ALMOST COMPLEX MANIFOLDS
AND CARTAN’S UNIQUENESS THEOREM

KANG-HYURK LEE

ABSTRACT. We present a generalization of Cartan’s uniqueness theorem to the almost complex manifolds.

1. INTRODUCTION

The primary goal of this article is to present a generalization to the almost complex manifolds of the following celebrated theorem of H. Cartan, which is usually called Cartan’s uniqueness theorem (see p. 66, [13]).

Theorem 1.1 (H. Cartan). Let $\Omega$ be a bounded domain in $\mathbb{C}^n$. If a holomorphic mapping $f : \Omega \to \Omega$ satisfies that $f(p) = p$ and $df_p = \text{Id}$ for some $p \in \Omega$, then $f$ is the identity mapping.

In order to state the main theorem of this article, we shall introduce the necessary terminology and concepts.

A pair $(M, J)$ is called an almost complex manifold if $M$ is a $C^\infty$-smooth real manifold and $J$ is a field of endomorphisms of the tangent bundle $TM$ with $J^2 = -\text{Id}$, i.e. for each $p \in M$, $J_p : T_p M \to T_p M$ is an endomorphism with $J_p^2 = -\text{Id}$. We call $J$ an almost complex structure on $M$. Throughout this paper, by a smooth almost complex manifold we mean a manifold with a $C^\infty$-smooth almost complex structure.

Given two almost complex manifolds $(M, J)$ and $(M', J')$, a $C^1$ mapping $f$ from $M$ to $M'$ is said to be $(J, J')$-holomorphic (or simply pseudo-holomorphic, so there is no danger of confusion) if its differential $df : TM \to TM'$ satisfies

$$df \circ J = J' \circ df \quad (1.1)$$

on $TM$. If $(M, J)$ is a Riemann surface, $f$ is called a pseudo-holomorphic curve. In the case $(M, J)$ is the unit disc $D$ in $\mathbb{C}$ with the standard complex structure $J_{st}$, we call $f$ a pseudo-holomorphic disc. We denote by $\mathcal{O}_{(J,J')}(M,M')$ the space of $(J, J')$-holomorphic mappings from $M$ to $M'$.

By the existence theorem of pseudo-holomorphic discs (Nijenhuis and Woolf [15]), we can define the Kobayashi pseudo-distance ([8]) and the Kobayashi-Royden pseudo-metric ([16]) for the almost complex manifolds.

Let $(M, J)$ be an almost complex manifold. Given two points $p$ and $q$ in $M$, a finite sequence of pseudo-holomorphic discs $c = \{\phi_j\}_{j=1,\ldots,k} \subset \mathcal{O}_{(J_{st}, J)}(D, M)$
is called a chain of pseudo-holomorphic discs from \( p \) to \( q \) if there are points \( p = p_0, p_1, \ldots, p_k = q \) in \( M \) and \( a_1, a_2, \ldots, a_k \) in \( D \) such that
\[
\phi_j(0) = p_{j-1} \quad \text{and} \quad \phi_j(a_j) = p_j
\]
for \( j = 1, \ldots, k \). For this chain, we define its length \( \ell(c) \) by
\[
\ell(c) = \log \frac{1 + |a_1|}{1 - |a_1|} + \ldots + \log \frac{1 + |a_k|}{1 - |a_k|}.
\]
Note that \( \log \frac{1 + |z|}{1 - |z|} \) is the Poincaré distance from 0 to \( z \) in \( D \). The Kobayashi pseudo-distance \( d_{(M, J)} \) on \((M, J)\) is then defined by
\[
d_{(M, J)}(p, q) = \inf \ell(c),
\]
where the infimum is taken over all chains of pseudo-holomorphic discs from \( p \) to \( q \).

The Kobayashi-Royden pseudo-metric \( F_{(M, J)} \) is the infinitesimal version of the Kobayashi pseudo-distance defined by
\[
F_{(M, J)}(p, v) = \inf \left\{ \frac{1}{|a|} : \phi \in \mathcal{O}_{(J, J)}(D, M) \text{ with } \phi(0) = p, \ d\phi(e) = av \right\},
\]
where \( e \) is the unit vector in \( T_0 D \) and \( p \in M \) and \( v \in T_p M \). We exploit from [10] and [11] the following properties that are exactly the same as in the integrable case ([8] and [10]):

(a) \( F_{(M, J)} \) is upper semi-continuous and
\[
d_{(M, J)}(p, q) = \inf \int_0^1 F_{(M, J)}(\gamma(t), \gamma'(t))dt,
\]
where the infimum is taken over all piecewise smooth paths \( \gamma : [0, 1] \rightarrow M \) with \( \gamma(0) = p \) and \( \gamma(1) = q \).

(b) Let \( f : (M, J) \rightarrow (M', J') \) be a pseudo-holomorphic mapping. For any points \( p \) and \( q \) in \( M \) and tangent vector \( v \in T_p M \), we have
\[
d_{(M', J')} (f(p), f(q)) \leq d_{(M, J)}(p, q)
\]
and
\[
F_{(M', J')} (f(p), df_p(v)) \leq F_{(M, J)} (p, v).
\]

(c) The Kobayashi pseudo-distance \( d_{(M, J)} \) is finite and continuous on \( M \times M \).

(d) If \( d_{(M, J)} \) is a distance, it induces the standard topology on \( M \).

We say that \((M, J)\) is (Kobayashi) hyperbolic if \( d_{(M, J)} \) is a proper distance. Note that for any neighborhood \( U \) of \( p \in M \), there is a constant \( r > 0 \) such that the Kobayashi ball \( B_{(M, J)}(p, r) = \{ q \in M : d_{(M, J)}(p, q) < r \} \) is contained in \( U \) when \((M, J)\) is hyperbolic.

Now we state our main theorem.

**Theorem 1.2.** Let \((M, J)\) be a \( C^\infty \)-smooth almost complex manifold. Moreover, \( M \) is connected and Kobayashi hyperbolic. Suppose that there is a pseudo-holomorphic mapping \( f : M \rightarrow M \) with \( f(p) = p \) and \( df_p = \text{Id} \). Then \( f \) is the identity mapping.

The proof of this theorem appears in Section 5. Sections 2, 3 and 4 contain a regularity theorem and derivative estimates for pseudo-holomorphic mappings which will be used in the proof of Theorem 1.2.
2. Regularity of pseudo-holomorphic mappings

We now study the smoothness of pseudo-holomorphic mappings. Since the problem is local, we assume that our manifold is a domain in a Euclidean space. Let \((\Omega, J) \subset \mathbb{R}^{2n}\) and \((\Omega', J') \subset \mathbb{R}^{2m}\) be domains with almost complex structures \(J \in C^\infty(\Omega)\) and \(J' \in C^\infty(\Omega')\). (If the underlying space of an almost complex manifold is a domain in a Euclidean space, we will call it the almost complex domain.) Assume that \(\Omega\) is bounded and has smooth boundary. Regard \(J\) and \(J'\) as matrix-valued functions on \(\Omega\) and \(\Omega'\), respectively. In this section \(j, k, \ldots = 1, 2, \ldots, 2n\) and \(\alpha, \beta, \gamma, \ldots = 1, 2, \ldots, 2m\).

Let \(f : \Omega \to \Omega'\) be a pseudo-holomorphic mapping of class \(C^1(\Omega)\). Then \(J'_f = J' \circ f\) is \(2m \times 2m\) matrix-valued function defined on \(\Omega\) of class \(C^1(\Omega)\). We will fix \(f\) and simply denote \(J'_f\) by \(J'\) for the rest of this section. Let \(J = (a^k_j)\) and \(J' = (b^\beta_j)\), where \(a^k_j \in C^\infty(\Omega)\) and \(b^\beta_j \in C^1(\Omega)\).

Denote by \(L^2(\Omega, \mathbb{R}^{2m})\) (resp. \(L^2(\Omega, M_{2m \times 2n}(\mathbb{R}))\)) the space of \(\mathbb{R}^{2m}\)-valued (resp. \(2m \times 2n\) matrix-valued) square integrable functions. For \(g \in L^2(\Omega, \mathbb{R}^{2m})\) and \(\varphi \in L^2(\Omega, M_{2m \times 2n}(\mathbb{R}))\), we write \(g = (g_\alpha)\) and \(\varphi = (\varphi^\beta_j)\). Define the inner products of \(L^2(\Omega, \mathbb{R}^{2m})\) and \(L^2(\Omega, M_{2m \times 2n}(\mathbb{R}))\) by

\[
(g, h) = \int_\Omega (\sum_\alpha g_\alpha h_\alpha),
\]

\[
(\varphi, \psi) = \int_\Omega \text{trace}(\varphi^a \psi + J(\varphi^a)J(\psi))
\]

\[
= \int_\Omega (\sum_\alpha \varphi^\alpha_j \psi^\alpha_j + \sum_{\alpha, \beta, \gamma, j} \varphi^\alpha_j b^\beta_j \psi^\beta_j),
\]

where \(g, h \in L^2(\Omega, \mathbb{R}^{2m})\) and \(\varphi, \psi \in L^2(\Omega, M_{2m \times 2n}(\mathbb{R}))\).

For fixed \(f\), we can define the densely defined linear differential operator \(\overline{\partial} : L^2(\Omega, \mathbb{R}^{2m}) \to L^2(\Omega, M_{2m \times 2n}(\mathbb{R}))\) by

\[
\overline{\partial}g = dg + J' dg J,
\]

where \(dg\) denotes the Jacobian matrix of \(g\). Since \(f\) satisfies equation (1.1), it follows that \(\overline{\partial} f = 0\). The \((\alpha, j)\)-th entry of \(\overline{\partial}g\) can be expressed by

\[
(\overline{\partial}g)^a_j = \frac{\partial g_\alpha}{\partial x_j} + \sum_{\beta, k} b^\alpha_j a^\beta_k \frac{\partial g^\beta_k}{\partial x_j}.
\]

We consider the following linear differential operator \(\partial : L^2(\Omega, M_{2m \times 2n}(\mathbb{R})) \to L^2(\Omega, \mathbb{R}^{2m})\) by

\[
(\partial \varphi)_\alpha = -\sum_j \frac{\partial \varphi^j_\alpha}{\partial x_j} + \sum_{\beta, j, k} b^j_\alpha a^\beta_k \frac{\partial \varphi^\beta_k}{\partial x_j}.
\]

In fact, the principal part of the formal adjoint operator of \(\overline{\partial}\) is of the form \((I + J'J')\partial\). Replacing \(\varphi\) by \(\overline{\partial}g\), we have

\[
(\partial \overline{\partial}g)_\alpha = -\sum_j \frac{\partial}{\partial x_j} (\overline{\partial}g)^a_j + \sum_{\beta, j, k} b^j_\alpha a^\beta_k \frac{\partial}{\partial x_j} (\overline{\partial}g)^\beta_j.
\]
Applying equation (2.1), we have that
\[
(\partial \overline{\partial} g)_\alpha = - \sum_j \frac{\partial^2 g_\alpha}{\partial x_j \partial x_j} - \sum_{\beta,j,k} b_\beta^j a^k_j \left( \frac{\partial^2 g_\beta}{\partial x_j \partial x_k} - \frac{\partial^2 g_\beta}{\partial x_k \partial x_j} \right) + \sum_{\beta,\gamma,j,k,l} b_\beta^\gamma a^k_\gamma b_j^{k,l} \frac{\partial^2 g_\gamma}{\partial x_k \partial x_l} + (Cg)_\alpha,
\]
where \((Cg)_\alpha\) is part of \((\partial \overline{\partial} g)_\alpha\) of lower order given by
\[
(Cg)_\alpha = - \sum_j \frac{\partial g_\beta}{\partial x_k} \frac{\partial}{\partial x_j} (b_\beta^j a^k_j) + \sum_{\beta,\gamma,j,k,l} b_\beta^\gamma a^k_\gamma \frac{\partial g_\gamma}{\partial x_k} \frac{\partial}{\partial x_j} (b_\gamma^j a^k_j) .
\]

Remark 2.1. Since \(a^j_\beta, b^j_\beta\) and its first derivatives are continuous on \(\Omega\), it follows that \((Cg)_\alpha \in L^2(\Omega)\) if \(g \in W^{1,2}(\Omega, \mathbb{R}^{2m}) = \bigoplus_{m=1}^2 W^{1,2}(\Omega)\). In particular, \((Cf)_\alpha \in L^p(\Omega)\) for any \(p \geq 1\).

Let \(p > 2n\). For any positive integer \(k\), we have \(kp > 2n\); hence by Theorem 5.23 in [1], \(W^{k,p}(\Omega)\) is a Banach algebra, i.e. \(uv \in W^{k,p}(\Omega)\) for any \(u\) and \(v\) in \(W^{k,p}(\Omega)\). Additionally, using the chain rule, \(b_\beta^j \in W^{k,p}(\Omega)\) whenever \(f_\alpha \in W^{k,p}(\Omega)\) for each \(\alpha\). Moreover, \((Cf)_\alpha \in W^{k-1,p}(\Omega)\).

For convenience, we let \(A^k_l = \sum_j a^k_j a^l_j \in C^\infty(\Omega)\). In fact, \(A^k_l\) is the \((k, l)\)-th entry of the matrix \(JJ^l\). Since \(\sum_{\beta} b_\beta^\gamma a^k_\gamma b_j^{k,l} = -\delta_{\alpha,\gamma}\), it follows that
\[
(\partial \overline{\partial} g)_\alpha = - \sum_j \frac{\partial g_\alpha}{\partial x_j} (\overline{\partial g})^\alpha_j + \sum_{\beta,j,k} b_\beta^j a^k_j \frac{\partial}{\partial x_k} (\overline{g})^\beta_j
\]
\[
= - \sum_j \frac{\partial^2 g_\alpha}{\partial x_j \partial x_j} - \sum_{k,l} A^k_l \frac{\partial^2 g_\alpha}{\partial x_k \partial x_l} + (Cg)_\alpha
\]
when each \(g_\alpha\) is of class \(C^\infty\). For any \(h \in C^1_0(\Omega)\), we obtain
\[
\int_{\Omega} (\partial \overline{\partial} g)_\alpha h = \sum_j \int_{\Omega} (\overline{\partial g})^\alpha_j \frac{\partial h}{\partial x_j} - \sum_{\beta,j,k} \int_{\Omega} (\overline{g})^\beta_j \frac{\partial}{\partial x_k} (b_\beta^j a^k_j h)
\]
\[
= \sum_j \int_{\Omega} \frac{\partial g_\alpha}{\partial x_j} \frac{\partial h}{\partial x_j} + \sum_{k,l} \int_{\Omega} \frac{\partial g_\alpha}{\partial x_l} \frac{\partial}{\partial x_k} (A^k_l h)
\]
\[
+ \int_{\Omega} (Cg)_\alpha h .
\]

Since \(C^\infty(\Omega)\) is dense in \(W^{1,2}(\Omega)\), we take a sequence \(f'\) in \(C^\infty(\Omega, \mathbb{R}^{2m})\) which converges to \(f\) in \(W^{1,2}(\Omega, \mathbb{R}^{2m})\). Then \((\partial \overline{\partial} f')_\alpha\), \((Cf')_\alpha\) and all the remaining first derivatives of \(f'\) converge to those of \(f\) in \(L^2(\Omega)\). Since \((\partial \overline{\partial} f')_\alpha = 0\), the sequence of
equations [2.2] for $f''$ converges to

\[(2.3) \quad - \sum_j \int_{\Omega} \partial f_{\alpha} \partial h \partial x_j - \sum_{k,l} \int_{\Omega} \partial f_{\alpha} \partial \partial x_l \partial x_k (A^k_l h) = \int_{\Omega} (Cf)_\alpha h\]

for any $h \in C^1_0(\Omega)$.

Take the linear partial differential operator $H = \sum_j \frac{\partial^2}{\partial x_j \partial x_j} + \sum_{k,l} A^k_l \frac{\partial^2}{\partial x_k \partial x_l}$. The symbol of $H$ is $\sum_j \zeta_j^2 + \sum_{k,l} \zeta_k A^k_l \zeta_l = |\zeta|^2 + |J\zeta|^2$. So $H$ is strictly elliptic on $\Omega$ with smooth coefficients. Equation (2.3) means that

\[
Hf = (Cf)_\alpha
\]

in the weak sense.

By our assumption, it follows that $(Cf)_\alpha \in L^2(\Omega)$ for each $\alpha$. By the elliptic regularity theorem (Theorem 8.8 in [5]), we have $f_\alpha \in W^{2,2}_{loc}(\Omega)$ for each $\alpha$.

Let $p > 2n$. Since $(Cf)_\alpha \in L^p(\Omega)$, by the uniqueness of solutions of the Dirichlet problem for the elliptic equation (Corollary 9.18 in [5]), it follows that $f_\alpha \in W^{2,p}_{loc}(\Omega) \cap C^0(\Omega)$ for each $\alpha$. From Remark 2.1 we have $(Cf)_\alpha \in W^{2,p}_{loc}(\Omega)$; hence Theorem 9.19 in [5] implies that $f_\alpha \in W^{k,p}_{loc}(\Omega)$ for each $\alpha$. Simultaneously, $(Cf)_\alpha \in W^{2,p}_{loc}(\Omega)$. Repeating our argument, we show that $f_\alpha \in W^{k,p}_{loc}(\Omega)$ for each positive integer $k$. By the Sobolev imbedding theorem, we have

**Proposition 2.2.** Let $(M^{2n}, J)$ and $(M^{2m}, J')$ be $C^\infty$-smooth almost complex manifolds. Any $C^1$ pseudo-holomorphic mapping from $M$ to $M'$ is of class $C^\infty$.

For the regularity of pseudo-holomorphic curves ($n = 1$), see Theorem 3.2.2 in [12] and Theorem 2.2.1 in [17].

### 3. First Order Estimate of Pseudo-Holomorphic Mappings

In this section, we derive the Cauchy estimate for pseudo-holomorphic mappings. For the first order estimate, it suffices to treat the case of pseudo-holomorphic discs.

**Proposition 3.1** (Sikorav [17]). Fix $r, \eta \in (0, 1)$. Let $W$ be a bounded domain in $\mathbb{C}^n$. Then there exist positive constants $\varepsilon$ and $C$ with the following property:

If $\phi : D \to W$ is a differentiable mapping such that

\[
\frac{\partial \phi}{\partial \overline{z}} + q(\phi) \frac{\partial \phi}{\partial z} = 0,
\]

where $q : W \to \text{End}_\mathbb{R}(\mathbb{C}^n)$ is of class $C^r$ and $\|q\|_{C^r} \leq \varepsilon$, then $\phi$ is of class $C^{1+r}$ on $D(1 - \eta)$. Moreover,

\[
\|\phi\|_{C^{1+r}(D(1-\eta))} \leq C\|\phi\|_{L^\infty}.
\]

The $C^0$ and $C^k$ norms for a $C^k$ mapping $f : U \subset \mathbb{R}^n \to \mathbb{R}^m$ is usually defined by $\|f\|_{C^0(U)} = \sum_{j=1}^m \sup_{x \in U} |f_j(x)|$ and $\|f\|_{C^k(U)} = \sum_{j=1}^m \sum_{|\alpha| \leq k} \|D^\alpha f_j\|_{C^0(U)}$, where $|\cdot|$ is a standard Euclidean norm. For $0 < r < 1$, the $C^{k+r}$ (Holder) norm is defined by

\[
\|f\|_{C^{k+r}(U)} = \|f\|_{C^k(U)} + \sum_{j=1}^m \sup_{\alpha=|\alpha|} \sup_{x \neq y \in U} \frac{|D^\alpha f_j(x) - D^\alpha f_j(y)|}{|x - y|^r}.
\]
Note that for a $C^1$ mapping $f : U \subset \mathbb{R}^n \to \mathbb{R}^m$, $\|f\|_{C^1(U)}$ is equivalent to
\[
\|f\|_{C^0(U)} + \sup_{v \in \mathbb{R}^m} |df(v)|.
\]

Now we present:

**Theorem 3.2.** Let $(\Omega, J) \subset \mathbb{R}^{2m}$ and $(\Omega', J') \subset \mathbb{R}^{2m}$ be almost complex domains. For each point $p \in \Omega'$, there is a bounded neighborhood $U$ of $p$ in $\Omega'$ such that \{\|f\|_{C^1(K)} : f \in \mathcal{O}(J,J')(\Omega,U)\} is uniformly bounded for any compact subset $K$ of $\Omega$.

**Proof.** First, let us study the pseudo-holomorphic discs in $\Omega'$. Applying a linear change of coordinates and a translation of $\mathbb{R}^{2m}$, we may assume that $p = 0$ and $J'$ coincides with the canonical complex structure at 0, i.e. $J'_0 = J_{st}$. Take a neighborhood $V$ of 0 such that $J' + J_{st}$ is invertible on $V$.

Suppose that $\phi : D \to V \subset \Omega'$ is a pseudo-holomorphic disc. Then the following equation holds:
\[
\frac{\partial \phi}{\partial x} = J'_a \frac{\partial \phi}{\partial y}.
\]

Since $\frac{\partial \phi}{\partial x} = \frac{\partial \phi}{\partial z} + \frac{\partial \phi}{\partial \bar{z}}$ and $\frac{\partial \phi}{\partial y} = J_{st} \left( \frac{\partial \phi}{\partial z} - \frac{\partial \phi}{\partial \bar{z}} \right)$, we have
\[
(J'_a + J_{st}) \frac{\partial \phi}{\partial \bar{z}} = -(J'_a - J_{st}) \frac{\partial \phi}{\partial z}.
\]

Defining the mapping $q : V \to \text{End}_\mathbb{R}(\mathbb{C}^m)$ by $q(a) = (J'_a + J_{st})^{-1} (J'_a - J_{st})$, we see that (3.1) can be written as
\[
\frac{\partial \phi}{\partial \bar{z}} + q(\phi) \frac{\partial \phi}{\partial z} = 0.
\]

Since $V$ is relatively compact in $\Omega'$, $q$ has the same (Hölder) regularity as that of $J'$ on $V$.

Define the renormalization $q_\beta$ of $q$ by $q_\beta : \beta^{-1}V = \{\beta^{-1}a : a \in V\} \to \text{End}_\mathbb{R}(\mathbb{C}^m)$ and $q_\beta(a) = q(\beta a)$ for an arbitrary real number $\beta > 0$. Take a sufficiently small $\beta$ such that $B(0,1) \subset \beta^{-1}V$, equivalently $B(0,\beta) \subset V$. Then for fixed $0 < r < 1$, we have
\[
\|q_\beta\|_{C^r(B(0,1))} = \|q\|_{C^r(B(0,1))} + \sup_{x \neq y \in B(0,1)} \frac{|q_\beta(x) - q_\beta(y)|}{|x - y|^r} = \|q\|_{C^r(B(0,\beta))} + \sup_{x \neq y \in B(0,1)} \frac{|q(x) - q(y)|}{|\beta x - \beta y|^r} \beta^r \leq \|q\|_{C^r(B(0,\beta))} + \sup_{x,y \in B(0,1)} \frac{|q(x) - q(y)|}{|x - y|^r} \beta^r.
\]

Since $q(0) = 0$, it follows that $\|q\|_{C^r(B(0,\beta))} \to 0$ as $\beta \to 0$. For a sufficiently small $\beta$, we have that $\|q_\beta\|_{C^r(B(0,1))} < \varepsilon$, where $\varepsilon$ is in Proposition 3.1 for the case $W = B(0,1)$. Now a new mapping $\phi_\beta = \beta^{-1}\phi$ satisfies
\[
\frac{\partial \phi_\beta}{\partial \bar{z}} + q_\beta(\phi_\beta) \frac{\partial \phi_\beta}{\partial z} = 0.
\]
Let $U = B(0, \beta)$. By Proposition 3.1 we can deduce that
\[
\|\phi\|_{C^1(D(1-\eta))} \leq C\|\phi\|_{C^{1+r}(D(1-\eta))} \\
\leq C\beta\|\phi\|_{L^\infty} \\
\leq C\|\phi\|_{L^\infty}
\]
for any $\phi \in \mathcal{O}(J_{\epsilon,J_r})(D,U)$.

By 5.4a in [15], there is a constant $R > 0$ such that for any vector $v \in T\Omega$ based on $K$ with $|v| \leq R$, there is a pseudo-holomorphic disc $\phi : D \to \Omega$ such that $d\phi(e) = v$, where $e$ is an unit vector in $T_0D$. For any $f \in \mathcal{O}(J_{\epsilon,J_r})(\Omega,U)$, $f \circ \phi : D \to U$ is pseudo-holomorphic; hence it follows that $|df(v)| = |d(f \circ \phi)(e)| \leq \|d(f \circ \phi)\|_0 \leq \|f \circ \phi\|_{C^1(D(1-\eta))} \leq C\|f \circ \phi\|_{L^\infty} \leq C\|f\|_{C^0}$. Therefore we have
\[
\|f\|_{C^1(K)} \sim \|f\|_{C^0(K)} + \sup_{x \in K} \sup_{|v| \leq R} \frac{1}{R} |df_x(v)| \\
\leq \|f\|_{C^0(\Omega)} + \frac{C}{R}\|f\|_{C^0(\Omega)} \\
\leq (1 + \frac{C}{R})\|f\|_{C^0(\Omega)} .
\]
This proves the theorem. \qed

4. Pseudo-holomorphic jet bundles

In order to prove Theorem 1.2 we need some information about the $\infty$-jet of a certain family of pseudo-holomorphic mappings at a given point. These can be obtained by jet bundles.

Gauduchon ([4]) has shown that there is a natural almost complex structure in a pseudo-holomorphic 1-jet bundle such that the lifting of the pseudo-holomorphic mapping is also pseudo-holomorphic. In the first two subsections, we follow Gauduchon’s work (see chapter 4 in [2] and [3]).

4.1. Horizontal distribution. Let $\pi : E \to M$ be a vector bundle with a linear connection $\nabla$. For any point $u \in E_x = \pi^{-1}(x)$, the vertical tangent space $T^v_uE$ at $u$ is a subspace of $T_uE$ whose elements are tangent to $E_x$. Let $T^vE = \bigcup_{u \in E} T^v_uE$.

Fix any section $\xi \in \Gamma(E)$ with $\xi(x) = u$. For each vector $X \in T_xM$, we define a lifting $\tilde{X}_u$ in $T_uE$ by
\[
\tilde{X}_u = d\xi_u(X) - \nabla_X\xi,
\]
where $\nabla_X\xi \in E_x$ is considered as an element of $T^v_uE$. This definition of $\tilde{X}_u$ is independent of the choices for $\xi$. Therefore, the horizontal subspace $H^\nabla_u$ at $u$ can be uniquely defined as a lifting subspace of $T_uM$ in $T_uE$ up to the linear connection $\nabla$. We call $H^\nabla = \bigcup_{u \in E} H^\nabla_u$ the horizontal distribution. It is easy to check that $H^\nabla$ is a smooth distribution and that the following properties hold:

(a) $T_uE = H^\nabla_u \oplus T^v_uE$ at each $u \in E$.

(b) Let $v^\nabla : H^\nabla \oplus T^vE \to T^vE$ be a natural projection (vertical projection). If $Y \in T_uE$ with $d\xi_u(X) = Y$ for some section $\xi$, then
\[
v^\nabla(Y) = \nabla_X\xi .
\]

(c) The vertical projection $v^\nabla$ is also smooth. This means that for any smooth vector field $X$ of $TE$, $v^\nabla(X)$ is a smooth vector field of $T^vE$. 
(d) Given \( Y \in T_u E \setminus T^*_u E \), there is a unique vector \( X \in T_x M \) such that 
\[ d\xi(X) = Y \] 
for some section \( \xi \). Therefore we have the natural projection from \( T_u E \) to \( T_x M \) and the canonical decomposition \( T_u E \approx T_{\pi(u)} M \times T^*_u E \).

### 4.2. Pseudo-holomorphic 1-jet bundle and its almost complex structure.

Given two smooth \((C^\infty)\) almost complex manifolds \((M^{2n}, J)\) and \((M'^{2m}, J')\), a \((J, J')\)-holomorphic (or pseudo-holomorphic) 1-jet bundle over \( M \times M' \) is defined by 
\[
\mathcal{J}^1_{(J, J')}(M, M') = \bigcup_{(x, y) \in M \times M'} \text{Hom}(J_x, J'_y)(T_x M, T_y M'),
\]
where \( \text{Hom}(J_x, J'_y)(T_x M, T_y M') \) is the space of \((J_x, J'_y)\)-linear transformations from \( T_x M \) to \( T_y M' \). Now \( \pi = \pi_1 \times \pi_2 : \mathcal{J}^1_{(J, J')}(M, M') \to M \times M' \) is a vector bundle of rank \( 2nm \). We will frequently use the notation \( \mathcal{J}^1(M, M') \) instead of \( \mathcal{J}^1_{(J, J')}(M, M') \) for simplicity.

Choose any linear connection \( \nabla \) on \( \mathcal{J}^1(M, M') \). We have the canonical identification
\[
T_u \mathcal{J}^1(M, M') \approx T_{\pi_1(u)} M \times T_{\pi_2(u)} M' \times T^*_u \mathcal{J}^1(M, M') \approx T_{\pi_1(u)} M \times T_{\pi_2(u)} M' \times \text{Hom}(\pi_1, \pi_2)^{(\gamma)}(T_{\pi_1(u)} M, T_{\pi_2(u)} M').
\]
By this, any tangent vector \( Y \in T_u \mathcal{J}^1(M, M') \) can be decomposed into
\[
Y = (X_1, X_2, v^\nabla(Y)),
\]
where:

i) \( X_1 \) and \( X_2 \) are images of the natural projection of \( Y \) into \( T_{\pi_1(u)} M \) and \( T_{\pi_2(u)} M' \), respectively,

ii) \( v^\nabla(Y) \) is considered as an element in \( \text{Hom}(\pi_1, \pi_2)^{(\gamma)}(T_{\pi_1(u)} M, T_{\pi_2(u)} M') \).

Now we can define an almost complex structure \( J^\nabla \) on \( \mathcal{J}^1(M, M') \) depending on \( \nabla \) by
\[
J^\nabla(Y) = (J_{\pi_1(u)} X_1, J'_{\pi_2(u)} X_2, J'_{\pi_2(u)} \circ v^\nabla(Y)).
\]

It is easy to see \( v^\nabla(J^\nabla(Y)) = J'_{\pi_2(u)} \circ v^\nabla(Y) \); hence \( J^\nabla \) is well defined. Furthermore, \( J^\nabla \) is a smooth almost complex structure. Hence \( (\mathcal{J}^1(M, M'), J^\nabla) \) is also a smooth almost complex manifold.

**Theorem 4.1** (Gauduchon [4]). There is a linear connection \( \nabla \) on \( \mathcal{J}^1(M, M') \) with following property:

For any pseudo-holomorphic mapping \( f : M \to M' \), its lifting \( L(f) : (M, J) \to (\mathcal{J}^1(M, M'), J^\nabla) \) is also pseudo-holomorphic.

### 4.3. Higher order jet bundles.

We can define the \( k \)-jet bundles over \( M \times M' \) inductively. But we need only the local information, so we shall consider the Euclidean case.

Let \((\Omega, J) \subset \mathbb{R}^{2n}\) and \((\Omega', J') \subset \mathbb{R}^{2m}\) be smooth almost complex domains. Let \((x_1, \ldots, x_{2n})\) and \((w_1, \ldots, w_{2m})\) be the standard coordinate systems for \( \mathbb{R}^{2n}\) and \( \mathbb{R}^{2m}\), respectively. Assume that
\[
\{\partial/\partial x_1, \ldots, \partial/\partial x_n\}
\]
is a complex basis of \( T_x \Omega \) for each \( x \in \Omega \).

Condition \((*)\) means that \( \{\partial/\partial x_1, \ldots, \partial/\partial x_n\} \) and its images under \( J_x \) form a real basis of \( T_x \Omega \).
By (⋆) a $(J,J')$-linear mapping from $T_2\Omega$ to $T_2\Omega'$ is completely determined by the images of $\{\partial/\partial x_1, \ldots, \partial/\partial x_n\}$; hence $J^1(\Omega, \Omega')$ is a trivial bundle. From now on, we consider $J^1(\Omega, \Omega')$ as an open set $\Omega \times \Omega' \times \mathbb{R}^{2nm}$ in $\mathbb{R}^{2(n + m + nm)}$. More precisely, a coordinate mapping is given by

$$(4.2) \quad \tau = \left( \pi_1(\tau), \pi_2(\tau), \left[ dw_n(\tau(\partial/\partial x_j))_{\alpha=1,\ldots,2m} \right]_{j=1,\ldots,n} \right).$$

The lifting $L(f)$ of a pseudo-holomorphic mapping $f$ is parameterized by

$$(4.3) \quad L(f)(x) = \left( x_1, \ldots, x_{2m}, f_1(x), \ldots, f_{2m}(x), \left[ \frac{\partial f_\alpha}{\partial x_j}(x) \right]_{\alpha=1,\ldots,2m} \right).$$

To compare $\|f\|_{C^l}$ with $\|L(f)\|_{C^{l-1}}$, we have to consider the partial derivatives of $f$ that are missing in the above expression of $L(f)(x)$. Solving the system of linear equations $J'_f \circ df = df \circ J$ with respect to $\{\partial f_\alpha/\partial x_j\}_{j>n}$, we have

$$\frac{\partial f_\beta}{\partial x_j}(x) = \sum_{\alpha=1}^{2m} \sum_{k=1}^{n} A_{jk}^{\alpha\beta}(x,f(x)) \frac{\partial f_\alpha}{\partial x_k}(x) \quad \text{on } \Omega$$

for $j > n$, where $A_{jk}^{\alpha\beta}$ is a globally defined $C^{\infty}$-smooth function on $\Omega \times \Omega'$. Therefore, for each compact subset $K$ in $\Omega$ and any positive integer $l$, there is a suitable constant $M_l$ depending on $K$ with

$$\left\| \frac{\partial f_\alpha}{\partial x_j} \right\|_{C^l(K)} \leq M_l \sum_{\beta=1}^{2m} \sum_{k=1}^{n} \left\| \frac{\partial f_\beta}{\partial x_k} \right\|_{C^{l-1}(K)}$$

for $j > n$. We may deduce that

$$(4.4) \quad \|f\|_{C^l(K)} \lesssim \|L(f)\|_{C^{l-1}(K)}$$

uniformly for $f \in \mathcal{O}(J,J')(\Omega, \Omega')$.

By the expression (4.3), we also obtain

**Proposition 4.2.** Let $f, g \in \mathcal{O}(J,J')(\Omega, \Omega')$ and $\nu \geq 1$. If $f$ and $g$ share the same $\nu$-jet at $p \in \Omega$, then $L(f)$ and $L(g)$ share the same $(\nu - 1)$-jet at $p$.

We now go to the 2-jet.

Take any linear connection $\nabla_1$ on $J^1(\Omega, \Omega')$. From our assumption (⋆) about $\Omega$, the pseudo-holomorphic 2-jet bundle over $\Omega \times \Omega'$ defined by

$$J^2(\Omega, \Omega') = J^1(\Omega, \Omega') \times (\Omega, J^1(\Omega, \Omega'))$$

is also trivial. Choosing $\nabla_\nu$, inductively, we can define a pseudo-holomorphic $(\nu+1)$-jet bundle by

$$J^{\nu+1}(\Omega, \Omega') = J^1(\Omega, J^\nu(\Omega, \Omega')).$$

For any choice of $\nabla_\nu$ at each step, $J^{\nu}(\Omega, \Omega')$ is always trivial.

From now on, we fix a suitable linear connection $\nabla_\nu$ as in Theorem 4.1 at each step. Then for a pseudo-holomorphic mapping $f : \Omega \to \Omega'$, its lifting $L^{\nu}(f) = L(L^{\nu-1}(f)) : \Omega \to J^{\nu}(\Omega, \Omega')$ is always $(J, J^{\nu})$-holomorphic.

Given $f \in \mathcal{O}(J,J')(\Omega, \Omega')$ and $p \in \Omega$, a family of mappings defined by

$$\mathcal{F}^{\nu}_p(f; \Omega, \Omega') = \{g \in \mathcal{O}(J,J')(\Omega, \Omega') : g \text{ has the same } \nu \text{-jet with } f \text{ at } p\}$$

has the following property.
Theorem 4.3. Let $(\Omega, J) \subset \mathbb{R}^{2n}$ and $(\Omega', J') \subset \mathbb{R}^{2m}$ be hyperbolic almost complex domains. Assume that $\Omega$ satisfies condition $(\ast)$. For any $f \in \mathcal{O}(I,J;\Omega,\Omega')$, there is a neighborhood $V_{\nu}$ of $p$ such that $\{L^\nu(g) : g \in F^\nu_p(f;\Omega,\Omega')\}$ is uniformly bounded on $V_{\nu}$. Moreover, we can find $V_{\nu}$ such that $V_{\nu+1} \subset V_{\nu}$ for each $\nu = 1, 2, \ldots$.

Proof. Choose $r > 0$ such that the Kobayashi ball $U = B_{\mathcal{O}(J,\Omega)}(f(p), r)$ is a bounded neighborhood of $f(p)$ as in Theorem 3.2. Denote $V = B_{\mathcal{O}(J,\Omega)}(p, r)$. Since $F^\nu_p(f;\Omega,\Omega') = \{g \in \mathcal{O}(J,\Omega;\Omega,\Omega') : g(p) = f(p)\}$, we have $g(V) \subset U$ for any $g \in F^\nu_p(f;\Omega,\Omega')$. Take any relatively compact neighborhood $V_1$ of $p$ in $V$. By Theorem 3.2, $\{\|g\|_{C^1(V_1)} : g \in F^\nu_p(f;\Omega,\Omega')\}$ is uniformly bounded so that $\{L(g) : g \in F^\nu_p(f;\Omega,\Omega')\}$ is uniformly bounded on $V_1$. This proves the case $\nu = 1$.

Since $(V, J)$ and $(U, J')$ are also Kobayashi hyperbolic, Theorem 3 in [11] implies that every bounded domain in $J^1(V, U)$ is hyperbolic with respect to $J^\nabla$. Therefore, we may assume that

$$
\bigcup_{g \in F^\nu_p(f;\Omega,\Omega')} L(g)(V_1) \subset \Omega_1,
$$

where $\Omega_1$ is a hyperbolic neighborhood of $L(f)(p)$ in $J^1(V, U)$.

Suppose that our theorem holds for the case $\nu \leq \lambda$. Since the pair $(V_1, J)$ and $(\Omega_1, J^\nabla)$ satisfy the assumption of the theorem, there are neighborhoods $V'_1, \ldots, V'_\lambda$ of $p$ in $V_1$ such that $\{L^\lambda(h) : h \in F^\nu_p(L(f); V_1, \Omega_1)\}$ is uniformly bounded on $V'_\nu$ for $\nu = 1, \ldots, \lambda$, and such that $V'_\lambda \subset V'_{\lambda-1} \subset \cdots \subset V'_1$. By Proposition 4.2, we have

$$L(F^\nu_p(f;\Omega,\Omega')) \subset F^{\nu-1}_p(L(f); V_1, \Omega_1)$$

for any $\nu$. Therefore $L^{\nu+1}(g) = L^\nu(L(g))$ is uniformly bounded on $V_{\nu+1} = V'_\nu$ for $g \in F^\nu_p(f;\Omega,\Omega')$ and for $\nu = 1, \ldots, \lambda$. This proves the theorem by the induction hypothesis. \hfill \Box

For this sequence $\{V_\nu\}$ of nested neighborhoods of $p$, we have

Corollary 4.4. $\{\|g\|_{C^\nu(V_\nu)} : g \in F^{\nu-1}_p(f;\Omega,\Omega')\}$ is uniformly bounded.

Proof. From (4.2), we have

$$\|g\|_{C^\nu(V_\nu)} \lesssim \|L(g)\|_{C^{\nu-1}(V_\nu)} \lesssim \cdots \lesssim \|L^\nu(g)\|_{C^0(V_\nu)}$$

uniformly for $g \in \mathcal{O}(J,\Omega;\Omega,\Omega')$. When $g \in F^{\nu-1}_p(f;\Omega,\Omega')$, the last term of this inequality is bounded by Theorem 4.3. \hfill \Box

5. Proof of Theorem 1.2

Let $(M, J)$ be a connected hyperbolic almost complex manifold of class $C^\infty$. Suppose that there is a pseudo-holomorphic self-mapping $f : M \rightarrow M$ with $f(p) = p$ and $df_p = 1$ for some $p \in M$. From Proposition 2.2, $f$ is of class $C^\infty$ and we can compare all partial derivatives of $f$ with those of the identity mapping. To prove that $f$ is the identity, we need the unique continuation property for pseudo-holomorphic mappings.

Proposition 5.1. Let $(M, J)$ and $(M', J')$ be smooth almost complex manifolds. Moreover $M$ is connected. Suppose that two pseudo-holomorphic mappings $f, g : M \rightarrow M'$ share the same $\infty$-jet at some point in $M$. Then $f \equiv g$ on $M$.
Proof. It is sufficient to prove that \( A = \{ p \in M : f \text{ and } g \text{ share the same } \infty\text{-jet at } p \} \) is open. Then our assertion follows, since \( A \) is open, closed and nonempty set.

Suppose that \( p \in A \). There is a neighborhood \( U_p \) of \( p \) such that any point \( q \) in \( U_p \) can be joined to \( p \) by a single pseudo-holomorphic disc (\cite{[6]} and \cite{[10]}). Take any \( q \) in \( U_p \) and suppose that there is a pseudo-holomorphic disc \( \phi : D \to M \) with \( \phi(0) = p \) and \( \phi(1/2) = q \). Since \( p \in A \), the two pseudo-holomorphic discs \( f \circ \phi, g \circ \phi : D \to M' \) share the same \( \infty\)-jet at \( 0 \). By the unique continuation property of pseudo-holomorphic curves (see \cite{[3]} and \cite{[12]}), it holds that \( f \circ \phi = g \circ \phi \). Furthermore \( f(q) = g(q) \). Since \( q \) is an arbitrary point in \( U_p \), we have \( f|_{U_p} \equiv g|_{U_p} \). Hence \( p \in U_p \subset A \), and \( A \) is open. This proves the proposition. \( \Box \)

By Proposition \( 5.1 \) it is sufficient to prove that \( D^\alpha f_j(p) = 0 \) for any \( j \) and any multi-indices \( |\alpha| \geq 2 \). Then \( f \) has the same \( \infty\)-jet with the identity mapping. Therefore \( f \) is the identity mapping.

Choose a local coordinate system \( \varphi : (V, 0) \to (M, p) \) about \( p \) with \( \varphi(V) \subset \subset M \). Since the Kobayashi distance function \( d_{(M, J)} \) is continuous, we can take a positive real number \( r < \min_{q \in \partial \varphi(V)} d_{(M, J)}(p, q) \). Then the Kobayashi ball \( B_{(M, J)}(p, r) \) is contained in \( \varphi(V) \). By the distance-decreasing property of the Kobayashi distance, we have \( f(B_{(M, J)}(p, r)) \subset B_{(M, J)}(p, r) \) for all \( r \). Now we identify \( p = 0 \), \( \varphi(V) = V \) is a bounded domain in \( \mathbb{R}^{2n} \) and \( J = \varphi^*J = (d\varphi)^{-1} \circ J \circ d\varphi \) is an induced almost complex structure on \( V \). For sufficiently small \( r \) we may assume that \( (U = \varphi^{-1}(B_{(M, J)}(p, r)), J) \) satisfies condition (\(*)\) in Section \( 4 \).

Consider an iterated family \( \{ f^m = f \circ f^{m-1} \}_{m=1, 2, \ldots} \) of \( f \). Note that \( f|_{U} \) is in \( \mathcal{O}(J, J)(U, U) \), so is \( f^m|_{U} \). Now we have

**Proposition 5.2.** \( (D^\alpha(f^m)_j)(0) = m(D^\alpha f_j)(0) \) for \( |\alpha| = 2 \).

Suppose that \( D^\alpha f_j(0) = 0 \) for any \( 2 \leq |\alpha| < \nu \) and \( j = 1, \ldots, 2n \). Then \( (D^\beta(f^m)_j)(0) = m(D^\beta f_j)(0) \) for each \( |\beta| = \nu \) and each \( j \).

**Proof.** Since \( d(f^m)_0 = (df_0)^m = \text{Id} \), we have

\[
\frac{\partial(f^m)_j}{\partial x_k}(0) = \delta_{j,k}
\]

for \( m = 1, 2, \ldots \).

Let \( D^\alpha = \frac{\partial^2}{\partial x_{\alpha_1} \partial x_{\alpha_2}} \). Since \( (f^m)_j = f_j \circ f^{m-1} \), we have

\[
\frac{\partial^2}{\partial x_{\alpha_1} \partial x_{\alpha_2}}(f^m)_j(0) = \frac{\partial}{\partial x_{\alpha_1}} \left( \sum_{k=1}^{2n} \frac{\partial f_j}{\partial x_k}(f^{m-1})(x) \frac{\partial(f^{m-1})_k}{\partial x_{\alpha_2}}(x) \right)(0) = \sum_{k=1}^{2n} \frac{\partial^2 f_j}{\partial x_{\alpha_1} \partial x_k}(f^{m-1})(0) \frac{\partial(f^{m-1})_k}{\partial x_{\alpha_2}}(0) + \sum_{k=1}^{2n} \frac{\partial f_j}{\partial x_k}(f^{m-1})(0) \frac{\partial^2(f^{m-1})_k}{\partial x_{\alpha_1} \partial x_{\alpha_2}}(0)
\]

where the last equality follows by \( (5.1) \). This equation proves the case of \( |\alpha| = 2 \) by induction.
Suppose that $D^\alpha f_j(0) = 0$ for any $2 \leq |\alpha| < \nu$ and $j = 1, \ldots, 2n$. Let $|\beta| = \nu$ and $D^\beta = \frac{\partial^\nu}{\partial x_{\beta_1} \cdots \partial x_{\beta_n}}$. From (5.3), we obtain

$$D^\beta (f^m_j)(0) = \sum_{\gamma_1, \ldots, \gamma_\nu = 1}^{2n} \frac{\partial^\nu f_j}{\partial x_{\gamma_1} \cdots \partial x_{\gamma_\nu}} (f^{m-1}(0)) \frac{\partial (f^{m-1})_{\gamma_1}}{\partial x_{\beta_1}} \cdots \frac{\partial (f^{m-1})_{\gamma_\nu}}{\partial x_{\beta_n}} (0)
+ \text{ (terms which contain } D^\alpha f_j \text{ for } 2 \leq |\alpha| < \nu)
+ \sum_{k=1}^{2n} \frac{\partial f_j}{\partial x_k} (f^{m-1}(0)) \frac{\partial^\nu (f^{m-1})_k}{\partial x_{\beta_1} \cdots \partial x_{\beta_n}} (0)
= \frac{\partial^\nu f_j}{\partial x_{\beta_1} \cdots \partial x_{\beta_n}} (0) + \frac{\partial^\nu (f^{m-1})_j}{\partial x_{\beta_1} \cdots \partial x_{\beta_n}} (0)
= D^\beta f_j(0) + D^\beta (f^{m-1})_j(0).$$

This proves the proposition. \qed

We are now ready to complete the proof of Theorem 1.2. Suppose that $D^\alpha f_j(0) \neq 0$ for some multi-index $\alpha$ with $|\alpha| = 2$ and some $j$. By Proposition 5.3, we have $|(D^\alpha (f^m)_j)(0)| = m|(D^\alpha f_j)(0)| \to \infty$ as $m \to \infty$. Since $f^m(0) = f(0) = 0$ and $d(f^m)_0 = df_0 = \text{Id}$, we have $f^m \in F^\nu_0(f; U, U)$ for each $m$. Corollary 4.4 implies that $\{|(D^\alpha (f^m)_j)(0)|\}_{m=1,2,\ldots}$ must be bounded. Therefore it follows that $D^\alpha f_j(0) = 0$ for each $|\alpha| = 2$ and $j$.

Inductively let us assume that $D^\beta f_j(0) \neq 0$ and $D^\nu f_k(0) = 0$ for $2 \leq |\alpha| < |\beta| = \nu$ and $k = 1, \ldots, 2n$. Proposition 5.2 implies that $(D^\alpha (f^m)_k)(0) = m|(D^\alpha f_k)(0)| = 0$ for $2 \leq |\alpha| < \nu$ and $k = 1, \ldots, 2n$. Hence it follows that $f^m \in F^{\nu-1}_0(f; U, U)$. But Proposition 5.2 also means that $|(D^\beta (f^m)_j)(0)| = m|(D^\beta f_j)(0)| \to \infty$ as $m \to \infty$. It is a contradiction to Corollary 4.4. Therefore we have $D^\alpha f_j(0) = 0$ for any $|\alpha| \geq 2$.

Consequently $f$ has same $\infty$-jet with the identity mapping at 0. This proves Theorem 1.2. \qed

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References


Department of Mathematics, Pohang University of Science and Technology, Pohang, 790-784, Republic of Korea

E-mail address: nyawoo@postech.ac.kr

Current address: Department of Mathematical Sciences, Seoul National University, Seoul, 151-747, Republic of Korea