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TWIST POINTS OF PLANAR DOMAINS

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Abstract. We establish a potential theoretic approach to the study of twist points in the boundary of simply connected planar domains.

1. Motivation and main results

We introduce a geometric, potential theoretic approach to the study of twist points in the boundary of simply connected planar domains. The study of the correspondence between geometric, potential theoretic properties of the domain and analytic properties of the conformal map of the unit disc onto the domain has deep roots in function theory (see, in particular, [2], [3], [5], [6], [7], [8], [10], [18], [28], [29], [30], [31], [32], [33], [34] and [35]) and usually yields inspiration for higher dimensional versions; cf. [4].

Background. Let $\mathbb{S}_b$ be the family of all bounded, connected, simply connected planar domains. If $D \in \mathbb{S}_b$, then let ∂D denote the boundary of D. Let $U \stackrel{\text{def}}{=} \{z \in \mathbb{R}^2 : |z| < 1\}$ be the unit disc of center 0. If $D \in \mathbb{S}_b$ and $x \in D$, then there exists an analytic isomorphism

$$f\colon U \to D$$

such that $f(0) = x$. The map f, determined modulo rotations of U around the origin, is called a *Riemann map of D with pole at x*. Let S be the class of all analytic, univalent functions $f : U \to \mathbb{C}$, normalized by $f(0) = 0$ and $f'(0) = 1$. According to Pommerenke [34], one of the main aims of the theory of the boundary behavior of conformal maps is the study of the correspondence between data expressible in terms of f, called *analytic*, and data that are expressible entirely in terms of the geometry of $D = f(U)$. Let $\mathcal{F}(f)$ be the set of all $\theta \in \partial U$ where the *angular limit* of f, denoted by $f_\flat(\theta)$, exists; see [34], p. 6. The set $\mathcal{F}(f)$ has full Lebesgue measure in ∂U, by a theorem of Fatou [15]. The correspondence between geometric and analytic data reveals itself in different, sometimes subtle guises. A special role is played by the analytic Bloch function

$$\log f'\colon U \to \mathbb{C}\,.$$

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This function has a remarkable property: For Lebesgue a.e. $\theta \in \partial U$, the angular boundary behavior of $\log f'$ at θ determines whether $f_\flat(\theta) \in \operatorname{twist}(D)$ or $f_\flat(\theta) \in \operatorname{sect}(D)$; see Section 2. The implication holds for almost every point, but *not* at every point; see [34] and [25]. A variant of $\log f'$ that appears in function theory is the *analytic functional of* $f \in S$ given by $\log \dfrac{\zeta f'(\zeta)}{f(\zeta)}$, $\zeta \in U$, where log is the branch equal to 0 at $\zeta = 0$; see [34] p. 123. Indeed, W. Seidel [36] showed that the imaginary part of this functional, given by

$$\arg \frac{\zeta f'(\zeta)}{f(\zeta)} \tag{1.1}$$

provides an analytic characterization of starlike domains. H. Grunsky [20] proved the sharp inequality

$$\left| \arg \frac{\zeta f'(\zeta)}{f(\zeta)} \right| \leq \log \frac{1+|\zeta|}{1-|\zeta|}$$

where $f \in S$ and $\zeta \in U$; cf. [18], p. 117, and [28], p. 168.

Main results. Given $D \in \mathbb{S}_b$ and $y \in D$, we define a harmonic Bloch function $h_D(y) : D \to \mathbb{R}$ playing the role of $\log f'$ and instrumental to a potential theoretic approach to twisting. The definition of $h_D(y)$ is purely geometric and potential theoretic, not being based on the Riemann map. Its harmonicity does not appear to be obvious from the point of view of potential theory. We also show that the function h_D yields a geometric, potential theoretic representation for the analytic quantity (1.1).

Indeed, let $\mathfrak{h}(D)$ be the space of real-valued functions defined on D and harmonic therein. For each $D \in \mathbb{S}_b$ we define and study a continuous function

$$h_D \colon D \to \mathfrak{h}(D)$$

that recaptures (1.1) directly in terms of geometric data of the domain $D \in \mathbb{S}_b$, where $D = f(U)$. The function h_D is

$$h_D(y)(z) \stackrel{\text{def}}{=} \int_{\partial D} \langle y, z\rangle_D(w)\, \mu_D(z, dw), \qquad y, z \in D.$$

Here $\mu_D(z, \cdot)$ is the harmonic measure with pole at z and $\langle y, z\rangle_D(w)$, called the *relative winding angle*, is the signed variation of the argument of $\xi - w$ as $\xi \in D$ goes from y to z along a continuous curve in D. The relative winding angle is defined in Section 2 without using complex analysis.

Here are the most salient properties of the function h_D.

Theorem 3.2. *$D \in \mathbb{S}_b$ and $y \in D$, then $h_D(y)$ is harmonic on D.*

The harmonicity of $h_D(y)(z)$ as a function of y is immediate, since it is the superposition of functions harmonic in y. A non-potential theoretic proof of Theorem 3.2 can be given. Indeed, if f is a Riemann map of D with pole at $x \in D$, then the composition

$$h_D(x) \circ f : U \to \mathbb{R}$$

turns out to be equal to the map (1.1); however, we also establish the harmonicity of $h_D(y)$ independently of this equality, since our proof is purely potential theoretic. The harmonicity of $h_D(x)$ will be seen to be the expression of an implicit symmetry under reflections.

Theorem 3.3. *For each $x \in D$, the function*

$$h_D(x) : D \to \mathbb{R}$$

is a harmonic Bloch function on D.

Theorem 3.5. *The behavior of $h_D(x)(z)$ as $z \to w \in \partial D$ determines whether $w \in \operatorname{sect}(D)$ or $w \in \operatorname{twist}(D)$, for a.e. w relative to harmonic measure.*

For NTA domains, we establish the previous assertion in a quantitative way, in Theorem 3.4.

Outline. In Section 2 we establish notation and the needed preliminary results. In Section 3 we introduce the h function of a domain in $\mathbb{S}_b$ and state its properties. In Section 4 we prove our results in the order of their mutual dependence.

Remarks. The third-named author has given a talk at the Potential Theory Workshop in Bucuresti, Romania, in September 2002. See also [1].

2. Notation and preliminary results

Henceforth we shall adopt the following notation: $D \in \mathbb{S}_b$, f is a Riemann map of D with pole at $x \in D$ and $\mathcal{F}(f)$ is the set of points $\theta \in \partial U$ where f has radial limit (hence angular limit). The set $\mathcal{F}(f)$ is a Borel set of full Lebesgue measure in ∂U; see [15] and [34]. Moreover, $G_D : D \times D \to (-\infty, \infty]$ is the Green function of D and $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$.

If $w \in \mathbb{R}^2$, we denote $\operatorname{dist}(w) : \mathbb{R}^2 \to [0, \infty)$ the function $\operatorname{dist}(w)(z) \stackrel{\text{def}}{=} |z - w|$ and by $\operatorname{dist}(\partial D)$ the function on $\mathbb{R}^2$ given by $\operatorname{dist}(\partial D) \stackrel{\text{def}}{=} \min_{w \in \partial D} \operatorname{dist}(w)$. We write $\operatorname{dist}(\partial D, z)$ for $\operatorname{dist}(\partial D)(z)$. Let $B(w, r) \stackrel{\text{def}}{=} \{z \in \mathbb{R}^2 : |z - w| < r\}$ and $\overline{B}(w, r) \stackrel{\text{def}}{=} \{z \in \mathbb{R}^2 : |z - w| \leq r\}$.

Let $C(\partial D)$ be the Banach space of real-valued continuous functions on ∂D, endowed with the uniform norm. If $A, B \subset \mathbb{C}$, the set of continuous maps from A to B is denoted $C(A, B)$. We let $J(A, B)$ be the subset of $C(A, B)$ consisting of injective maps. If $g \in C(A, \mathbb{C})$ and $w \in \mathbb{C}$, then $g - w \in C(A, \mathbb{C})$ is the function $s \mapsto g(s) - w$.

The unweighted average of $u \in C(\partial B(w, r))$ over $\partial B(w, r)$ is denoted $L(u, w, r)$, as in [14], Section 1.I.2.

We use $\Im(z)$ to denote the imaginary part of $z \in \mathbb{C}$, $\Re(z)$ for its real part and $z^\star = \overline{z}$ for its complex conjugate. If $A \subset \mathbb{R}^2$, then $1_A : \mathbb{R}^2 \to \{0, 1\}$ is the indicator function of A.

Curves. If $B \subset \mathbb{C}$, the elements of $C([0, 1], B)$ are called *curves in B*. The points $c(0)$, $c(1)$ are called the *endpoints of* the curve c; $c(0)$ is the *initial point of* c, $c(1)$ its *final point* and c is said to be a curve *from* $c(0)$ *to* $c(1)$. The image of c is denoted by $\operatorname{st}(c)$. The elements of $J([0, 1], B)$ are called *Jordan curves in B*. We may assume, after a change of parameter, that the parameter space is equal to $[0, \tau]$ where $0 < \tau < \infty$; see [24]. A curve in B whose endpoints coincide is called a *closed curve in B*. After a change of parameter, closed curves in B can be seen as elements of $C(\partial U, B)$. Let $\Sigma_B(y, z)$ be the set of smooth (C^∞) curves in B from $y \in B$ to $z \in B$.

Half-open arcs. If $B \subset \mathbb{C}$, the elements of $C([0,1), B)$ are called *half-open arcs in* B. We say that the half-open arc c *ends at* $w \in \mathbb{C}$ or that c is a half-open arc *from* $c(0)$ *to* w if the limit $\lim_{s\to 1} c(s)$ exists and is equal to w. Elements of $J([0,1), B)$ are called *Jordan half-open arcs in* B. We write $J(B)$ for $J([0,1), B)$. Let $J_w(B)$ be the set of Jordan half-open arcs in B ending at $w \in \mathbb{C}$.

Accessibility. If $\theta \in \partial U$ and $s \in [0,1)$ let $\rho_\theta(s) \overset{\text{def}}{=} s\theta$. Then $\rho_\theta \in J_\theta(U)$. If $c \in J(U)$ ends at some point of ∂U, then it is not necessarily true that $f \circ c$ ends at some point of ∂D. However, if $c \in J(D)$ ends at some point of ∂D, then $f^{-1} \circ c$ ends at some point of ∂U; see [24]. Indeed, the set $\partial_\flat D$, image of $f_\flat$, can be described geometrically.

Lemma 2.1. *If $w \in \partial D$, then the following conditions are equivalent:*
(i) *there is $\theta \in \mathcal{F}(f)$ such that $f_\flat(\theta) = w$;*
(ii) *there is $\theta \in \partial U$ such that $f \circ \rho_\theta \in J_w(D)$;*
(iii) *there exist $\theta \in \partial U$ and $c \in J_\theta(U)$ such that $f \circ c \in J_w(D)$;*
(iv) $J_w(D) \neq \emptyset$.

Proof. Apply [34], Theorem 4.3, and [24], Theorems III.2.6 and III.2.7. □

Liftings. If A is a topological space, $g \in C(A, \mathbb{C}^*)$, $\gamma \in C(A, \mathbb{C})$ and $e^\gamma = g$, then we say that γ is a *lifting of* g. Let $\mathfrak{L}(g)$ denote the set of all liftings of g. If A is connected and simply connected, then $\mathfrak{L}(g) \neq \emptyset$ for each $g \in C(A, \mathbb{C}^*)$ and, if $\gamma_1, \gamma_2 \in \mathfrak{L}(g)$, then there is a unique $n \in \mathbb{Z}$ such that $\gamma_1 = \gamma_2 + 2\pi i n$; see [19].

Twisting half-open arcs. Let $c \in C([0,1), \mathbb{C}^*)$ and assume that there is a $\gamma \in \mathfrak{L}(c)$ such that $\Im(\gamma)$ is unbounded above and below on $[0,1)$. Then each lifting of c has the same property and we say that c is *twisting*. Let $\mathfrak{T}$ be the set of twisting half-open arcs in $\mathbb{C}^*$.

Proposition 2.2. *Assume that $c_1, c_2 \in J_0(\mathbb{C}^*)$, $c_1 \in \mathfrak{T}$ and $c_1(s_1) = c_2(s_2)$ if and only if $s_1 = s_2 = 0$. Then $c_2 \in \mathfrak{T}$.*

Proof. Let $\gamma_k \in \mathfrak{L}(c_k)$, $k = 1, 2$. Then $\lim_{s\to 1} \Re(\gamma_k(s)) = -\infty$ and, for each $n \in \mathbb{Z}$, $\mathrm{st}(\gamma_2)$ does not intersect the vertical translation of $\mathrm{st}(\gamma_1)$ by $2\pi n i$. It follows that, for s close enough to 1, $\gamma_2(s)$ belongs to the strip between two successive vertical translates of $\mathrm{st}(\gamma_1)$ and therefore its imaginary part must be unbounded above and below as well. □

Harmonic measure. The notions and results of potential theory we shall use are explained in [14], Chapter 1.VIII. Let $D \in \mathbb{S}_b$. The boundary ∂D is regular, hence resolutive. If $\phi : \partial D \to \mathbb{R}$ is a real-valued function defined on ∂D, we denote by $\phi^\natural = \phi^\natural(D) : D \to [-\infty, \infty]$ the upper PWB solution for ϕ (in [14] the notation $\overline{H}_\phi$ is used with the same meaning). If ϕ is defined on a superset of ∂D, then $\phi^\natural(D)$ is to be interpreted as $(\phi|_{\partial D})^\natural(D)$. The σ algebra of Borel subsets of ∂D is denoted by $\mathcal{B}_D$. The σ algebra of subsets of ∂D whose indicator function is resolutive is denoted by $\mathcal{R}_D$. Then $\mathcal{B}_D \subset \mathcal{R}_D$. If $\xi \in D$ and $A \in \mathcal{R}_D$, define

$$\mu_D(\xi, A) \overset{\text{def}}{=} (1_A)^\natural(\xi).$$

Then $\mu_D(\xi, \cdot)$ is a measure on $\mathcal{R}_D$, called *harmonic measure with pole at* ξ, and $\mu_D(\xi, \cdot)$ is the completion of the restriction of $\mu_D(\xi, \cdot)$ to $\mathcal{B}_D$. We usually write

$\mu_D(A)$ for $\mu_D(\xi, A)$ if the ambiguity about ξ is not relevant. If ϕ is a real-valued function defined on ∂D, we write $\mu_D(\cdot, \phi)$ for

$$\int_{\partial D} \phi(w)\mu_D(\cdot, dw)$$

whenever this integral is defined. If ϕ is bounded and $\mathcal{R}_D$-measurable, then $\mu_D(\cdot, \phi) \in \mathfrak{h}(D)$ and $\phi^\natural = \mu_D(\cdot, \phi)$. If ϕ is resolutive, then $\phi^\natural = \mu_D(\cdot, \phi)$. Thus,

$$\phi^\natural(z) = \int_{\partial D} \phi(w)\, \mu_D(z, dw), \quad \forall\, z \in D,\ \phi \in C(\partial D) \tag{2.1}$$

and

$$\lim_{z \to w} \mu_D(z, \cdot) = \delta_w \tag{2.2}$$

in the sense of vague convergence of measures, where δ_w is the probability measure supported by $\{w\}$. Harmonic measures with different poles are mutually absolutely continuous. If $B = B(x, r)$ and $u \in C(\partial B)$, then

$$L(u, x, r) = \int_{\partial B} u(z)\, \mu_B(x, dz)\,.$$

Indeed, $\mathcal{R}_U$ is the σ algebra of Lebesgue measurable subsets of ∂U and $\mu_U(0, \cdot)$ is normalized Lebesgue measure on ∂U identified with $\mathbb{R}/2\pi\,\mathbb{Z}$. Let $\partial_\flat D$ denote the image of the map

$$f_\flat : \mathcal{F}(f) \to \partial D\,.$$

In general, the map $f_\flat$ is not onto. The set $\partial_\flat(D)$ is independent of f, dense and of full harmonic measure in ∂D. Indeed, if $w \in \partial D$, then $w \in \partial_\flat D$ if and only if there is a Jordan half-open arc in D ending at w. The set $\partial_\flat(D)$ is analytic; see [34]. Since $(\partial D, \mathcal{R}_D, \mu_D)$ is a complete measure space of finite measure, a theorem of Lusin implies that $\partial_\flat(D)$ belongs to $\mathcal{R}_D$; see [14], Theorem A.II.4. See also [5]. If $A \subset \mathcal{F}(f)$ and $B \subset \partial D$, we write $f_\flat(A)$ for $\{f_\flat(\theta) : \theta \in A\}$ and $f_\flat^{-1}(B)$ for $\{\theta \in \mathcal{F}(f) : f_\flat(\theta) \in B\}$. If $A \subset \partial D$, then $A \in \mathcal{R}_D$ if and only if $f_\flat^{-1}(A) \in \mathcal{R}_U$ and

$$\mu_D(x, A) = \mu_U(0, f_\flat^{-1}(A)) \qquad \text{for every } A \in \mathcal{R}_D\,. \tag{2.3}$$

Let $\mathbb{S}_\flat^\infty \subset \mathbb{S}_b$ be the set of all domains in $\mathbb{S}_b$ with C^∞ boundary. For each $D \in \mathbb{S}_b$ there are domains $D_n \in \mathbb{S}_\flat^\infty$, $n \geq 1$, such that $\overline{D_n} \subset D_{n+1}$ and $\bigcup_{n=1}^\infty D_n = D$. Any such sequence $\{D_n\}_n$ is called a *regular exhaustion of* D. The next result is basically due to Wiener [39].

Lemma 2.3. *Let $D \in \mathbb{S}_b$ and fix a regular exhaustion $\{D_n\}_n$ of D. Let $\phi \in C(\partial D)$. If $\Psi : O \to \mathbb{R}$ is any continuous extension of ϕ to an open neighborhood O of ∂D, then there exists n_0 such that*

$$\Psi^\natural(D_n) : D_n \to \mathbb{R}$$

is defined and harmonic for $n \geq n_0$ and converges to

$$\phi^\natural(D) : D \to \mathbb{R}$$

uniformly on compact subsets of D, as $n \to \infty$.

Proof. When O contains D, the statement is proved in [39]. The restriction of Ψ to a smaller neighborhood of ∂D has a continuous extension $\Phi : D \to \mathbb{R}$, by a theorem of Tietze. Since the restrictions of Φ and Ψ on ∂D_n coincide for n big enough, we may apply the special case. □

Sectorial accessibility for the Riemann map. The set $\mathrm{Sect}(f) \subset \partial U$, defined in [34], p. 144, is a subset of the boundary of the unit disc.

Sectorial accessibility for the domain. Define $\mathrm{sect}(D) \subset \partial D$ as follows: A point $w \in \partial D$ belongs to $\mathrm{sect}(D)$ if and only if D contains an open triangle having a vertex at w. The correspondence between $\mathrm{Sect}(f)$ and $\mathrm{sect}(D)$ is given by the following relation:

$$f_\flat(\mathrm{Sect}(f)) = \mathrm{sect}(D)\,. \tag{2.4}$$

It may happen that $f_\flat(\theta) \in \mathrm{sect}(D)$ while $\theta \notin \mathrm{Sect}(f)$.

Twist points of the Riemann map. Let $\mathrm{Twist}(f) \subset \partial U$ be the set of points $\theta \in \mathcal{F}(f)$ such that $f \circ c - f_\flat(\theta) \in \mathfrak{T}$ for each $c \in J_\theta(U)$.

Lemma 2.4. *If $\theta \in \mathcal{F}(f)$, then the following conditions are equivalent:*
(i) $\theta \in \mathrm{Twist}(f)$*;*
(ii) $f \circ \rho_\theta - f_\flat(\theta) \in \mathfrak{T}$*;*
(iii) *for each $c \in J_\theta(U)$ and each $g \in \mathfrak{L}(f - f_\flat(\theta))$, $\Im(g) \circ c$ is unbounded above and below.*

Proof. Apply [34], Proposition 4.4, p. 77; cf. [25]. ☐

Twist points of the domain. Let $\mathrm{twist}(D)$ be the set of points $w \in \partial_\flat D$ such that $c - w \in \mathfrak{T}$ for each $c \in J_w(D)$. The following result clarifies the relation between $\mathrm{Twist}(f)$ and $\mathrm{twist}(D)$. Since we could not find it in the literature, we record it for future reference.

Proposition 2.5. *If $w \in \partial_\flat D$, then the following conditions are equivalent:*
(1) $w \in \mathrm{twist}(D)$*;*
(2) $\{\theta \in \mathcal{F}(f) : f_\flat(\theta) = w\} \subset \mathrm{Twist}(f)$*;*
(3) $\{\theta \in \mathcal{F}(f) : f_\flat(\theta) = w\} \cap \mathrm{Twist}(f) \neq \emptyset$.

Proof. Apply Proposition 2.2 and [34], Theorem 4.3. ☐

Proposition 2.6. *If $D \in \mathbb{S}_b$, then $\mu_D(\mathrm{sect}(D) \cup \mathrm{twist}(D)) = 1$.*

Proof. Apply (2.4), Proposition 2.5, and the Twist Point Theorem [25], [34]. ☐

Prime ends. For background about the theory of prime ends, due to Carathéodory, see [34]. Let $\partial_{\mathrm{c}}\mathrm{D}$ be the set of prime ends of D and denote by $f_{\mathrm{c}} : \partial U \to \partial_{\mathrm{c}}\mathrm{D}$ the homeomorphism given in the Prime End Theorem (in [34], the notation $\hat{f}$ is used for f_{c} and $P(D)$ for $\partial_{\mathrm{c}}\mathrm{D}$). We claim that it is possible to define subsets $\partial_{\mathrm{c}}^{\mathrm{t}}\mathrm{D}$ and $\partial_{\mathrm{c}}^{\mathrm{s}}\mathrm{D}$ of $\partial_{\mathrm{c}}\mathrm{D}$, in a purely geometric fashion, in such a way that (1) $\partial_{\mathrm{c}}^{\mathrm{t}}\mathrm{D}$ and $\partial_{\mathrm{c}}^{\mathrm{s}}\mathrm{D}$ consist of prime ends of the first or the second kind; (2) if $\theta \in \partial U$, then $f_c(\theta) \in \partial_{\mathrm{c}}^{\mathrm{t}}\mathrm{D} \Leftrightarrow \theta \in \mathrm{Twist}(f)$ and $f_c(\theta) \in \partial_{\mathrm{c}}^{\mathrm{s}}\mathrm{D} \Leftrightarrow \theta \in \mathrm{Sect}(f)$. The proof is left to the reader. We shall not need these results in the rest of this paper.

NTA domains. NTA domains in $\mathbb{R}^n$ were introduced in [22]. Let NTA_2 be the collection of all planar NTA domains. Then $\mathrm{NTA}_2 \subset \mathbb{S}_b$ with proper inclusion, since a planar NTA domain is a quasidisc; see [22] and [34]. The von Koch snowflake is an NTA domain [37]. If $D \in \mathrm{NTA}_2$, $w \in \partial D$ and $\alpha > 0$, then $\Gamma_\alpha(w) \subset D$ is defined as

$$\Gamma_\alpha(w) = \{z \in D : |z - w| < (1+\alpha)\mathrm{dist}(\partial D, z)\}\,.$$

If D has smooth boundary, then the approach regions $\Gamma_\alpha(w)$ are comparable to open triangles having vertex at w and contained in D. In general, $\Gamma_\alpha(w)$ may not contain any triangle having vertex at w.

Theorem 2.7. *If $D \in \mathrm{NTA}_2$ and $u \in \mathfrak{h}(D)$, then $\partial D = N \cup P \cup L$ where*

(i) *N has harmonic measure zero;*

(ii) *If $w \in P$, then u is unbounded above and below in $\Gamma_\alpha(w) \cap B(w, r)$ for each positive α, r;*

(iii) *If $w \in L$, then $u(z)$ has a finite limit as $z \to w$ and $z \in \Gamma_\alpha(w)$ for each $\alpha > 0$.*

Proof. Apply [22], Theorem 6.4, paying due care to measurability issues. □

Remark. The previous result is a real-variable higher dimensional extension of a theorem of Plessner on functions analytic on U; see [27].

The quasihyperbolic metric and harmonic Bloch functions. The *quasihyperbolic distance* $k_D(z, y)$ in D from z to y is defined as the minimum of the arc length integrals

$$\int_c \frac{1}{\operatorname{dist}(\partial D)}\, ds,$$

evaluated along all rectifiable paths c from z to y contained in D. The quasihyperbolic distance is a geometric quantity; cf. [34]. It was introduced in [17]; see also [16] and [34], p. 92. Observe that if $z, y \in D$ and $|z - y| < \frac{1}{2}\operatorname{dist}(\partial D, z)$, then

$$\frac{1}{2}\frac{|z-y|}{\operatorname{dist}(\partial D, z)} < k_D(z, y) < 2\frac{|z-y|}{\operatorname{dist}(\partial D, z)}\,.$$

Indeed, $k_D(z, y) \geq \log(1 + \frac{|z-y|}{\operatorname{dist}(z,\partial D)})$; see [17], Lemma 2.1. It follows that a function $U : D \to \mathbb{R}$ is Lipschitz relative to the metric (D, k_D), i.e.,

$$\sup_{z,z' \in D, z \neq z'} \frac{|U(z) - U(z')|}{k_D(z, z')} < \infty\,, \tag{2.5}$$

if and only if U satisfies

$$\sup_D \operatorname{dist}(\partial D) |\operatorname{grad} U| < \infty\,. \tag{2.6}$$

If $U \in \mathfrak{h}(D)$ satisfies (2.5), then U is called a *harmonic Bloch function*; cf. [23].

The hyperbolic metric and the Green function. Let λ_U be the hyperbolic metric of the unit disc, normalized so as to yield

$$\lambda_U(0, \zeta) = \frac{1}{2}\log\frac{1 + |\zeta|}{1 - |\zeta|}$$

as in [34], p. 6. Other normalizations can be found in the literature. If $z, y \in D$, then $\lambda_D(z, y) \stackrel{\text{def}}{=} \lambda_U(f^{-1}(z), f^{-1}(y))$ is the *hyperbolic distance* in D between z and y. The Koebe distortion theorem implies that

$$\lambda_D(z, y) \leq k_D(z, y) \leq 4\lambda_D(z, y) \qquad \forall\, z, y \in D\,. \tag{2.7}$$

See [34], p. 92. Given $z \in D \setminus \{x\}$, we denote by $g_D(x, z)$ the hyperbolic geodesic in D from x to z. If $f(\zeta) = z$, then a parametric representation of $g_D(x, z)$ is given by $s \mapsto f(s\zeta)$, $0 \leq s \leq 1$. The curve $g_D(x, z)$ is precisely the integral curve from x to z of the gradient of the Green function $G_D(x, \cdot)$.

The winding angle. Fix an orientation in $\mathbb{R}^2$. Let $c \in J([0,1],\mathbb{R}^2)$ and assume that c is smooth. Let $c_\star$ be the potential of the double layer of constant unit density over c. The function $c_\star$ is harmonic on the open set $\mathbb{R}^2 \setminus \mathrm{st}(c)$, complement of the image of c, and it extends by continuity at the endpoints of c but not at the other points of c, since therein it is subject to a jump; see [11]. Recall that the value of $c_\star$ at w is given by the arc length integral

$$c_\star(w) \stackrel{\text{def}}{=} \int_c \frac{\partial}{\partial n} \log \frac{1}{\mathrm{dist}(w)}\, ds \tag{2.8}$$

where n is the positively oriented normal to c. We call $c_\star(w)$ the *winding angle of c as seen from w*, since it is the signed variation of the argument of $y-w$, when y goes from x to z along c, as can be seen via the Green formula; see [11]. Observe that $c_\star(w) = (c-w)_\star(0)$. The following estimate will be useful:

$$|c_\star(w)| \le \int_c \frac{1}{\mathrm{dist}(w)}\, ds\,. \tag{2.9}$$

Indeed,

$$\left|\frac{\partial}{\partial n} \log \frac{1}{\mathrm{dist}(w)}\right| \le \left|\mathrm{grad} \log \frac{1}{\mathrm{dist}(w)}\right| = \frac{1}{\mathrm{dist}(w)}\,.$$

If $0 \notin \mathrm{st}(c)$, then $c_\star(0)$ equals the integral along c of the closed differential form $\frac{xdy-ydx}{x^2+y^2}$ — modulo a sign that depends on the orientation of the plane. Since a closed differential form can be integrated along any curve, without assuming any smoothness on the curve (see [9], p. 58), then $c_\star(w)$ can be defined for any curve c in $\mathbb{R}^2$ and

$$|c_\star(w)| = \int_{c-w} \frac{xdy-ydx}{x^2+y^2}\,.$$

Similarly, since $\frac{d\zeta}{z}$ is a closed differential form in $\mathbb{C}^*$, if c is curve in $\mathbb{R}^2$, then $c_\star(w)$ is equal (modulo a sign) to the imaginary part of the complex line integral $\displaystyle\int_c \frac{d\zeta}{\zeta - w}$.

If $\gamma \in \mathfrak{L}(c-w)$, then $c_\star(w) = \Im(\gamma)(1) - \Im(\gamma)(0)$; see [26]. Indeed, if c is given by the parametric representation $\phi(s), 0 \le s \le 1$ and we let $\Phi(s) = \frac{\phi(s)-w}{|\phi(s)-w|}$, then $\Phi : [0,1] \to \partial U$. Now, let $\Phi^\sim : [0,1] \to \mathbb{R}$ be any lifting of Φ via the universal covering map $\mathbb{R} \to \partial U$, $r \mapsto e^{ir}$. Then $c_\star(w) = \Phi^\sim(1) - \Phi^\sim(0)$ (the right-hand side being independent on the choice of the lifting).

The relative winding angle. Consider the continuous function

$$\langle\,\cdot\,,\cdot\,\rangle_{\mathrm{D}} \colon D \times D \to C(\partial D)$$

defined by evaluating the winding angle along curves in D: If $y, z \in D$, $w \in \partial D$ and $c \in \Sigma_D(y,z)$, then

$$\langle y, z\rangle_D(w) \stackrel{\text{def}}{=} c_\star(w).$$

Since D is simply connected and $c_\star(w)$ is given by integrating along c a differential 1-form defined on the punctured plane $\mathbb{R}^2 \setminus \{w\}$ and closed therein, Stokes' theorem implies that the function $\langle\,\cdot\,,\cdot\,\rangle_{\mathrm{D}}$ is well defined. Moreover, $\langle y,z\rangle_D(w)$ is separately harmonic in $y, z \in D$ for fixed w.

Thus, $\langle y,z\rangle_D(w)$, called the *winding angle relative to D*, measures the signed variation of the argument of $y'-w$ as y' goes from y to z staying within D. Observe that $\langle x,x\rangle_D = 0$, $\langle x,z\rangle_D = -\langle z,x\rangle_D$, and in fact

$$\langle x,y\rangle_D + \langle y,z\rangle_D = \langle x,z\rangle_D \qquad \forall\, x,y,z \in D. \tag{2.10}$$

Lemma 2.8. *If $w \in \partial_\flat D$, then the following conditions are equivalent:*

(i) $w \in \operatorname{twist}(D)$*;*

(ii) $\limsup_{s\to 1} \langle x, c(s)\rangle_D(w) = +\infty$ *and* $\liminf_{s\to 1} \langle x, c(s)\rangle_D(w) = -\infty$ *for each half-open arc in D ending at w.*

Remark. Property (ii) is independent of the choice of x.

Starlike domains. A domain D is called *starlike with respect to $x \in D$* if

$$|\langle x, z\rangle_D(w)| < \pi$$

for each $w \in \partial D$ and $z \in D$. An equivalent definition is that the line segment from x to z is entirely contained in D for each $z \in D$.

The harmonic winding angle. If $y_1, y_2 \in D$, then the function $\langle y_1, y_2\rangle_D : \partial D \to \mathbb{R}$ has the harmonic extension $\langle y_1, y_2\rangle_D^\natural \colon D \to \mathbb{R}$. Following (2.1), $\langle y_1, y_2\rangle_D^\natural$ is given by

$$\langle y_1, y_2\rangle_D^\natural(z) = \int_{\partial D} \langle y_1, y_2\rangle_D(w)\mu_D(z, dw), \qquad z \in D.$$

Moreover, $\langle y_1, y_2\rangle_D^\natural(z)$ is harmonic in each variable separately, and

$$\lim_{z\to w} \langle y_1, y_2\rangle_D^\natural(z) = \langle y_1, y_2\rangle_D(w) = c_\star(w) \tag{2.11}$$

for each $w \in \partial D$ and $c \in \Sigma_D(y_1, y_2)$.

Lemma 2.9.

$$\left|\langle y_1, y_2\rangle_D^\natural(z)\right| \le k_D(y_1, y_2), \qquad \forall\, z \in D. \tag{2.12}$$

Proof. The maximum principle for harmonic functions implies that it suffices to verify (2.12) when $z = w \in \partial D$, in which case (2.11) and (2.9) imply that

$$\left|\langle y_1, y_2\rangle_D^\natural(w)\right| \le \int_c \frac{1}{\operatorname{dist}(\partial D)}\, ds$$

for each rectifiable arc in D from y_1 to y_2. Thus, $|\langle y_1, y_2\rangle_D^\natural(w)| \le k_D(y_1, y_2)$ and now (2.12) follows at once. □

Proposition 2.10. *If x and y belong to $D \in \mathbb{S}_b$ and $\{D_n\}_n$ is a regular exhaustion of D, then $\langle x, y\rangle_{D_n}^\natural$ converges to $\langle x, y\rangle_D^\natural$ uniformly on compact subsets of D.*

Proof. Let K be a compact subset of D and pick $c \in \Sigma_D(x, y)$. Then $c_\star$ is a continuous (indeed, harmonic) extension of $\langle x, y\rangle_D$ on $\mathbb{R}^2 \setminus \operatorname{st}(c)$, neighborhood of ∂D. If n is big enough, then x, y, c and K are all contained in D_n. Moreover, if $w \in \partial D_n$, then $\langle x, y\rangle_{D_n}(w) = c_\star(w)$, since $c \in \Sigma_{D_n}(x, y)$. Thus, if n is big enough, $\langle x, y\rangle_{D_n}$ is the restriction to ∂D_n of a continuous extension of $\langle x, y\rangle_D$ to a neighborhood of ∂D. Now apply Lemma 2.3. □

3. The function h_D

The original potential theoretic definition. We let $h_D \colon D \times D \to \mathbb{R}$ be the function

$$h_D(y, z) \overset{\text{def}}{=} \langle y, z\rangle_D^\natural(z) = \int_{\partial D} \langle y, z\rangle_D(w)\, \mu_D(z, dw), \qquad y, z \in D.$$

Notation. Whenever convenient, we shall write $h_D(y)(z)$ for $h_D(y,z)$, so that $h_D(y) : D \to \mathbb{R}$ denotes $h_D(y,\cdot)$ as a function of the *second* variable.

Remarks. The functional h_D is covariant under translations, rotations and dilations of D. We shall see that it uniquely determines the domain D (apart from the scale) since it determines (the inverse of) its Riemann map.

Since $h_D(y,z)$ is a superposition of functions harmonic in y, its harmonicity in y follows from standard arguments. Its harmonicity with respect to z does not seem to be equally immediate from the viewpoint of potential theory; we now show that it is related to a certain (hidden) symmetry under reflections (in the direct proof of Corollary 3.9 this symmetry is explicit).

Proposition 3.1. *If $D \in \mathbb{S}_b$, then the following conditions are equivalent:*

(a) *For each $x \in D$ the function $h_D(x)$ is harmonic at each point of D;*

(b) *For each $x \in D$ the function $h_D(x)$ has vanishing mean-value over each sufficiently small ball centered at the point x;*

(c) *For each $x \in D$, if z' denotes $x + (z-x)^\star$ and $D' \stackrel{def}{=} \{z' : z \in D\}$, then*

$$L(h_D(x), x, r) = L(h_{D'}(x), x, r)$$

for each $r > 0$ such that $\overline{B}(x,r) \subset D$.

Proof. (a) $\Rightarrow$ (b) Observe that $h_D(x,x) = 0$ and apply [14], Section 1.I.3.

(b) $\Rightarrow$ (a) If $x_1 \in D$, $h_D(x_1)$ is continuous on D and, by (2.10),

$$h_D(x_1) = \langle x_1, x_2\rangle_D^\natural + h_D(x_2) \qquad \forall\, x_1, x_2 \in D. \tag{3.1}$$

Since $\langle x_1, x_2\rangle_D^\natural$ is harmonic on D, the conclusion follows from (3.1).

(c) $\Rightarrow$ (b) Observe that

$$L(h_D(x), x, r) + L(h_{D'}(x), x, r) = 0 \tag{3.2}$$

since

$$h_D(x)(z) + h_{D'}(x)(z') = 0 \qquad \text{for every} \quad z \in D,$$

thus (c) implies that $L(h_D(x), x, r) = 0$.

(b)$\Rightarrow$(c) We deduce from (3.2) that $L(h_{D'}(x), x, r)$ equals $-L(h_D(x), x, r)$ and from (b) that $L(h_D(x), x, r)$ is equal to 0. □

Theorem 3.2. *If $D \in \mathbb{S}_b$ and $y \in D$, then $h_D(y)$ is harmonic on D.*

Theorem 3.3. *If $D \in \mathbb{S}_b$ and $y \in D$, then $h_D(y)$ is a harmonic Bloch function on D.*

The following two results deal with a natural question: Determine whether and in what sense the approximation

$$h_D(x,z) \sim \langle x, z\rangle_D(w)\,, \qquad z \to w \in \partial_\flat D, \tag{3.3}$$

holds. Indeed, motivated by (2.2), the definition of h_D was inspired precisely by the hope that (3.3) would hold, in some sense. Cf. Proposition 3.15 and [6]. Observe that, once we choose x, $\langle x, z\rangle_D(w)$ is a family of functions harmonic in z—one for each boundary point w—and $h_D(x)$ is a harmonic function only depending on D.

Theorem 3.4. *If $D \subset \mathbb{R}^2$ is NTA, then for each $\alpha > 0$ and each $w \in \partial D$,*

$$\sup_{z \in \Gamma_\alpha(w)} |h_D(x,z) - \langle x, z\rangle_D(w)| < \infty. \tag{3.4}$$

Observe that (3.4) is a general, intrinsic form of the estimate (1) given in [13] in the proof of Lemma 3 of that paper. Examples show that (3.4) may fail if D is not NTA at w; see [25]. However, the following result shows that (3.3), *if properly interpreted*, holds for every $D \in \mathbb{S}_b$. The interpretation is qualitative, in the sense that the boundary behaviour of one side reflects the boundary behaviour of the other.

Theorem 3.5. *If $D \in \mathbb{S}_b$, then for a.e. point $w \in \partial D$, relative to harmonic measure, the boundary behaviour of $h_D(x)$ at w predicts whether w is a twist point or sectorially accessible.*

We shall see that the proof of the following result is independent of (3.14). Thus, we recapture the well-known analytic characterization of starlike domains, due to W. Seidel [36].

Theorem 3.6. *A domain $D \in \mathbb{S}_b$ is starlike with respect to $x \in D$ if and only if $|h_D(x,z)| < \frac{\pi}{2}$ for all $z \in D$.*

The twist function of a smoothly bounded domain. The function h_D also admits a second, less direct, potential theoretic description, based on an approximation argument. If $D \in \mathbb{S}_{\mathrm{b}}^{\infty}$, then there is $r_D > 0$ such that if $0 < r \leq r_D$ and $w \in \partial D$, then there is a point in D, denoted $n_D^r(w)$, at distance r from w, on the inner normal to ∂D at w. If $D \in \mathbb{S}_{\mathrm{b}}^{\infty}$ and $x \in D$, we define

$$t_D \colon D \to C(\partial D)$$

as follows: for $0 < r \leq r_D$,

$$t_D(x)(w) \stackrel{\text{def}}{=} \langle x, n_D^r(w) \rangle_D(w), \qquad w \in \partial D. \tag{3.5}$$

The values of $t_D(x)$ are independent of r. The function t_D, called the *twist function of D*, measures the twisting of the domain around its boundary and it gauges the difference between, say, a disc and a domain winding around itself, like the one in [18], Figure 2, p. 36. Indeed, these domains have no twist points but the disc twists much less around its boundary than the other.

Whenever convenient, we shall write $t_D(x,w)$ for $t_D(x)(w)$.

Lemma 3.7. *If $D \in \mathbb{S}_{\mathrm{b}}^{\infty}$, then the following conditions are equivalent:*
(a) $h_D(x) = t_D(x)^\natural$ *for each* $x \in D$;
(b) $t_D(x)^\natural(x) = 0$ *for each* $x \in D$.

Proof. Observe that (2.10) implies

$$h_D(x,z) = t_D(x)^\natural(z) - t_D(z)^\natural(z); \tag{3.6}$$

hence the result. □

Theorem 3.8. *If $D \in \mathbb{S}_{\mathrm{b}}^{\infty}$, then*

$$h_D(x) = t_D(x)^\natural \qquad \forall x \in D\,.$$

Corollary 3.9. *If $D \in \mathbb{S}_{\mathrm{b}}^{\infty}$, then for each $w \in \partial D$*

$$\lim_{z\to w} h_D(x,z) = t_D(x,w)\,. \tag{3.7}$$

If the boundary of D is not smooth, the function $t_D(x)$ cannot be directly defined as in (3.5). In particular, the function $h_D(x,\cdot)$ may not possess boundary values as in (3.7). Consider, for example, the von Koch snowflake [37], whose boundary consists of twist points, apart from a set of harmonic measure zero [13], [34]. However, the following theorem provides another (less direct but equally natural) description of h_D. Indeed, its statement is inspired by the construction of the generalized solution of Dirichlet's problem in arbitrary domains in $\mathbb{R}^n$, due to N. Wiener [39].

Theorem 3.10. *If $D \in \mathbb{S}_b$, $\{D_n\}_n$ is a regular exhaustion of D and $x \in D_1$, then the sequence of harmonic functions*

$$\{t_{D_n}(x)^\natural\}_{n\in\mathbb{N}}$$

converges, as $n \to \infty$, uniformly on compact subsets of D, to a function that is independent of the choice of the regular exhaustion of D; indeed, the limit function is precisely $h_D(x)$.

We now show that if we choose a special exhaustion of D, then there is no need to use a limiting argument.

The Green exhaustion and hyperbolic geodesics. We now choose the regular exhaustion given by the superlevel sets of the Green function of the domain. If $z \in D$ and $z \neq x$, then the subdomain $D_x^z \subset D$ defined by

$$D_x^z \stackrel{\text{def}}{=} \{\zeta \in D \colon G_D(x,\zeta) > G_D(x,z)\}$$

has smooth boundary and contains x. Moreover, $z \in \partial D_x^z$ and $\overline{D_x^z} \subset D$. In particular, $t_{D_x^z}$ is defined at the point (x,z). Recall that $h_D(x,x) \equiv 0$.

Theorem 3.11. *If $D \in \mathbb{S}_b$ and $x \in D$, then*

$$h_D(x,z) = t_{D_x^z}(x,z) \qquad \forall z \in D \setminus \{x\}\,. \tag{3.8}$$

Thus, the restriction of $h_D(x)$ to the subdomain D_x^z is equal to $h_{D_x^z}(x)$.

Theorem 3.11 implies that h_D can be described in terms of the hyperbolic geodesics of the domain. In Section 2 we denoted by $g_D(x,z) \in \Sigma_D(x,z)$ the hyperbolic geodesic in D between x and z. Recall also that, if $c \in \Sigma_D(x,z)$, then $c_\star$ is also defined at the endpoints x and z of c. If $c = c(s)$ is a smooth curve, we let $\dot c = \frac{dc}{ds}$ denote its tangent vector. Thus, $\dot c = \dot c(s)$ is itself a curve in $\mathbb{R}^2$.

Corollary 3.12. *If $D \in \mathbb{S}_b$, $x,z \in D$ and $x \neq z$, then*

$$h_D(x,z) = g_D(x,z)_\star(z), \tag{3.9}$$

and therefore,

$$h_D(x,z) - h_D(z,x) = \dot g_D(x,z)_\star(0)\,. \tag{3.10}$$

Thus, $h_D(x,z) - h_D(z,x)$ is the winding angle of $\dot g_D(x,z)$ as seen from 0.

An analytic description. We now give a purely analytic description of h_D. Let $(z_1,z_2,z_3,z_4) \stackrel{\text{def}}{=} \frac{z_1-z_2}{z_1-z_4}\frac{z_3-z_4}{z_3-z_2}$ denote the cross ratio of the points z_j in the Riemann sphere, as in [21], p. 58. In particular, $(\infty,a,c,b) = \frac{b-c}{a-c}$.

Definition. Let $f_\star$ be the nonvanishing analytic function on the polydisc

$$f_\star : U \times U \times U \to \mathbb{C}^*$$

given by

$$f_\star(\zeta,\xi,\eta) \stackrel{\text{def}}{=} \frac{(\infty, f(\eta), f(\xi), f(\zeta))}{(\infty,\eta,\xi,\zeta)}\,. \tag{3.11}$$

Remarks. Note that $f_\star(0,0,0) = 1$. Observe that the functional $f \mapsto f_\star$ is invariant under translations, rotations and dilations.

If $\zeta = \xi$, $\eta = \theta \in \partial U$ and $\theta \in \mathcal{F}(f)$, then (3.11), defined by continuity, is

$$f_\star(\zeta,\zeta,\theta) = \frac{f'(\zeta)(\zeta-\theta)}{f(\zeta) - f_\flat(\theta)}$$

also known as the Visser-Ostrowski quotient; cf. [34].

If $\zeta = \xi$ and $\eta = 0$, then (3.11) yields

$$f_\star(\zeta,\zeta,0) = \frac{\zeta f'(\zeta)}{f(\zeta) - f(0)}\,,$$

the analytic quantity used by W. Seidel to characterize starlike domains; see [36] p. 206; cf. [34] p. 66.

Let arg be the branch equal to zero for $\zeta = 0$.

Theorem 3.13. *Under the hypotheses given above, the following identities hold for all $\zeta, \xi \in U$, with $x = f(0)$, $z = f(\zeta)$, $w = f_\flat(\theta)$ and $z' = f(\xi)$:*

$$\arg f_\star(\zeta,\theta,0) = \langle x,z\rangle_D(w) - \langle 0,\zeta\rangle_U(\theta), \tag{3.12}$$

$$\arg f_\star(\zeta,\xi,0) = \langle x,z\rangle_D^\natural(z') - \langle 0,\zeta\rangle_U^\natural(\xi), \tag{3.13}$$

$$\arg f_\star(\zeta,\zeta,0) = h_D(x,z)\,. \tag{3.14}$$

Remarks. Theorem 3.2 follows from (3.14). In Section 4, we shall give a different proof of Theorem 3.2, independent of (3.14).

From (3.9) and (3.14) we obtain

$$\arg f_\star(\zeta,\zeta,0) = g_D(x,z)_\star(z)$$

where $x = f(0)$ and $z = f(\zeta)$. The previous identity can also be proved directly, as in [38], p. 672, without employing our function h_D. See also [12].

Lemma 3.14. *If f is the Riemann map of D and $\theta \in \mathcal{F}(f)$, then*

$$\frac{f_\star(\zeta,\zeta,\theta) f_\star(0,0,\zeta) f_\star(\zeta,\theta,0)}{f_\star(\zeta,\zeta,0)} = f_\star(0,0,\theta) \qquad \forall\, \zeta \in U\,. \tag{3.15}$$

Proof. Left to the reader. □

Proposition 3.15. *If f is the Riemann map of D and $\theta \in \mathcal{F}(f)$, then*

$$h_D(x,z) - \langle x,z\rangle_D(w) = \arg f_\star(\zeta,\zeta,\theta) + h_D(z,x) + \epsilon(z) \tag{3.16}$$

for all $\zeta \in U$, where $z = f(\zeta)$, $x = f(0)$ and $|\epsilon(z)| < 3\pi/2$.

Proof. Choose a value for $\arg f_\star(0,0,\theta)$, such that $|\arg f_\star(0,0,\theta)| < \pi$. This choice uniquely determines a branch of $\arg f_\star(\zeta,\zeta,\theta)$ with the given initial condition for $\zeta = 0$. Now, select the branch of $\arg f_\star(\zeta,\zeta,0)$, equal to 0 for $\zeta = 0$ and do the same for $\arg f_\star(\zeta,0,0)$ and $\arg f_\star(\zeta,\theta,0)$. The result follows from (3.15) and Theorem 3.13, with $\epsilon(z) = -\langle 0,\zeta\rangle_U(\theta) - \arg f_\star(0,0,\theta)$. □

Remark. The quantity $h_D(z,x)$ does not necessarily remain bounded when $z \to w$, not even assuming that $w \in \partial_\flat D$. An example is a domain approaching the origin spiraling around it infinitely many times.

4. Proofs

Proof of Theorem 3.8. Recall that $t_{D_x^z}(x,z)$ has been defined so far only for $z \neq x$. Observe that if $z \in D \setminus \{x\}$, then $t_{D_x^z}(x,z)$ is locally equal to the difference between the argument of the gradient of $G_D(x,\cdot)$ and the argument of $x-z$, up to an additive constant. Thus, $z \mapsto t_{D_x^z}(x,z)$ is harmonic on $D \setminus \{x\}$. The level curves of $G_D(x,\cdot)$ near x are close to circles of center x. Therefore,

$$\lim_{D \ni z \to x} t_{D_x^z}(x,z) = 0,$$

thus x is a removable singularity of $z \mapsto t_{D_x^z}(x,z)$; see [14], Theorem 1.V.5, p. 60. It follows that, if $t_{D_x^z}(x,z)$ is defined to be 0 for $z = x$, then it is harmonic on D. So far, we have not used the hypothesis that D has smooth boundary. Now, since $D \in \mathbb{S}_\flat^\infty$, the Green function $G_D(x,\cdot)$ is smooth on ∂D, thus the function $t_{D_x^z}(x,z)$ extends continuously to ∂D, where it equals $t_D(x,\cdot)$. It follows that

$$\int_{\partial D} t_D(x,w)\mu_D(z,dw) = t_{D_x^z}(x,z) \qquad \text{for every } x,z \in D \tag{4.1}$$

and

$$\int_{\partial D} t_D(x,w)\mu_D(x,dw) = 0 \qquad \text{for every } x \in D. \tag{4.2}$$

Now apply Lemma 3.7.

Proof of Theorem 3.10. Let $K \subset D$ be compact. Fix $\delta > 0$ such that if n is large enough, then $|\xi - w| \geq \delta$ for all $\xi \in K$ and $w \in \partial D \cup \partial D_n$. If $0 < \epsilon < 1$, then there is a finite subset $Y \subset K$ such that the balls of center $y \in Y$ and radius $\delta\epsilon/4$ cover K. For $y \in K$ let $B[y]$ be the ball of center y and radius $\delta\epsilon/4$. Therefore, for each $z \in K$ there is $y_z \in Y$ such that $z \in B[y_z]$. Since Y is finite, we may apply Proposition 2.10 to each ball $B[y]$, $y \in Y$, so that, if n is large enough, then

$$\sup_{z \in B[y]} \left| \langle x,y\rangle_D^\natural(z) - \langle x,y\rangle_{D_n}^\natural(z) \right| < \epsilon/3 \qquad \text{for every } y \in Y. \tag{4.3}$$

Let $z \in K$. Then $|h_D(x,z) - h_{D_n}(x,z)| \leq I + II + III$, where

$$\begin{aligned} I &= \left| \langle x,z\rangle_D^\natural(z) - \langle x,y_z\rangle_D^\natural(z) \right|, \\ II &= \left| \langle x,y_z\rangle_D^\natural(z) - \langle x,y_z\rangle_{D_n}^\natural(z) \right|, \\ III &= \left| \langle x,y_z\rangle_{D_n}^\natural(z) - \langle x,z\rangle_{D_n}^\natural(z) \right| . \end{aligned}$$

Now, $II < \epsilon/3$, by (4.3). The straight line segment from y_z to z is entirely contained in D. Thus (2.10) and (2.12) yield

$$I = \left| \langle y_z,z\rangle_D^\natural(z) \right| \leq k_D(y_z,z) < \frac{\delta\epsilon/4}{\delta - \delta\epsilon/4} < \frac{\epsilon}{3}.$$

Similar reasoning shows that $III < \epsilon/3$ for n large enough. Thus, if n is large enough, then $|h_D(x,z) - h_{D_n}(x,z)| < \epsilon$ for each $z \in K$.

Proof of Theorem 3.11. If $D \in \mathbb{S}_{\mathrm{b}}^{\infty}$, then (3.8) follows from (3.6), (4.1) and (4.2). If $D \in \mathbb{S}_b$, we consider the domains $D_n = \{\zeta \in D \colon G_D(x,\zeta) > \frac{1}{n}\}$. They form a regular exhaustion of D. If n is big enough, then $(D_n)_x^z = D_x^z$, since $G_{D_n}(x,\cdot) = G_D(x,\cdot) - \frac{1}{n}$ on D_n. Thus $t_{[D_x^z]}(x,z) = t_{[(D_n)_x^z]}(x,z) = h_{D_n}(x,z)$, since $D_n \in \mathbb{S}_{\mathrm{b}}^{\infty}$ and $h_{D_n}(x,z)$ converges to $h_D(x,z)$, by Theorem 3.10. Thus, (3.8) is proved. Now, let $\zeta \in D_x^z$. Then (3.8) implies that $h_D(x,\zeta) = t_{[D_x^\zeta]}(x,\zeta)$ and $h_{[D_x^z]}(x,\zeta) = t_{[(D_x^z)_x^\zeta]}(x,\zeta)$.

Now observe, as before, that $(D_x^z)_x^\zeta = D_x^\zeta$, thus concluding the proof.

Proof of Theorem 3.2. The restriction of $h_D(x,\cdot)$ to the subdomain D_x^z is equal to $h_{[D_x^z]}(x,\cdot)$, by Theorem 3.11. Thus, $h_D(x,\cdot)$ is harmonic on D_x^z, by Theorem 3.8, since D_x^z has smooth boundary. Since each open subset of D is contained in a subdomain of the form D_x^z, the theorem is proved.

Proof of Theorem 3.5. Let $w \in \partial_\flat D$. Theorem 3.11 shows that if $w \in \operatorname{sect}(D)$, then $h_D(x,z)$ remains bounded as $z \to w$ and, likewise, that if $w \in \operatorname{twist}(D)$, then $h_D(x,z)$ is unbounded above and below as $z \to w$. Thus, the conclusion follows from Proposition 2.6.

Proof of Theorem 3.6. If D has smooth boundary, apply the maximum principle for harmonic functions and Theorem 3.8. An approximation argument completes the proof for a general domain in $\mathbb{S}_b$.

Proof of Theorem 3.13. Let $F \in C(U^2, \mathbb{C}^*)$ be the nonvanishing analytic function given by $F(\zeta,\xi) \stackrel{\text{def}}{=} f_\star(\zeta,\xi,0)$, where $(\zeta,\xi) \in U^2 = U \times U$. Let $\ell \in \mathfrak{L}(F)$ be such that $\ell((0,0)) = 0$. Denote $\arg F$ the imaginary part of ℓ. If c is a curve in U^2 from $(0,0)$ to (ζ,ξ), then $F \circ c$ is a curve in $\mathbb{C}^*$ from 1 to $F(\zeta,\xi)$ and there is a unique $l \in \mathfrak{L}(F \circ c)$ with $l(0) = 0$. It follows that $\ell(\zeta,\xi) = l(1)$. For fixed $\zeta \in U$, the harmonic function $\xi \mapsto \arg F(\zeta,\xi)$ is bounded on U, and, therefore, it is determined by its angular boundary values given, for a.e. $\theta \in \partial U$, by

$$\lim_{\xi \to \theta} \arg F(\zeta,\xi) = \langle f(0), f(\zeta)\rangle_D (f_\flat(\theta)) - \langle 0,\zeta\rangle_U(\theta).$$

Since $\langle 0,\zeta\rangle_U^\natural(\zeta) = 0$, then $\arg F(\zeta,\zeta) = h_D(f(0), f(\zeta))$ and (3.14) follows.

Proof of Theorem 3.3. In view of (3.14), the statement follows from the Koebe distortion theorem. Indeed, it suffices to show that the analytic function $\log f_\star(\zeta,\zeta,0)$ is a Bloch analytic on the unit disc, by [34], Corollary 1.4, p. 9. Thus, we need to show that

$$\sup_{\zeta \in U} (1 - |\zeta|) \frac{d}{d\zeta} \log f_\star(\zeta,\zeta,0) < \infty\,.$$

We may assume $|\zeta| > 1/2$. A calculation shows that $\frac{d}{d\zeta} \log f_\star(\zeta,\zeta,0)$ is equal to

$$\frac{1}{\zeta} + \frac{f''(\zeta)}{f'(\zeta)} - \frac{f'(\zeta)}{f(\zeta) - f(0)}\,.$$

The conclusion follows from [34], Proposition 1.2, and Theorem 1.3, p. 9.

Proof of Theorem 3.4. A preliminary reduction is based on the fact that the function $h_D(x,\cdot)$ is Bloch harmonic. Thus, it suffices to show that

$$\sup_{z\in g_D(x,w)} |h_D(x,z) - \langle x,z\rangle_D(w)| < \infty\,,$$

where $g_D(x,w)$ is the hyperbolic geodesic in D from x to w. Now, observe that $h_D(x,z) - \langle 0,z\rangle_D(w) = I + II + III$ where $I = h_D(x,z) - h_D(x,z_1)$, $II = \langle z,z_1\rangle_D^\natural(z_1)$ and $III = \langle 0,z\rangle_D^\natural(z_1) - \langle 0,z\rangle_D(w)$. It is convenient to choose $z_1 = f(\zeta_1)$ where $\zeta_1 = \frac{1}{2}(\zeta + \frac{\zeta}{|\zeta|})$. Here, as before, we let $z = f(\zeta)$ and $w = f_\flat(\theta)$. Moreover, we let $\theta = \frac{\zeta}{|\zeta|}$. Now, $|I| \le C < \infty$ follows from the fact that $h_D(x,\cdot)$ is a harmonic Bloch function and $|II| \le C < \infty$ from (2.12). Thus, the main goal is to obtain a uniform bound for $|III|$. In view of (3.13), we let $u(\xi) \stackrel{\text{def}}{=} \arg f_\star(\zeta,\xi,0)$ and seek a bound for $|u(\zeta_1) - u(\theta)| = |III|$, uniform in ζ. Since the distance between ζ_1 and θ is equal to $\frac{1}{2}\operatorname{dist}(\zeta,\partial U)$, the mean value theorem shows that it suffices to seek a good bound for $|\operatorname{grad} u(\xi)|$, when ξ belongs to the line segment between ζ_1 and θ. Indeed,

$$u(\zeta_1) - u(\theta) = \int_{r_1}^{r_\infty} \frac{d}{dr} u(\gamma(r))\, dr$$

where $r_1 = \frac{1}{2}\operatorname{dist}(\zeta,\partial U)$, $r_\infty = \operatorname{dist}(\zeta,\partial U)$ and $\gamma(r) \stackrel{\text{def}}{=} \zeta + r\theta$. Now, $\left|\frac{d}{dr}\gamma(r)\right| = 1$ and $\left|\operatorname{grad}_\xi u\right|$ is equal to $|\frac{d}{d\xi}\log f_\star(\zeta,\xi,0)|$. Now,

$$\left|\frac{d}{d\xi}\log f_\star(\zeta,\xi,0)\right| = \left|\frac{1}{f_\star(\zeta,\xi,0)}\frac{d}{d\xi} f_\star(\zeta,\xi,0)\right| \le I + II$$

where $I = \frac{|\zeta|}{|\zeta-\xi||\xi|} \le 2\frac{1}{\operatorname{dist}(\zeta,\partial U)}$ and

$$II = \frac{|f'(\xi)||f(\zeta)-f(0)|}{|f(\xi)-f(0)||f(\zeta)-f(\xi)|} \le C_1\frac{|f'(\xi)|}{|f(\zeta)-f(\xi)|}$$

if $|\zeta| \ge 1/2$. Since the domain D is NTA, there is a constant C_2 such that $\operatorname{dist}(\partial D, f(\zeta)) \le \frac{1}{C_2}|f(\zeta)-f(\xi)|$.

Thus,

$$\int_{r_1}^{r_\infty} II\, dr \le C_1C_2\frac{1}{\operatorname{dist}(\partial D, f(\zeta))}\int_{r_1}^{r_\infty} |f'(\gamma(r))|\, dr.$$

The integral on the right side of the previous inequality is bounded by the length of the hyperbolic geodesic in D from ζ to w and therefore by $C_3\operatorname{dist}(\partial D, f(\zeta))$, since D is NTA. Thus $|u(\zeta_1) - u(\theta)|$ is bounded above by $1 + C_1C_2C_3$.

A direct proof of Corollary 3.9. We shall study h_D near a boundary point, using the Strong Markov Property of harmonic measure, given in [14], (8.3), p. 117. Indeed, let us assume for simplicity that the boundary of D is polygonal and that $w_0 \in \partial D$ is not a corner. Let $B(w_0,r)$ be the Euclidean ball in $\mathbb{R}^2$ of center w_0 and radius r. We may assume that $x \in D(w_0,r)$. Let $D(w_0,r) \stackrel{\text{def}}{=} D \cap B(w_0,r)$. Then $h_D(x,z) = \beta(z) + R(z)$ where

$$\beta(z) \stackrel{\text{def}}{=} \int_{\partial D\cap\partial D(w_0,r)} \langle x,z\rangle_D(w)\mu_{D(w_0,r)}(z,dw)$$

and

$$R(z) = \int_{\partial D(w_0,r)\setminus\partial D}\left(\int_{\partial D}\langle x,z\rangle_D(w)\mu_D(y,dw)\right)\mu_{D(w_0,r)}(z,dy)\,.$$

Now, $R(z) \to 0$ if $z \to w_0$, since

$$|R(z)| \leq C\mu_{D(w_0,r)}\left(z, \partial D(w_0,r) \setminus \partial D\right);$$

thus it suffices to show that $\beta(z) \to \langle x, n_D^r(w_0)\rangle_D(w_0)$ as $z \to w_0$. Since the relative winding angle is covariant under translations and dilations, we may assume that $w_0 = 0$ and that $\Omega \cap B(w_0,r) = D(w_0,r)$ where Ω is the half plane $\{\Im > 0\}$. The conclusion follows from a calculation that exploits the symmetries of the half-plane.

Proof of (3.14) *based on Theorem* 3.11. We have $\frac{\zeta f'(\zeta)}{f(\zeta)-f(0)} = \frac{i\frac{\partial}{\partial\theta}f(\zeta)}{x-f(\zeta)}$. Indeed, for $\zeta = re^{i\theta}$, $\dfrac{\partial}{\partial\theta}f(\zeta) = i\zeta f'(\zeta)$. Since $G_D(x, f(\zeta)) = -\log|\zeta|$, the conclusion follows from Theorem 3.11.

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