be a diffeomorphism which is the identity on the boundary. We define the *variation map* $\Delta_h : H_2(M, \partial M) \to H_2(M)$ as follows. For a homology class $\gamma \in H_2(M, \partial M)$, take a 2-cycle $(D, \partial D)$ in $(M, \partial M)$ representing γ. Then $D \cup (-h(D))$ is a 2-cycle in M and we define $\Delta_h(\gamma)$ to be the class represented by $D \cup (-h(D))$. Note that this does not depend on the choice of $(D, \partial D)$ and that Δ_h is a homomorphism. See also [8].

Note that if two diffeomorphisms of M which are the identity on the boundary are isotopic relative to boundary, then their variation maps coincide with each other. The main result of this section is that the converse is also true "stably" as follows.

Theorem 2.2. *Let M be a compact 1-connected 4-manifold with nonempty connected boundary. Suppose that h_0 and $h_1 : M \to M$ are two diffeomorphisms with $h_0|_{\partial M} = h_1|_{\partial M}$ being the identity map. Then h_0 and h_1 are stably isotopic relative to boundary if and only if $\Delta_{h_0} = \Delta_{h_1} : H_2(M, \partial M) \to H_2(M)$.*

Proof. In the following, for an integer $k \geq 0$, M_k will denote the 4-manifold $M \sharp k(S^2 \times S^2)$.

If h_0 and h_1 are stably isotopic relative to boundary, then it is easy to see that their variation maps coincide with each other, since on the direct summand of $H_2(M_k, \partial M_k)$ corresponding to $H_2(\sharp^k S^2 \times S^2)$, the variation maps of $h_j \sharp k(\mathrm{id})$, $j = 0, 1$, are the zero homomorphisms.

In order to prove the converse, let us consider the following construction, which is called an open book construction (see [8]). For a diffeomorphism $h : M \to M$ which is the identity on the boundary, let L_h be the 5-dimensional manifold with boundary obtained from $M \times [0, 1]$ by identifying $M \times \{1\}$ with $M \times \{0\}$ using $h : M \times \{1\} \to M \times \{0\}$. Note that ∂L_h is canonically diffeomorphic to $\partial M \times S^1$. Then, let N_h be the closed 5-dimensional manifold obtained by attaching $\partial M \times D^2$ to L_h along the boundary. Note that N_h is 1-connected and that M is naturally identified with $M \times \{0\} \subset N_h$.

Then, by an argument similar to that in [9, §2], we easily obtain the following.

Lemma 2.3. *A diffeomorphism $h : M \to M$ which is the identity on the boundary is pseudo-isotopic to the identity relative to boundary if and only if N_h bounds a compact 1-connected 6-dimensional manifold W such that $\pi_j(W, M) = 0$ for all j.*

We also have the following.

Lemma 2.4. *Let $h : M \to M$ be a diffeomorphism which is the identity on the boundary. Then we have the following exact sequence:*

$$0 \to H_3(N_h) \to H_2(M, \partial M) \xrightarrow{\Delta_h} H_2(M) \to H_2(N_h) \to 0.$$

Proof. Consider the exact sequence for the pair (N_h, M):

$$0 \to H_3(N_h) \to H_3(N_h, M) \xrightarrow{\partial} H_2(M) \to H_2(N_h) \to H_2(N_h, M). \tag{2.1}$$

Let $\widetilde{M} \cong M \times [0, 1]$ be a normal 1-disk bundle neighborhood of M in N_h. Note that the closure of $N_h \smallsetminus \widetilde{M}$ is again diffeomorphic to $M \times [0, 1]$. Hence, by excision together with Poincaré-Lefschetz duality, we have

$$\begin{aligned} H_i(N_h, M) &\cong H_i(N_h, \widetilde{M}) \cong H_i(M \times [0, 1], \partial(M \times [0, 1])) \\ &\cong H^{5-i}(M \times [0, 1]) \cong H^{5-i}(M) \cong H_{i-1}(M, \partial M). \end{aligned}$$

In particular, we have $H_2(N_h, M) \cong H_1(M, \partial M) = 0$, since M is simply connected and ∂M is connected. Furthermore, under the isomorphism $H_3(N_h, M) \cong H_2(M, \partial M)$ given above, the boundary homomorphism $\partial : H_3(N_h, M) \to H_2(M)$ appearing in (2.1) is identified with the variation map $\Delta_h : H_2(M, \partial M) \to H_2(M)$. This completes the proof of Lemma 2.4. □

Let us go back to the proof of Theorem 2.2. Let h_0 and $h_1 : M \to M$ be diffeomorphisms which are the identity on the boundary such that $\Delta_{h_0} = \Delta_{h_1}$. Set $h = h_0 \circ (h_1)^{-1}$. Then it is easy to show that Δ_h is the zero homomorphism (for example, see [16, §9]). Then by following the same argument as in [9, §2] and by replacing the Wang exact sequence for a bundle over S^1 with that of Lemma 2.4, we can construct a 6-dimensional manifold W as in Lemma 2.3. Therefore, h is pseudo-isotopic to the identity relative to boundary (see also the last paragraph of [10]).

Thus h_0 and h_1 are pseudo-isotopic relative to boundary. Then by Quinn [15], they are stably isotopic relative to boundary. This completes the proof of Theorem 2.2. □

Compare Theorem 2.2 with [16, Conjecture 9.6].

Remark 2.5. In the above situation, we can prove that $h_0 \sharp k(\mathrm{id})$ and $h_1 \sharp k(\mathrm{id})$ are pseudo-isotopic relative to boundary for some $k \geq 0$ also by using an argument based on the theory of open books on 5-dimensional manifolds [16, 17].

3. Stable mapping class group and variational homomorphisms

In this section, we define the stable mapping class group of a compact 4-manifold with boundary and give a result which describes it algebraically.

Definition 3.1. Let M be a compact 4-manifold with boundary. A *stable diffeomorphism* of M is a diffeomorphism of $M_k = M\sharp k(S^2 \times S^2)$ for some k. Two stable diffeomorphisms $h_0 : M_{k_0} \to M_{k_0}$ and $h_1 : M_{k_1} \to M_{k_1}$ which are the identity on the boundary are said to be *stably isotopic* relative to boundary if $h_0\sharp k_0'(\mathrm{id}) : M_{k_0+k_0'} \to M_{k_0+k_0'}$ and $h_1\sharp k_1'(\mathrm{id}) : M_{k_1+k_1'} \to M_{k_1+k_1'}$ are isotopic relative to boundary for some k_0' and k_1' with $k_0 + k_0' = k_1 + k_1'$. Finally, the set of all stable isotopy classes relative to boundary of stable diffeomorphisms of M which are the identity on the boundary is called the *stable mapping class group* of $(M, \partial M)$ and is denoted by $\mathcal{SM}(M, \partial M)$. Note that if we denote by $\mathcal{M}(M, \partial M)$ the usual mapping class group relative to boundary, then we can naturally identify $\mathcal{SM}(M, \partial M)$ with the inductive limit

$$\lim_{k\to\infty} \mathcal{M}(M_k, \partial M_k).$$

This obviously forms a group with respect to the composition of maps.

Let M be as above and let $h : M_k \to M_k$ be a stable diffeomorphism of M which is the identity on the boundary. Let us denote the *variation map* of h by $\Delta_h : H_2(M_k, \partial M_k) \to H_2(M_k)$. We also denote the induced map $h_* : H_2(M_k) \to H_2(M_k)$ by Λ_h. Since the intersection form

$$H_2(M_k) \times H_2(M_k, \partial M_k) \to \mathbf{Z} \tag{3.1}$$

of M_k is unimodular, we may regard $H_2(M_k)$ and $H_2(M_k, \partial M_k)$ dual to each other. For a homomorphism ρ involving these spaces, ρ^* will denote its dual homomorphism. Let $j : M_k \to (M_k, \partial M_k)$ denote the inclusion map.

In order to give an algebraic description of the stable mapping class group, let us prepare some lemmas.

Lemma 3.2. *We have the following:*

(1) $(j_*)^* = j_* : H_2(M_k) \to H_2(M_k, \partial M_k)$.
(2) $\Lambda_h = \mathrm{id} - \Delta_h \circ j_* : H_2(M_k) \to H_2(M_k)$.
(3) $j_* = \Lambda_h^* \circ j_* \circ \Lambda_h : H_2(M_k) \to H_2(M_k)$.
(4) $\Delta_h + \Delta_h^* = \Delta_h \circ j_* \circ \Delta_h^* : H_2(M_k, \partial M_k) \to H_2(M_k)$.
(5) *For two stable diffeomorphisms h and $h' : M_k \to M_k$ which are the identity on the boundary, we have*

$$\Delta_{h\circ h'} = \Delta_h + \Lambda_h \circ \Delta_{h'} : H_2(M_k, \partial M_k) \to H_2(M_k).$$

Proof. (1) This follows from the fact that $\alpha \cdot j_*\beta = \beta \cdot j_*\alpha$ for all $\alpha, \beta \in H_2(M_k)$, where "$\cdot$" denotes the intersection form (3.1).

(2) For each $\alpha \in H_2(M_k)$, we have

$$(\mathrm{id} - \Delta_h \circ j_*)(\alpha) = \alpha - (\alpha - h_*\alpha) = h_*\alpha = \Lambda_h\alpha.$$

Thus the result follows.

(3) This follows from the fact that $h_* : H_2(M_k) \to H_2(M_k)$ is an isometry of $(H_2(M_k), \cdot)$, where "$\cdot$" denotes the intersection form of M_k on $H_2(M_k)$.

(4) We have

$$\begin{aligned} \Delta_h + \Delta_h^* = \Delta_h \circ j_* \circ \Delta_h^* \quad &\Longleftrightarrow \quad (\mathrm{id} - \Delta_h \circ j_*) \circ \Delta_h^* = -\Delta_h \\ &\Longleftrightarrow \quad \Lambda_h \circ \Delta_h^* = -\Delta_h \qquad (3.2) \\ &\Longleftrightarrow \quad \Delta_h^* = -\Lambda_h^{-1} \circ \Delta_h, \qquad (3.3) \end{aligned}$$

where (3.2) follows from (2). This is equivalent to

$$\Delta_h\beta \cdot \alpha = -(\Lambda_h^{-1} \circ \Delta_h\alpha) \cdot \beta \tag{3.4}$$

for all $\alpha, \beta \in H_2(M_k, \partial M_k)$. Let a and b be 2-cycles in $(M_k, \partial M_k)$ representing α and β, respectively. Then the right-hand side of the above equation is equal to

$$\begin{aligned} -h^{-1}(a - ha) \cdot b &= (a - h^{-1}a) \cdot b \\ &= a \cdot b - h^{-1}a \cdot b \\ &= b \cdot a - hb \cdot a \\ &= (b - hb) \cdot a. \end{aligned}$$

Thus we have (3.4) and have proved the required equality.

(5) For every 2-cycle a of $(M_k, \partial M_k)$, we have

$$a - (h \circ h')a = a - ha + ha - h(h'a) = a - ha + h(a - h'a).$$

Thus we have

$$\Delta_{h\circ h'} = \Delta_h + \Lambda_h \circ \Delta_{h'}.$$

This completes the proof. □

Definition 3.3. Let M be a compact 4-manifold with boundary. A homomorphism $\Delta : H_2(M, \partial M) \to H_2(M)$ is *variational* if $\Delta + \Delta^* = \Delta \circ j_* \circ \Delta^*$. Let $\mathcal{V}(M, \partial M)$ denote the set of all variational homomorphisms.

Lemma 3.4. *Let $\Delta : H_2(M, \partial M) \to H_2(M)$ be a variational homomorphism. Then $\Lambda = \mathrm{id} - \Delta \circ j_*$ is an isometry of the inner product space $(H_2(M), \cdot)$, where "$\cdot$" denotes the intersection form of M on $H_2(M)$.*

Proof. Since $\Delta + \Delta^* = \Delta \circ j_* \circ \Delta^*$ holds by our assumption, we have $j_* \circ (\Delta + \Delta^*) = j_* \circ \Delta \circ j_* \circ \Delta^*$. This is equivalent to the equality

$$(\mathrm{id} - j_* \circ \Delta) \circ (\mathrm{id} - j_* \circ \Delta^*) = \mathrm{id} : H_2(M) \to H_2(M).$$

Since $H_2(M)$ is free, we see that $\mathrm{id} - j_* \circ \Delta$ and $\Lambda^* = \mathrm{id} - j_* \circ \Delta^*$ are isomorphisms which are inverses to each other. Thus Λ is an isomorphism and

$$(\mathrm{id} - j_* \circ \Delta^*) \circ (\mathrm{id} - j_* \circ \Delta) = \mathrm{id}$$

also holds. In particular, we have

$$(\mathrm{id} - j_* \circ \Delta^*) \circ (\mathrm{id} - j_* \circ \Delta) \circ j_* = j_*,$$

which is equivalent to $\Lambda^* \circ j_* \circ \Lambda = j_*$. Thus Λ is an isometry. This completes the proof. □

Lemma 3.5. *The set $\mathcal{V}(M, \partial M)$ of all variational homomorphisms forms a group under the multiplication given by*

$$\Delta_1 * \Delta_2 = \Delta_1 + (\mathrm{id} - \Delta_1 \circ j_*) \circ \Delta_2,$$

for $\Delta_1, \Delta_2 \in \mathcal{V}(M, \partial M)$.

Proof. Put $\mathcal{V} = \mathcal{V}(M, \partial M)$ for simplicity. Let us first show that $\Delta_1 * \Delta_2 \in \mathcal{V}$. Putting $\Lambda_i = \mathrm{id} - \Delta_i \circ j_*$, $i = 1, 2$, we have

$$\begin{aligned} (\Delta_1 * \Delta_2) + (\Delta_1 * \Delta_2)^* &= (\Delta_1 + \Lambda_1 \circ \Delta_2) + (\Delta_1 + \Lambda_1 \circ \Delta_2)^* \\ &= \Delta_1 + \Delta_1^* + \Lambda_1 \circ \Delta_2 + (\Lambda_1 \circ \Delta_2)^*. \end{aligned} \tag{3.5}$$

On the other hand, we have

$$
\begin{aligned}
&(\Delta_1 * \Delta_2) \circ j_* \circ (\Delta_1 * \Delta_2)^* \\
&= (\Delta_1 + \Lambda_1 \circ \Delta_2) \circ j_* \circ (\Delta_1 + \Lambda_1 \circ \Delta_2)^* \\
&= \Delta_1 \circ j_* \circ \Delta_1^* + \Delta_1 \circ j_* \circ \Delta_2^* \circ \Lambda_1^* + \Lambda_1 \circ \Delta_2 \circ j_* \circ \Delta_1^* + \Lambda_1 \circ \Delta_2 \circ j_* \circ \Delta_2^* \circ \Lambda_1^* \\
&= (\Delta_1 + \Delta_1^*) + \Delta_1 \circ j_* \circ \Delta_2^* \circ \Lambda_1^* + \Lambda_1 \circ \Delta_2 \circ j_* \circ \Delta_1^* + \Lambda_1 \circ (\Delta_2 + \Delta_2^*) \circ \Lambda_1^* \\
&= (\Delta_1 + \Delta_1^*) + (\Delta_1 \circ j_* + \Lambda_1) \circ \Delta_2^* \circ \Lambda_1^* + \Lambda_1 \circ \Delta_2 \circ (j_* \circ \Delta_1^* + \Lambda_1^*) \\
&= (\Delta_1 + \Delta_1^*) + \Delta_2^* \circ \Lambda_1^* + \Lambda_1 \circ \Delta_2.
\end{aligned}
$$

Thus by (3.5), we see that $\Delta_1 * \Delta_2 \in \mathcal{V}$.

The associativity holds, since we have

$$
\begin{aligned}
\mathrm{id} - (\Delta_1 * \Delta_2) \circ j_* &= \mathrm{id} - (\Delta_1 + (\mathrm{id} - \Delta_1 \circ j_*) \circ \Delta_2) \circ j_* \\
&= (\mathrm{id} - \Delta_1 \circ j_*) \circ (\mathrm{id} - \Delta_2 \circ j_*)
\end{aligned} \tag{3.6}
$$

and hence

$$
\begin{aligned}
(\Delta_1 * \Delta_2) * \Delta_3 &= (\Delta_1 + \Lambda_1 \circ \Delta_2) + (\Lambda_1 \circ \Lambda_2) \circ \Delta_3 \\
&= \Delta_1 + \Lambda_1 \circ (\Delta_2 + \Lambda_2 \circ \Delta_3) \\
&= \Delta_1 * (\Delta_2 * \Delta_3).
\end{aligned}
$$

It is obvious that the zero homomorphism is variational and is the identity element.

Finally, for $\Delta \in \mathcal{V}$, its inverse Δ^{-1} is given by

$$\Delta^{-1} = -\Lambda^{-1} \circ \Delta,$$

where $\Lambda = \mathrm{id} - \Delta \circ j_*$ is an isometry of $(H_2(M), \cdot)$ by Lemma 3.4. This is seen as follows. We have

$$
\begin{aligned}
(-\Lambda^{-1} \circ \Delta) \circ j_* \circ (-\Lambda^{-1} \circ \Delta)^* &= \Lambda^{-1} \circ \Delta \circ j_* \circ \Delta^* \circ (\Lambda^{-1})^* \\
&= \Lambda^{-1} \circ (\Delta + \Delta^*) \circ (\Lambda^{-1})^* \\
&= \Lambda^{-1} \circ \Delta \circ (\Lambda^{-1})^* + \Lambda^{-1} \circ \Delta^* \circ (\Lambda^{-1})^* \\
&= -\Delta^* \circ (\Lambda^{-1})^* - \Lambda^{-1} \circ \Delta \\
&= (-\Lambda^{-1} \circ \Delta) + (-\Lambda^{-1} \circ \Delta)^*
\end{aligned}
$$

by an argument similar to (3.3). Thus we have $\Delta^{-1} = -\Lambda^{-1} \circ \Delta \in \mathcal{V}$. Furthermore, since

$$\mathrm{id} - (-\Lambda^{-1} \circ \Delta) \circ j_* = \Lambda^{-1} \circ (\Lambda + \Delta \circ j_*) = \Lambda^{-1},$$

we have

$$\Delta * (-\Lambda^{-1} \circ \Delta) = \Delta + \Lambda \circ (-\Lambda^{-1} \circ \Delta) = 0$$

and

$$(-\Lambda^{-1} \circ \Delta) * \Delta = -\Lambda^{-1} \circ \Delta + \Lambda^{-1} \circ \Delta = 0.$$

Thus $\mathcal{V}$ forms a group. This completes the proof. □

Definition 3.6. A *stable variational homomorphism* of M is a variational homomorphism $\Delta : H_2(M_k, \partial M_k) \to H_2(M_k)$ for some $k \geq 0$. Two stable variational homomorphisms $\Delta_0 : H_2(M_{k_0}, \partial M_{k_0}) \to H_2(M_{k_0})$ and $\Delta_1 : H_2(M_{k_1}, \partial M_{k_1}) \to H_2(M_{k_1})$ of M are said to be *stably equivalent* if

$$
\begin{aligned}
\Delta_0 \oplus 0_{k_0'} : H_2(M_{k_0+k_0'}, \partial M_{k_0+k_0'}) &\cong H_2(M_{k_0}, \partial M_{k_0}) \oplus H_2(\sharp^{k_0'}(S^2 \times S^2)) \\
&\to H_2(M_{k_0+k_0'}) \cong H_2(M_{k_0}) \oplus H_2(\sharp^{k_0'}(S^2 \times S^2))
\end{aligned}
$$

coincides with

$$\Delta_1 \oplus 0_{k_1'} : H_2(M_{k_1+k_1'}, \partial M_{k_1+k_1'}) \cong H_2(M_{k_1}, \partial M_{k_1}) \oplus H_2(\sharp^{k_1'}(S^2 \times S^2)) \\ \to H_2(M_{k_1+k_1'}) \cong H_2(M_{k_1}) \oplus H_2(\sharp^{k_1'}(S^2 \times S^2))$$

for some k_0' and k_1' with $k_0 + k_0' = k_1 + k_1'$, where

$$0_{k'} : H_2(\sharp^{k'}(S^2 \times S^2)) \to H_2(\sharp^{k'}(S^2 \times S^2))$$

stands for the zero map. The set of all equivalence classes of stable variational homomorphisms $H_2(M_k, \partial M_k) \to H_2(M_k)$, $k \geq 0$, of M is called the *stable variational group* and is denoted by $\mathcal{SV}(M, \partial M)$. It is not difficult to see that $\mathcal{SV}(M, \partial M)$ naturally forms a group, which is nothing but the inductive limit

$$\lim_{k\to\infty} \mathcal{V}(M_k, \partial M_k),$$

where we may naturally regard $\mathcal{V}(M_k, \partial M_k) \subset \mathcal{V}(M_{k+1}, \partial M_{k+1})$ for each $k \geq 0$.

We have a natural map

$$\Theta : \mathcal{SM}(M, \partial M) \to \mathcal{SV}(M, \partial M),$$

which maps each class of a stable diffeomorphism to the class of its variation map. This is a homomorphism of groups by Lemma 3.2(5).

Theorem 3.7. *If M is a smooth compact 1-connected 4-manifold with nonempty connected boundary, then the above correspondence Θ gives an isomorphism of groups.*

Proof. The well-definedness and the injectivity follow from Theorem 2.2.

Let $\Delta : H_2(M_k, \partial M_k) \to H_2(M_k)$ be a stable variational homomorphism of M. Note that

$$\Lambda = \mathrm{id} - \Delta \circ j_* : H_2(M_k) \to H_2(M_k)$$

is an isometry of $(H_2(M_k), \cdot)$ by Lemma 3.4.

The following lemma has been proved implicitly in the proof of Lemma 3.2 (see (3.4)).

Lemma 3.8. $\Delta\beta \cdot \alpha = -(\Lambda^{-1} \circ \Delta\alpha) \cdot \beta$ *for all* $\alpha, \beta \in H_2(M_k, \partial M_k)$.

Let $V = M^0 \cup_{\mathrm{id}} (-M^1)$ with $M^0 = M^1 = M_k$ be the double of M_k, which is a smooth closed 1-connected 4-manifold. Furthermore, let $\delta : H_2(M_k, \partial M_k) \to H_2(V)$ be the doubling homomorphism, which is given by

$$\delta([a]) = [i_0 a - i_1 a],$$

where a is an arbitrary 2-cycle of $(M_k, \partial M_k)$ and $i_0 : M^0 \to V$ and $i_1 : M^1 \to V$ are the inclusion maps. Then put

$$J' = \mathrm{im}(\delta + i_{1*} \circ \Delta : H_2(M_k, \partial M_k) \to H_2(V)).$$

Furthermore, let $G(-\Lambda)$ denote the subgroup of $H_2(M^0) \oplus H_2(M^1)$ consisting of the elements of the form $(\alpha, -\Lambda\alpha)$, $\alpha \in H_2(M^0)$, where we regard Λ as an isomorphism from $H_2(M^0)$ to $H_2(M^1)$. Let $\iota : H_2(M^0) \oplus H_2(M^1) \to H_2(V)$ be the natural homomorphism induced by the inclusions. Then we have the following.

Lemma 3.9. $\iota G(-\Lambda) \subset J'$.

Proof. For every $\alpha \in H_2(M_k)$, we have

$$\begin{aligned}(\delta + i_{1*} \circ \Delta)(j_* \alpha) &= (i_{0*}\alpha - i_{1*}\alpha) + i_{1*} \circ \Delta \circ j_* \alpha \\ &= (i_{0*}\alpha - i_{1*}\alpha) + i_{1*}(\alpha - \Lambda\alpha) \\ &= i_{0*}\alpha - i_{1*} \circ \Lambda\alpha.\end{aligned}$$

This shows our required inclusion. □

Lemma 3.10. *The subgroup J' of $H_2(V)$ is isotropic.*

Proof. For every $\alpha, \beta \in H_2(M_k, \partial M_k)$, we have

$$\begin{aligned}&(\delta + i_{1*} \circ \Delta)(\alpha) \cdot (\delta + i_{1*} \circ \Delta)(\beta) \\ &= \delta\alpha \cdot \delta\beta + \delta\alpha \cdot (i_{1*} \circ \Delta\beta) + (i_{1*} \circ \Delta\alpha) \cdot \delta\beta + (i_{1*} \circ \Delta\alpha) \cdot (i_{1*} \circ \Delta\beta) \\ &= \Delta\beta \cdot \alpha + \Delta\alpha \cdot \beta - \Delta\alpha \cdot (j_* \circ \Delta\beta) \\ &= (\Delta\beta \cdot \alpha + \Delta^*\beta \cdot \alpha) - (\Delta^* \circ j_* \circ \Delta\beta) \cdot \alpha = 0,\end{aligned}$$

since $\delta\alpha \cdot \delta\beta = 0$ and Δ is variational. This completes the proof. □

Consider the Mayer-Vietoris exact sequence for the pair (M^0, M^1):

$$0 \longrightarrow H_2(\partial M_k) \longrightarrow H_2(M^0) \oplus H_2(M^1) \xrightarrow{\iota} H_2(V) \xrightarrow{\partial'} H_1(\partial M_k) \longrightarrow 0.$$

Lemma 3.11. *We have $\partial' J' = H_1(\partial M_k)$.*

Proof. For every $\alpha \in H_2(M_k, \partial M_k)$, we have

$$\partial' \circ (\delta + i_{1*} \circ \Delta)\alpha = \partial' \circ \delta\alpha = \partial\alpha,$$

where $\partial : H_2(M_k, \partial M_k) \to H_1(\partial M_k)$ is the boundary homomorphism in the exact sequence of the pair $(M_k, \partial M_k)$:

$$0 \longrightarrow H_2(\partial M_k) \longrightarrow H_2(M_k) \longrightarrow H_2(M_k, \partial M_k) \xrightarrow{\partial} H_1(\partial M_k) \longrightarrow 0.$$

Since ∂ is surjective, we have the conclusion. □

Let J be a maximal isotropic subgroup of $H_2(V)$ containing J'. Then by [3, (4.2)] (see also [18]), there exists a smooth h-cobordism W relative to boundary between M^0 and M^1 such that

$$\ker(\kappa_* : H_2(V) \to H_2(W)) = J,$$

where $\kappa : V \to W$ is the inclusion map. Let $\eta : M^0 \to M^1$ be the homotopy equivalence relative to boundary induced by the h-cobordism W. We can define the variation map $\Delta_\eta : H_2(M^0, \partial M^0) \to H_2(M^1)$ of η as before.

Lemma 3.12. *The variation map Δ_η coincides with Δ.*

Proof. Since $J' \subset \ker \kappa_*$, we have

$$-\kappa_* \circ \delta = \kappa_* \circ i_{1*} \circ \Delta : H_2(M^0, \partial M^0) \to H_2(W).$$

Let a be an arbitrary 2-cycle of $(M^0, \partial M^0)$. Then we have

$$\begin{aligned}\kappa_* \circ i_{1*} \circ \Delta_\eta[a] &= \kappa_* \circ i_{1*}([a - \eta a]) \\ &= [\kappa \circ i_1 a - \kappa \circ i_0 a] \\ &= \kappa_*(-\delta[a]) \\ &= \kappa_* \circ i_{1*} \circ \Delta[a].\end{aligned}$$

Since $\kappa_* \circ i_{1*} : H_2(M^1) \to H_2(V)$ is an isomorphism, we have $\Delta_h = \Delta$. This completes the proof. $\square$

Now taking the connected sum along cobordisms of the smooth h-cobordism W with the trivial cobordism $(\sharp^{k'} S^2 \times S^2) \times [0,1]$ for some k', we see that Δ is realized stably by a stable diffeomorphism of M [11], [14]. Hence the correspondence Θ is surjective. This completes the proof of Theorem 3.7. $\square$

Remark 3.13. Using an argument similar to the above together with the topological h-cobordism theorem of Freedman [5], we can also obtain some results about the topological mapping class group of topological 4-manifolds with boundary without stabilization. More precisely, let M be a compact 1-connected *topological* 4-manifold with nonempty connected boundary. Let $\mathcal{M}^{\mathrm{TOP}}(M, \partial M)$ denote the group of topological isotopy classes relative to boundary of the *homeomorphisms* of M which are the identity on the boundary. Then we have the natural map

$$\Theta^{\mathrm{TOP}} : \mathcal{M}^{\mathrm{TOP}}(M, \partial M) \to \mathcal{V}(M, \partial M)$$

which maps each isotopy class of a homeomorphism to its variation map. Then the argument in the proof of Theorem 3.7 shows that Θ^{TOP} is surjective. We do not know if it is injective or not.

4. Relation to stable isometry group

In this section, we study the relationship between the stable variational group and the stable isometry group of a 4-manifold with boundary.

Definition 4.1. Let M be a compact 4-manifold with boundary. We denote by $\mathcal{I}(M)$ the isometry group of the inner product space $(H_2(M), \cdot)$, where "$\cdot$" denotes the intersection form. A *stable isometry* of M is an isometry of $(H_2(M_k), \cdot)$ for some k. Two stable isometries $\Lambda_0 : H_2(M_{k_0}) \to H_2(M_{k_0})$ and $\Lambda_1 : H_2(M_{k_1}) \to H_2(M_{k_1})$ of M are said to be *stably equivalent* if the isometry $\Lambda_0 \oplus \mathrm{id}$ of $(H_2(M_{k_0+k'_0}), \cdot) = (H_2(M_{k_0}) \oplus H_2(\sharp^{k'_0} S^2 \times S^2), \cdot)$ and the isometry $\Lambda_1 \oplus \mathrm{id}$ of $(H_2(M_{k_1+k'_1}), \cdot) = (H_2(M_{k_1}) \oplus H_2(\sharp^{k'_1} S^2 \times S^2), \cdot)$ coincide with each other for some k'_0 and k'_1 with $k_0 + k'_0 = k_1 + k'_1$. The set of all equivalence classes of stable isometries of M is called the *stable isometry group* of M and is denoted by $\mathcal{SI}(M)$, which can naturally be identified with the inductive limit

$$\lim_{k\to\infty} \mathcal{I}(M_k).$$

This obviously forms a group with respect to the composition of isometries.

Let M be a 1-connected 4-manifold with nonempty connected boundary $\partial M = K$. For each variational homomorphism $\Delta \in \mathcal{V}(M, \partial M)$, the endomorphism $\Lambda = \mathrm{id} - \Delta \circ j_*$ is an isometry of $(H_2(M), \cdot)$ by Lemma 3.4. Let us define

$$\Xi : \mathcal{V}(M, \partial M) \to \mathcal{I}(M)$$

by $\Xi\Delta = \Lambda$. By (3.6), Ξ is a homomorphism of groups.

Let us denote by $\wedge^2 H_1(K)^*$ the set of all skew-symmetric bilinear forms $H_1(K) \times H_1(K) \to \mathbf{Z}$. Note that this naturally forms an additive group, which is finitely generated and free abelian. Note that an element $\kappa \in \wedge^2 H_1(K)^*$ can also be regarded as a homomorphism $H_1(K) \to \mathrm{Hom}(H_1(K), \mathbf{Z}) \cong H^1(K) \cong H_2(K)$. For such an element κ, define $\widetilde{\kappa} : H_2(M, \partial M) \to H_2(M)$ by the composition

$$H_2(M, \partial M) \xrightarrow{\ \partial\ } H_1(K) \xrightarrow{\ \kappa\ } H_2(K) \xrightarrow{\ \iota_*\ } H_2(M),$$

where $\partial : H_2(M, \partial M) \to H_1(K)$ is the boundary homomorphism and $\iota : K \to M$ is the inclusion. Then we can easily check that $\widetilde{\kappa}$ is a variational homomorphism and that the map $S : \wedge^2 H_1(K)^* \to \mathcal{V}(M, \partial M)$ defined by $S\kappa = \widetilde{\kappa}$ is an injective homomorphism.

The main result of this section is the following.

Proposition 4.2. *For a compact* 1*-connected* 4*-manifold M with nonempty connected boundary $\partial M = K$, the sequence*

$$0 \longrightarrow \wedge^2 H_1(K)^* \xrightarrow{\ S\ } \mathcal{V}(M, \partial M) \xrightarrow{\ \Xi\ } \mathcal{I}(M)$$

is exact.

Proof. We have already observed that S is injective.

It is easy to check that $\mathrm{im}\, S \subset \ker \Xi$. Suppose that Δ is an element of $\ker \Xi$. Thus $\Xi\Delta = \mathrm{id} - \Delta \circ j_*$ is the identity of $H_2(M)$ and hence $\Delta = 0$ on $\mathrm{im}\, j_* = \ker \partial$. Thus we may regard Δ as a homomorphism

$$H_2(M, \partial M)/\ker \partial \cong H_1(K) \to H_2(M).$$

On the other hand, since Δ is variational, we have $\Delta + \Delta^* = \Delta \circ j_* \circ \Delta^* = 0$. Hence, we have $0 = \Delta \circ j_* + \Delta^* \circ j_* = \Delta^* \circ j_*$, which implies that $j_* \circ \Delta = 0$. Thus $\mathrm{im}\, \Delta \subset \ker j_* = H_2(K)$. Hence Δ can be regarded as a homomorphism $\kappa : H_1(K) \to H_2(K)$. Now it is easy to check that $\kappa \in \wedge^2 H_1(K)^*$ and $\Delta = S\kappa$. This completes the proof. □

We can characterize the image of Ξ as follows. Let us first take a basis

$$\{\alpha_1, \alpha_2, \ldots, \alpha_u, \alpha_{u+1}, \ldots, \alpha_{u+v}\}$$

of $H_2(M)$ over the integers, where $u = \mathrm{rank}\, H_2(K)$, $u + v = \mathrm{rank}\, H_2(M)$, and

$$\{\alpha_1, \alpha_2, \ldots, \alpha_u\}$$

is a basis of $\iota_* H_2(K)$. This is possible, since $\iota_* H_2(K)$ is a direct summand of $H_2(M)$. For $H_2(M, \partial M)$, we take the dual basis

$$\{\alpha_1^*, \alpha_2^*, \ldots, \alpha_u^*, \alpha_{u+1}^*, \ldots, \alpha_{u+v}^*\}$$

such that the intersection number in M satisfies $\alpha_k \cdot \alpha_l^* = \delta_{kl}$, where

$$\delta_{kl} = \begin{cases} 1, & k = l, \\ 0, & k \neq l. \end{cases}$$

Let

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

be the matrix representation of a homomorphism $\Delta : H_2(M, \partial M) \to H_2(M)$ with respect to the above basis, where A, B, C and D are $u \times u$, $u \times v$, $v \times u$ and $v \times v$ integral matrices, respectively. Furthermore, we see easily that the matrix representation of j_* with respect to the above basis must be of the form

$$\widetilde{Q} = \begin{pmatrix} 0 & 0 \\ 0 & Q \end{pmatrix}$$

for some nonsingular $v \times v$ integral matrix Q. Then it is easy to see that Δ is variational if and only if

$$D + {}^tD = D\,Q\,{}^tD,\ A + {}^tA = B\,Q\,{}^tB,\ B + {}^tC = B\,Q\,{}^tD.$$

Thus, an isometry $\Lambda \in \mathcal{I}(M)$ is in the image of Ξ if and only if, with respect to the above basis, Λ is represented by a matrix of the form

$$\begin{pmatrix} E_u & -BQ \\ 0 & E_v - DQ \end{pmatrix}$$

such that

$$D + {}^tD = D\,Q\,{}^tD,\ A + {}^tA = B\,Q\,{}^tB,\ B + {}^tC = B\,Q\,{}^tD$$

for some integral matrices A, B, C and D, where E_w denotes the $w \times w$ unit matrix.

If the diagonal entries of Q are even, or if $H_2(K) = 0$, then we can show that this condition is equivalent to that the following diagram is commutative:

(4.1)

$$\begin{array}{ccccccccccc} 0 & \to & H_2(K) & \xrightarrow{\iota_*} & H_2(M) & \xrightarrow{j_*} & H_2(M,\partial M) & \xrightarrow{\partial} & H_1(K) & \to & 0 \\ & & \big\downarrow \mathrm{id} & & \big\downarrow \Lambda & & \big\uparrow \Lambda^* & & \big\downarrow \mathrm{id} & & \\ 0 & \to & H_2(K) & \xrightarrow{\iota_*} & H_2(M) & \xrightarrow{j_*} & H_2(M,\partial M) & \xrightarrow{\partial} & H_1(K) & \to & 0. \end{array}$$

In this case, we have the exact sequence

$$0 \longrightarrow \wedge^2 H_1(K)^* \xrightarrow{\ S\ } \mathcal{V}(M,\partial M) \xrightarrow{\ \Xi\ } \mathcal{I}(M) \xrightarrow{\ \partial\ } A_{\widetilde{Q}}(K) \longrightarrow 0,$$

where ∂ and $A_{\widetilde{Q}}(K)$ are a homomorphism and a group as defined in [3, §1].

If some diagonal entries of Q are not even and $H_2(K) \neq 0$, then the commutativity of the diagram (4.1) is still necessary, but it is not sufficient in general.

In fact, using Boyer's obstruction class θ [3], we can show the following.

Proposition 4.3. *An isometry $\Lambda \in \mathcal{I}(M)$ is in the image of $\Xi : \mathcal{V}(M,\partial M) \to \mathcal{I}(M)$ if and only if $\partial\Lambda = \mathrm{id} \in A_{\widetilde{Q}}(K)$ and Boyer's obstruction $\theta(\mathrm{id},\Lambda) \in I^1(K)$ vanishes, where $I^1(K) = \mathrm{im}(H^1(K) \to H^1(K;\mathbf{Z}_2))$.*

Proof. Suppose $\partial\Lambda = \mathrm{id}$ and $\theta(\mathrm{id},\Lambda) = 0$. Then by Boyer [3], there exists a homeomorphism $h : M \to M$ such that $h|_{\partial M} = \mathrm{id}$ and $h_* = \Lambda$. Let $\Delta = \Delta_h \in \mathcal{V}(M,\partial M)$ be the variation map of h. Then it is easy to see that $\Xi\Delta = \Lambda$.

Conversely, if $\Lambda = \Xi\Delta$ for some Δ, then we have already seen that $\partial\Lambda = \mathrm{id}$. Furthermore, by Remark 3.13, there exists a homeomorphism $h : M \to M$ such that $h|_{\partial M} = \mathrm{id}$ and $\Delta_h = \Delta$. Hence by Boyer [3], the obstruction $\theta(\mathrm{id},\Lambda_h)$ vanishes, where $\Lambda_h = h_*$. Since $\Lambda_h = \Xi\Delta_h = \Lambda$ by Lemma 3.2(2), we have $\theta(\mathrm{id},\Lambda) = 0$. This completes the proof. □

Note that when the diagonal entries of Q are even or $H_2(K) = 0$, the obstruction $\theta(\mathrm{id},\Lambda)$ always vanishes (see [3, (0.8) Proposition]).

The above proposition can also be proved purely algebraically by using the above-mentioned characterization of the image of Ξ in terms of matrices, together with the definition of the obstruction θ.

All the above results hold stably as well. More precisely, we can naturally define the homomorphisms

$$\mathcal{S}S : \wedge^2 H_1(K)^* \to \mathcal{S}\mathcal{V}(M,\partial M),\ \mathcal{S}\Xi : \mathcal{S}\mathcal{V}(M,\partial M) \to \mathcal{S}\mathcal{I}(M),$$

and the sequence

$$0 \longrightarrow \wedge^2 H_1(K)^* \xrightarrow{\mathcal{S}S} \mathcal{SV}(M, \partial M) \xrightarrow{\mathcal{S}\Xi} \mathcal{SI}(M)$$

is exact. The characterization of the image of Ξ holds similarly for $\mathcal{S}\Xi$ as well.

5. Boundary-free diffeomorphisms

In this section, we study the stable mapping class group of boundary-free diffeomorphisms of compact 4-manifolds whose boundaries are certain spherical 3-manifolds.

Definition 5.1. Let M be a compact 4-manifold with boundary. Two stable diffeomorphisms $h_0 : M_{k_0} \to M_{k_0}$ and $h_1 : M_{k_1} \to M_{k_1}$ of M, which may not necessarily be the identity on the boundary, are said to be *stably isotopic* if the diffeomorphisms $h_0 \sharp k_0'(\mathrm{id}) : M_{k_0+k_0'} \to M_{k_0+k_0'}$ and $h_1 \sharp k_1'(\mathrm{id}) : M_{k_1+k_1'} \to M_{k_1+k_1'}$ are smoothly isotopic for some k_0' and k_1' with $k_0 + k_0' = k_1 + k_1'$. Note that the isotopy should leave the boundary invariant as a set, but can move points on the boundary. The set of all stable isotopy classes of stable diffeomorphisms of M is called the *stable mapping class group of boundary free diffeomorphisms* of M and is denoted by $\mathcal{SM}(M)$. This obviously forms a group with respect to the composition of maps.

Let $\Lambda : H_2(M) \to H_2(M)$ be an isometry of a compact 4-manifold M with boundary K. Then we have the commutative diagram
(5.1)

$$\begin{array}{ccccccccccc}
0 & \to & H_2(K) & \xrightarrow{\iota_*} & H_2(M) & \xrightarrow{j_*} & H_2(M, \partial M) & \xrightarrow{\partial} & H_1(K) & \to & 0 \\
 & & \downarrow{\scriptstyle \varphi_2} & & \downarrow{\scriptstyle \Lambda} & & \uparrow{\scriptstyle \Lambda^*} & & \downarrow{\scriptstyle \varphi_1} & & \\
0 & \to & H_2(K) & \xrightarrow{\iota_*} & H_2(M) & \xrightarrow{j_*} & H_2(M, \partial M) & \xrightarrow{\partial} & H_1(K) & \to & 0
\end{array}$$

for some isomorphisms φ_1 and φ_2, where $\iota : \partial M \to M$ and $j : M \to (M, \partial M)$ denote the inclusions. Let $A(K)$ denote the group of automorphisms of $H_*(K)$, where an automorphism is required to preserve the torsion linking pairing $T_1(K) \times T_1(K) \to \mathbf{Q}/\mathbf{Z}$ on the torsion subgroup $T_1(K)$ of $H_1(K)$ together with the intersection pairing $H_1(K) \times H_2(K) \to \mathbf{Z}$ (for details, see [3, §1]). Note that the isomorphism of $H_*(K)$ determined by (φ_1, φ_2) in (5.1) is an automorphism and that the map

$$\partial : \mathcal{I}(M) \to A(K)$$

defined by $\partial\Lambda = (\varphi_1, \varphi_2)$ is a homomorphism.

Furthermore, let $H_+(K)$ denote the subgroup of $A(K)$ consisting of those automorphisms which are realized by an orientation preserving homeomorphism of K.

Set $\mathcal{I}^0(M) = \partial^{-1}(H_+(K))$, which is a subgroup of $\mathcal{I}(M)$. It is easy to observe that we naturally have

$$\mathcal{I}^0(M) \subset \mathcal{I}^0(M_1) \subset \cdots \subset \mathcal{I}^0(M_k) \subset \mathcal{I}^0(M_{k+1}) \subset \cdots .$$

Thus we can define the inductive limit

$$\mathcal{SI}^0(M) = \lim_{k\to\infty} \mathcal{I}^0(M_k),$$

which is also a group.

Note that if $H_1(K) = 0$, then we have $\mathcal{SI}^0(M) = \mathcal{SI}(M)$. Note also that if $h : M_k \to M_k$ is an orientation preserving stable diffeomorphism of M, then $h_* : H_2(M_k) \to H_2(M_k)$ lies in $\mathcal{SI}^0(M)$.

The main result of this section is the following.

Theorem 5.2. *Let M be a smooth compact 1-connected 4-manifold with boundary ∂M homeomorphic to a lens space $L(p, q)$ ($p \geq 2$), the 3-sphere S^3, or the Poincaré homology 3-sphere $\Sigma(2, 3, 5)$. Then the correspondence*

$$\Theta^{\mathrm{fr}} : \mathcal{SM}(M) \to \mathcal{SI}^0(M)$$

which maps each class of a stable diffeomorphism to the class of its induced isometry of the second homology group gives an isomorphism of groups.

Proof. Suppose that $h : M_k \to M_k$ is a stable diffeomorphism of M such that $\Theta^{\mathrm{fr}} h$ is the identity. Then the automorphism $(h|_{\partial M})_*$ of $H_*(\partial M)$ must be the identity by a commutative diagram similar to (5.1). Then by results of [1], [2], [6], and [7] together with our assumption on the boundary, we see that $h|_{\partial M}$ is smoothly isotopic to the identity. Thus, by extending the isotopy to the whole of M using the collar neighborhood, we may assume that $h|_{\partial M}$ is the identity. On the other hand, since $H_1(\partial M)$ is finite, the homomorphism $\Xi : \mathcal{V}(M_k, \partial M_k) \to \mathcal{I}(M_k)$ is injective by Proposition 4.2. Hence the variation map of h must be the zero homomorphism. Thus by Theorem 2.2, h is stably isotopic to the identity diffeomorphism (relative to boundary). Thus the homomorphism Θ^{fr} is injective.

Let $\Lambda : H_2(M_k) \to H_2(M_k)$ be a stable isometry of M such that $\Lambda \in \mathcal{I}^0(M_k)$. Then by the definition of $\mathcal{I}^0(M_k)$, there exists an orientation preserving homeomorphism $f : \partial M_k \to \partial M_k$ such that $\partial\Lambda = f_*$. We may assume that f is a diffeomorphism (see [12]). Then by [3, (0.8) Proposition] together with the fact that $H_*(\partial M; \mathbf{Q}) = 0$, we see that Boyer's obstruction $\theta(f, \Lambda)$ vanishes, which implies that there exists a homeomorphism $\widetilde{f} : M_k \to M_k$ such that $\widetilde{f}_* = \Lambda$ and $\widetilde{f}|_{\partial M} = f$. Then by using the stable h-cobordism theorem [11], [14] together with an argument as in [3, §4] for constructing a smooth h-cobordism, we can show that there exists a diffeomorphism $\widetilde{f}'$ of $M_{k+k'}$ for some $k' \geq 0$ such that $\widetilde{f}'|_{\partial M} = \widetilde{f}|_{\partial M}$ and $\widetilde{f}'_*$ and $\widetilde{f}_*$ are stably equivalent. Hence Θ^{fr} is surjective. This completes the proof. $\square$

Remark 5.3. For the lens space $L(p, q)$, $p \geq 2$, the structure of $H_+(L(p, q))$ has been determined. In fact, according to [2], $H_+(L(p, q))$ coincides with

(1)	$\{\mathrm{id}\}$,	if $p = 2$,
(2)	$\{\pm\mathrm{id}, \pm q\}$,	if $q^2 \equiv 1 \pmod p$ and $q \not\equiv \pm 1 \pmod p$,
(3)	$\{\pm\mathrm{id}\}$,	otherwise,

where $\pm q$ stands for the multiplication by $\pm q$. Thus, if ∂M is diffeomorphic to $L(p, q)$ which satisfies (1) or (3), then $\mathcal{SI}^0(M)$ coincides with $\mathcal{SI}(M)$.

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