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EXISTENCE AND REGULARITY OF ISOMETRIES

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ABSTRACT. We use local harmonic coordinates to establish sharp results on the regularity of isometric maps between Riemannian manifolds whose metric tensors have limited regularity (e.g., are Hölder continuous). We also discuss the issue of local flatness and of local isometric embedding with given first and second fundamental form, in the context of limited smoothness.

1. INTRODUCTION

We discuss some very classical topics in differential geometry, in particular:

(I) Is a distance-preserving map actually a smooth, metric-tensor preserving diffeomorphism?

(II) When can you say a Riemannian manifold is locally isometric to flat Euclidean space?

(III) Are the Gauss-Codazzi equations sufficient for isometric embedding with given first and second fundamental form?

Affirmative answers, in the case where the metric tensors are all smooth, are available in many texts, but precise results become a little more subtle when the metric tensors (and second fundamental form, in question (III)) have only a limited degree of regularity.

Precise answers have been worked out for C^k metric tensors, particularly in the papers [HW1] and [CH] by P. Hartman and his collaborators, A. Wintner and E. Calabi. These papers also have results for Hölder continuous metric tensors, and for metric tensors whose kth derivatives are Hölder continuous.

Here we put forward an alternative method of establishing these results, making use of well-known results on the existence and regularity of harmonic coordinates. This allows for fairly short and simple demonstrations of such regularity results. Furthermore, use of harmonic coordinates allows one to obtain variants of these results, in other function spaces, that do not seem so accessible by the older techniques. The use of harmonic coordinates to establish regularity of geometrical objects was pioneered in [DTK]. It is interesting that Hartman and Wintner themselves proved some important results about harmonic coordinates, but somehow the idea to make use of them in this way had to await [DTK].

While this work is in part a commentary on the papers of Hartman and collaborators mentioned above, its significance as an alternative approach has been highlighted by recent work of A. Lytchak and A. Yaman [LY]. It is shown in [LY]

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that a key lemma of [CH], regarding the regularity of geodesics given a Hölder continuous metric tensor, is in error, and that the corrected version of this result would yield a less than optimal answer to question (I).

We discuss questions (I)–(III) in succession, in §§2–4. Section 5 considers the phenomenon that harmonic coordinates on a submanifold of class C^2 in $\mathbb{R}^n$ are a little smoother than one might think, and also makes a remark on the geodesic flow on such a submanifold.

We make extensive use of the function spaces $C^r(\Omega)$. For $r = k \in \mathbb{Z}^+$, this space consists of functions whose kth order derivatives are continuous; for $r = k + \sigma,\ k \in \mathbb{Z}^+,\ 0 < \sigma < 1$, $C^r(\Omega)$ consists of functions whose kth order derivatives are Hölder continuous, of exponent σ. We also make occasional reference to L^p-Sobolev spaces $H^{s,p}(\Omega)$, to Zygmund spaces $C^r_*(\Omega)$, and to bmo. Material on these function spaces can be found in [T1] and [T2], amongst many other places.

2. Regularity of isometries

Our goal in this section is to prove the following result. As mentioned in the Introduction, different techniques were applied to this problem in [CH].

Theorem 2.1. *Let $\mathcal{O}$, Ω be open in $\mathbb{R}^n$ and carry metric tensors $g = (g_{jk})$ and $h = (h_{jk})$, respectively. Assume $r > 0$ and $g_{jk} \in C^r(\mathcal{O})$, $h_{jk} \in C^r(\Omega)$. Let $\varphi : \mathcal{O} \to \Omega$. The following are equivalent:*

$$\varphi \ \text{ is a distance-preserving homeomorphism,} \tag{2.1}$$

$$\varphi \ \text{ is bi-Lipschitz and } \varphi^* h(x) = g(x), \ \text{ for a.e. } x \in \mathcal{O}, \tag{2.2}$$

$$\varphi \ \text{ is a } C^1 \text{ diffeomorphism and } \varphi^* h = g, \tag{2.3}$$

$$\varphi \ \text{ is a diffeomorphism of class } C^{r+1} \text{ and } \varphi^* h = g. \tag{2.4}$$

Proof. First assume (2.1) holds. Even given g_{jk}, h_{jk} continuous, this implies φ is bi-Lipschitz (with respect to the Euclidean distance). Thus Rademacher's theorem implies φ is differentiable almost everywhere, with derivative $D\varphi(x) \in \mathrm{End}(\mathbb{R}^n)$. Since φ maps Lipschitz curves in $\mathcal{O}$ to Lipschitz curves in Ω of the same length, it follows that $D\varphi(x)$ is a linear isometry from $(\mathbb{R}^n, g(x))$ to $(\mathbb{R}^n, h(\varphi(x)))$, for almost all $x \in \mathcal{O}$. Hence (2.1) $\Rightarrow$ (2.2).

Next assume (2.2) holds. Given $p \in \Omega$, pick a local harmonic coordinate system $(u_1, \dots, u_n)$ on a neighborhood of p (which we rename Ω). The hypothesis $h_{jk} \in C^r(\Omega)$ yields

$$u_i \in C^{1+r}_*(\Omega), \quad 1 \le i \le n. \tag{2.5}$$

We also have $v_i = u_i \circ \varphi \in \mathrm{Lip}(\mathcal{O})$. We claim each v_i is weakly harmonic with respect to the metric tensor g, i.e., we claim

$$\int_{\mathcal{O}} \langle df, dv_i \rangle \, dV_{\mathcal{O}} = 0, \quad \forall\ f \in C_0^\infty(\mathcal{O}). \tag{2.6}$$

Written out in coordinates, the left side of (2.6) is equal to

$$\int_{\mathcal{O}} \partial_j f(x)\, \partial_k v_i(x)\, g^{jk}(x) \sqrt{g(x)}\, dx_1 \cdots dx_n. \tag{2.7}$$

The change of variable formula for integrals is valid for bi-Lipschitz maps, and since $\varphi^* h = g$ a.e., we have that (2.7) is equal to

$$\int_\Omega \partial_j (f \circ \varphi^{-1})(y)\, \partial_k u_i(y)\, h^{jk}(y)\, \sqrt{h(y)}\, dy_1 \cdots dy_n, \tag{2.8}$$

which vanishes for all $f \circ \varphi^{-1}$, Lipschitz on Ω and compactly supported. Thus each function v_i is weakly harmonic on $\mathcal{O}$, and hence the hypothesis $g_{jk} \in C^r(\mathcal{O})$ gives

$$v_i \in C^{1+r}_*(\mathcal{O}), \quad 1 \le i \le n. \tag{2.9}$$

Now, with $u = (u_1, \dots, u_n)$ and $v = (v_1, \dots, v_n)$, we have

$$\varphi = u^{-1} \circ v \in C^{1+r}(\mathcal{O}), \tag{2.10}$$

at least as long as $r \in (0, \infty) \setminus \mathbb{N}$. Interchanging the roles of $\mathcal{O}$ and Ω gives the same result for φ^{-1}. (That (2.2) is preserved under this interchange follows from the validity a.e. of the chain rule for the composition of two bi-Lipschitz maps.)

This shows that (2.2) $\Rightarrow$ (2.3), and we even already have (2.2) $\Rightarrow$ (2.4), except for $r \in \mathbb{N} = \{1, 2, 3, \dots\}$. We mention that standard elliptic results do yield

$$\varphi \in H^{1+r,p}(\mathcal{O}), \quad \forall\, p < \infty, \tag{2.11}$$

when $r = k \in \mathbb{N}$ in the hypotheses of Theorem 2.1.

Following [CH], we finish the proof that (2.2) $\Rightarrow$ (2.4) when $r \in \mathbb{N}$ via the following device. Let

$$\Gamma^a{}_{bj} = \frac{1}{2} g^{a\ell} \Big(\frac{\partial g_{b\ell}}{\partial x_j} + \frac{\partial g_{\ell j}}{\partial x_b} - \frac{\partial g_{bj}}{\partial x_\ell} \Big) \tag{2.12}$$

denote the connection coefficients of (g_{jk}) and $\widetilde{\Gamma}^a{}_{bj}$ denote those of (h_{jk}). Then the standard formula relating these under a change of variable is

$$\Gamma^k{}_{ij}(x) \frac{\partial \varphi_m}{\partial x_k} - \widetilde{\Gamma}^m{}_{k\ell} \frac{\partial \varphi_k}{\partial x_i} \frac{\partial \varphi_\ell}{\partial x_j} = \frac{\partial^2 \varphi_m}{\partial x_i \partial x_j}. \tag{2.13}$$

This is typically demonstrated under the hypothesis that $g_{jk}, h_{jk} \in C^1$ and $\varphi \in C^2$, but the argument extends readily to the case $\varphi \in C^1 \cap H^{2,p}$. Now under our current hypotheses we certainly have the left side of (2.13) in $C^{r-1}(\mathcal{O})$, hence the right side belongs to $C^{r-1}(\mathcal{O})$. This gives (2.4).

Finally, it is clear that (2.4) $\Rightarrow$ (2.1), so the proof of Theorem 2.1 is complete.

In the case $r = 0$, the argument above breaks down, since the Schauder estimates do not produce C^1 harmonic functions. In this connection we mention that [HW2] gives an example of a continuous metric tensor for which there are no local C^1 harmonic coordinates. Furthermore, [CH] produces an example of continuous metric tensors and a map satisfying (2.1) but not (2.3).

It is shown in [CH] that (2.3) follows from (2.1), provided g_{jk} and h_{jk} satisfy a Dini condition. An extension of the harmonic coordinate argument given above also establishes such a conclusion. Given h_{jk} Dini continuous, there exist C^1 harmonic coordinates $(u_1, \dots, u_n)$. See [HW3] for a proof (at least in dimension 2); see also Proposition III.9.2 of [T2] for a related result. Then the argument above shows that $v_i = u_i \circ \varphi$ are g-harmonic, and hence $v_i \in C^1$ if also g_{jk} are Dini continuous, giving $\varphi = u^{-1} \circ v \in C^1$.

A number of texts present a weak version of Theorem 2.1, due to [MS]. The proof there makes use of "distance coordinates," which are derived from a collection of exponential coordinates. For these coordinates to be C^1, one needs $g_{jk} \in C^2$. Actually, [MS] mistakenly asserted that their argument works for C^1 metric tensors; this lapse was pointed out in [CH].

3. Flat metrics

Let $\Omega \subset \mathbb{R}^n$ be open and carry a metric tensor (g_{jk}). A classical question is: how to tell if there is a (local) isometric map

$$\varphi : (\Omega, g_{jk}) \to (\mathcal{O}, \delta_{jk}), \tag{3.1}$$

with $\mathcal{O} \subset \mathbb{R}^n$ open and δ_{jk} the standard Euclidean metric. The Riemann tensor was constructed to study this question. We recall that the Riemann tensor is constructed from the connection 1-form $\Gamma = \sum \Gamma^a{}_{bj}\, dx_j$, with coefficients given by (2.12), as

$$\mathcal{R} = d\Gamma + \Gamma \wedge \Gamma. \tag{3.2}$$

This represents $\mathcal{R}$ as a matrix-valued 2-form, with components $R^a{}_{bjk}$. A necessary condition for a (sufficiently smooth) isometry of the form (3.1) to exist is that $R^a{}_{bjk} \equiv 0$. The converse is also true. Well-known ODE techniques (involving Frobenius' theorem) give the following.

Theorem 3.1. *If g_{jk} is a C^∞ metric tensor and $R^a{}_{bjk} \equiv 0$, then there exists a (local) C^∞ isometry of the form* (3.1).

If we assume g_{jk} has some finite regularity, one could keep careful track of the steps in this ODE argument, but that is not what we want to do, although we do want to establish some flatness results when g_{jk} is fairly rough. In particular, we will establish the following.

Proposition 3.2. *Assume g_{jk} is a metric tensor of class C^σ, $\sigma \geq 1$, and $R^a{}_{bjk} \equiv 0$. Then there exists a (local) isometry of the form* (3.1), *and $\varphi \in C^{\sigma+1}$.*

This was proven, by different means, in [CL], when $\sigma \in \mathbb{N}$ and $\sigma \geq 2$. Let us explain how the Riemann tensor is defined if $g_{jk} \in C^\sigma$ and $\sigma \in [1, 2)$. In fact, $\mathcal{R}$ is well defined even under the weaker hypothesis

$$g_{jk} \in C(\Omega) \cap H^{1,2}(\Omega). \tag{3.3}$$

In such a case (2.12) gives $\Gamma^a{}_{bj} \in L^2(\Omega)$, and then (3.2) gives

$$R^a{}_{bjk} \in H^{-1,2}(\Omega) + L^1(\Omega). \tag{3.4}$$

Prior to proving Proposition 3.2, we will obtain a flatness result under a hypothesis just slightly stronger than (3.3).

Proposition 3.3. *Let g_{jk} be a metric tensor on Ω, satisfying*

$$g_{jk} \in C^\sigma(\Omega) \cap H^{1,2}(\Omega), \quad \sigma \in (0, 1). \tag{3.5}$$

Assume $R^a{}_{bjk} \equiv 0$. Then there exists a (local) isometry of the form (3.1), *and $\varphi \in C^{\sigma+1}$.*

To prove Proposition 3.3, we first take local harmonic coordinates $(u_1, \dots, u_n)$ in a neighborhood of a point $p \in \Omega$. These exist as long as $\sigma > 0$, and we have

$$u_j \in C^{1+\sigma} \cap H^{2,2}, \quad 1 \le j \le n, \tag{3.6}$$

as long as $\sigma \in (0,1)$ (cf. Proposition 9.4 in Chapter 3 of [T2]). Denote by $\tilde{g}_{jk}$ the components of the metric tensor in these harmonic coordinates. A priori we have $\tilde{g}_{jk} \in C^\sigma \cap H^{1,2}$, but in fact these functions are much more regular than that, as we now demonstrate.

This uses the method put forward in [DTK]. In harmonic coordinates the Ricci tensor takes the form

$$\Delta \tilde{g}_{\ell m} + B_{\ell m}(\tilde{g}_{jk}, \nabla \tilde{g}_{jk}) = \mathrm{Ric}_{\ell m}\,. \tag{3.7}$$

Here Δ is the Laplace-Beltrami operator (for the metric tensor $\tilde{g}$), applied componentwise to the components $\tilde{g}_{\ell m}$, and $B_{\ell m}$ is a quadratic form in $\nabla \tilde{g}_{jk}$. Our hypothesis implies that the right side of (3.7) *vanishes*. Hence regularity results for this elliptic PDE (cf. Proposition 12B.2 in Chapter 14 of [T1]) yield

$$\tilde{g}_{\ell m} \in C^\infty, \quad 1 \le \ell, m \le n. \tag{3.8}$$

(If one assumes $g_{jk} \in H^{1,p},\ p > n$, then one has first that $\tilde{g}_{jk} \in H^{1,p}$ in (3.7), and then simpler elliptic regularity arguments apply, such as Proposition 4.10 in Chapter 14 of [T1], to yield (3.8).)

Now Theorem 3.1 applies to $(\widetilde{\Omega}, \tilde{g}_{jk})$, yielding a C^∞ isometric map

$$\widetilde{\varphi} : (\widetilde{\Omega}, \tilde{g}_{jk}) \longrightarrow (\mathcal{O}, \delta_{jk}). \tag{3.9}$$

With $u = (u_1, \dots, u_n) : \Omega \to \widetilde{\Omega}$, we then have the desired isometry

$$\varphi = \widetilde{\varphi} \circ u : (\Omega, g_{jk}) \longrightarrow (\mathcal{O}, \delta_{jk}), \tag{3.10}$$

and (3.6) plus (3.8) imply $\varphi \in C^{\sigma+1}$, so this proves Proposition 3.3.

Now we can prove Proposition 3.2. The desired isometry has just been obtained, and its stated regularlty, $\varphi \in C^{\sigma+1}$ (given $g_{jk} \in C^\sigma,\ \sigma \ge 1$), follows from Theorem 2.1.

4. Isometric immersions onto hypersurfaces

Let $\Omega \subset \mathbb{R}^n$ be an open set with metric tensor (g_{jk}). Let (s_{jk}) be a symmetric tensor field on Ω. A classical problem is to determine when there is a (local) isometric immersion $\varphi : \Omega \to \mathbb{R}^{n+1}$, giving an n-dimensional surface $M \subset \mathbb{R}^{n+1}$ with metric tensor (g_{jk}) and second fundamental form (s_{jk}). As is well known, if $\varphi \in C^r,\ r \ge 3$, and if these tensor fields are the metric tensor and second fundamental form, then $g_{jk} \in C^{r-1},\ s_{jk} \in C^{r-2}$, and the classical Gauss-Codazzi equations must hold. In view of our previous remarks on curvature, it is clear that this extends to $r \ge 2$, and even a bit further. Here we discuss the converse. We recover some results of [HW1]. We use a pretty approach taken in [CL]. Our results are more general and precise than those of [CL], but we do not obtain all the results of [HW1]. For simplicity we concentrate on local results, though the setup is well suited for applying familiar rigidity techniques to pass to global results, under appropriate hypotheses.

The approach of [CL] begins with the following observation. Say $\varphi : \Omega \to \mathbb{R}^{n+1}$ is an isometric immersion of class C^r; let N denote the unit normal to $M = \varphi(\Omega)$.

Then define

$$\psi : \Omega \times (-\varepsilon, \varepsilon) \longrightarrow \mathbb{R}^{n+1}, \quad \psi(x,y) = \varphi(x) + yN(x). \tag{4.1}$$

This is a local diffeomorphism for small $\varepsilon > 0$ (maybe upon shrinking Ω), and the flat Euclidean metric tensor of $\mathbb{R}^{n+1}$ pulls back to (G_{jk}), defined by

$$G_{jk}(x,y) = g_{jk}(x) - 2ys_{jk}(x) + y^2 g^{\ell m}(x) s_{\ell j}(x) s_{mk}(x), \tag{4.2}$$

for $1 \le j,k \le n$, and

$$G_{j,n+1}(x,y) = \delta_{j,n+1}. \tag{4.3}$$

Now turn this around. Given the metric tensor (g_{jk}) and the symmetric tensor field (s_{jk}) on Ω, form G_{jk} using (4.2)–(4.3); this will be a metric tensor on $\Omega \times (-\varepsilon, \varepsilon)$ (maybe upon shrinking Ω). If $g_{jk},\ s_{jk} \in C^\sigma$, then $G_{jk} \in C^\sigma$. As long as $\sigma \ge 1$, then the Riemann tensor $\mathcal{R}$ of (G_{jk}) is well defined, as discussed in §3. Furthermore (as noted in [CL]), the Gauss-Codazzi equations for (g_{jk}, s_{jk}) are equivalent to the vanishing of the Riemann tensor $\mathcal{R}$. Thus results of §3 yield the following.

Proposition 4.1. *Let (g_{jk}) be a metric tensor on $\Omega \subset \mathbb{R}^n$, (s_{jk}) a symmetric tensor field, and assume $g_{jk}, s_{jk} \in C^\sigma,\ \sigma \ge 1$. If g_{jk}, s_{jk} satisfy the Gauss-Codazzi equations, then there is a (local) isometric immersion*

$$\varphi : \Omega \longrightarrow \mathbb{R}^{n+1}, \quad \varphi \in C^{\sigma+1}, \tag{4.4}$$

for which (g_{jk}) and (s_{jk}) are the first and second fundamental forms.

Such a result was established in [CL], though in that paper σ was restricted to $\{2,3,4,\dots\}$. This result is weaker than that of [HW1] in several respects, some of which we address below. In preparation for improving Proposition 4.1, we draw some further conclusions from what we have so far.

If $N(x)$ denotes the unit normal to $M = \varphi(\Omega)$ at $\varphi(x)$, we have

$$\frac{\partial N}{\partial x_j} = -g^{k\ell} s_{jk} \frac{\partial \varphi}{\partial x_\ell}, \tag{4.5}$$

which belongs to $C^\sigma(\Omega)$, hence $N \in C^{\sigma+1}(\Omega)$. Let us now parametrize M as the graph of a real-valued function over the tangent plane to some point $\varphi(x_0) \in M$. Rotating coordinates so that $N(x_0) = (0,\dots,0,1)$, we have M represented near $\varphi(x_0)$ as

$$x_{n+1} = f(x), \quad x = (x_1,\dots,x_n) \in \mathcal{O}, \tag{4.6}$$

with $\mathcal{O} \subset \mathbb{R}^n$ open. The implicit function theorem then yields $f \in C^{\sigma+1}(\mathcal{O})$, but in fact we have more. Note that also, in these new coordinates, $N \in C^{\sigma+1}(\mathcal{O})$. With this we can establish the following.

Proposition 4.2. *Under the hypotheses of Proposition* 4.1*,*

$$f \in C^{\sigma+2}(\mathcal{O}). \tag{4.7}$$

Proof. We have

$$\frac{\partial_j f}{\sqrt{1+|\nabla f|^2}} = N_j \in C^{\sigma+1}(\mathcal{O}), \quad 1 \le j \le n, \tag{4.8}$$

where $|\nabla f|^2 = (\partial_1 f)^2 + \cdots + (\partial_n f)^2$. Squaring and summing gives

$$\frac{|\nabla f|^2}{1+|\nabla f|^2} = \sum_{j=1}^n N_j^2 \in C^{\sigma+1}(\mathcal{O}). \tag{4.9}$$

Since $\psi(s) = s/(1+s)$ is a smooth diffeomorphism of $\mathbb{R}$ onto $(-1, 1)$, this implies $|\nabla f|^2 \in C^{\sigma+1}(\mathcal{O})$, and using this in (4.8) then gives $\partial_j f \in C^{\sigma+1}(\mathcal{O})$, as desired.

Hence, if we define

$$\tilde{\varphi} : \mathcal{O} \longrightarrow \mathbb{R}^{n+1}, \quad \tilde{\varphi}(x) = (x, f(x)), \tag{4.10}$$

we have a $C^{\sigma+2}$ immersion with the same image, $\tilde{\varphi}(\mathcal{O}) = M = \varphi(\Omega)$ (locally). The metric tensor on M pulls back to a metric tensor $\tilde{g} = \tilde{\varphi}^*(ds^2) \in C^{\sigma+1}(\mathcal{O})$, and the map $\tilde{\varphi}^{-1} \circ \varphi : \Omega \to \mathcal{O}$ is an isometry from (Ω, g) to $(\mathcal{O}, \tilde{g})$. We record the resulting complement to Proposition 4.1.

Corollary 4.3. *Under the hypotheses of Proposition* 4.1*, the surface obtained as the image of the immersion* (4.4) *is actually a surface of class* $C^{\sigma+2}$.

With this observation in hand, we can establish the following more "natural" variant of Proposition 4.1.

Proposition 4.4. *Let* (g_{jk}) *be a metric tensor,* (s_{jk}) *a symmetric tensor field, and assume* $g_{jk} \in C^{\sigma+1}$, $s_{jk} \in C^{\sigma}$, $\sigma \geq 1$. *If* g_{jk}, s_{jk} *satisfy the Gauss-Codazzi equations, then there exists a (local) isometric immersion*

$$\varphi : \Omega \longrightarrow \mathbb{R}^{n+1}, \quad \varphi \in C^{\sigma+2}. \tag{4.11}$$

Proof. We have the map $\tilde{\varphi} : \mathcal{O} \to \mathbb{R}^{n+1}$, $\tilde{\varphi} \in C^{\sigma+2}$, so it suffices to prove that $\tilde{\varphi}^{-1} \circ \varphi : \Omega \to \mathcal{O}$ is of class $C^{\sigma+2}$. Since this is an isometry from (Ω, g) to $(\mathcal{O}, \tilde{g})$ and $\tilde{g} \in C^{\sigma+1}$, the hypothesis $g \in C^{\sigma+1}$ guarantees this, by Theorem 2.1.

This result was established in [HW1], using different techniques. Furthermore, [HW1] also treated the case $\sigma = 0$. While the technique of bringing in the metric tensor (4.2)–(4.3) is quite pretty, it does not seem applicable to the case $\sigma = 0$. It can be extended somewhat beyond the limits of Propositions 4.1 and 4.4. For example, one could assume

$$g_{jk},\ s_{jk} \in C^{\sigma} \cap H^{1,2}, \quad \sigma > 0, \tag{4.12}$$

and show by these techniques that if the Gauss-Codazzi equations hold, then there is a local isometric immersion satisfying (4.4). Such a result is neither stronger nor weaker than the $\sigma = 0$ case established by [HW1]. To end this section we show how this result of [HW1] can be combined with an argument involving harmonic coordinates to produce a result that strengthens them both.

Let (g_{jk}) and (s_{jk}) be as in Proposition 4.4, but assume

$$g_{jk} \in C^{\sigma} \cap H^{1,2}, \quad 0 < \sigma < 1, \quad s_{jk} \in C^0. \tag{4.13}$$

Continue to assume that the Gauss-Codazzi equations are satisfied. Now one has local harmonic coordinates u_j, satisfying (3.6), and the pull-back $\tilde{g} = (u^{-1})^* g$ has components $\tilde{g}_{jk}$, a priori in $C^{\sigma} \cap H^{1,2}$, satisfying (3.7). Also $\tilde{s} = (u^{-1})^* s$ has components $\tilde{s}_{jk}$ in C^0. Hence the Gauss-Codazzi equations imply $\mathrm{Ric}_{\ell m} \in C^0$ in (3.7), so elliptic regularity gives

$$\tilde{g}_{jk} \in C^{2-\varepsilon} \tag{4.14}$$

in this context. Thus [HW1] applies to $\tilde{g}_{jk}, \tilde{s}_{jk}$, to yield a C^2 immersion

$$\tilde{\varphi} : \mathcal{O} \longrightarrow \mathbb{R}^{n+1}, \tag{4.15}$$

with metric tensor $(\tilde{g}_{jk})$ and second fundamental form $(\tilde{s}_{jk})$. Composing with $u : \Omega \to \mathcal{O}$ we have the following proposition.

Proposition 4.5. *Assume* (g_{jk}) *and* (s_{jk}) *are as in Proposition* 4.4*, but with the regularity hypotheses weakened to* (4.13)*. Then there exists a (local) isometric immersion*

$$\varphi : \Omega \longrightarrow \mathbb{R}^{n+1}, \quad \varphi \in C^{\sigma+1}. \tag{4.16}$$

The surface obtained as the image of this immersion is actually a surface of class C^2*.*

5. Harmonic coordinates on surfaces in $\mathbb{R}^n$

Let M be an m-dimensional submanifold of $\mathbb{R}^n$. Assume M is of class C^2, hence locally the graph of a map $f : \Omega \to \mathbb{R}^{n-m}$, with $\Omega \subset \mathbb{R}^m$ open and $f \in C^2(\Omega)$. So Ω is embedded via $\varphi(x) = (x, f(x))$, and the Euclidean metric tensor on $\mathbb{R}^n$ pulls back to $g_{jk} \in C^1(\Omega)$. Thus any $p \in \Omega$ has a neighborhood $\mathcal{O}$ on which there are local harmonic coordinates, $u = (u_1, \dots, u_m)$, giving a diffeomorphism $u : \mathcal{O} \to \widetilde{\mathcal{O}} \subset \mathbb{R}^m$. These coordinates satisfy the PDE

$$\partial_j (g^{1/2} g^{jk} \partial_k u_\ell) = 0, \quad 1 \le \ell \le m. \tag{5.1}$$

Standard elliptic estimates give

$$u_\ell \in H^{2,p}(\mathcal{O}), \quad \forall\ p < \infty, \tag{5.2}$$

but for a general metric tensor $g_{jk} \in C^1(\Omega)$ one does not expect to obtain $u_\ell \in C^2(\mathcal{O})$. Thus it is of interest that the special structure of $M \subset \mathbb{R}^n$ allows us to show the following.

Proposition 5.1. *Under the hypotheses given above, we have* $u_\ell \in C^2(\mathcal{O})$*.*

Proof. Since M is a C^2 manifold, its Gauss map is C^1 and hence the Weingarten map is a well-defined continuous section of $\mathrm{Hom}(\nu(M) \otimes TM, TM)$. Thus the Riemann curvature tensor of M is continuous, and hence the Ricci tensor is continuous. Now the metric tensor $(\tilde{g}_{jk})$, the pullback of (g_{jk}) via u^{-1}, clearly belongs to $C^{1-\varepsilon}(\widetilde{\mathcal{O}})$, and it satisfies the PDE (3.7). This time the right side of (3.7) is continuous; hence we have

$$\tilde{g}_{jk} \in C^{2-\varepsilon}(\widetilde{\mathcal{O}}). \tag{5.3}$$

Since $u : (\mathcal{O}, g) \to (\widetilde{\mathcal{O}}, \tilde{g})$ is an isometry, it follows from Theorem 2.1 that $u \in C^2$, as asserted.

Remark 1. In fact elliptic regularity results yield

$$\partial^2 \tilde{g}_{jk} \in \mathrm{bmo}(\widetilde{\mathcal{O}}); \tag{5.4}$$

cf. [T2], Chapter III, Proposition 10.2. Thus $\nabla \tilde{g}_{jk}$ has a log-Lipschitz modulus of continuity. This implies that the geodesic flow on M is uniquely defined by Osgood's theorem. There are more elementary proofs of this fact in [H] and in [P], but the approach here lends insight into how this phenomenon is a special case of non-branching of geodesics when the Ricci tensor is bounded. See [AKLT] for a further development of this idea.

Remark 2. Suppose that M is a surface of class $C^{\sigma+2}$ in $\mathbb{R}^n$, with $\sigma \in [0, \infty)$, given as above with $f \in C^{\sigma+2}(\Omega)$. The proof of Proposition 5.1 extends to yield $u_\ell \in C^{\sigma+2}(\mathcal{O})$ and also, if $\sigma \notin \mathbb{Z}^+$, $\tilde{g}_{jk} \in C^{\sigma+2}(\widetilde{\mathcal{O}})$, while if $\sigma \in \mathbb{Z}^+$, then $\partial^{\sigma+2} \tilde{g}_{jk} \in \mathrm{bmo}(\widetilde{\mathcal{O}})$.

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