TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 358, Number 6, Pages 2373–2398
S 0002-9947(05)04035-3
Article electronically published on December 20, 2005

KOSZUL DUALITY AND EQUIVALENCES OF CATEGORIES

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Abstract. Let A and $A^!$ be dual Koszul algebras. By Positselski a filtered algebra U with $\operatorname{gr} U = A$ is Koszul dual to a differential graded algebra $(A^!, d)$. We relate the module categories of this dual pair by a $\otimes-$Hom adjunction. This descends to give an equivalence of suitable quotient categories and generalizes work of Beilinson, Ginzburg, and Soergel.

Introduction

Koszul algebras are graded associative algebras which have found applications in many different branches of mathematics. A prominent feature of Koszul algebras is that there is a related dual Koszul algebra. Many of the applications of Koszul algebras concern relating the module categories of two dual Koszul algebras and this is the central topic of this paper.

Let A and $A^!$ be dual Koszul algebras which are quotients of the tensor algebras $T(V)$ and $T(V^*)$ for a finite-dimensional vector space V and its dual V^*. The classical example being the symmetric algebra $S(V)$ and the exterior algebra $E(V^*)$. Bernstein, Gel'fand, and Gel'fand [3] related the module categories of $S(V)$ and $E(V^*)$, and Beilinson, Ginzburg, and Soergel [2] developed this further for general pairs A and $A^!$.

Now Positselski [21] considered filtered deformations U of A such that the associated graded algebra $\operatorname{gr} U$ is isomorphic to A. He showed that this is equivalent to giving $A^!$ the structure of a *curved differential graded* algebra (cdga), i.e. $A^!$ has an anti-derivation d and a distinguished cycle c in $(A^!)_2$ such that $d^2(a) = [c, a]$. When U is augmented then $c = 0$ and $A^!$ is a differential graded algebra. The typical example of this is when U is the enveloping algebra of a Lie algebra V. A Lie algebra structure on V is equivalent to giving $E(V^*)$ the structure of a differential graded algebra.

We take this situation as our basic setting and relate the module categories of U and the cdga $(A^!, d, c)$. More precisely let $\mathrm{Kom}(U)$ be the category of complexes of left U-modules. We define a natural category $\mathrm{Kom}(A^!, d, c)$ of *curved differential graded* (cdg) left modules over the cdga $(A^!, d, c)$ (when $c = 0$ these are the differential graded modules), and relate them by a pair of adjoint functors

$$\mathrm{Kom}(A^!, d, c) \underset{G}{\overset{F}{\rightleftarrows}} \mathrm{Kom}(U). \tag{1}$$

The vector space V may be equipped with a grading of an abelian group Λ such that U and $(A^!, d, c)$ have Λ-gradings. Then these categories consist of Λ-graded

Received by the editors January 26, 2004.
2000 *Mathematics Subject Classification.* Primary 16S37, 16D90.

modules. When $\Lambda = \mathbf{Z}$ and $\deg_\Lambda V = 1$ (then c and d are zero) we get the classical setting of [2] of categories of complexes of *graded* modules over $A^!$ and A respectively, giving adjoint functors

$$\mathrm{Kom}(A^!) \underset{G}{\overset{F}{\rightleftarrows}} \mathrm{Kom}(A). \tag{2}$$

Now we would like to find suitable quotient categories of (1) such that these functors descend to an equivalence of categories. The functors do descend to functors between homotopy categories

$$K(A^!, d, c) \underset{G}{\overset{F}{\rightleftarrows}} K(U). \tag{3}$$

However, they do not in general descend to an equivalence of derived categories. In [2] this is remedied for (2) by restricting it to suitable subcategories

$$\mathrm{Kom}^{\downarrow}(A^!) \underset{G}{\overset{F}{\rightleftarrows}} \mathrm{Kom}^{\uparrow}(A) \tag{4}$$

(actually in [2] F and G are only defined in this case), and these functors do descend to an equivalence of *derived* categories.

We show instead that when $c = 0$ then (3) can be descended to an equivalence of categories

$$D(A^!, d) \underset{\overline{G}}{\overset{\overline{F}}{\rightleftarrows}} D(U). \tag{5}$$

These categories are not in general derived categories. Rather they are "between" the homotopy categories and the derived categories. This is a fuller result than [2] since the categories in (5) consist of *all* complexes in contrast to the categories in (4) (and it is of course also a more general setting). Another approach to getting an equivalence, involving a derived category on one side, may be found in [15].

Another feature we establish is that the adjunction (1) is basically a $\otimes -$ Hom adjunction. I.e. there exists a bimodule T which is a complex of left U-modules and a right cdg-module over $(A^!, d, c)$, such that the functors F and G are given by

$$F(M) = T \otimes_{A^!} M, \quad G(N) = \mathrm{Hom}_U(T, N).$$

This gives a more compact and conceptual definition of the Koszul functors than the more explicit descriptions in [2]. It also establishes that there is a close link between Koszul duality and tilting theory [14], [17], [16]. See also [9] for another approach to this.

In order to better understand the categories in (5) it is natural to determine which complexes are isomorphic to zero. Instead of the setting (5) we consider the categories of complexes of free left U-modules and also a correspondingly defined category of cofree dg left $(A^!, d)$ modules. The adjunction (1) restricts to these categories and descends to an equivalence of categories

$$DCoF(A^!, d) \underset{\overline{G}}{\overset{\overline{F}}{\rightleftarrows}} DF(U), \tag{6}$$

and these categories are equivalent to the categories in (5). We then give simple criteria for which complexes in (6) are isomorphic to zero. For example a complex

P of free left U-modules is isomorphic to zero in $DF(U)$ iff both P and $k \otimes_U P$ are acyclic.

The organization of the paper is as follows. In Section 1 we recall basic definitions and results on Koszul algebras, in particular the work of Positselski on cdga structures on $A^!$ and filtered deformations U of A and related work of Braverman and Gaitsgory [4]. Section 2 defines the categories $\mathrm{Kom}(A^!, d, c)$, and the adjunctions (1) and (3). Section 3 considers some variations and specializations of this. In particular we consider the variation of Koszul duality by Goresky, Kottwitz, and MacPherson [10] and the specialization of [2] in the $\mathbf{Z}$-graded case.

Section 4 first gives a complex for the pair U and $(A^!, d)$ which generalizes the Koszul complex for A and $A^!$ and the Chevalley-Eilenberg complex for a Lie algebra. It then shows that the natural maps $FG(N) \to N$ and $M \to GF(M)$ coming from the adjunction are quasi-isomorphisms. In the end we establish the equivalence (5). In Section 5 we consider complexes of free left U-modules and cofree dg-modules over $(A^!, d)$. We establish the equivalence of the categories in (5) and (6), and we give criteria for complexes in (6) to be isomorphic to zero. Also, if N is in $\mathrm{Kom}(U)$, then $G(N)$ is in general a rather "large" complex. We show that if N is bounded above, then there is actually a minimal version of $G(N)$, the latter being homotopic to a minimal cofree dg-module over $(A^!, d)$. In the end, inspired by [2], we discuss t-structures on the categories in (5) and (6).

1. Koszul algebras

1.1. **Basic definitions and properties.** We recall the definitions and some basic facts about quadratic algebras and Koszul algebras. For fuller surveys see [2], [7], and [8] in a more general version. Let V be a finite-dimensional vector space over a field k, and let $T(V) = \bigoplus_{n \geq 0} V^{\otimes n}$ be the tensor algebra on V. For a subspace R of $V \otimes V$ let (R) be the two-sided ideal in $T(V)$ generated by R. Then $A = T(V)/(R)$ is called a *quadratic* algebra. This algebra is called a *Koszul* algebra if the module $k = A_0$ has a graded resolution

$$k \leftarrow A \leftarrow A(-1) \otimes V_1 \leftarrow A(-2) \otimes V_2 \leftarrow \cdots \leftarrow A(-i) \otimes V_i \leftarrow \cdots .$$

Here and in the rest of the paper all tensor products are over k unless otherwise indicated.

Example 1.1. The symmetric algebra $S = S(V)$ is quadratic since it is defined by the subspace $(x \otimes y - y \otimes x)_{x,y \in V}$ of $V \otimes V$. It is also Koszul since the Koszul complex

$$k \leftarrow S \leftarrow S(-1) \otimes V \leftarrow S(-2) \otimes \wedge^2 V \leftarrow \cdots$$

gives a resolution of k.

Given a quadratic algebra A defined by $R \subseteq V \otimes V$, we may dualize this inclusion and get an exact sequence

$$0 \to R^{\perp} \to V^* \otimes V^* \to R^* \to 0.$$

The algebra $A^! = T(V^*)/(R^{\perp})$ is called the *quadratic dual* algebra of A.

Example 1.2. The quadratic dual algebra of $S(V)$ is the exterior algebra $E(V^*)$ defined by the relations $(x \otimes x)_{x \in V^*}$ in $V^* \otimes V^*$.

From [2] we recall the following basic facts about Koszul algebras:

1. If A is Koszul, then $A^!$ is Koszul [Prop. 2.9.1]. Hence by the example above $E(V^*)$ is a Koszul algebra.
2. If A is Koszul, the opposite ring A^{op} is also Koszul [Prop. 2.2.1].
3. Let $E(A)$ be the algebra of extensions $\bigoplus_{i\geq 0} \mathrm{Ext}^i_A(k,k)$. Then $E(A)$ is canonically isomorphic to $(A^!)^{op}$ [Prop. 2.10.1].

Remark 1.3. All of the above, with proper care taken, holds in the more general setting where k is a semi-simple ring and V a k-bimodule which is finitely generated as a left k-module. This is the setting of [2]. In fact all the results of the present paper, with proper care taken, should hold in this setting. However for simplicity of presentation we shall just assume k is a field.

1.2. **Non-homogeneous quadratic and Koszul algebras.** Let P be a subspace of $k \oplus V \oplus (V \otimes V)$ such that $P \cap (k \oplus V)$ is zero. We then get $U = T(V)/(P)$, a *non-homogenous quadratic* algebra. The filtration $F^pT(V)$ on $T(V)$ given by $\bigoplus_{i\leq p} V^{\otimes i}$ induces a filtration of U, and so we get an associated graded algebra gr$U$. Let $R = p_2(P)$ be the projection onto the quadratic factor. We get the quadratic algebra A defined by R and a natural epimorphism $A \to \mathrm{gr}U$. We say that U is of *Poincaré-Birkhoff-Witt* (PBW) type if this map is an isomorphism.

Example 1.4. Let A be the symmetric algebra $S(V)$. In this case R is

$$(x \otimes y - y \otimes x)_{x,y\in V}.$$

If V has the structure of a Lie algebra and P is $(x \otimes y - y \otimes x - [x,y])_{x,y\in V}$, then U will be the enveloping algebra of V, and by the Poincaré-Birkhoff-Witt theorem it is of PBW type.

Now since $P \cap (k \oplus V)$ is zero, the subspace P can be described by the maps

$$R \xrightarrow{\alpha} V, \quad R \xrightarrow{\beta} k \tag{7}$$

such that

$$P = \{x + \alpha(x) + \beta(x) \mid x \in R\}.$$

Following Positselski [21], a triple (B,d,c), where B is a positively graded algebra $\bigoplus_{i\geq 0} B_i$ with an (anti-)derivation d and an element c in B_2 such that 1. $d(c) = 0$ and 2. $d^2(b) = [c,b]$ for b in B, is called a *curved differential graded algebra* (cdga).

Now let $A^!$ be the quadratic dual algebra of A. Dualizing the maps α and β in (7), we get (note that R^* is equal to $A^!_2$)

$$V^* \xrightarrow{\alpha^*} A^!_2, \quad k \xrightarrow{\beta^*} A^!_2. \tag{8}$$

Positselski loc.cit. now shows the following.

Theorem 1.5. *Assume A is Koszul. Then U is of PBW-type if and only iff the map α^* extends to an (anti-)derivation d on $A^!$ such that, letting $c = \beta^*(1)$, $(A^!, d, c)$ is a cdg-algebra.*

In particular, when $c = 0$, giving $A^!$ the structure of a differential graded algebra is equivalent to giving a subspace P of $V \oplus (V \otimes V)$ with $p_2(P) = R$ such that $grU = A$.

The condition that U is of PBW-type is also investigated by Braverman and Gaitsgory in [4]. They show the following.

Theorem 1.6. *Assume A is Koszul. Then U is of PBW-type if and only if*

1. $im(\alpha\otimes id - id\otimes\alpha) \subseteq R \subseteq V\otimes V$ *(this map is defined on $(R\otimes V)\cap(V\otimes R)$).*
2. $\alpha\circ(\alpha\otimes id - id\otimes\alpha) = \beta\otimes id - id\otimes\beta$.
3. $\beta\circ(\alpha\otimes id - id\otimes\alpha) = 0$.

Thus these conditions are precisely equivalent to giving $A^!$ starting from (8) the structure of a curved differential graded algebra.

Example 1.7. If R is $\wedge^2 V \subseteq V\otimes V$ and $\beta = 0$, then 1. and 2. say that $\alpha : \wedge^2 V \to V$ given by $x\wedge y \mapsto [y,x]$ satisfies the Jacobi identity. Thus U is of PBW-type if and only if α satisfies the Jacobi identity making V into a Lie algebra. By Theorem 1.5 above this is also equivalent to giving $E(V^*)$ the structure of a differential graded algebra. See [11, III] for more on this.

Example 1.8. If R is $V\otimes V$ and β is zero, then the conditions above reduce to 2. which says that $\alpha : V\otimes V \to V$ makes V an associative algebra. So we get the classical fact that giving V the structure of an associative algebra is equivalent to give a differential graded algebra structure on $T(V^*)$.

We end this section with a lemma which will be used subsequently. Let x_α be a basis for V and let $\check{x}_\alpha$ be a dual basis for V^*.

Lemma 1.9. *The element in $U\otimes_k A^!$*

$$\sum x_\alpha x_\beta \otimes \check{x}_\beta \check{x}_\alpha + \sum x_\alpha \otimes d(\check{x}_\alpha) + 1\otimes c, \tag{9}$$

and the element in $A^!\otimes_k U$

$$\sum \check{x}_\beta \check{x}_\alpha \otimes x_\alpha x_\beta + d(\check{x}_\alpha)\otimes x_\alpha + c\otimes 1,$$

are both zero.

Proof. Consider the pairing

$$(U\otimes A^!_2)\otimes (A^!_2)^* \to U.$$

Denoting the element in (9) by m, we show that $\langle m, -\rangle : (A^!_2)^* \to U$ is zero:

$$\langle m, r\rangle = \sum \langle x_\alpha x_\beta \otimes \check{x}_\beta \check{x}_\alpha, r\rangle + \sum \langle x_\alpha \otimes d(\check{x}_\alpha), r\rangle + \langle 1\otimes c, r\rangle.$$

Now note that for an element r in $R = (A^!_2)^*$ we have

$$\begin{aligned} \sum x_\alpha x_\beta \langle \check{x}_\beta \check{x}_\alpha, r\rangle &= r, \\ \sum x_\alpha \langle d(\check{x}_\alpha), r\rangle &= \alpha(r), \\ \langle c, r\rangle &= \beta(r). \end{aligned}$$

Since $r + \alpha(r) + \beta(r) = 0$ in U, we get the lemma. □

2. Functors between module categories

We shall define appropriate module categories of U and $(A^!, d, c)$. Following the usual formalism of Koszul duality we shall relate these module categories by a pair of adjoint functors. In doing so several new features are introduced. First an appropriate definition of the module category of $(A^!, d, c)$. Second we give a new compact definition of the functors between module categories compared to the more explicit versions of say [2] or [10], showing that the adjunction between them is simply a $\otimes - \mathrm{Hom}$ adjunction. Third the functors will be defined on the category of all complexes of modules rather than a subcategory of these consisting of bounded above or bounded below complexes.

2.1. **Categories of modules.** The category $\mathrm{Kom}(U)$ is the category of complexes of left U-modules. The category $\mathrm{Kom}(A^!, d, c)$ consists of graded left $A^!$-modules N with a module anti-derivation d_N of degree 1, i.e.

$$d_N(an) = d_{A^!}(a)n + (-1)^{|a|} a d_N(n)$$

such that $d_N^2(n) = cn$.

We call the objects of $\mathrm{Kom}(A^!, d, c)$ *curved differential graded* (cdg) modules over $(A^!, d, c)$. Note that when $c = 0$ then $\mathrm{Kom}(A^!, d, 0)$ is just the category of differential graded modules over $(A^!, d)$. Also note that when $c \neq 0$ then $A^!$ is in general not a cdg-module over itself. Similarly we can define a category $\mathrm{Kom}(-A^!, d, c)$ of right $A^!$-modules N with

$$\begin{aligned} d_N(na) &= d_N(n)a + (-1)^{|n|} n d_{A^!}(a), \\ d_N^2(n) &= -nc. \end{aligned}$$

The vector space V may also be graded by an abelian group Λ giving A and $A^!$ natural Λ-gradings. In this case we assume that the subspace P of $k \oplus V \oplus (V \otimes V)$ is homogeneous for Λ so that U gets a Λ-grading, or, equivalently, that c has Λ-degree 0 and the derivation d is homogeneous for the Λ-grading.

We then assume that the categories $\mathrm{Kom}(U)$ and $\mathrm{Kom}(A^!, d, c)$ above consist of Λ-graded complexes with homogeneous module derivations and that the morphisms in these categories are also homogeneous.

Example 2.1. Suppose $\Lambda = \mathbf{Z}$ and $\deg_\Lambda V = 1$ (so c and d are 0). Then $\mathrm{Kom}(A)$ consists of complexes of $\mathbf{Z}$-graded modules and $\mathrm{Kom}(A^!, 0, 0)$ is equivalent to the category $\mathrm{Kom}(A^!)$ of complexes of $\mathbf{Z}$-graded $A^!$-modules. So in this case we have the setting of [2]. We elaborate more on this in Section 3.

Example 2.2. Suppose $(A^!, d)$ is $(E(V^*), d)$, where $V = \mathfrak{g}$ is a Lie algebra and so U is the enveloping algebra of $\mathfrak{g}$. When $\mathfrak{g}$ is semi-simple it is graded by weights Λ, and so $\mathrm{Kom}(U)$ becomes the category of complexes of modules over U graded by weights.

We shall now relate the categories $\mathrm{Kom}(U)$ and $\mathrm{Kom}(A^!, d, c)$ by giving functors between them. First let $T = U \otimes A^!$. This is a $U - A^!$ bimodule which we consider as graded by the grading of $A^!$. The following makes it an object of $\mathrm{Kom}(-A^!, d, c)$.

Lemma 2.3. *The linear endomorphism d of $U \otimes_k A^!$ given by*

$$u \otimes a \mapsto \sum u x_\alpha \otimes \check{x}_\alpha a + u \otimes d(a)$$

gives $U \otimes_k A^!$ the structure of right cdg-module over $(A^!, d, c)$.

Proof. It is straightforward to check that

$$d((u \otimes a) \cdot b) = d(u \otimes a) \cdot b + (-1)^{|a|} (u \otimes a) \cdot d(b).$$

Let us now prove that $d^2(1\otimes a) = -1\otimes ac$. We find

$$\begin{aligned} & d^2(1\otimes a) \\ = \quad & \sum x_\alpha x_\beta \otimes \check{x}_\beta \check{x}_\alpha a + \sum x_\alpha \otimes \check{x}_\alpha d(a) + \sum x_\alpha \otimes d(\check{x}_\alpha a) + 1 \otimes d^2(a) \\ = \quad & \sum x_\alpha x_\beta \otimes \check{x}_\beta \check{x}_\alpha a + \sum x_\alpha \otimes d(\check{x}_\alpha) a + 1\otimes ca - 1 \otimes ac, \end{aligned}$$

and we conclude by Lemma 1.9. □

We now have functors

$$\mathrm{Kom}(A^!, d, c) \underset{G}{\overset{F}{\rightleftarrows}} \mathrm{Kom}(U)$$

given by

$$F(N) = T \otimes_{A^!} N, \quad G(M) = \mathrm{Hom}_U(T, M). \tag{10}$$

The meaning of these expressions needs elaboration.

2.2. **Total complexes.** First note that if R is a ring and P and Q are complexes of right and left R-modules respectively, then we get a complex $P \otimes_R Q$ of abelian groups by taking the total direct sum complex of the double complex $P^p \otimes_R Q^q$ with horizontal differential $d \otimes 1$ and vertical differential $(-1)^p \otimes d$. As a graded module, $T \otimes_{A^!} N$ is the cokernel of the map of complexes:

$$T \otimes_k A^! \otimes_k N \xrightarrow{\mu} T \otimes_k N \tag{11}$$

given by

$$t \otimes a \otimes n \mapsto ta \otimes n - t \otimes an.$$

Now T and N do not have differentials but linear endomorphisms d of degree 1. But we may use the same procedure and equip the graded U-modules in (11) with a linear endomorphism of degree 1 and let $F(N)$ be the cokernel of μ. Below we show that $F(N)$ is actually in $\mathrm{Kom}(U)$, i.e. $d^2 = 0$.

Now if P and Q are complexes of left R-modules we get a complex of abelian groups $\mathrm{Hom}_R(P, Q)$ by taking the total direct product complex of the double complex $\mathrm{Hom}_R(P^p, Q^q)$ with horizontal differential $(-1)^p d \otimes 1$ and vertical differential $d \otimes (-1)^p$. (Note the unusual convention. This is however more natural and correct than the ordinary convention of letting the horizontal differential be $d \otimes 1$ and the vertical differential be $(-1)^{p+q} \otimes d$.) Consider the graded $A^!$-module $\mathrm{Hom}_U(T, M)$. Now T has a linear endomorphism of degree 1 instead of a differential, but we may use the same procedure as above and equip $\mathrm{Hom}_U(T, M)$ with a linear endomorphism of degree 1. We show below that it is an object of $\mathrm{Kom}(A^!, d, c)$.

2.3. **Explicit descriptions.** Explicitly the definition of the functor $F(N)$ becomes

$$F(N)^p = U \otimes_k N^p \tag{12}$$

with linear endomorphism d given by

$$u \otimes n \xrightarrow{d} \sum u x_\alpha \otimes \check{x}_\alpha n + u \otimes d(n). \tag{13}$$

The definition of $G(M)$ becomes

$$G(M)^p = \prod_{r \geq 0} \mathrm{Hom}_k(A^!_r, M^{p+r}) \cong \prod_{r\geq 0} ((A^!_r)^* \otimes_k M^{p+r}) \tag{14}$$

with linear endomorphism d given by

$$d(f)(a) = (-1)^{|a|} \sum x_\alpha f(\check{x}_\alpha a) + (-1)^{|a|} f(d_{A^!}(a)) + (-1)^{|a|} d_M(f(a)) \tag{15}$$

or alternatively, if d^* on $(A^!)^*$ is the dual of $d_{A^!}$,

$$a^* \otimes m \xrightarrow{d} (-1)^{|a^*|} \sum a^* \check{x}_\alpha \otimes x_\alpha m + (-1)^{|a^*|} d^*(a^*) \otimes m + (-1)^{|a^*|} a^* \otimes d_M(m). \tag{16}$$

Lemma 2.4. *The U-module homomorphism d in* (13) *satisfies $d^2(u \otimes n) = 0$. The module anti-derivation d in* (16) *satisfies $d^2(a^* \otimes m) = ca^* \otimes m$. Hence $F(N)$ is in Kom(U) and $G(M)$ is in Kom$(A^!, d, c)$.*

Proof. The first statement follows from Lemma 1.9. Calculating $d^2(a^* \otimes m)$ explicitly, this also follows from Lemma 1.9 by noting that

$$d^*(a^* \check{x}_\alpha) = -d^*(a^*) \check{x}_\alpha + a^* d^*(\check{x}_\alpha)$$

and that

$$(d^*)^2(a^*) = [a^*, c].$$

□

Example 2.5. Let U be $k[x]/(x^2 - (a+b)x + ab)$. Then $A^! = k[\check{x}]$ with $c = ab\check{x}^2$ and

$$d(\check{x}^n) = \begin{cases} -(a+b)\check{x}^{n+1}, & n \text{ odd}, \\ 0, & n \text{ even}. \end{cases}$$

Now U has two simple modules, k_a and k_b, each of dimension 1 over k. For v in k_a we have $x.v = av$ and similarly for k_b. We see that $G(k_a)$ is the cdg-module

$$\dots (x^3) \xrightarrow{\cdot a\check{x}} (x^2) \xrightarrow{\cdot b\check{x}} (x) \xrightarrow{\cdot a\check{x}} (1) \to 0 \to \cdots.$$

Similarly $G(k_b)$ is the module

$$\dots (x^3) \xrightarrow{\cdot b\check{x}} (x^2) \xrightarrow{\cdot a\check{x}} (x) \xrightarrow{\cdot b\check{x}} (1) \to 0 \to \cdots.$$

2.4. **Adjunction.**

Proposition 2.6. *For N in Kom$(A^!, d, c)$ and M in Kom(U) there is a canonical isomorphism of differential graded vector spaces*

$$Hom_U(F(N), M) = Hom_{A^!}(N, G(M)).$$

Remark 2.7. The first Hom-complex is formed as described in Subsection 2.2. The second Hom-complex is, as a graded vector space, just the graded Hom, and the differential is induced by the inclusion of $\mathrm{Hom}_{A^!}(N, G(M))$ in $\mathrm{Hom}_k(N, G(M))$.

Proof. Since $F(N)$ is $T \otimes_{A^!} N$ and $G(M)$ is $\mathrm{Hom}_U(T, M)$, this is just the standard $\otimes - \mathrm{Hom}$ adjunctions. Explicitly both complexes have p'th term equal to

$$\prod_{r \in \mathbf{Z}} \mathrm{Hom}_k(N^r, M^{p+r})$$

and differential δ given as follows. Let f be in the above product and n in N^r. Then

$$n \overset{\delta(f)}{\mapsto} (-1)^r d_M f(n) + (-1)^{r+1} f d_N(n) + (-1)^{r+1} \sum_\alpha x_\alpha f(\check{x}_\alpha n).$$

□

Corollary 2.8. *The functors F and G are adjoint, i.e.*

$$Hom_{Kom(U)}(F(N), M) = Hom_{Kom(A^!,d,c)}(N, G(M)).$$

Proof. Both sides are the cycles of degree 0 in the complexes above. □

2.5. **Exact sequences.** We note that the functors F and G are exact in the sense that if

$$0 \to N_1 \to N_2 \to N_3 \to 0$$

is an exact sequence of cdg-modules (i.e. componentwise exact), then

$$0 \to F(N_1) \to F(N_2) \to F(N_3) \to 0$$

is exact.

2.6. **Homotopy categories.** If $M \xrightarrow{f} N$ is a morphism in $\mathrm{Kom}(U)$ or in $\mathrm{Kom}(A^!, d, c)$ we can form the cone $C(f)$ as we do ordinarily. It is straightforward to check from the explicit descriptions given in (13) and (15) that the functors F and G take cones to cones. Also given two objects M and N in $\mathrm{Kom}(A^!, d, c)$ (resp. $\mathrm{Kom}(U)$), we say that two morphisms $f, g : M \to N$ are *homotopic* if the difference $f - g$ is in the image of $\mathrm{Hom}^{-1}_{A^!}(M, N)$ (resp. $\mathrm{Hom}^{-1}_U(M, N)$), where $\mathrm{Hom}_{A^!}(M, N)$ (resp. $\mathrm{Hom}_U(M, N)$) is the morphism complex, formed as in Remark 2.7.

Explicitly, with the conventions in Subsection 2.2, this means that

$$f^n - g^n = (-1)^n d_N^{n-1} \circ s^n + (-1)^{n+1} s^{n+1} \circ d_M^n$$

for a morphism $s : M \to N[-1]$ of graded $A^!$-modules (resp. U-modules). We may then form the homotopy categories $K(U)$ and $K(A^!, d, c)$ and these will be triangulated categories with distinguished triangles those isomorphic to

$$0 \to N \to C(f) \to M[1] \to 0$$

for a morphism $M \xrightarrow{f} N$.

Proposition 2.9. *The functors F and G descend to give functors between (triangulated) homotopy categories*

$$K(A^!, d, c) \underset{G}{\overset{F}{\rightleftarrows}} K(U). \tag{17}$$

Proof. We must show that F and G take homotopic morphisms to homotopic morphisms. But $M \xrightarrow{f} N$ is nullhomotopic if and only if the inclusion $N \to C(f)$ is split and this condition is preserved by the additive functors F and G. Since F and G preserve cones we are done. □

2.7. **Filtrations of functors.** We can compose the functor F from $\mathrm{Kom}(A^!, d, c)$ to $\mathrm{Kom}(U)$ with the forgetful functor $\mathrm{Kom}(U) \to \mathrm{Kom}(k)$. This composition comes with a natural filtration of functors. Namely, let

$$F_i : \mathrm{Kom}(A^!, d, c) \to \mathrm{Kom}(k)$$

be given by

$$F_i(N) : \ \cdots \to U_i \otimes_k N^0 \to U_{i+1} \otimes_k N^1 \to \cdots$$

with the differentials just the restrictions of the differentials of F. Note that $F(N)$ is $\varinjlim F_i(N)$.

There is also a filtration of the composition GF by functors $(GF)_i$ from $\mathrm{Kom}(A^!, d, c)$ to $\mathrm{Kom}(A^!, d, c)$ given by

$$(GF)_i(N)^p = \prod_{r \geq 0} \mathrm{Hom}_k(A^!_r, U_{p+i} \otimes_k N^{r+p}) \tag{18}$$

with linear endomorphism the restriction of the linear endomorphism of GF.

3. Variations and specializations

This section discusses some variations and specializations of the categories and functors defined in the previous section. In particular we discuss the variation of Koszul duality used by Goresky, Kottwitz, and MacPherson in [10]. Also we consider in more detail the case when our algebras and categories are graded by the abelian group $\Lambda = \mathbf{Z}$ so we get complexes of graded modules. This is the case considered by Beilinson, Ginzburg and Soergel in [2].

3.1. **Categories.** Let $B = \bigoplus_{i \geq 0} B_i$ be a positively graded algebra. Given an integer r we let rB be the algebra given by $(rB)_{ri} = B_i$ and $(rB)_d = 0$ if d is not divisible by r. Then rB may be equipped with an anti-derivation d of degree 1 (which can be non-zero only if $r = -1$ or 1) and a cycle c in $(rB)_2$ (which can be non-zero only if $r = 1$ or 2) such that $d^2(b) = [c, b]$.

We may then define the category $\mathrm{Kom}_r(B, d, c)$ as consisting of graded left rB-modules N with a module derivation d_N of degree 1, i.e.

$$d_N(bn) = d(b)n + (-1)^{r|b|} b d_N(n)$$

such that $d_N^2(n) = cn$. When $r \neq 0$ we shall consider the case when B is a Koszul algebra A or $A^!$, and when $r = 0$ we may also allow B to be a non-homogeneous Koszul algebra so $(rB)_0 = B = U$ (if the associated graded Koszul algebra is $A^!$ we vary a bit and write $(rB)_0 = B = U'$).

We may now define functors

$$\mathrm{Kom}_r(A^!) \underset{G}{\overset{F}{\rightleftarrows}} \mathrm{Kom}_{1-r}(A) \tag{19}$$

for all integers r. In some particular cases we may also define

$$\mathrm{Kom}_2(A^!, 0, c) \rightleftarrows \mathrm{Kom}_{-1}(A, d), \text{ when } r = 2, \tag{20}$$

$$\mathrm{Kom}_1(A^!, d, c) \rightleftarrows \mathrm{Kom}_0(U), \text{ when } r = 1, \tag{21}$$

$$\mathrm{Kom}_0(U') \rightleftarrows \mathrm{Kom}_1(A, d, c), \text{ when } r = 0, \tag{22}$$

$$\mathrm{Kom}_{-1}(A^!, d) \rightleftarrows \mathrm{Kom}_2(A, 0, c), \text{ when } r = -1. \tag{23}$$

In case (19) these are defined as in (10) by using the $A - A^!$ bimodule $T = A \otimes A^!$ with differential $a \otimes b \mapsto \sum_\alpha a x_\alpha \otimes \check{x}_\alpha b$ and defining

$$F(N) = T \otimes_{A^!} N, \quad G(M) = \mathrm{Hom}_A(T, M).$$

The other functors are defined similarly. In (20) the differential d gives a map from $(-A)_{-1} = V$ to $(-A)_0 = k$ and so a map $k \xrightarrow{d^*} V^*$ and c is $d^*(1)$; the case of (23) is similar. Note that in (19) there is no symmetry between r and $1 - r$ since the functors F and G are defined differently.

The case considered by [10] is the case (19) for $r = -1$, with A the symmetric algebra $S(V)$ and $A^!$ the exterior algebra $E(V^*)$ (and $d, c = 0$). They do however only define the functors F and G on the subcategory of bounded above (resp. bounded below) complexes.

One might wonder if the cases in (19) are all equivalent. However, these different categories do not seem to be equivalent, so they are most likely genuinely different cases. However, this changes when we consider the categories of complexes of graded modules, as we see in the next subsection.

3.2. **The graded case.** Assume that $\Lambda = \mathbf{Z}$ and $\deg_\Lambda V = 1$. Since d is homogeneous and c has degree 0, we must then have d and c zero. We are then in case (19) of the preceding subsection.

Proposition 3.1. *When* $\Lambda = \mathbf{Z}$ *and* $\deg_\Lambda V = 1$ *then the categories and functors in* (19) *for various* r *are all isomorphic.*

Proof. Given a dg-module M in $\mathrm{Kom}_{1-r}(A)$, denote the piece of cohomological degree p and Λ-degree q by M_q^p. Then

$$M_q^p \xrightarrow{d} M_q^{p+1},$$

and for x in V

$$M_q^p \xrightarrow{\cdot x} M_{q+1}^{p+(1-r)}.$$

Now let

$$p' = p + (r-1)q, \quad q' = q.$$

With the cohomological grading of M given by p' and the Λ-grading of M given by q', then M becomes an object both of $\mathrm{Kom}_0(A)$. Similarly if N is in $\mathrm{Kom}_r(A^!)$, then with exactly the same change of grading, N becomes an object of $\mathrm{Kom}_1(A^!)$. Via these isomorphisms of categories it is furthermore easy to check that the functors F and G correspond. □

The commonly used variation of this is the following. For a positively graded ring $B = \bigoplus_{i \geq 0} B_i$ let $\mathrm{Kom}(B)$ be the category of complexes of graded modules. An object $M = \bigoplus M_q^p$ in $\mathrm{Kom}_{1-r}(A)$ becomes an object of $\mathrm{Kom}(A)$ by letting it have cohomological grading $p' = p + (r-1)q$ and internal grading $q' = q$. Similarly an object $N = \bigoplus N_q^p$ of $\mathrm{Kom}_r(A^!)$ becomes an object of $\mathrm{Kom}(A^!)$ by letting it have cohomological grading $p' = p + rq$ and internal grading $q' = q$. We then get the setting of [2] and the functors

$$\mathrm{Kom}(A^!) \underset{G}{\overset{F}{\rightleftarrows}} \mathrm{Kom}(A).$$

There is however a slight inconvenience in this, because by letting $T = A \otimes A^!$, this complex does not have a natural grading such that naturally $F(N) = T \otimes_{A^!} N$ and $G(M) = \mathrm{Hom}_A(T, M)$. However we may give the functor F explicitly as the total *direct sum* complex of the double complex (unbounded in all directions, also down and to the left)

$$\begin{array}{ccccc}
A(2)\otimes N_2^0 & \longrightarrow & \cdots & & \\
\uparrow & & \uparrow & & \\
A(1)\otimes N_1^0 & \longrightarrow & A(1)\otimes N_1^1 & \longrightarrow & \cdots \\
{\scriptstyle d_v}\uparrow & & \uparrow & & \uparrow \\
A\otimes N_0^0 & \xrightarrow{d_h} & A\otimes N_0^1 & \longrightarrow & A\otimes N_0^2
\end{array}$$

where the horizontal differential is given by $a \otimes n \mapsto a \otimes d_N(n)$ and the vertical differential by $a \otimes n \mapsto \sum_\alpha ax_\alpha \otimes \check{x}_\alpha n$.

Similarly the functor G is given explicitly as the total *direct product* complex of the double complex (unbounded in all directions, also down and to the left)

$$\begin{array}{ccccc}
\mathrm{Hom}_k(A^!(-2), M_2^0) & \longrightarrow & \cdots & & \\
\uparrow & & \uparrow & & \\
\mathrm{Hom}_k(A^!(-1), M_1^0) & \longrightarrow & \mathrm{Hom}_k(A^!(-1), M_1^1) & \longrightarrow & \cdots \\
{\scriptstyle d_v}\uparrow & & \uparrow & & \uparrow \\
\mathrm{Hom}_k(A^!, M_0^0) & \xrightarrow{d_h} & \mathrm{Hom}_k(A^!, M_0^1) & \longrightarrow & \mathrm{Hom}_k(A^!, M_0^2)
\end{array}$$

where the horizontal differential d_h is given by $f \mapsto d_M \circ f$ and the vertical differential d_v by $f(-) \mapsto \sum x_\alpha f(\check{x}_\alpha \cdot -)$.

In [2] the setting is however only a subsetting of this. They define the subcategories $\mathrm{Kom}^\uparrow(A)$ of $\mathrm{Kom}(A)$ consisting of complexes M such that there exist a and b (depending on M) such that $M_q^p \neq 0$ only if $p \leq a$ and $p + q \geq b$. Similarly $\mathrm{Kom}^\downarrow(A^!)$ is the subcategory of $\mathrm{Kom}(A^!)$ consisting of complexes N_q^p such that there exists a and b (depending on N) with $N_q^p \neq 0$ only if $p \geq a$ and $p + q \leq b$. They then define the functors F and G only on these categories

$$\mathrm{Kom}^\downarrow(A^!) \underset{G}{\overset{F}{\rightleftarrows}} \mathrm{Kom}^\uparrow(A). \tag{24}$$

The reason for only considering these subcategories is that the functors then descend to functors of derived categories. See the next section.

A more natural way to consider these categories and functors from the point of gradings is as follows. For a positively graded ring $B = \bigoplus_{d\geq 0} B_d$ let $\mathrm{Kom}(1, B)$ (resp. $\mathrm{Kom}(2, B)$) be the category of differential bigraded modules $M = \bigoplus M^{p,q}$ such that the differential has bigrade $(1, 1)$ and $B_d \otimes M^{p,q}$ maps to $M^{p+d,q}$ (resp. $M^{p,q+d}$). There is then a more "symmetric" way of describing the categories above. Namely if $M = \bigoplus M_q^p$ is in $\mathrm{Kom}_{1-r}(A)$, then consider it as an object of $\mathrm{Kom}(1, A)$ by letting it have bigrade $(p + rq, p + (r - 1)q)$. Similarly consider $N = \bigoplus N_q^p$ in $\mathrm{Kom}_r(A^!)$ as an object of $\mathrm{Kom}(2, A^!)$ by giving it a bigrade in exactly the same way. Then $T = A \otimes A^!$ with its natural bigrading becomes an object of both $\mathrm{Kom}(1, A)$ and $\mathrm{Kom}(2, -A^!)$, the category of differential bigraded right $A^!$-modules, and one gets functors

$$\mathrm{Kom}(1, A) \underset{G}{\overset{F}{\rightleftarrows}} \mathrm{Kom}(2, A^!)$$

defined by $F(N) = T \otimes_{A^!} N$ and $G(M) = \mathrm{Hom}_A(T, M)$.

The subcategory $\mathrm{Kom}^{\uparrow}(A)$ of $\mathrm{Kom}(A)$ then corresponds to the subcategory $\mathrm{Kom}^{\lrcorner}(1, A)$ of $\mathrm{Kom}(1, A)$ consisting of complexes $M = \bigoplus M^{p,q}$ supported in a second quadrant type region, i.e. there exist a and b (depending on M) such that $M^{p,q}$ is non-zero only if $p \leq a$ and $q \geq b$. Similarly the subcategory $\mathrm{Kom}^{\downarrow}(A^!)$ of $\mathrm{Kom}(A^!)$ corresponds to the subcategory $\mathrm{Kom}^{\lrcorner}(2, A^!)$ of $\mathrm{Kom}(2, A^!)$ consisting of complexes supported in a second quadrant type region, and the adjunction (24) corresponds to an adjunction

$$\mathrm{Kom}^{\lrcorner}(1, A) \underset{G}{\overset{F}{\rightleftarrows}} \mathrm{Kom}^{\lrcorner}(2, A^!).$$

3.3. **Categories of right modules.** Again consider the adjunction

$$\mathrm{Kom}(A^!, d, c) \underset{G}{\overset{F}{\rightleftarrows}} \mathrm{Kom}(U)$$

given by $F(N) = T \otimes_{A^!} N$ and $G(M) = \mathrm{Hom}_U(T, M)$.

We may also get an adjunction between the categories of right modules

$$\mathrm{Kom}(-U) \underset{-G}{\overset{-F}{\rightleftarrows}} \mathrm{Kom}(-A^!, d, c),$$

where $-F(M) = M \otimes_U T$ and $-G(N) = \mathrm{Hom}_{-A^!}(T, N)$.

By considering the $A^! - U$ bimodule $T' = A^! \otimes U$ we similarly get functors

$$\mathrm{Kom}(U) \underset{G'}{\overset{F'}{\rightleftarrows}} \mathrm{Kom}(A^!, d, c)$$

and

$$\mathrm{Kom}(-A^!, d, c) \underset{-G'}{\overset{-F'}{\rightleftarrows}} \mathrm{Kom}(U).$$

Proposition 3.2. *Let N be in $\mathrm{Kom}(A^!, d, c)$ and let M be in $\mathrm{Kom}(-U)$. Then*

$$M \otimes_U F(N) = -F(M) \otimes_{A^!} N.$$

Proof. This is clear. □

4. Quasi-isomorphisms and equivalences

Throughout the rest of the paper we assume $c = 0$, so $(A^!, d)$ is simply a differential graded algebra and $\mathrm{Kom}(A^!, d)$ the category of differential graded modules over $(A^!, d)$.

The categories $\mathrm{Kom}(U)$ and $\mathrm{Kom}(A^!, d)$ are related by the adjoint functors F and G. We shall now find suitable quotient categories of these categories such that the functors F and G descend to give an adjoint *equivalence* of categories. In [2], where the adjunction is between certain subcategories of the above categories in the $\Lambda = \mathbf{Z}$-graded case, the quotient categories are derived categories. This is however not true in general. Rather, the desired quotient categories are "between" the homotopy category and the derived category.

4.1. **Generalized Koszul and Chevalley-Eilenberg complex.**

Lemma 4.1. *Let M be a U-module considered as a complex concentrated in degree 0. Then $FG(M) \to M$ is a quasi-isomorphism.*

Note that $FG(M)$ is the complex

$$\cdots \to U \otimes (A^!_p)^* \otimes M \to U \otimes (A^!_{p-1})^* \otimes M \to \cdots \to U \otimes M$$

where the differential is given by

$$\begin{aligned} u \otimes a^* \otimes m \quad \mapsto \quad & \sum u x_\alpha \otimes \check{x}_\alpha a^* \otimes m + (-1)^{|a^*|} \sum u \otimes a^* \check{x}_\alpha \otimes x_\alpha m \\ & + (-1)^{|a^*|} u \otimes d^*(a^*) \otimes m. \end{aligned}$$

So the lemma says that this is a resolution of M. Letting $M = k$ we get a complex generalizing the Koszul complex, the case when the derivation d in $A^!$ is zero, and the Chevalley-Eilenberg complex, the case when U is an enveloping algebra $U(\mathfrak{g})$. This complex already appears in [22].

Proof of Lemma 4.1. The complex $FG(M)$ has a filtration $F_iG(M)$. We claim that for $i \geq 0$ then $H^p(F_iG(M)) = M$ for $p = 0$ and zero otherwise. This follows by induction from the exact sequence

$$0 \to F_{i-1}G(M) \to F_iG(M) \to F_iG(M)/F_{i-1}G(M) \to 0$$

by noting that the right term is a homogeneous part of the Koszul complex for A and $A^!$ tensored with M (over k)

$$A_0 \otimes (A^!_i)^* \otimes M \to \cdots \to A_i \otimes (A^!_0)^* \otimes M$$

with differential

$$u \otimes a^* \otimes m \mapsto \sum u x_\alpha \otimes \check{x}_\alpha a^* \otimes m.$$

Now since $FG(M)$ is $\varinjlim F_iG(M)$ and $\varinjlim$ is exact in the category of vector spaces, we get the lemma. □

4.2. **Ext's and Tor's.** The following gives interpretations of the cohomology of the complexes $G(M)$ and $F'(M)$ (see Subsection 3.3).

Proposition 4.2. *Let M be a U-module considered as a complex concentrated in degree* 0.

a. $Tor^U_p(k, M) = H^{-p}G(M)$.
b. $Ext^p_U(k, M) = H^pF'(M)$.

Proof. a. The complex $G(M)$ is $\mathrm{Hom}_{A^!}(A^!, G(M))$ which by the adjunction of Corollary 2.8 is $\mathrm{Hom}_U(F(A^!), M)$. Since $F(A^!)$ is a complex of free U-modules of finite rank, this is equal to

$$\mathrm{Hom}_U(F(A^!), U) \otimes_U M.$$

Here $\mathrm{Hom}_U(F(A^!), U)$ is a free resolution of k since it is isomorphic as a complex of vector spaces to $\mathrm{Hom}_{A^!}(A^!, G(U))$ which is $GF(k)$, and this is quasi-isomorphic to k.

b. $FG(k) \to k$ is a resolution of k of finite rank free modules. Therefore

$$\mathrm{Hom}_U(FG(k), M) \cong \mathrm{Hom}_U(FG(k), U) \otimes_U M.$$

But $\mathrm{Hom}_U(FG(k), U)$ is the complex $A^! \otimes U$ which is $-F'(A^!)$. By analogy of Proposition 3.2

$$-F'(A^!) \otimes_U M = A^! \otimes_{A^!} F'(M) = F'(M).$$

□

4.3. **Quasi-isomorphisms.**

Proposition 4.3. *The natural morphisms coming from the adjunction*

$$FG(M) \to M, \quad N \to GF(N)$$

are quasi-isomorphisms.

Proof. We start with the first one. For a complex M let $\sigma^{\leq p} M$ be the truncation $\cdots \to M^{p-1} \to M^p \to 0$ and $\sigma^{\geq p} M$ be the truncation $0 \to M^p \to M^{p+1} \to \cdots$. We have a short exact sequence

$$0 \to \sigma^{>p} M \to M \to \sigma^{\leq p} M \to 0. \tag{25}$$

i) If M is a bounded complex it follows by induction, using the above sequence, the fact that F and G are exact on short exact sequences, and Lemma 4.1, that $FG(M) \to M$ is a quasi-isomorphism.

ii) Now suppose that M is bounded above so $M = \varinjlim \sigma^{\geq p} M$ where the $\sigma^{\geq p} M$ are bounded. By (14) we see that G commutes with such colimits, i.e.

$$G(M) = G(\varinjlim \sigma^{\geq p} M) = \varinjlim G(\sigma^{\geq p} M).$$

Since F is a left adjoint it also commutes with colimits so

$$FG(M) = FG(\varinjlim \sigma^{\geq p} M) = \varinjlim FG(\sigma^{\geq p} M) \to \varinjlim \sigma^{\geq p} M = M$$

is a quasi-isomorphism since $\varinjlim$ is exact on the category of vector spaces.

Before proceeding note that for a module M over U, $F_i G(M)$ is exact in cohomological degrees < 0 by the proof of Lemma 4.1.

iii) Now suppose that M is bounded below, say $M = \sigma^{>0} M$. By the remark just above we see that $F_i G(\sigma^{\leq p} M)$ is exact in cohomological degrees ≤ 0. The functor G commutes with $\varprojlim$ since it is a right adjoint, and the F_i also commutes with $\varprojlim$ since the U_i are finite dimensional. So

$$F_i G(M) = F_i G(\varprojlim \sigma^{\leq p} M) = \varprojlim F_i G(\sigma^{\leq p} M) \to \varprojlim \sigma^{\leq p} M = M$$

is a quasi-isomorphism in cohomological degrees ≤ 0 and so $F_i G(M)$ is exact in this range. Since $FG(M)$ is $\varinjlim F_i G(M)$, the same also holds true for $FG(M)$.

iv) Now let M be an arbitrary complex. From the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & FG(\sigma^{>p} M) & \longrightarrow & FG(M) & \longrightarrow & FG(\sigma^{\leq p} M) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \sigma^{>p} M & \longrightarrow & M & \longrightarrow & \sigma^{\leq p} M & \longrightarrow & 0 \end{array}$$

we get that $FG(M) \to M$ is a quasi-isomorphism in cohomological degrees $< p$. Since p can be chosen arbitrary we are done.

We now prove the second part. The complex $GF(k)$ is the complex

$$\cdots \to (A^!_p)^* \otimes U \to (A^!_{p-1})^* \otimes U \to \cdots \to U.$$

By the same argument as in Lemma 4.1 the map $k \to GF(k)$ is a quasi-isomorphism. So if $N = N^0$ we have $GF(N) = GF(k) \otimes_k N$, and so $N \to GF(N)$ is also a quasi-isomorphism.

By induction using truncations, we get that $N \to GF(N)$ is a quasi-isomorphism for bounded N.

If N is bounded above, then $N = \varinjlim \sigma^{>p} N$. Since $F(N)$ is $\varinjlim F(\sigma^{>p} N)$ and G is seen to commute with this colimit (N is bounded above), we get that $N \to GF(N)$ is a quasi-isomorphism.

If N is any differential graded $A^!$-module, then $N = \varprojlim \sigma^{\leq p} N$, and since F commutes with this inverse limit and G also does so, we get that $N \to GF(N)$ is a quasi-isomorphism. □

4.4. **Equivalences of categories.** We now show how we may descend the adjunction (17) to an equivalence of categories. For generalities on triangulated categories and null systems we refer to [13]. We first abstract the situation. Consider the adjoint functors between two triangulated categories

$$\mathbf{A} \underset{G}{\overset{F}{\rightleftarrows}} \mathbf{B} \tag{26}$$

and let $Z_{\mathbf{A}}$ and $Z_{\mathbf{B}}$ be null systems of $\mathbf{A}$ and $\mathbf{B}$, respectively, such that the natural maps coming from the adjunction $a \to GF(a)$ and $FG(b) \to b$ are in the multiplicative systems determined by $Z_{\mathbf{A}}$ and $Z_{\mathbf{B}}$, respectively. This means that when completing to a triangle

$$a \to GF(a) \to E \to a[1], \tag{27}$$

then E is in $Z_{\mathbf{A}}$ and the other case is similar.

In our case $Z_{\mathbf{A}}$ will be the acyclic complexes in $K(A^!, d)$ and $Z_{\mathbf{B}}$ the acyclic complexes in $K(U)$.

Now let $N_{\mathbf{A}}$ be the subcategory of $Z_{\mathbf{A}}$ consisting of objects a such that $F(a)$ is in $Z_{\mathbf{B}}$. Correspondingly we have $N_{\mathbf{B}}$. Then $N_{\mathbf{A}}$ and $N_{\mathbf{B}}$ are easily seen to be null systems.

Proposition 4.4. *The adjunction* (26) *descends to give an adjoint equivalence of categories*

$$\mathbf{A}/N_{\mathbf{A}} \underset{\overline{G}}{\overset{\overline{F}}{\rightleftarrows}} \mathbf{B}/N_{\mathbf{B}}.$$

Proof. Consider the composition

$$\mathbf{A} \xrightarrow{F} \mathbf{B}/N_{\mathbf{B}}.$$

To show that this factors through $\mathbf{A}/N_{\mathbf{A}}$ let a be in $N_{\mathbf{A}}$. Then $F(a)$ is in $Z_{\mathbf{B}}$ and we need to show in addition that $GF(a)$ is in $Z_{\mathbf{A}}$. But this follows from the triangle (27), since both a and E are in $Z_{\mathbf{A}}$ and hence also $GF(a)$ must be in $Z_{\mathbf{A}}$. Hence we get the functor $\overline{F}$ and similarly $\overline{G}$.

To show that there is an adjoint equivalence of categories we show (see [18, IV.4]) i) that the functors are adjoint and ii) that the canonical morphism $\overline{F}\,\overline{G}(b) \to b$ and $a \to \overline{G}\,\overline{F}(a)$ are isomorphisms.

i) The elements of $\mathrm{Hom}_{\mathbf{B}/N_{\mathbf{B}}}(\overline{F}(a), b)$ are equivalence classes of diagrams in $\mathbf{B}$

$$\begin{array}{ccc} F(a) & & b \\ & \searrow \quad \overset{s}{\swarrow} & \\ & c & \end{array} \tag{28}$$

where s is in the multiplicative system determined by $N_{\mathbf{B}}$. Such a diagram will, via the adjunction between F and G, correspond to a diagram

$$
\begin{array}{ccccc}
a & & & & G(b) \\
& \searrow & & \swarrow^{G(s)} & \\
& & G(c) & &
\end{array}
\tag{29}
$$

where $G(s)$ is in the multiplicative system determined by $N_{\mathbf{A}}$. Equivalent diagrams (28) give equivalent diagrams (29). Hence we get a map

$$\mathrm{Hom}_{\mathbf{B}/N_{\mathbf{B}}}(\overline{F}(a), b) \xrightarrow{\phi} \mathrm{Hom}_{\mathbf{A}/N_{\mathbf{A}}}(a, \overline{G}(b)).$$

Correspondingly we get a map ψ in the other direction by letting a diagram

$$
\begin{array}{ccccc}
& & d & & \\
& \swarrow^{s'} & & \searrow & \\
a & & & & G(b)
\end{array}
$$

with s' in the multiplicative system determined by $N_{\mathbf{A}}$ map to a diagram

$$
\begin{array}{ccccc}
& & F(d) & & \\
& \swarrow^{F(s')} & & \searrow & \\
F(a) & & & & b
\end{array}
$$

That $\psi \circ \phi$ is the identity follows by axiom (S3) in [13, Def. 1.6.1] of multiplicative systems, i.e. we can complete (29) to a diagram

$$
\begin{array}{ccccc}
& & d & & \\
& \swarrow^{t} & & \searrow & \\
a & & & & G(b) \\
& \searrow & & \swarrow^{G(s)} & \\
& & G(a) & &
\end{array}
$$

Similarly $\phi \circ \psi$ is the identity.

ii) Consider b in $\mathbf{B}$. We get a triangle

$$FG(b) \xrightarrow{\epsilon(b)} b \to E \to FG(b)[1],$$

where E is in $Z_{\mathbf{B}}$. We then get a triangle in $\mathbf{A}$

$$GFG(b) \xrightarrow{G(\epsilon(b))} G(b) \to G(E) \to GFG(b)[1].$$

From the adjunction between F and G there is also a natural transformation η

$$G(b) \xrightarrow{\eta(G(b))} GFG(b) \xrightarrow{G(\epsilon(b))} G(b),$$

and the composition here is an isomorphism. Since the first map is in the multiplicative system determined by $Z_{\mathbf{A}}$, the second map $G(\epsilon(b))$ is also. Hence $G(E)$ is in $Z_{\mathbf{A}}$. Therefore E is in $N_{\mathbf{B}}$ and so $\overline{F}\,\overline{G}(b) \to b$ is an isomorphism in $\mathbf{B}/N_{\mathbf{B}}$. Similarly $a \to \overline{G}\,\overline{F}(a)$ is an isomorphism in $\mathbf{A}/N_{\mathbf{A}}$. □

Letting $D(A^!, d))$ be the quotient category $\mathbf{A}/N_{\mathbf{A}}$ and $D(U)$ the quotient category $\mathbf{B}/N_{\mathbf{B}}$ we get the following.

Corollary 4.5. *The adjunction* (17) *descends to give an adjoint equivalence of categories*

$$D(A^!, d) \underset{\overline{G}}{\overset{\overline{F}}{\rightleftarrows}} D(U).$$

Remark 4.6. These are not in general derived categories. Confer Example 4.8. Rather they are "between" the homotopy category and the derived category.

4.5. **The Z-graded case.** Consider again the case $\Lambda = \mathbf{Z}$ with $\deg_{\mathbf{Z}} V = 1$ so the differential of $A^!$ is zero. In this case $\mathrm{Kom}(A)$ is the category of complexes of graded modules. We shall denote $D(U)$ above as $D^l(A)$ to make it clear that it depends on how the right adjoint functor G transforms objects of the homotopy category $K(A)$. (The l stands for left.) The category $\mathrm{Kom}(A^!, 0)$ or $\mathrm{Kom}_1(A^!)$ in the notation of Subsection 3.1 is now isomorphic to $\mathrm{Kom}_0(A^!)$. We may thus identify $D(A^!, d)$ above with a quotient of the homotopy category $K(A^!)$ and denote this by $D^r(A^!)$. It depends on how the left adjoint functor F transforms objects of the homotopy category $K(A^!)$.

Corollary 4.7. *There is an adjoint equivalence of categories*

$$D^r(A^!) \underset{\overline{G}}{\overset{\overline{F}}{\rightleftarrows}} D^l(A).$$

Also, interchanging A and $A^!$ there is an adjoint equivalence of categories

$$D^r(A) \underset{\overline{G}}{\overset{\overline{F}}{\rightleftarrows}} D^l(A^!).$$

(The latter functors are defined analogously to the former.)

Example 4.8. As said above these are not in general derived categories. Let $A^!$ be the exterior algebra $E = E(V^*)$. Then acyclic complexes of free modules are easily constructed as follows. Take any finitely generated graded module M over E and a minimal free resolution of it (which is infinite unless M is free). Since E is injective as a module over itself, we may also form an injective resolution of M consisting of free E-modules. Splicing these together gives an acyclic complex T of free E-modules. However neither $F(T)$ nor $G(T)$ is acyclic. Hence T is not zero in either $D^l(E)$ or $D^r(E)$. This example is closely related to the BGG-correspondence stemming from [3]; see [5] and [6] for more on this.

Remark 4.9. In [2] they show that for the adjunction (24), F and G take acyclic complexes to acyclic complexes, and so we get an adjoint equivalence of *derived* categories

$$D^{\downarrow}(A^!) \underset{\overline{G}}{\overset{\overline{F}}{\rightleftarrows}} D^{\uparrow}(A).$$

5. Complexes of free and cofree modules

We define categories of complexes of free U-modules and cofree dg-modules over $(A^!, d)$ and establish that the inclusion descended to quotient categories

$$DF(U) \xrightarrow{\bar{i}} D(U)$$

is an equivalence, and similarly in the case of cofree dg-modules. In order to get a better understanding of what subcategories we divide out by when we form the quotient categories, we describe the complexes in $DF(U)$ which are isomorphic to zero. We also show that if N is in $\mathrm{Kom}(U)$ and bounded above, then $G(N)$ is homotopic to a minimal cofree dg-module.

5.1. **Equivalences of categories of free and cofree complexes.** Let $\mathrm{KomF}(U)$ be the subcategory of $\mathrm{Kom}(U)$ such that each module in the complex has the form $U \otimes_k N$, where N is a vector space over k, and let $KF(U)$ be the corresponding homotopy category.

Similarly let $\mathrm{KomCoF}(A^!, d)$ be the full subcategory of $\mathrm{Kom}(A^!, d)$ consisting of modules which, forgetting the differential, have the form (using graded $\mathrm{Hom}_k(-,-)$)

$$\prod_{p \in \mathbf{Z}} \mathrm{Hom}_k(A^!, N^p(-p)),$$

where the N^p are vector spaces, and let $KCoF(A^!, d)$ be the corresponding homotopy category.

We can now form the quotient categories $DF(U)$ and $DCoF(A^!, d)$ of $KF(U)$ and $KCoF(A^!, d)$ respectively in exactly the same way that we form $D(U)$ and $D(A^!, d)$. We again abstract the situation as in Subsection 4.4 and consider a subcategory $\mathbf{B}_0 \overset{i}{\hookrightarrow} \mathbf{B}$ which in our case is the inclusion of $KF(U)$ into $K(U)$. Let $Z_0 = \mathbf{B}_0 \cap Z_{\mathbf{B}}$ and $N_0 = \mathbf{B}_0 \cap N_{\mathbf{B}}$. Suppose the $F(a)$ is in $\mathbf{B}_0$ for each a in $\mathbf{A}$, so we get adjoint functors

$$\mathbf{A} \underset{G}{\overset{F}{\rightleftarrows}} \mathbf{B}_0.$$

Proposition 5.1. *The functors*

$$\mathbf{B}_0/N_0 \underset{\overline{F}\,\overline{G}}{\overset{\overline{i}}{\rightleftarrows}} \mathbf{B}/N_{\mathbf{B}}$$

are both equivalences of categories.

Proof. Proposition 4.4 applies so we get an adjoint equivalence

$$\mathbf{A}/N_{\mathbf{A}} \underset{\overline{G}}{\overset{\overline{F}}{\rightleftarrows}} \mathbf{B}_0/N_0.$$

Hence $\mathbf{B}/N_{\mathbf{B}} \xrightarrow{\overline{F}\,\overline{G}} \mathbf{B}_0/N_0$ is an equivalence. Now the natural transformation η given by $FG(b) \xrightarrow{\eta(b)} b$ becomes an isomorphism on objects in $\mathbf{B}_0/N_0$. Hence the functor $\mathbf{B}_0/N_0 \xrightarrow{\overline{i}} \mathbf{B}/N_{\mathbf{B}}$ when composed with the equivalence $\overline{F}\,\overline{G}$ becomes an equivalence. Thus $\overline{i}$ becomes an equivalence. □

Similarly of course

$$\mathbf{A}_0/N_0 \underset{\overline{G}\,\overline{F}}{\overset{\overline{i}}{\rightleftarrows}} \mathbf{A}/N_{\mathbf{A}}$$

are equivalences of categories.

Corollary 5.2. *The functors*

$$DF(U) \underset{\overline{F}\,\overline{G}}{\overset{\overline{i}}{\rightleftarrows}} D(U), \quad DCoF(A^!, d) \underset{\overline{G}\,\overline{F}}{\overset{\overline{i}}{\rightleftarrows}} D(A^!, d)$$

all give equivalences of categories.

5.2. **Complexes isomorphic to zero.** We would like to give a transparent description for when a complex is in the null system in $KF(U)$ or $KCoF(A^!, d)$ that we divide out by, that is for complexes in $DF(U)$ and $DCoF(A^!, d)$ to be isomorphic to zero. This is achieved in Corollaries 5.6 and 5.9.

Lemma 5.3. *I in $Kom(A^!, d)$ is nullhomotopic iff $Hom_{A^!}(M, I)$ is acyclic for all M in $Kom(A^!, d)$.*

Proof. If $\mathrm{Hom}_{A^!}(I, I)$ is acyclic, the identity map in $\mathrm{Hom}^0_{A^!}(I, I)$ is in the image of $\mathrm{Hom}^{-1}_{A^!}(I, I)$, and hence I is nullhomotopic.

Conversely, if I is nullhomotopic, let $I \xrightarrow{s} I(-1)$ be a homotopy. Then a cycle z in $\mathrm{Hom}^p_{A^!}(M, I)$ is the image of $s \circ z$ in $\mathrm{Hom}^{p-1}_{A^!}(M, I)$, as an easy calculation shows. □

Lemma 5.4. *Let I in $KomCoF(A^!, d)$ be (forgetting the differential)*

$$I = \prod_{p>0} Hom_k(A^!, N^p(-p)).$$

Then $H^p F(I)$ is zero for $p \leq 0$.

Proof. There is a sequence

$$0 \to F_{i-1}(I) \to F_i(I) \to F_i(I)/F_{i-1}(I) \to 0.$$

Now we can easily verify from (13) that $F_i(I)/F_{i-1}(I)$ is a direct product of homogeneous parts of the Koszul complex for A and $A^!$

$$A_0 \otimes (A^!_{i+p})^* \otimes N^p \to \cdots \to A_{i+p} \otimes (A^!_0)^* \otimes N^p$$

taking the product over all p greater than or equal to maximum of $-i$ and 1. Hence $F_i(I)$ is acyclic in all cohomological degrees ≤ 0. Since $F(I)$ is $\varinjlim F_i(I)$ we get $F(I)$ exact in cohomological degrees ≤ 0. □

If I is in $\mathrm{KomCoF}(A^!, d)$ we may form a truncated complex $t^{\leq p} I$. Let K^p be the kernel of N^p in the complex $\mathrm{Hom}_{A^!}(k, I)$. Then as a graded module

$$t^{\leq p} I = \mathrm{Hom}_k(A^!, K^p(-p)) \bigoplus \prod_{i<p} \mathrm{Hom}_k(A^!, N^i(-i)).$$

It is easily verified that this becomes a subcomplex of I. Also let $t_{>p} I$ be the cokernel of the inclusion $t^{\leq p} I \hookrightarrow I$.

Proposition 5.5. *Let I be a bounded above complex in $KomCoF(A^!, d)$. The following are equivalent:*

i. *I is nullhomotopic.*
ii. *$Hom_{A^!}(k, I)$ is acyclic.*
iii. *$F(I)$ is acyclic.*

Proof. Clearly i implies ii and iii. We next show that ii implies i. Let I be $\prod_{q \leq q_0} \mathrm{Hom}_k(A^!, N^q(-q))$ such that $\mathrm{Hom}_{A^!}(k, I)$ is an acyclic complex

$$\cdots \to N^q \xrightarrow{d^q} N^{q+1} \to \cdots \to N^{q_0}.$$

Then $\mathrm{Hom}_{A^!}(M, I)$ is the total direct product complex of a double complex with terms $\mathrm{Hom}_k(M^p, N^q)$. The vertical differentials are given by

$$\cdots \to \mathrm{Hom}_k(M^p, N^q) \xrightarrow{\mathrm{Hom}_k(M^p, d^q)} \mathrm{Hom}_k(M^p, N^{q+1}) \to \cdots ,$$

so the columns are bounded above and exact. Then by [23, Lemma 2.7.3] the total direct product complex is acyclic and we conclude by Lemma 5.3.

We now show that iii implies ii. Assume $\mathrm{Hom}_{A^!}(k, I)$ is not acyclic with non-zero cohomology in maximal degree p which we assume to be 0. By the sequence

$$0 \to t^{\leq 0}I \to I \to t_{>0}I \to 0,$$

we get $\mathrm{Hom}_{A^!}(k, t_{>0}I)$ acyclic. Hence $t_{>0}I$ is nullhomotopic and $F(t^{\leq 0}I)$ and $F(I)$ have isomorphic cohomology. Replacing I with $t^{\leq 0}I$, we may assume that $\mathrm{Hom}_{A^!}(k, I)$ has non-zero cohomology in degree 0, that I^p is zero for $p > 0$, and we want to show that $F(I)$ is not acyclic. Now $N^{-1} \xrightarrow{d^{-1}} N^0$ is not surjective. But then it is easily checked explicitly by (13) that $F(I)$ has non-zero cohomology in degree 0. □

Corollary 5.6. *Let I be any complex in KomCoF$(A^!, d)$.*

i. $H^p F(I) = H^p \mathit{Hom}_{A^!}(k, I)$.
ii. *I is in $N_{K(A^!,d)}$ iff I and $\mathit{Hom}_{A^!}(k, I)$ are acyclic.*

Proof. ii follows from i. To prove i, the triangle

$$I \to GF(I) \to E \to I(1)$$

in KCoF$(A^!, d)$ gives a triangle

$$\mathrm{Hom}_{A^!}(k, I) \to \mathrm{Hom}_{A^!}(k, GF(I)) \to \mathrm{Hom}_{A^!}(k, E) \to \mathrm{Hom}_{A^!}(k, I)[1].$$

Since $\mathrm{Hom}_{A^!}(k, GF(I))$ is equal to $\mathrm{Hom}_U(F(k), F(I))$ which is $F(I)$, it will be enough to show that $\mathrm{Hom}_{A^!}(k, E)$ is acyclic. Replace E with I, now assumed to be in $N_{K(A^!,d)}$. We will show that $\mathrm{Hom}_{A^!}(k, I)$ is acyclic, so assume to the contrary that it has non-zero cohomology in degree p which we assume to be 0. Consider

$$0 \to t^{\leq 0}I \to I \to t_{>0}I \to 0.$$

Now since $\mathrm{Hom}_{A^!}(k, t^{\leq 0}I)$ has non-zero cohomology, by Proposition 5.5, $F(t^{\leq 0}I)$ is not acyclic and so has non-zero cohomology in some degree ≤ 0. Since $F(I)$ is acyclic, $F(t_{>0}I)$ gets non-zero cohomology in some degree ≤ -1. But this contradicts Lemma 5.4. □

For later use we note the following.

Proposition 5.7. *Suppose that $A^!_d$ is zero for $d \gg 0$ and I is any complex in KomCoF$(A^!, d)$. Then I is nullhomotopic iff $\mathit{Hom}_{A^!}(k, I)$ is acyclic.*

Proof. When $A^!_d$ is zero for $d \gg 0$, then $I = \varinjlim t^{\leq p}I$. Hence

$$\mathrm{Hom}_{A^!}(I, I) = \varprojlim \mathrm{Hom}_{A^!}(t^{\leq p}I, I).$$

Since all $t^{\leq p}I$ are nullhomotopic, $\mathrm{Hom}_{A^!}(I, I)$ will be acylic and hence I is nullhomotopic. □

There are analogs of the above theorems for complexes of free modules over U. The following may be proved in an analogous way to Proposition 5.5.

Proposition 5.8. *Suppose P in KomF(U) is bounded above. The following are equivalent:*

i. *P is nullhomotopic.*
ii. *$k \otimes_U P$ is acyclic.*
iii. *$G(P)$ is acyclic.*

Corollary 5.9. *Let P be any complex in KomF(U).*

a. $H^p G(P) = H^p(k \otimes_U P)$.
b. *P is in $N_{K(U)}$ iff P and $k \otimes_U P$ are acyclic.*

Proof. This is analogous to Corollary 5.6. In order to prove the analogue of Lemma 5.4 one should use that P is $\varprojlim \sigma^{\leq p} P$ and that G commutes with inverse limits. □

5.3. **Equivalences of homotopy categories.** We may now pose the following.

Problem 5.10. *If P in KomF(U) is in $N_{K(U)}$, i.e. P and $k \otimes_U P$ are acyclic, is P then nullhomotopic?*

(One may of course ask a similar question for I in KomCoF$(A^!, d)$.) Note that when this has a positive answer, then $DF(U)$ will be the homotopy category $KF(U)$ and by Corollary 5.2 becomes an excellent category for doing homological algebra of right exact functors on U-modules.

This problem clearly has a positive answer when U has finite global dimension. In this case P acyclic suffices to make it nullhomotopic.

By Proposition 5.7 we now get the following.

Proposition 5.11. *If U has finite global dimension and $A^!_d$ is zero for $d \gg 0$, then there is an equivalence of homotopy categories*

$$KF(U) \cong KCoF(A^!, d).$$

Remark 5.12. One may show that these conditions on U and $A^!$ are equivalent.

In the **Z**-graded case we get the following.

Corollary 5.13. *There is an equivalence of homotopy categories of complexes of graded modules*

$$KF(S(V)) \cong KCoF(E(V^*)).$$

5.4. **Minimal versions.**

Lemma 5.14. a. *If $I_1 \to I_2$ is a map of bounded above complexes in KomCoF$(A^!, d)$ which is a quasi-isomorphism after applying Hom$_{A^!}(k, -)$, then it is a homotopy equivalence.*

b. *If R is a bounded above graded $A^!$-module (forget differentials) such that Ext$^1_{A^!}(k, R)$ is zero, then R is of the form*

$$\prod_{p \leq p_0} Hom_k(A^!, N^p(-p)).$$

Proof. a. Apply Proposition 5.5 to the cone of $I_1 \to I_2$.

b. Let $\mathrm{Hom}_{A^!}(k, R)$ be $\bigoplus_{p \leq p_0} N^p(-p)$. There is an injective map

$$R \to \prod_{p \leq p_0} \mathrm{Hom}_k(A^!, N^p(-p)).$$

Let Q be the cokernel. Then Q is a bounded above $A^!$-module with $\mathrm{Hom}_{A^!}(k, Q)$ zero. But then Q must be zero. □

The following gives the existence of minimal versions of the complexes $G(M)$.

Theorem 5.15. *Assume M in $Kom(U)$ is bounded above. Then there exists a homotopy equivalence $G(M) \to I$, where I is in $KomCoF(A^!, d)$ such that $Hom_{A^!}(k, I)$ is the complex with zero differential*

$$\cdots \to H^{p-1}(M) \xrightarrow{0} H^p(M) \xrightarrow{0} H^{p+1}(M) \to \cdots.$$

Proof. There are short exact sequences of complexes

$$0 \to K^pM \to \tau_{\geq p}M \to \tau_{\geq p+1}M \to 0,$$

where $\tau_{\geq p}M$ is the truncation of M and K^pM is the kernel complex

$$0 \to \operatorname{im} d^{p-1} \to \ker d^p \to 0$$

which is quasi-isomorphic to $H^p(M)$. We get a sequence

$$0 \to G(K^pM) \to G(\tau_{\geq p}M) \to G(\tau_{\geq p+1}M) \to 0.$$

Now assume by induction that there is a homotopy equivalence

$$G(\tau_{\geq p+1}M) \xrightarrow{\eta} I_{p+1}$$

with $\mathrm{Hom}_{A^!}(k, I_{p+1})$ having zero differential. Since $\mathrm{Hom}_{A^!}(k, \eta)$ is a surjective map, η must be surjective because of the form of $G(\tau_{\geq p+1}M)$. Let R_p be the kernel of the composite surjection

$$G(\tau_{\geq p}M) \to G(\tau_{\geq p+1}M) \to I_{p+1}.$$

Then $\mathrm{Ext}^1_{A^!}(k, R_p)$ is zero and so by Lemma 5.14, R_p is in $\mathrm{KomCoF}(A^!, d)$. Also the map $G(K^pM) \to R_p$ is a quasi-isomorphism after applying $\mathrm{Hom}_{A^!}(k, -)$. Hence it is a homotopy equivalence. Composing its inverse with $G(K^pM) \to G(H^p(M))$ we get a pushout diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & R_p & \longrightarrow & G(\tau_{\geq p}M) & \longrightarrow & I_{p+1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \| \\
0 & \longrightarrow & G(H^pM) & \longrightarrow & I_p & \longrightarrow & I_{p+1} & \longrightarrow & 0
\end{array}$$

Applying $\mathrm{Hom}_{A^!}(k, -)$ we see that $\mathrm{Hom}_{A^!}(k, I_p)$ also has zero differential and the middle vertical map is a quasi-isomorphism after applying $\mathrm{Hom}_{A^!}(k, -)$. We now get diagrams

$$\begin{array}{ccc}
G(\tau_{\geq p}M) & \longrightarrow & G(\tau_{\geq p+1}M) \\
\downarrow & & \downarrow \\
I_p & \longrightarrow & I_{p+1}
\end{array}$$

Taking inverse limits we get a map $G(M) \to I$ which becomes a quasi-isomorphism when applying $\mathrm{Hom}_{A^!}(k, -)$. This proves the theorem. □

5.5. **t-structures.** For a ring R the *derived* category $Der(R)$ comes along with a standard t-structure (by truncating complexes), inducing the cohomology functor on complexes. In the same way our category $D(U)$ comes with a t-structure. However the category $D(A^!, d)$ does not have this standard t-structure. However being equivalent to $D(U)$ it gets transported a t-structure from $D(U)$ which we shall describe. Recall [13] that a t-structure on a triangulated category D consists of two full subcategories $D^{\leq 0}$ and $D^{\geq 0}$ such that the following axioms hold:

1. $D^{\leq -1} = D^{\leq 0}[1] \subseteq D^{\leq 0}$ and $D^{\geq 1} = D^{\geq 0}[-1] \subseteq D^{\geq 0}$.
2. For X in $D^{\leq 0}$ and Y in $D^{\geq 1}$ we have $\mathrm{Hom}_D(X, Y)$ zero.

3. For each X in D there is a triangle $X' \to X \to X'' \to X'[1]$, where X' is in $D^{\leq 0}$ and X'' is in $D^{\geq 1}$.

In the case of D being $DCoF(A^!, d)$ let $D^{\leq 0}$ consist of complexes I isomorphic to (forgetting the differential)

$$\prod_{p \leq 0} \mathrm{Hom}_k(A^!, N^p(-p)) \tag{30}$$

and $D^{\geq 0}$ consisting of complexes isomorphic to

$$\prod_{p \geq 0} \mathrm{Hom}_k(A^!, N^p(-p)). \tag{31}$$

Proposition 5.16. *$(D^{\leq 0}, D^{\geq 0})$ gives a t-structure on $DCoF(A^!, d)$.*

Proof. Axiom 1 clearly holds. Axiom 3 follows by the exact sequence

$$0 \to t^{\leq 0}X \to X \to t_{>0}X \to 0$$

which induces a triangle. Now $t_{>0}X$ is a product like (31), and we would like to have a product over $p \geq 1$. However using Proposition 5.5ii it is easily shown that $t_{>0}X$ is actually isomorphic to a product like (31) over $p \geq 1$.

For axiom 2 suppose we are given a diagram

$$\begin{array}{ccccc} & & Z & & \\ & \overset{s}{\swarrow} & & \searrow & \\ X & & & & Y \end{array}$$

in KomCoF$(A^!, d)$ where s is in the multiplicative system defined by the null system. We may assume X has the form (30) and Y has the form (31) for $p \geq 1$. It will be sufficient to prove that if Z is in $D^{\leq 0}$, then

$$t^{\leq 0}Z \hookrightarrow Z \tag{32}$$

is an isomorphism.

Now suppose $Z \xrightarrow{s} X$ is an isomorphism where X as in (30). Then the cone $C(s)$ is in $N_{K(A^!,d)}$ and by Corollary 5.6, $\mathrm{Hom}_{A^!}(k, C(s))$ is acyclic. Hence $\mathrm{Hom}_{A^!}(k, -)$ is a quasi-isomorphism when applied to s. Then it is also a quasi-isomorphism when applied to

$$t^{\leq 0}Z \to t^{\leq 0}X = X. \tag{33}$$

But then $\mathrm{Hom}_{A^!}(k, -)$ applied to its cone is acyclic, and its cone is therefore null-homotopic by Proposition 5.5. Thus (33) becomes an isomorphism and therefore also (32). □

Proposition 5.17. *In the equivalence*

$$DCoF(A^!, d) \underset{\overline{G}}{\overset{\overline{F}}{\rightleftarrows}} D(U)$$

the t-structure on the left side given by Proposition 5.16 *corresponds to the standard t-structure on the right side.*

Proof. It is clear that G takes $D^{\leq 0}(U)$ to $D^{\leq 0}$ and $D^{\geq 0}(U)$ to $D^{\geq 0}$. □

In the $\mathbf{Z}$-graded case $D^r(A^!)$ now comes equipped with two t-structures (and hence two cohomological functors ${}^{I}H$ and ${}^{II}H$). One non-standard t-structure from the isomorphism of $D^r(A^!)$ with $D(A^!, 0)$, which is equivalent to $DCoF(A^!, 0)$, and it also has the standard t-structure by truncation. Similarly $D^l(A)$ has two t-structures and via the equivalence

$$D^r(A^!) \underset{\overline{G}}{\overset{\overline{F}}{\rightleftarrows}} D^l(A)$$

the standard t-structure on one corresponds to the non-standard on the other. It is noteworthy that a complex X in $D^r(A^!)$ is zero iff all cohomology groups ${}^{I}H^p(X)$ and ${}^{II}H^p(X)$ vanish.

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