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ON THE EIGENVALUE PROBLEM FOR PERTURBED NONLINEAR MAXIMAL MONOTONE OPERATORS IN REFLEXIVE BANACH SPACES

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ABSTRACT. Let X be a real reflexive Banach space with dual X^* and $G \subset X$ open and bounded and such that $0 \in G$. Let $T : X \supset D(T) \to 2^{X^*}$ be maximal monotone with $0 \in D(T)$ and $0 \in T(0)$, and $C : X \supset D(C) \to X^*$ with $0 \in D(C)$ and $C(0) \neq 0$. A general and more unified eigenvalue theory is developed for the pair of operators (T, C). Further conditions are given for the existence of a pair $(\lambda, x) \in (0, \infty) \times (D(T + C) \cap \partial G)$ such that

$$(**) \qquad Tx+\lambda Cx \ni 0.$$

The "implicit" eigenvalue problem, with $C(\lambda, x)$ in place of λCx, is also considered. The existence of continuous branches of eigenvectors of infinite length is investigated, and a Fredholm alternative in the spirit of Necas is given for a pair of homogeneous operators $T,\ C$. No compactness assumptions have been made in most of the results. The degree theories of Browder and Skrypnik are used, as well as the degree theories of the authors involving densely defined perturbations of maximal monotone operators. Applications to nonlinear partial differential equations are included.

1. INTRODUCTION-PRELIMINARIES

Unless otherwise stated, the symbol X stands for a real reflexive Banach space which has been renormed so that it and its dual X^* are locally uniformly convex. The symbol $\|\cdot\|$ stands for the norm of $X,\ X^*$ and $J : X \to X^*$ is the normalized duality mapping. In what follows, "continuous" means "strongly continuous" and the symbol "$\to$" ("$\rightharpoonup$") means strong (weak) convergence.

The symbol $\mathcal{R}$ ($\mathcal{R}_+$) stands for the set $(-\infty, \infty)$ ($[0, \infty)$) and the symbols ∂D, $\overline{D}$ denote the strong boundary and closure of the set D, respectively. We denote by $B_r(0)$ the open ball of X or X^* with center at zero and radius $r > 0$.

For an operator $T : X \to 2^{X^*}$ we denote by $D(T)$ the effective domain of T, i.e. $D(T) = \{x \in X \ : \ Tx \neq \emptyset\}$. We denote by $G(T)$ the graph of T, i.e. $G(T) = \{(x, y) \ : \ x \in D(T),\ y \in Tx\}$. An operator $T : X \supset D(T) \to 2^{X^*}$ is called "monotone" if for every $x,\ y \in D(T)$ and every $u \in Tx,\ v \in Ty$ we have

$$\langle u - v, x - y\rangle \geq 0.$$

A monotone operator T is "maximal monotone" if $G(T)$ is maximal in $X \times X^*$, when $X \times X^*$ is partially ordered by inclusion. In our setting, a monotone operator

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T is maximal if and only if $R(T+\lambda J)=X^*$ for all $\lambda\in(0,\infty)$. If T is maximal monotone, then the operator $T_t\equiv(T^{-1}+tJ^{-1})^{-1}:X\to X^*$ is bounded, demicontinuous, maximal monotone and such that $T_tx\rightharpoonup T^{\{0\}}x$ as $t\to 0^+$ for every $x\in D(T)$, where $T^{\{0\}}x$ denotes the element $y^*\in Tx$ of minimum norm, i.e. $\|T^{\{0\}}x\|=\inf\{\|y^*\| \ : \ y^*\in Tx\}$. In our setting, this infimum is always attained and $D(T^{\{0\}})=D(T)$. Also, $T_tx\in TJ_tx$, where $J_t\equiv I-tJ^{-1}T_t:X\to X$ and satisfies $\lim_{t\to 0}J_tx=x$ for all $x\in\overline{\text{co}D(T)}$, where co$A$ denotes the convex hull of the set A. In addition, $x\in D(T)$ and $t_0>0$ imply $\lim_{t\to t_0}T_tx=T_{t_0}x$. The operators $T_t,\ J_t$ were introduced by Brézis, Crandall and Pazy in [2]. For their basic properties, we refer the reader to [2] as well as Pascali and Sburlan [18, pp. 128-130]. In our setting, the duality mapping J is single-valued and bicontinuous.

An operator $T:X\supset D(T)\to Y$, with Y another real Banach space, is "bounded" if it maps bounded subsets of $D(T)$ onto bounded sets. It is "compact" if it is continuous and maps bounded subsets of $D(T)$ onto relatively compact subsets of Y. It is "demicontinuous" ("completely continuous") if it is strong-weak (weak-strong) continuous on $D(T)$.

Given an operator $T:X\supset D(T)\to 2^{X^*}$, we say that T has the property $\mathcal{P}$ "locally" on $G\subset X$ if for every $x_0\in D(T)\cap G$ there exists a closed ball $\overline{B_r(x_0)}\subset G$ such that T has the property $\mathcal{P}$ on $D(T)\cap\overline{B_r(x_0)}$. If $G=X$, then we simply say that T has "locally" the property $\mathcal{P}$.

We say that an operator $T:X\supset D(T)\to 2^{X^*}$ satisfies condition "(S)" on $B\subset D(T)$ if $\{x_n\}\subset B,\ x_n\rightharpoonup x_0$ and

$$(*)\qquad \lim_{n\to\infty}\langle u_n,x_n-x_0\rangle=0,$$

for some $u_n\in Tx_n$, imply $x_n\to x_0$.

We say that an operator $T:X\supset D(T)\to 2^{X^*}$ satisfies condition "(S_+)" on $B\subset D(T)$ if $\{x_n\}\subset B,\ x_n\rightharpoonup x_0$ and

$$(**)\qquad \limsup_{n\to\infty}\langle u_n,x_n-x_0\rangle\le 0,$$

for some $u_n\in Tx_n$, imply $x_n\to x_0$.

Let L be a dense subspace of X. An operator $T:X\supset D(T)\to X^*$, with $L\subset D(T)$, is said to satisfy condition $(\widetilde{S}_+)_L$ if $\{u_n\}\subset L,\ u_n\rightharpoonup u_0,\ Tu_n\rightharpoonup h_0^*$ and

$$\limsup_{n\to\infty}\langle Tu_n,u_n-u_0\rangle\le 0$$

imply $u_n\to u_0,\ u_0\in D(T)$ and $Tu_0=h_0^*$.

An operator $T:X\supset D(T)\to X^*$ with $L\subset D(T)$ is said to satisfy condition $(\widetilde{S}_+)$ if $\{u_n\}\subset D(T),\ u_n\rightharpoonup u_0,\ Tu_n\rightharpoonup h_0^*$ and

$$\limsup_{n\to\infty}\langle Tu_n,u_n-u_0\rangle\le 0$$

imply $u_n\to u_0,\ u_0\in D(T)$ and $Tu_0=h_0^*$.

We say that the operator $T:X\supset D(T)\to 2^{X^*}$ satisfies condition (S_q) on a set $A\subset D(T)$ if for every sequence $\{x_n\}\subset A$ such that $x_n\rightharpoonup x_0\in X$ and any $y_n^*\in Tx_n$, with $y_n^*\to$ (some) $y^*\in X^*$, we have $x_n\to x_0$. If $A=D(T)$, then we say that T satisfies (S_q).

Obviously, if an operator is of type (S) on $A\subset D(T)$, then it is also of type (S_q) on A. The following lemma can be found in Zeidler [24, p. 915].

Lemma A. *Let $T : X \supset D(T) \to 2^{X^*}$ be maximal monotone. Then the following are true:*

(i) $\{x_n\} \subset D(T)$, $x_n \to x_0$ *and* $Tx_n \ni y_n \rightharpoonup y_0$ *imply* $x_0 \in D(T)$ *and* $y_0 \in Tx_0$.
(ii) $\{x_n\} \subset D(T)$, $x_n \rightharpoonup x_0$ *and* $Tx_n \ni y_n \to y_0$ *imply* $x_0 \in D(T)$ *and* $y_0 \in Tx_0$.

From Lemma A we see that either one of (i) or (ii) implies that the graph $G(T)$ of the operator T is closed, i.e. $G(T) \equiv \{(u,x)\ ;\ x \in D(T),\ u \in Tx\}$ is a closed subset of $X \times X^*$.

Unless otherwise stated, the symbol $d(T,G,p)$ denotes the Leray-Schauder degree of the mapping T on the closed and bounded set G w.r.t. $p \notin T(\partial G)$.

For facts involving monotone operators, and other related concepts, the reader is referred to Barbu [1], Browder [3], Cioranescu [7], Pascali and Sburlan [18], Simons [21], Skrypnik [23], and Zeidler [24]. We cite the books of Browder [3], Lloyd [17], Petryshyn [19], Rothe [20], Skrypnik [23] and the papers of Browder [4]–[6] and Kartsatos and Skrypnik [13] as references to degree theories.

For recent nonlinear eigenvalue results we refer the reader to Guan and Kartsatos [9], Kartsatos [11], the authors [12], Li and Huang [16], and Skrypnik [23, p. 124]. For a recent "invariance of domain" theory, we cite the paper of the authors [14].

It is our main intention here to establish an eigenvalue theory for various classes of operators $T+C$, with T maximal monotone, acting from the space X to its dual X^*. Guan and Kartsatos gave a series of results in [9] involving the existence of eigenvalues and eigenvectors of inclusions of the type

$$Tx + \lambda Cx \ni 0 \tag{$*$}$$

containing maximal monotone or m-accretive operators T and perturbing operators C. Roughly speaking, these results use or imply conditions of the type $\|Cx\| \geq \alpha,\ x \in \partial G$, where G is an open and bounded subset of X. The reason for such conditions is that when they hold they guarantee that the degree of a certain mapping associated with problem ($*$) is zero (cf., e.g., Guan and Kartsatos [9, Proof of Theorem A]). These considerations were substantially improved by the authors in [12]. In fact, we considered in [12] implicit eigenvalue problems of the type

$$Tx + C(\lambda, x) = 0, \tag{$**$}$$

for many combinations of operators $T,\ C$ with T m-accretive, or maximal monotone. Our results in [12] were based on various compactness assumptions on the operator C or the resolvents of the operator T. We also showed in [12] that one can even obtain normalized eigenvectors x for such problems, which are lying on the boundaries of sets which may be unbounded in the norm of the underlying energy space.

This paper can be considered to be a continuation of all the above mentioned eigenvalue papers. In particular, we assume here that problems like

$$Tx + \lambda Cx + \varepsilon Jx \ni 0 \tag{$***$}$$

have no solutions in $D(T) \cap G$ for some $\lambda > 0$ and all small $\varepsilon > 0$.

In Section 1 we give a result that guarantees the existence of eigenvalues for problems of type ($**$)

$$Tx + C(\lambda, x) \ni 0,$$

where T is maximal monotone and $C(\lambda, x)$ is demicontinuous, bounded and of type (S_+).

In Section 2 we use Browder's degree theory in [6] involving multi-valued maximal monotone operators T and demicontinuous operators C defined on the closures of bounded open sets in X.

In Section 3 we first consider the problem

$$Tx + \lambda Cx = 0$$

for single-valued densely defined operators $T,\ C$. Our approach here uses the degree theory that was developed by the authors in [13].

Section 4 is devoted to the existence of eigenvalues for operators $T,\ C$, where T is maximal monotone and C is densely defined. Here, we use the new degree theory developed by the authors in [15].

In Section 5 we give a Fredholm alternative result in the spirit of Necas [8, p. 61] concerning the surjectivity of operators $\lambda T + C$ whenever $\lambda\ (\geq 1)$ is not an eigenvalue for the pair (T, C). In this result both operators $T,\ C$ are positively homogeneous of degree $\gamma \geq 1$.

Section 6 is devoted to continuous and bounded operators C defined on $\overline{D(T)}$, and maximal monotone operators T with compact resolvents.

In Section 7 we demonstrate the fact that our eigenvalue results can give rise to the existence of continuous branches of eigenvectors of infinite length.

Applications to partial differential equations are given in Section 8.

2. Demicontinuous operators C of type (S_+)

Our main purpose in this section is to prove Theorem 1 below about the implicit eigenvalue problem

$$\text{(I)} \qquad Tx + C(\lambda, x) \ni 0.$$

Let $G \subset X$ be open and bounded, $\Lambda > 0$. An operator $C : [0, \Lambda] \times \overline{G} \to X^*$ is "demicontinuous" if $[0, \Lambda] \times \overline{G} \ni (t_n, x_n) \to (t_0, x_0)$ implies $C(t_n, x_n) \rightharpoonup C(t_0, x_0)$. A demicontinuous operator $C(t, x)$ as above is continuous in t "uniformly w.r.t. $x \in \overline{G}$" if $[0, \Lambda] \ni t_n \to t_0$ implies $C(t_n, x) \to C(t_0, x)$ uniformly w.r.t. $x \in \overline{G}$. A demicontinuous operator C as above is said to satisfy condition "(S_+)" if for every $\lambda \in (0, \Lambda]$ and every sequence $\{x_n\} \subset \overline{G}$ with $x_n \rightharpoonup x_0$ and

$$\limsup_{n \to \infty} \langle C(\lambda, x_n), x_n - x_0 \rangle \leq 0$$

we have $x_n \to x_0$.

Theorem 1. *Let $G \subset X$ be open and bounded. Let $T : X \supset D(T) \to 2^{X^*}$ be maximal monotone with $0 \in D(T) \cap G$ and $0 \in T(0)$. Let $C : [0, \Lambda] \times \overline{G} \to X^*$ be demicontinuous, bounded, of type (S_+), and such that $C(0, x) = 0,\ x \in \overline{G}$, and $C(t, x)$ is continuous in t uniformly w.r.t. $x \in \overline{G}$. Let $\varepsilon,\ \varepsilon_0$ be positive numbers. Assume that*

($\mathcal{P}$) *there exists $\lambda \in (0, \Lambda]$ such that the inclusion*

$$(1) \qquad Tx + C(\lambda, x) + \varepsilon Jx \ni 0$$

has no solution $x \in D(T) \cap G$. Then

(i) *there exists $(\lambda_0, x_0) \in (0, \Lambda] \times (D(T) \cap \partial G)$ such that*

$$(2) \qquad Tx_0 + C(\lambda_0, x_0) + \varepsilon Jx_0 \ni 0;$$

(ii) *if* $0 \notin T(D(T) \cap \partial G)$, T *satisfies condition* (S_q) *on* ∂G, *and property* $(\mathcal{P})$ *is satisfied for every* $\varepsilon \in (0, \varepsilon_0]$, *then there exists* $(\lambda_0, x_0) \in (0, \Lambda] \times (D(T) \cap \partial G)$ *such that* $Tx_0 + C(\lambda_0, x_0) \ni 0$.

Theorem 1 was proved by Li and Huang [16, Theorem 3.1] under the assumption that $C(\lambda, x) \equiv \lambda Cx$ and C is a compact operator. We should note here that these authors should have assumed that $0 \in T(0)$. In fact, their proof is based on the homotopy invariance of the Leray-Schauder degree for the homotopy function $H \equiv I - (T_s + \varepsilon J)^{-1}(-\lambda C)$ on $\overline{G}$. However, this function H is not generally homotopic to the identity I for $\lambda = 0$, a fact that was used in [15]. It is homotopic to I if we assume that $0 \in T(0)$. Properties like $(\mathcal{P})$ were assumed by Guan and Kartsatos in 9.

Proof of Theorem 1. (i) Assume that (2) is not true. Then for every $\lambda \in (0, \Lambda]$ the equation

$$Tx + C(\lambda, x) + \varepsilon Jx \ni 0$$

has no solution $x \in D(T) \cap \partial G$. We note that this is also true for $\lambda = 0$ because $(T + \varepsilon J)G \ni 0$ and the operator $T + \varepsilon J$ is injective by the strict monotonicity of the duality mapping. We set $H(\lambda, x) \equiv Tx + C(\lambda, x) + \varepsilon Jx$ and observe that

$$H(\lambda, D(T) \cap \partial G) \not\ni 0, \quad \lambda \in [0, \Lambda]. \tag{3}$$

We are now going to show that there exist $s_0 > 0,\ \lambda_0 \in (0, \Lambda]$ such that for every $s \in (0, s_0],\ \lambda \in (0, \lambda_0]$ we have $0 \notin H_1(s, \lambda, \partial G)$, where

$$H_1(s, \lambda, x) \equiv T_s x + C(\lambda, x) + \varepsilon Jx. \tag{4}$$

Assume that this is not true. Then there exist $s_n \downarrow 0,\ \lambda_n \downarrow 0,\ x_n \in \partial G$ with $x_n \rightharpoonup x_0,\ Jx_n \rightharpoonup j^*$, for some $x_0 \in X$ and $j^* \in X^*$, and such that

$$T_{s_n} x_n + C(\lambda_n, x_n) + \varepsilon Jx_n = 0. \tag{5}$$

This implies

$$\begin{aligned} \langle T_{s_n} x_n, x_n - x_0 \rangle &= -\langle C(\lambda_n, x_n), x_n - x_0 \rangle - \varepsilon \langle Jx_n, x_n - x_0 \rangle \\ &\le \|C(\lambda_n, x_n)\| \|x_n - x_0\| - \varepsilon \langle Jx_n - Jx_0, x_n - x_0 \rangle \\ &\quad - \varepsilon \langle Jx_0, x_n - x_0 \rangle \\ &\le \|C(\lambda_n, x_n)\| \|x_n - x_0\| - \varepsilon \langle Jx_0, x_n - x_0 \rangle. \end{aligned} \tag{6}$$

Thus,

$$\limsup_{n\to\infty} \langle T_{s_n} x_n, x_n - x_0 \rangle \le \lim_{n\to\infty} [\|C(\lambda_n, x_n)\| \|x_n - x_0\|] - \varepsilon \lim_{n\to\infty} \langle Jx_0, x_n - x_0 \rangle = 0. \tag{7}$$

Here, we have used the fact that $|\|C(\lambda_n, x_n)\| - \|C(0, x_n)\|| = \|C(\lambda_n, x_n)\| \to 0$ by the continuity of $C(t, x)$ in t which is uniform w.r.t. $x \in \overline{G}$. Also, (5) implies

$$T_{s_n} x_n \rightharpoonup -\varepsilon j^*,$$

which in turn gives

$$\limsup_{n\to\infty} \langle T_{s_n} x_n, x_n \rangle \le \langle -\varepsilon j^*, x_0 \rangle. \tag{8}$$

Now, fix $x \in D(T),\ x^* \in Tx$. Then, as in Browder [4, Proof of Theorem 12],

$$\begin{aligned} \liminf_{n\to\infty} \langle T_{s_n} x_n, x_n \rangle &\ge \liminf_{n\to\infty} \langle T_{s_n} x_n, x \rangle + \langle x^*, x_0 - x \rangle \\ &= \langle -\varepsilon j^*, x \rangle + \langle x^*, x_0 - x \rangle. \end{aligned} \tag{9}$$

Combining (8) and (9) we get

$$\langle -\epsilon j^* - x^*, x_0 - x\rangle \geq 0, \tag{10}$$

which, by the maximal monotonicity of T, implies $x_0 \in D(T)$. Letting $x = x_0$ in (9) and using (8) we obtain

$$\lim_{n\to\infty} \langle T_{s_n} x_n, x_n\rangle = \lim_{n\to\infty} \langle T_{s_n} x_n, x_0\rangle = \langle -\varepsilon j^*, x_0\rangle.$$

This implies

$$\lim_{n\to\infty} \langle T_{s_n} x_n, x_n - x_0\rangle = 0. \tag{11}$$

Using this with (5) we get

$$\varepsilon\langle Jx_n, x_n - x_0\rangle = -\langle T_{s_n} x_n, x_n - x_0\rangle - \langle C(\lambda_n, x_n), x_n - x_0\rangle,$$

which gives

$$\lim_{n\to\infty} \langle Jx_n, x_n - x_0\rangle = 0.$$

Since the duality mapping J is of type (S), we have $x_n \to x_0 \in \partial G$, $Jx_n \to Jx_0 = j^*$. Now, $J_{s_n} x_n = x_n - s_n J^{-1} T_{s_n} x_n \to x_0$. Here, we have used the fact that $\{T_{s_n} x_n\}$ and J^{-1} are bounded and $s_n \downarrow 0$. Consequently, $T_{s_n} x_n \in TJ_{s_n} x_n$, $J_{s_n} x_n \to x_0$, $T_{s_n} x_n \rightharpoonup -\varepsilon j^* = -\varepsilon Jx_0$ and the closedness of T (see Lemma A) imply that $Tx_0 + \varepsilon Jx_0 \ni 0$. However, $x_0 \in D(T) \cap \partial G$ is a contradiction because we already have $0 \in (T + \varepsilon J)G$, and the operator $T + \varepsilon J$ is injective by the strict monotonicity of the duality mapping. Thus, our assertion is true.

Now, we fix $s \in (0, s_0]$, $\lambda \in (0, \lambda_0]$ and consider the homotopy function

$$H_2(t, x) \equiv T_s x + C(t\lambda, x) + \varepsilon Jx. \tag{12}$$

Using the fact that $(T_s + \varepsilon J)(0) = 0$, we note that $0 \notin H_2(t, \partial G)$ for any $t \in [0, 1]$. Following Browder [4], $H_2(t, x)$ is a homotopy of class (S_+) if the following condition holds: for any sequence $\{u_j\} \subset \overline{G}$ with $u_j \rightharpoonup u_0$ and any sequence $\{t_j\} \subset [0, 1]$ with $t_j \to t_0$ for which we have

$$\limsup_{j\to\infty} \langle H_2(t_j, u_j), u_j - u_0\rangle \leq 0, \tag{13}$$

we also have $u_j \to u_0$ and $H_2(t_j, u_j) \rightharpoonup H_2(t_0, u_0)$. We are going to show that $H_2(t, x)$ is actually a homotopy of class (S_+). To this end, we let $\{t_j\}$, $\{u_j\}$ be as above. Then

$$\limsup_{j\to\infty} \langle H_2(t_j, u_j), u_j - u_0\rangle = \limsup_{j\to\infty} \langle T_s u_j + C(t_j\lambda, u_j) + \varepsilon Ju_j, u_j - u_0\rangle \leq 0. \tag{14}$$

We observe that

$$\begin{aligned} &\langle H_2(t_j, u_j), u_j - u_0\rangle \\ &\qquad = \langle T_s u_j, u_j - u_0\rangle + \langle C(t_j\lambda, u_j), u_j - u_0\rangle + \varepsilon\langle Ju_j, u_j - u_0\rangle \\ &\qquad = \langle T_s u_j - T_s u_0, u_j - u_0\rangle + \langle T_s u_0, u_j - u_0\rangle \\ &\qquad\quad + \langle C(t_j\lambda, u_j), u_j - u_0\rangle \\ &\qquad\quad + \varepsilon\langle Ju_j - Ju_0, u_j - u_0\rangle + \varepsilon\langle Ju_0, u_j - u_0\rangle \\ &\qquad \geq \langle T_s u_0, u_j - u_0\rangle + \langle C(t_j\lambda, u_j), u_j - u_0\rangle + \varepsilon\langle Ju_0, u_j - u_0\rangle. \end{aligned} \tag{15}$$

Using this in (14) we obtain

$$\limsup_{j\to\infty} \langle C(t_j\lambda, u_j), u_j - u_0\rangle \leq 0. \tag{16}$$

If $t_0 = 0$, then $C(t_j\lambda, u_j) \to 0$ and

$$\lim_{j\to\infty} \langle C(t_j\lambda, u_j), u_j - u_0\rangle = 0.$$

Using this and the monotonicity of T_s in the first equality of (15) we obtain

$$\limsup_{j\to\infty} \langle Ju_j, u_j - u_0\rangle \le 0. \tag{17}$$

Since J is of type (S_+), we have $u_j \to u_0$, which implies $T_s u_j \rightharpoonup T_s u_0,\ C(t_j\lambda, u_j) \to C(0, u_0) = 0$ and $Ju_j \to Ju_0$. This says that

$$H_2(t_j, u_j) \rightharpoonup H_2(0, u_0) = T_s u_0 + \varepsilon J u_0,$$

and the proof for the case $t_0 = 0$ is complete.

Now, let $t_0 > 0$. We have

$$\langle C(t_j\lambda, u_j), u_j - u_0\rangle = \langle C(t_j\lambda, u_j) - C(t_0\lambda, u_j), u_j - u_0\rangle + \langle C(t_0\lambda, u_j), u_j - u_0\rangle,$$

which implies

$$\begin{aligned}\limsup_{j\to\infty} \langle C(t_0\lambda, u_j), u_j - u_0\rangle \le{}& \limsup_{j\to\infty} \langle C(t_j\lambda, u_j), u_j - u_0\rangle \\ &+ \limsup_{j\to\infty} \{-\langle C(t_j\lambda, u_j) - C(t_0\lambda, u_j), u_j - u_0\rangle\}.\end{aligned}$$

By (16), this yields

$$\limsup_{j\to\infty} \langle C(t_0\lambda, u_j), u_j - u_0\rangle \le \lim_{j\to\infty} \|C(t_j\lambda, u_j) - C(t_0\lambda, u_j)\| \|u_j - u_0\| = 0.$$

By the (S_+)-property of C, we get $u_j \to u_0,\ T_s u_j \rightharpoonup T_s u_0,\ C(t_j\lambda, u_j) \rightharpoonup C(t_0\lambda, u_0)$ and $Ju_j \to Ju_0$. Consequently, $H_2(t_j, u_j) \rightharpoonup H_2(t_0, u_0)$. This finishes the proof of the fact that H_2 is a homotopy of class (S_+). Thus,

$$\begin{aligned} d_S(H_2(t,\cdot), G, 0) &= d_S(H_2(1,\cdot), G, 0) \\ &= d_S(H_2(0,\cdot), G, 0) = d_S(T_s + \varepsilon J, G, 0) = 1, \end{aligned} \tag{18}$$

where d_S denotes the Skrypnik degree (cf. [22]–[23]). The last equality in (18) comes from [6, Theorem 3, (iv)]. In fact, the mapping $T_s + \varepsilon J$ is demicontinuous, injective and of type (S_+) on $\overline{G}$, and such that

$$\langle T_s x + \varepsilon J x, x\rangle \ge 0, \quad x \in \partial G.$$

For Browder's degree d_B in [5] we have, in our setting,

$$d_B(H(\lambda,\cdot), G, 0) = \lim_{s\downarrow 0} d_S(H_1(s,\lambda,\cdot), G, 0) = \lim_{s\downarrow 0} d_S(H_2(1,\cdot), G, 0) = 1 \tag{19}$$

because $H_1(s,\lambda,x) = H_2(1,x)$. Thus, by Browder's degree theory,

$$0 \in (T + C(\lambda,\cdot) + \varepsilon J)(D(T) \cap G), \tag{20}$$

which contradicts our assumed property $(\mathcal{P})$. Therefore, (2) is true.

(ii) Let the sequences $\{x_n\} \subset D(T) \cap \partial G,\ u_n^* \in Tx_n,\ \lambda_n \in (0, \Lambda]$ be such that

$$u_n^* + C(\lambda_n, x_n) + (1/n)Jx_n = 0. \tag{21}$$

We may assume that $\lambda_n \to \lambda_0 \in [0,\Lambda],\ x_n \rightharpoonup x_0,\ C(\lambda_n, x_n) \rightharpoonup c^*$ and $Jx_n \rightharpoonup j^*$. We consider two cases:

(j) $\lambda_0 = 0$;

(jj) $\lambda_0 > 0$.

(j) Since for some $u_n^* \in Tx_n$ we have $u_n^* = -C(\lambda_n, x_n) - (1/n)Jx_n \to 0$ and T satisfies condition (S_q), we have $x_n \to x_0 \in \partial G$. The closedness of T (see Lemma A) now implies $0 \in Tx_0$, which contradicts $0 \notin T(D(T) \cap \partial G)$.

(jj) We are going to show first that

$$\limsup_{n\to\infty} \langle C(\lambda_n, x_n), x_n - x_0 \rangle \le 0. \tag{22}$$

Assume the contrary. Then we may also choose $\{x_n\}$, or a subsequence of it denoted again by $\{x_n\}$, so that

$$\lim_{n\to\infty} \langle C(\lambda_n, x_n), x_n - x_0 \rangle > 0. \tag{23}$$

We have

$$\langle u_n^*, x_n - x_0 \rangle = -\langle C(\lambda_n, x_n), x_n - x_0 \rangle - \langle (1/n)Jx_n, x_n - x_0 \rangle,$$

which says

$$\limsup_{n\to\infty} \langle u_n^*, x_n - x_0 \rangle < 0. \tag{24}$$

Since, by (21), $u_n^* \rightharpoonup -c^*$, we also have

$$\langle u_n^*, x_n \rangle = \langle u_n^*, x_n - x_0 \rangle + \langle u_n^*, x_0 \rangle$$

and

$$\limsup_{n\to\infty} \langle u_n^*, x_n \rangle < \langle -c^*, x_0 \rangle. \tag{25}$$

Now, we fix $(x, x^*) \in G(T)$ and examine

$$\langle u_n^* - x^*, x_n - x \rangle \ge 0.$$

We obtain

$$\langle u_n^*, x_n \rangle \ge \langle u_n^*, x \rangle + \langle x^*, x_n - x \rangle,$$

which implies

$$\liminf_{n\to\infty} \langle u_n^*, x_n \rangle \ge \langle -c^*, x \rangle + \langle x^*, x_0 - x \rangle.$$

Combining this and (25), we find that

$$\langle -c^* - x^*, x_0 - x \rangle > 0. \tag{26}$$

Since T is maximal monotone and $(x, x^*) \in G(T)$ is arbitrary, we get $x_0 \in D(T)$ and $-c^* \in Tx_0$. However, letting $x = x_0$ in (26) we get a contradiction. Thus, (22) is true. We observe that

$$\langle C(\lambda_n, x_n), x_n - x_0 \rangle = \langle C(\lambda_n, x_n) - C(\lambda_0, x_n), x_n - x_0 \rangle + \langle C(\lambda_0, x_n), x_n - x_0 \rangle.$$

Using again the fact that $C(\lambda_n, x_n) - C(\lambda_0, x_n) \to 0$, we obtain

$$\limsup_{n\to\infty} \langle C(\lambda_0, x_n), x_n - x_0 \rangle \le 0.$$

Since C is of type (S_+), we have $x_n \to x_0 \in \partial G$, $C(\lambda_n, x_n) \rightharpoonup C(\lambda_0, x_0) = c^*$ and $u_n^* \rightharpoonup -C(\lambda_0, x_0)$. The demiclosedness of T (see Lemma A) implies $Tx_0 + C(\lambda_0, x_0) \ni 0$, and the proof of the theorem is complete. $\square$

Theorem 1 has the following important corollary.

Corollary 1. *Let $G \subset X$ be open and bounded. Let $T : X \supset D(T) \to 2^{X^*}$ be maximal monotone with $0 \in D(T) \cap G$ and $0 \in T(0)$. Let $C : \overline{G} \to X^*$ be demicontinuous, bounded and of type (S_+). Let Λ, ε and ε_0 be positive numbers. Assume that*

($\mathcal{P}$) *there exists $\lambda \in (0, \Lambda]$ such that the inclusion*

$$Tx + \lambda Cx + \varepsilon Jx \ni 0$$

has no solution in $D(T) \cap G$. Then

(i) *there exists $(\lambda_0, x_0) \in (0, \Lambda] \times (D(T) \cap \partial G)$ such that*

$$Tx_0 + \lambda_0 Cx_0 + \varepsilon Jx_0 \ni 0;$$

(ii) *if $0 \notin T(D(T) \cap \partial G)$, T satisfies condition (S_q) of ∂G, and property ($\mathcal{P}$) is satisfied for every $\varepsilon \in (0, \varepsilon_0]$, then there exists $(\lambda_0, x_0) \in (0, \Lambda] \times (D(T) \cap \partial G)$ such that $Tx_0 + \lambda_0 Cx_0 \ni 0$.*

Proof. It suffices to note that the operator $C(\lambda, x) \equiv \lambda Cx$ has all the properties assumed for it in Theorem 1. □

3. Densely defined operators T, C

In this section we apply the authors' degree theory from [13] for densely defined operators T, C.

Let L be a subspace of X and let $T : X \supset D(T) \to X^*$ be maximal monotone and $C : X \supset D(C) \to X^*$. Let $\mathcal{F}(L)$ be the set of all finite-dimensional subspaces of L. For the operator T we consider the following assumptions:

t_1) T is monotone, i.e.

$$\langle Tu - Tv, u - v\rangle \geq 0, \tag{27}$$

for every u, $v \in D(T)$. Moreover,

$$L \subset D(T), \quad \overline{L} = X; \tag{28}$$

t_2) for every $(u_0, h_0) \in X \times X^*$ with

$$\langle Tu - h_0, u - u_0\rangle \geq 0, \quad \text{for } u \in L, \tag{29}$$

we have $u_0 \in D(T)$ and $Tu_0 = h_0$;

t_3) for any $u_0 \in D(T)$ we have

$$\inf\{\langle Tv - Tu_0, v - u_0\rangle \ : \ v \in L\} = 0; \tag{30}$$

t_4) for every $F \in \mathcal{F}(L)$, $v \in L$ the mapping $t(F, v) \ : \ F \to \mathcal{R}$, defined by $t(F, v)u = \langle Tu, v\rangle$ is continuous.

For the operator C we have the following assumptions:

c_1)

$$L \subset D(C) \tag{31}$$

and C is quasi-bounded with respect to T, i.e. for every number $S > 0$ there exists a number $K(S) > 0$ such that from the inequalities

$$\langle Tu + Cu, u\rangle \leq 0, \quad \|u\| \leq S, \quad u \in L, \tag{32}$$

we have $\|Cu\| \leq K(S)$;

c_2) the operator C satisfies the following generalized (S_+) condition with respect to T: for every sequence $\{u_n\} \subset L$ such that $u_n \rightharpoonup u_0,\ Cu_n \rightharpoonup h_0$ and

$$\limsup_{n\to\infty} \langle Cu_n, u_n - u_0\rangle \le 0, \quad \langle Tu_n + Cu_n, u_n\rangle \le 0, \tag{33}$$

for some $u_0 \in X,\ h_0 \in X^*$, we have $u_n \to u_0,\ u_0 \in D(C)$ and $Cu_0 = h_0$;

c_3) for every $F \in \mathcal{F}(L),\ v \in L$ the mapping $c(F,v) : F \to \mathcal{R}$, defined by $c(F,v)(u) = \langle Cu, v\rangle$, is continuous.

Note that the conditions $t_2),\ t_3)$ are satisfied for a maximal monotone operator T whose domain $D(T) = L$.

Remark 1. We should note here that the degree theory developed in [13] used the number S in place of 0 in the first inequality of (32). A careful study on the development in [13] reveals that all we need is our present assumption. The same remark applies to the homotopy assumption $a_t^{(1)}$ in [13, p. 432]: we can replace S in the first inequality there by 0.

An operator $C : X \supset D(C) \to X^*$ with $L \subset D(C)$ is called "L-quasibounded" if for every $S > 0$ there exists $K(S) > 0$ such that $u \in L$ with $\|u\| \le S,\ \langle Cu, u\rangle \le 0$ implies $\|Cu\| \le K(S)$.

Theorem 2. *Let $G \subset X$ be open and bounded with $0 \in G$. Assume that the operator T is single-valued and maximal monotone, $D(T) = L,\ T(0) = 0$ and T satisfies t_4), while the operator $C : X \supset D(C) \to X^*$ is L-quasibounded and satisfies $(\widetilde{S}_+)_L$ and c_3). Let $\varepsilon,\ \varepsilon_0$ and Λ be positive numbers. Assume that*

($\mathcal{P}$) *there exists $\lambda \in (0, \Lambda]$ such that the equation*

$$Tx + \lambda Cx + \varepsilon Jx = 0 \tag{34}$$

has no solution in $L \cap G$. Then

(i) *there exists $(\lambda_0, x_0) \in (0, \Lambda] \times (L \cap \partial G)$ such that*

$$Tx_0 + \lambda_0 Cx_0 + \varepsilon Jx_0 = 0; \tag{35}$$

(ii) *if $0 \notin T(L \cap \partial G)$, T satisfies (S_q) on ∂G, and property ($\mathcal{P}$) is satisfied for every $\varepsilon \in (0, \varepsilon_0]$, then there exists $(\lambda_0, x_0) \in (0, \Lambda] \times (L \cap \partial G)$ such that $Tx_0 + \lambda_0 Cx_0 = 0$.*

Before we prove this theorem, we notice that both conditions $c_1),\ c_2)$ are satisfied, because $\langle Tu, u\rangle \ge 0$ for all $u \in D(T)$.

Proof. Assume that (35) is not true. Then for every $\lambda \in (0, \Lambda]$ the equation

$$Tx + \lambda Cx + \varepsilon Jx = 0 \tag{36}$$

has no solution $x \in L \cap \partial G$. We consider the operators: $T_t \equiv T + (\varepsilon/2)J$ and $C_t \equiv t\Lambda C + (\varepsilon/2)J,\ t \in [0,1]$. We need to show that $T_t + C_t$ is an admissible homotopy in the sense of Definition 4.3 in [13]. To this end, we show first the uniform quasiboundedness property of C_t w.r.t. T_t. This is the property $a_t^{(1)}$ in [13] with the first occurrence of S in (4.11) there replaced by 0 (see Remark 1 above). Assume that for some $S > 0$ we have

$$\langle T_t u + C_t u, u\rangle \le 0, \quad \|u\| \le S, \quad \text{for some } u \in L,\ t \in [0,1]. \tag{37}$$

If $t = 0$, then

$$\|C_0 u\| = (\varepsilon/2)\|Ju\| \le (\varepsilon/2)S. \tag{38}$$

If $t > 0$, then

$$\Lambda\langle Cu, u\rangle \le \langle (1/t)Tu + \Lambda Cu, u\rangle \le -\langle (1/t)\varepsilon Ju, u\rangle \le 0 \tag{39}$$

and the L-quasiboundedness of C gives $\|Cu\| \le K(S)$. This implies

$$\|C_t u\| \le K(S) + (\varepsilon/2)S.$$

Combining this with (37) we obtain an obvious uniform quasiboundedness constant $\widetilde{K}(S)$ for the operator C_t.

We now show the uniform generalized condition (S_+) of C_t w.r.t. T_t, which is condition $a_t^{(2)}$ in [13]. To this end, assume that $\{t_n\} \subset [0,1]$, $\{u_n\} \subset L$ are such that $u_n \rightharpoonup u_0$, $C_{t_n} u_n \rightharpoonup h_0^*$, $t_n \to t_0$ and

$$\limsup_{n\to\infty} \langle C_{t_n} u_n, u_n - u_0\rangle \le 0,\ \langle T_{t_n} + C_{t_n} u_n, u_n\rangle \le 0. \tag{40}$$

We rewrite (40) as follows:

$$\limsup_{n\to\infty} \langle t_n \Lambda C u_n + (\varepsilon/2)Ju_n, u_n - u_0\rangle \le 0,\ \langle Tu_n + t_n\Lambda Cu_n + \varepsilon Ju_n, u_n\rangle \le 0. \tag{41}$$

If $t_n = 0$ for all large n, then the first of (41) implies

$$\limsup_{n\to\infty} \langle Ju_n, u_n - u_0\rangle \le 0. \tag{42}$$

Since J is of type (S_+), this says that $u_n \to u_0$ and $Ju_n \to Ju_0$. Thus, $u_0 \in D(C_0) = X$ and $C_0 u_0 = (\varepsilon/2)Ju_0 = h_0^*$.

Let $t_n = 0$ for infinitely many n, but not all large n. Then $t_n \to 0$. Denote by $\{t_n\}$ again the subsequence of $\{t_n\}$ of positive terms. Then $t_n \to 0$ and, from the second part of (41) (since $t_n \le 1$),

$$\Lambda\langle Cu_n, u_n\rangle \le \langle (1/t_n)Tu_n + \Lambda Cu_n + (\varepsilon/t_n)Ju_n, u_n\rangle \le 0. \tag{43}$$

Since C is L-quasibounded, there exists a constant $K > 0$ such that $\|Cu_n\| \le K$ for all n. It follows that, for the original sequence $\{t_n\}$, we have

$$\lim_{n\to\infty} t_n \Lambda C u_n = 0 \tag{44}$$

and

$$w - \lim_{n\to\infty} C_{t_n} u_n = w - \lim_{n\to\infty} [t_n\Lambda Cu_n + (\varepsilon/2)Ju_n] = w - \lim_{n\to\infty} (\varepsilon/2)Ju_n = h_0^*, \tag{45}$$

where "w" denotes weak limit. Also, the first part of (41) implies (42), which implies again $u_n \to u_0$ and $Ju_n \to Ju_0$. Once again, we have

$$u_0 \in D(C_0) = X \text{ and } C_0 u_0 = (\varepsilon/2)Ju_0 = h_0^*. \tag{46}$$

It remains to consider the case $t_n > 0$ for all large n. We assume that $t_n > 0$ for all n. If $t_n \to 0$, we repeat the above argument to obtain (46). Let us assume $t_0 > 0$. Then

$$\begin{aligned}\langle t_n\Lambda Cu_n, u_n - u_0\rangle &= \langle t_n\Lambda Cu_n + (\varepsilon/2)Ju_n, u_n - u_0\rangle - (\varepsilon/2)\langle Ju_n, u_n - u_0\rangle \\ &= \langle t_n\Lambda Cu_n + (\varepsilon/2)Ju_n, u_n - u_0\rangle - (\varepsilon/2)\langle Ju_n - Ju_0, u_n - u_0\rangle \\ &\quad - (\varepsilon/2)\langle Ju_0, u_n - u_0\rangle, \\ &\le \langle t_n\Lambda Cu_n + (\varepsilon/2)Ju_n, u_n - u_0\rangle - (\varepsilon/2)\langle Ju_0, u_n - u_0\rangle,\end{aligned} \tag{47}$$

which, in view of (41), implies

$$\begin{aligned}\limsup_{n\to\infty}\langle t_n\Lambda Cu_n, u_n-u_0\rangle &\le \limsup_{n\to\infty}\langle t_n\Lambda Cu_n+(\varepsilon/2)Ju_n, u_n-u_0\rangle\\ &\quad+\limsup_{n\to\infty}\{-(\varepsilon/2)\langle Ju_0, u_n-u_0\rangle\}\\ &=\limsup_{n\to\infty}\langle t_n\Lambda Cu_n+(\varepsilon/2)Ju_n, u_n-u_0\rangle\\ &\quad+\lim_{n\to\infty}\{-(\varepsilon/2)\langle Ju_0, u_n-u_0\rangle\}\le 0.\end{aligned}\tag{48}$$

If we assume that

$$q\equiv\limsup_{n\to\infty}\langle Cu_n, u_n-u_0\rangle>0,\tag{49}$$

then there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that

$$\lim_{k\to\infty}\langle Cu_{n_k}, u_{n_k}-u_0\rangle=q.$$

Then

$$0<t_0\Lambda q=\lim_{n\to\infty}t_{n_k}\langle\Lambda Cu_{n_k}, u_{n_k}-u_0\rangle\le\limsup_{n\to\infty}\langle t_n\Lambda Cu_n, u_n-u_0\rangle\le 0,$$

i.e. a contradiction. Consequently,

$$\limsup_{n\to\infty}\langle Cu_n, u_n-u_0\rangle\le 0.\tag{50}$$

Since (43) and the L-quasiboundedness property of C imply that $\{\|Cu_n\|\}$ is bounded, we may assume that $Cu_n\rightharpoonup h_1^*$. Since from the second part of (41) we obtain (43), we use the $(\widetilde{S}_+)_L$-property of C to obtain that $u_n\to u_0\in D(C)$ and $Cu_0=h_1^*$. Thus, $t_0\Lambda Cu_0=h_0^*-(\varepsilon/2)Ju_0$. Thus, we have actually shown the following: every subsequence of $\{u_n\}$ contains a further subsequence, denoted again by $\{u_n\}$, such that $u_n\to u_0,\ u_0\in D(C_{t_0})$ and $C_{t_0}u_0=h_0^*$. This implies that the original sequence $\{u_n\}$ has this property. The rest of the required conditions for the admissibility of our homotopy are trivially true. If follows that T_t+C_t is an admissible homotopy for our degree d in [13]. This implies

$$d(T_t+C_t, G, 0)=d(T+t\Lambda C+\varepsilon J, G, 0)=d(T+\varepsilon J, G, 0)=1.\tag{51}$$

The last equality follows from the fact that

$$d(t(T+\varepsilon J)+(1-t)\varepsilon J, G, 0)=d(T+\varepsilon J, G, 0)=d(\varepsilon J, G, 0)=1.\tag{52}$$

The first degree above is well defined because

$$(t(T+\varepsilon J)+(1-t)\varepsilon J)\,(L\cap\partial G)=(tT+\varepsilon J)(L\cap\partial G)\not\ni 0\tag{53}$$

due to the fact that the mapping $tT+J$ is strictly and maximal monotone with $0\in(tT+J)(L\cap G)$. This degree is also constant. In fact, the homotopy $H(t,x)=t(T+\varepsilon J)+(1-t)\varepsilon J$ has already been used in [10, Proof of Theorem 3.1]. The operator $B+(1/n)J-s$ there can be easily replaced by the operator $B=(\varepsilon/2)J$ plus the operator $(\varepsilon/2)J$. The last equality in (52) follows from Browder's Theorem 3, (iv) in [6] because $0\in\varepsilon J(G)$, and the mapping εJ is demicontinuous, bounded, of type (S_+), and such that $\langle\varepsilon Jx, x\rangle\ge 0,\ x\in\partial G$. From our degree theory in [13] we obtain that equation (34) has a solution (λ, x) in $L\cap G$ for every $\lambda\in(0,\Lambda]$. This contradicts $(\mathcal{P})$ and proves (i).

(ii) We have

$$Tx_n+\lambda_nCx_n+(1/n)Jx_n=0,\tag{54}$$

where $\{x_n\} \subset L \cap \partial G,\ \{\lambda_n\} \subset (0, \Lambda]$. We may assume that $\lambda_n \to \lambda_0 \in [0, \Lambda],\ x_n \rightharpoonup x_0$ and $Jx_n \rightharpoonup j^*$. We consider two cases:

(s) $\lambda_0 = 0$;
(ss) $\lambda_0 > 0$.

We first note that we cannot have $\lambda_n = 0$ for any n. This is due to the fact that $0 \in G$, $0 = (T + (1/n)J)(0)$ and the operator $T + (1/n)J$ is strictly monotone. Thus, $\lambda_n > 0$ for all n. Using this in (51) we obtain $\langle Cx_n, x_n \rangle \le 0$ and the boundedness of $\{\|Cx_n\|\}$.

(s) From (54) we also obtain

$$Tx_n = -\lambda_n Cx_n - (1/n)Jx_n \to 0.$$

Since T is of type (S_q), we have $x_n \to x_0 \in \partial G$. Since T is closed (see Lemma A), $x_0 \in D(T) = L$ and $Tx_0 = 0$. Since $x_0 \in L \cap \partial G$, we have a contradiction to our assumption on T. Thus, case (s) is impossible.

(ss) We may assume that $Cx_n \rightharpoonup c^* \in X^*$. We have $Tx_n \rightharpoonup -\lambda_0 c^*$. We can now establish (50) exactly as in the last part of the proof of Theorem 1. The $(\widetilde{S}_+)_L$-property of C implies $x_n \to x_0 \in \partial G,\ x_0 \in D(C)$ and $Cx_n \rightharpoonup Cx_0 = c^*$. Since $Tx_n \rightharpoonup -\lambda_0 Cx_0$, the demiclosedness of T implies $Tx_0 + \lambda_0 Cx_0 = 0$ with $\lambda_0 > 0$ and $x \in L \cap \partial G$. This completes the proof. □

Actually, if $\Lambda \le 1$ in Theorem 2, then that theorem holds under the assumptions $c_1) - c_3)$ for the operator C. This is the content of Theorem 3 below.

Theorem 3. *Let $G \subset X$ be open and bounded with $0 \in G$. Assume that the operator T is single-valued and maximal monotone, $D(T) = L,\ T(0) = 0$ and T satisfies $t_4)$, while the operator C satisfies $c_1) - c_3)$. Let $\varepsilon,\ \varepsilon_0$ and Λ be positive numbers with $\Lambda \in (0, 1]$. Assume that*

($\mathcal{P}$) *there exists $\lambda \in (0, \Lambda]$ such that the equation*

$$Tx + \lambda Cx + \varepsilon Jx = 0 \tag{55}$$

has no solution in $L \cap G$. Then

(i) *there exists $(\lambda_0, x_0) \in (0, \Lambda] \times (L \cap \partial G)$ such that*

$$Tx_0 + \lambda_0 Cx_0 + \varepsilon Jx_0 = 0; \tag{56}$$

(ii) *if $0 \notin T(L \cap \partial G)$, T satisfies (S_q) on ∂G, and property ($\mathcal{P}$) is satisfied for every $\varepsilon \in (0, \varepsilon_0]$, then there exists $(\lambda_0, x_0) \in (0, \Lambda] \times (L \cap \partial G)$ such that $Tx_0 + \lambda_0 Cx_0 = 0$.*

Proof. We just mention here that when $\Lambda \in (0, 1]$ we may replace C by $T + C$ in the relevant inequalities in the proof of Theorem 2. For example, the inequality (39) will now be replaced by

$$\langle Tu + Cu, u \rangle \le \langle [1/(t\Lambda)]Tu + Cu, u \rangle \le -\langle [1/(t\Lambda)]\varepsilon Ju, u \rangle \le 0, \tag{57}$$

while (43) will be changed to

$$\langle Tu_n + Cu_n, u_n \rangle \le \langle [1/(t\Lambda)]Tu_n + Cu_n, u_n \rangle \le -\langle [1/(t\Lambda)]\varepsilon Ju_n, u_n \rangle \le 0. \tag{58}$$ □

4. Densely defined perturbations C

Theorem 4 below uses a new degree that was introduced by the authors in [14]. In particular, this degree applies to certain generalized pseudomonotone perturbations of multivalued maximal monotone operators.

The following definitions are needed for the application of the new degree. We recall that L is a fixed dense subspace of the space X.

An operator $C : X \supset D(C) \to X^*$ is called "quasibounded" if for every $S > 0$ there exists $K(S) > 0$ such that $u \in D(C)$ with $\|u\| \le S,\ \langle Cu, u\rangle \le 0$ implies $\|Cu\| \le K(S)$.

An operator $C : X \supset D(C) \to X^*$ with $L \subset D(C)$ is said to be "generalized pseudomonotone" if $\{u_n\} \subset D(C),\ u_n \rightharpoonup u_0,\ Cu_n \rightharpoonup h_0^*$ and

$$\limsup_{n\to\infty} \langle Cu_n, u_n - u_0\rangle \le 0 \tag{59}$$

imply $u_0 \in D(C),\ Cu_0 = h_0^*$ and $\langle Cu_n, u_n\rangle \to \langle h_0^*, u_0\rangle$.

It is easy to see that if an operator C satisfies $(\widetilde{S}_+)$, then it is generalized pseudomonotone.

We denote by J_ψ the duality mapping with gauge function ψ. The function $\psi : \mathcal{R}_+ \to \mathcal{R}_+$ is continuous, strictly increasing and such that $\psi(0) = 0$ and $\psi(r) \to \infty$ at $r \to \infty$. This mapping J_ψ is continuous, bounded, surjective, strictly and maximal monotone, and satisfies condition (S_+). Also,

$$\langle J_\psi x, x\rangle = \psi(\|x\|)\|x\| \text{ and } \|J_\psi x\| = \psi(\|x\|),\ x \in X. \tag{60}$$

For these facts we refer to Petryshyn [18, pp. 32-33 and 132].

Theorem 4. *Let $G \subset X$ be open and bounded with $0 \in G$. Assume that the operator $T : X \supset D(T) \to 2^{X^*}$ is maximal monotone with $0 \in D(T)$ and $0 \in T(0)$. Assume that the operator $C : X \supset D(C) \to X^*$ is quasibounded, with $L \subset D(C)$, and satisfies $(\widetilde{S}_+)$ and c_3). Let $\varepsilon,\ \varepsilon_0$ and Λ be positive numbers. Assume that*

($\mathcal{P}$) *there exists $\lambda \in (0, \Lambda]$ such that the inclusion*

$$Tx + \lambda Cx + \varepsilon J_\psi x \ni 0 \tag{61}$$

has no solution in $D(T + C) \cap G$. Then

(i) *there exists $(\lambda_0, x_0) \in (0, \Lambda] \times (D(T + C) \cap \partial G)$ such that*

$$Tx_0 + \lambda_0 Cx_0 + \varepsilon J_\psi x_0 \ni 0; \tag{62}$$

(ii) *if $0 \notin T(D(T) \cap \partial G)$, T satisfies (S_q) on ∂G, and property ($\mathcal{P}$) is satisfied for every $\varepsilon \in (0, \varepsilon_0]$, then there exists $(\lambda_0, x_0) \in (0, \Lambda] \times (D(T + C) \cap \partial G)$ such that $Tx_0 + \lambda_0 Cx_0 \ni 0$.*

Before we prove this result, we should mention that conditions $t_1) - t_4)$ have now been replaced by the condition that T is maximal monotone (possible multivalued), $L \subset D(T)$ and $0 \in T(0)$. Also, the conditions on C are no longer involving the space L or the operator T. The degree mapping to be applied here (see [15]) comes from

$$d(T + C, G, 0) = \lim_{s\downarrow 0} d(T_s + C, G, 0), \tag{63}$$

where the degree mapping on the right-hand side is our degree from [13], which is fixed for all small values of $s > 0$. Finally, the domain of the operator T is not necessarily just the subspace L. The reader will have no trouble in extending Theorems 2 and 3 to other situations suggested by Theorem 4.

Proof of Theorem 4. (i) Assume that $\mathcal{P}$ is true and that the conclusion is false. Then (61) has no solution $(\lambda, x) \in (0, \Lambda] \times (D(T+C) \cap \partial G)$. We consider the homotopy inclusion

$$H(t,x) \equiv Tx + t\Lambda Cx + \varepsilon J_\psi x \ni 0, \quad t \in [0,1]. \tag{64}$$

This inclusion has no solution $x \in D(H(t,\cdot)) \cap \partial G$ for $t \in (0,1]$. This is also true for $t = 0$ because 0 is already in the set $(T + \varepsilon J_\psi)(0)$ and the operator $T + \varepsilon J_\psi$ is strictly monotone (and hence one-to-one). We are going to show that $H(t,x)$ is an admissible homotopy for this degree. We do this because this homotopy was not studied in [15].

We set $T^t = T$ and recall from the Introduction the properties of the operator $T_{t,s} = T_s \equiv (T^{-1} + sJ^{-1})^{-1} \ : \ X \to X^*,\ s > 0$. The operator T^t here should not be confused, for $t = 0$, with the operator $T^{\{0\}}$ in the Introduction. We also set $J_{t,s} = J_s \equiv I - sJ^{-1}T_{t,s} = I - sJ^{-1}T_s \ : \ X \to X$ and $C^t \equiv t\Lambda C + J_\psi$. We have $D(H(0,\cdot)) = D(T)$ and $D(H(t,\cdot)) = D(T+C),\ t \in (0,1]$. We also set $D^t = D(t\Lambda C) = D(tC)$. We have $D^0 = X$ and $D^t = D(C)$ for $t \in (0,1]$. Let G be an open and bounded subset of X.

We know that the equation

$$T^t x + C^t x \ni 0 \tag{65}$$

has no solution $x \in D(H(t,\cdot)) \cap \partial G$ for any $t \in [0,1]$. We consider the equation

$$T_s x + t\Lambda Cx + \varepsilon J_\psi = 0, \tag{66}$$

and show that there exists $s_1 > 0$ such that

$$0 \notin (T_s + t\Lambda C + \varepsilon J_\psi)(D^t \cap \partial G), \quad (s,t) \in (0, s_1] \times [0,1]. \tag{67}$$

Assume that this is not true, and let $\{s_n\} \subset (0,\infty),\ \{t_n\} \subset [0,1],\ \{x_n\} \subset \partial G$ be such that $s_n \downarrow 0,\ t_n \to t_0$ and $x_n \rightharpoonup x_0$, where $t_0 \in [0,1]$ and $x_0 \in X$, and

$$T_{s_n} x_n + t_n \Lambda C x_n + \varepsilon J_\psi x_n = 0. \tag{68}$$

Obviously, we cannot have $t_n = 0$ for any n, because $(T_{s_n} + J_\psi)(0) = 0$ and the operator $T_{s_n} + J_\psi$ is strictly monotone (hence one-to-one). Thus, $t_n > 0$ for all n. From (68) we see that

$$\langle Cx_n, x_n\rangle = -(1/[t_n\Lambda])\langle T_{s_n} x_n + \varepsilon J_\psi x_n, x_n\rangle \le 0.$$

This and the quasiboundedness of C imply that $\{Cx_n\}$ is bounded. We may thus assume that $Cx_n \rightharpoonup h_0^*$.

If $t_0 = 0$, then from

$$\varepsilon\psi(\|x_n\|)\|x_n\| = \langle \varepsilon J_\psi x_n, x_n\rangle \le -t_n\Lambda\langle Cx_n, x_n\rangle \to 0$$

we obtain $x_n \to 0 \in \partial G$, which is a contradiction to $0 \in G$. It follows that $t_0 > 0$. Since $\{T_{s_n} x_n\},\ \{J_\psi x_n\}$ are bounded, we may assume that $T_{s_n} x_n \rightharpoonup h_1^*$ and $J_\psi x_n \rightharpoonup h_2^*$ with $t_0\Lambda h_0^* = -h_1^* - \varepsilon h_2^*$.

We now claim that

$$\limsup_{n\to\infty} \langle t_n\Lambda Cx_n + \varepsilon J_\psi x_n, x_n - x_0\rangle \le 0. \tag{69}$$

If this is not true, then there is a subsequence of $\{t_n\}$, denoted again by $\{t_n\}$, such that

$$\lim_{n\to\infty} \langle t_n\Lambda Cx_n + \varepsilon J_\psi x_n, x_n - x_0\rangle > 0.$$

This implies

$$\lim_{n\to\infty}\langle T_{s_n}x_n, x_n - x_0\rangle < 0.$$

This and $T_{s_n}x_n \rightharpoonup h_1^*$ imply

$$\lim_{n\to\infty}\langle T_{s_n}x_n, x_n\rangle < \langle h_1^*, x_0\rangle. \tag{70}$$

We can now repeat the relevant part of the proof of Theorem 3, (ii) in [15] in order to obtain a contradiction. In fact, as in [15], we arrive at

$$\langle h_1^* - y, x_0 - x\rangle > 0, \quad \text{for every } x \in D(T),\ y \in Tx. \tag{71}$$

Since T is maximal monotone, we have $x_0 \in D(T)$ and $h_1^* = -t_0\Lambda h_0^* - \varepsilon h_2^* \in Tx_0$. However, this is a contradiction because (71) does not hold for $x = x_0,\ y = h_1^*$. Thus, (69) is true.

From (69) we easily obtain that there is a subsequence of $\{n\}$, denoted again by $\{n\}$, such that one of the following is true:

$$\limsup_{n\to\infty}\langle t_n\Lambda C_n x_n, x_n - x_0\rangle \le 0, \quad \limsup_{n\to\infty}\langle \varepsilon J_\psi x_n, x_n - x_0\rangle \le 0. \tag{72}$$

Assume that the first one is true. Then, by the $(\widetilde{S}_+)$-property of C, $x_n \to x_0,\ x_0 \in D(C)$ and $Cx_0 = h_0^*$. Then since

$$\lim_{n\to\infty}\langle t_n\Lambda C x_n + \varepsilon J_\psi x_n, x_n - x_0\rangle = 0,$$

we also get

$$\lim_{n\to\infty}\langle T_{s_n}x_n, x_n - x_0\rangle = 0,$$

which implies (70), but with an equality sign. Working again as in the argument following (70) (see [15], proof of Theorem 3, (ii)), we see that (71) holds now but for the "$\ge$" sign. It follows that $x_0 \in D(T)$ and $Tx_0 \ni -t_0\Lambda h_0^* - \varepsilon h_2^* = -t_0\Lambda C x_0 - \varepsilon J_\psi x_0$. This is a contradiction again because, by $x_n \to x_0$, we have $x_0 \in \partial G$. We have shown the validity of (67). An analogous proof holds when the second part of (72) is true.

We have shown that $H(t,x)$ is an admissible homotopy for our degree. We can now work as in Theorem 3 of [15] in order to show that $d(H(t,\cdot), G, 0) =$const. In fact, our case here is easier because the operator $T_{t,s}$ in [15] is now independent of t. Thus,

$$d(H(t,\cdot), G, 0) = d(T_s + \varepsilon J_\psi, G, 0) = 1.$$

The last equality above follows from Theorem 3, (i) of [15]. Consequently, the inclusion $H(t,x) \ni 0$ has a solution in G for each $t \in [0,1]$. In particular, this says that $Tx + \lambda Cx + \varepsilon J_\psi x \ni 0$ has a solution in G for every $\lambda \in (0,\Lambda]$. This is a contradiction to $(\mathcal{P})$ and finishes the proof of (i).

(ii) Let $\lambda_n \in (0,\Lambda],\ x_n \in D(C)\cap\partial G$ be such that, for some $u_n^* \in Tx_n$,

$$u_n^* + \lambda_n C x_n + (1/n)J_\psi x_n \ni 0. \tag{73}$$

Again, we cannot have $\lambda_n = 0$ for any n. Since $\lambda_n > 0$, we have $\langle Cx_n, x_n\rangle \le 0$, which implies the boundedness of $\{Cx_n\}$. We may assume that $\lambda_n \to \lambda_0,\ x_n \rightharpoonup x_0,\ Cx_n \rightharpoonup h_0^*$. Then $u_n^* \rightharpoonup -\lambda_0 h_0^*$. If $\lambda_0 = 0$, then (73) implies

$$\lim_{n\to\infty} u_n^* = \lim_{n\to\infty}[-\lambda_n C x_n - (1/n)J_\psi x_n] = 0.$$

Since T satisfies (S_q), this says that $x_n \to x_0 \in \partial G$. Now, we can invoke the demiclosedness of T (see Lemma A) to obtain $x_0 \in D(T)$ and $0 \in Tx_0$. This however contradicts $0 \notin T(D(T)\cap\partial G)$. Consequently, $\lambda_0 > 0$.

At this point we can repeat the method of proof of part (i) in order to get the inequality

$$\limsup_{n\to\infty}\langle Cx_n, x_n - x_0\rangle \le 0.$$

Since C satisfies $(\widetilde{S}_+)$, this implies $x_n \to x_0,\ x_0 \in D(C)$ and $Cx_0 = h_0^*$. Using the demiclosedness of T (see Lemma A), we obtain $x_0 \in D(T)\cap\partial G$ and $Tx_0+\lambda_0 Cx_0 \ni 0$. The proof is complete. □

5. A Fredholm alternative

The function $\widetilde{J}_\gamma$ below is the duality mapping of X with gauge function $\phi(r) \equiv r^\gamma$, where $\gamma > 0$. We have

$$\langle \widetilde{J}_\gamma x, x\rangle = \|x\|^{\gamma+1}, \quad \|\widetilde{J}_\gamma x\| = \|x\|^\gamma.$$

Let us assume that $T : X \supset D(T) \to X^*,\ C : X \supset D(T) \to X^*$ are such that $0 \in D(T) \cap D(C)$ and $T(0) = C(0) = 0$. Then a number $\lambda \in \mathcal{R}$ is called an "eigenvalue" of the pair (T,C) if the equation $\lambda Tx+Cx = 0$ has a nonzero solution in $D(T) \cap D(C)$. We denote by $\Lambda(T,C)$ the set of all eigenvalues of (T,C).

Our purpose in this section is to give a Fredholm alternative result in the sense of Necas (cf. [8, p. 61]). The operators $T,\ C$ are now homogeneous of degree $\gamma \ge 1$. This result has an analogue for linear operators C and $T = I$ mapping a Hilbert space X into itself. In that setting our result implies that if λ is not an eigenvalue of C (i.e. λ does not belong to the point spectrum of C), then the resolvent operator exists on all of X and is bounded.

Theorem 5. *Assume that L is a dense subspace of X. Assume that $T : L \to X^*$ is maximal monotone and satisfies t_4). Assume that $C : X \supset D(C) \to X^*,\ L \subset D(C)$ and C satisfies $c_1)-c_3$), but c_2) is satisfied with 0 in the second part of (33) replaced by any, but fixed, number $S > 0$. Assume that $T(0) = 0,\ C(0) = 0$, and that for every $x \in L$ and every $r > 0$ we have $T(rx) = r^\gamma Tx$ and $C(rx) = r^\gamma Cx$, where $\gamma \ge 1$ is fixed. Assume that $\lambda \ge 1$ and that the equation*

$$\lambda Tx + Cx + \mu\widetilde{J}_\gamma x = 0 \tag{E}$$

has only the zero solution for any $\mu > 0$. Then if $\lambda \notin \Lambda(T,C)$, the operator $\lambda T + C$ is surjective.

Proof. We first show that there exists a constant $\nu > 0$ and $\varepsilon_0 > 0$ such that

$$\|\lambda Tx + Cx + \varepsilon\widetilde{J}_\gamma x\| \ge \nu\|x\|^\gamma, \quad \text{for all } x \in L,\ \varepsilon \in (0,\varepsilon_0]. \tag{74}$$

If this is not true, there exist sequences $\{x_n\} \subset L$ and $\{\varepsilon_n\} \downarrow 0$ such that

$$\lim_{n\to\infty} \frac{1}{\|x_n\|^\gamma}\|\lambda Tx_n + Cx_n + \varepsilon_n\widetilde{J}_\gamma x_n\| = \lim_{n\to\infty}\left\|\lambda T\left(\frac{x_n}{\|x_n\|}\right) + C\left(\frac{x_n}{\|x_n\|}\right) + \varepsilon\widetilde{J}_\gamma\left(\frac{x_n}{\|x_n\|}\right)\right\| = 0.$$

Letting $u_n = x_n/\|x_n\|$, we have $\|u_n\| = 1$ and

$$\lim_{n\to\infty}\|\lambda Tu_n + Cu_n + \varepsilon_n\widetilde{J}_\gamma u_n\| = 0. \tag{75}$$

Since u_n is bounded, we may assume that $u_n \rightharpoonup u_0$. We have

$$v_n^* \equiv \lambda Tu_n + Cu_n \to 0. \tag{76}$$

Also, the boundedness of $\{u_n\}$, the inequality

$$\langle Tu_n + Cu_n, u_n\rangle \le \langle \lambda Tu_n + Cu_n, u_n\rangle = \langle v_n^*, u_n\rangle \le \|v_n^*\|\|u_n\| = \|v_n^*\|$$

and the quasiboundedness of C w.r.t. T imply the boundedness of the operator C. Thus, we may assume that $Cu_n \rightharpoonup c^* \in X^*$. We now claim that

$$\limsup_{n\to\infty}\langle Cu_n, u_n - u_0\rangle \le 0. \tag{77}$$

If this is not true, then there exists a subsequence of $\{u_n\}$, denoted by $\{u_n\}$ again such that

$$\lim_{n\to\infty}\langle Cu_n, u_n - u_0\rangle > 0. \tag{78}$$

This and (76) imply

$$\lambda \limsup_{n\to\infty}\langle Tu_n, u_n - u_0\rangle = \limsup_{n\to\infty}[\langle v_n^*, u_n - u_0\rangle - \langle Cu_n, u_n - u_0\rangle] < 0.$$

Once again, at this point we invoke our argument starting at (22) in the proof of Theorem 1 in order to obtain a contradiction to (78) and the validity of (77). Now, in view of (77) and the inequality following (76), we invoke the generalized (S_+)-property of C w.r.t. T in order to obtain $u_n \to u_0 \in D(C)$, $Cu_0 = c^*$. Since $\lambda Tu_n \rightharpoonup -Cu_0$, the demiclosedness of λT implies $\lambda Tu_0 + Cu_0 = 0$. However, since $\|u_0\| = 1$, we obtain a contradiction to our assumption $\lambda \notin \Lambda(T, C)$. It follows that (74) is true.

We now fix $p^* \in X^*$ and look at the homotopy equation

$$H(t, x) \equiv t(\lambda Tx + Cx + \varepsilon\widetilde{J}_\gamma x - p^*) + (1 - t)\varepsilon\widetilde{J}_\gamma x = 0. \tag{79}$$

All the solutions of (79) are bounded for $t = 1$. In fact, let $\{x_n\} \subset L$, $\|x_n\| \to \infty$ and

$$\lambda Tx_n + Cx_n + \varepsilon\widetilde{J}_\gamma x_n - p^* = 0.$$

We may assume that $\|x_n\| > 0$. Dividing by $\|x_n\|^\gamma$ we obtain

$$\lambda Tu_n + Cu_n + \varepsilon\widetilde{J}_\gamma u_n = p^*/\|x_n\| \equiv v_n^* \to 0.$$

Repeating the argument above about (76), we obtain again that $u_n \to u_0 \in L$ and $\lambda Tu_0 + Cu_0 + \varepsilon\widetilde{J}_\gamma u_0 = 0$. Since $\|u_0\| = 1$, this is a contradiction to (74). We also note that the only solution of (79) for $t = 0$ is $x = 0$. We show first that all solutions $x = x_t$ of equation (79) are bounded independently of $t \in [0, 1]$. By what we have just showed, we may assume that $t \in (0, 1)$.

Assume that our assertion is not true. Then there exist sequences $\{t_n\} \subset (0, 1)$, $\{x_n\} \subset L$ such that $\|x_n\| \to \infty$, $t_n \to t_0 \in [0, 1]$ and

$$t_n(\lambda Tx_n + Cx_n + \varepsilon\widetilde{J}_\gamma x_n - p^*) + (1 - t_n)\varepsilon\widetilde{J}_\gamma x_n = 0. \tag{80}$$

We distinguish two cases:

(j) $t_0 = 0$;

(jj) $t_0 > 0$.

(j) Since $t_n > 0$, we have

$$\lambda Tx_n + Cx_n - p^* + \frac{1}{t_n}\varepsilon\widetilde{J}_\gamma x_n = 0. \tag{81}$$

Assuming, without loss of generality, that $\|x_n\| \geq 1$, we have

$$\begin{aligned}(82)\qquad &\frac{1}{\|x_n\|^\gamma}(\lambda T x_n + C x_n - p^*) + \frac{1}{\|x_n\|^\gamma}\frac{1}{t_n}\varepsilon \widetilde{J}_\gamma x_n \\ &= \lambda T\left(\frac{x_n}{\|x_n\|}\right) + C\left(\frac{x_n}{\|x_n\|}\right) - \frac{p^*}{\|x_n\|^\gamma} + \frac{1}{t_n}\varepsilon \widetilde{J}_\gamma \left(\frac{x_n}{\|x_n\|}\right) = 0.\end{aligned}$$

Letting $u_n \equiv x_n/\|x_n\|$ and $q_n \equiv 1/t_n$ in (82), we obtain $q_n > 0,\ q_n \to +\infty$ and

$$(83)\qquad \lambda T u_n + C u_n - (p^*/\|x_n\|^\gamma) + q_n \widetilde{J}_\gamma u_n = 0.$$

Since $T(0) = 0$, we have $\langle T u_n, u_n\rangle \geq 0$. We also have $\langle \widetilde{J}_\gamma u_n, u_n\rangle = \|u_n\|^{\gamma+1} = 1$. Thus,

$$\begin{aligned}(84)\qquad \langle T u_n + C u_n, u_n\rangle &\leq \langle \lambda T u_n + C u_n, u_n\rangle \\ &= \langle p^*/\|x_n\|^\gamma, u_n\rangle - q_n\langle \widetilde{J}_\gamma u_n, u_n\rangle \leq \|p^*\|\|u_n\| = \|p^*\|.\end{aligned}$$

Since C is quasibounded, $\|u_n\| = 1$ and (84) imply that $\{C u_n\}$ and $\langle C u_n, u_n\rangle$ are bounded. Using this in (84) we obtain

$$0 \leq \langle T u_n, u_n\rangle \leq -\langle C u_n, u_n\rangle + \|p^*\| - q_n \to -\infty.$$

This contradiction covers the case (j).

(jj) We are again working with (83) with $u_n \rightharpoonup u_0,\ C u_n \rightharpoonup c^*$. We now have $q_n \to q_0 \equiv 1/t_0 \geq 1$. Once again, we claim that

$$(85)\qquad \limsup_{n\to\infty} \langle C u_n, u_n - u_0\rangle \leq 0.$$

If this is not true, then (83) implies

$$\limsup_{n\to\infty} \langle T u_n, u_n - u_0\rangle \leq 0.$$

Following the argument about (22) in the proof of Theorem 1, we get $u_0 \in L$, which, along with (83), implies (85), i.e. a contradiction. Since C satisfies the generalized condition (S_+) w.r.t. T, we have $u_n \to u_0$ and $C u_n \rightharpoonup C u_0$. We also have $\widetilde{J}_\gamma u_n \to \widetilde{J}_\gamma u_0$ and $\lambda T u_n \rightharpoonup -C u_0 - q_0 \varepsilon \widetilde{J}_\gamma u_0$. By the demiclosedness of λT, we get $\lambda T u_0 + C u_0 + q_0 \varepsilon \widetilde{J}_\gamma u_0 = 0$. Since $u_0 \in \partial B_1(0)$, we have a contradiction to our assumption about (E). It follows that all possible solutions of the homotopy equation (79) are bounded independently of $t \in [0,1]$. Assume that they all lie inside the ball $B_K(0)$, for some $K > 0$. We remark that $H(t,x)$ is an admissible homotopy for our degree in [13]. Because of this,

$$\begin{aligned} d(H(t,\cdot), B_K(0), 0) &= d(H(1,\cdot), B_K(0), 0) \\ &= d(H(0,\cdot), B_K(0), 0) = d(\widetilde{J}_\gamma, B_K(0), 0) = 1.\end{aligned}$$

The last equality above follows from the fact that $\widetilde{J}_\gamma$ is demicontinuous, bounded, satisfies (S_+), is one-to-one on $\overline{B_K(0)}$ and such that $\langle \widetilde{J}_\gamma x, x\rangle \geq 0$ for every $x \in \partial B_K(0)$. Here, we quote Browder [6, Theorem 3, (iv)]. It follows that the equation

$$(86)\qquad \lambda T x + C x + (1/n)\widetilde{J}_\gamma x = p^*$$

is solvable for all large n. We may assume that this is true for all $n \geq 1$. Let $\{x_n\} \subset L$ solve (86). Then

$$(87)\qquad \lambda T x_n + C x_n + (1/n)\widetilde{J}_\gamma x_n = p^*.$$

If we assume that $\{x_n\}$, or a subsequence of it denoted again by $\{x_n\}$, is such that $\|x_n\| \to \infty$, we can divide (87) by $\|x_n\|$ and arrive at (76) with

$$v_n^* \equiv \lambda T u_n + C u_n = -(1/n)\widetilde{J}_\gamma u_n + p^*/\|x_n\| \to 0, \tag{88}$$

where $u_n = x_n/\|x_n\|$. Assuming that $u_n \rightharpoonup u_0$, we use (88) to arrive again at $u_n \to u_0$ with $\lambda T u_0 + C u_0 = 0$, which is a contradiction to $\lambda \notin \Lambda(T, C)$.

It follows that $\{x_n\}$ in (87) is bounded. Since

$$\langle T x_n + C x_n, x_n \rangle \le \langle \lambda T x_n + C x_n, x_n \rangle \le -(1/n)\|x_n\|^{\gamma+1} + \|p^*\|\|x_n\|,$$

the quasiboundedness of C w.r.t. T implies that $\{Cx_n\}$ is bounded. We may assume that $x_n \rightharpoonup x_0$ and $Cx_n \rightharpoonup c^*$. Again repeating the argument about (22) as in the proof of Theorem 1, we obtain (85). With (85), we use the generalized (S_+)-property of C w.r.t. T in order to obtain $x_n \to x_0$, $x_0 \in D(C)$ and $Cx_n \rightharpoonup Cx_0$. Again using the demiclosedness of λT, we obtain $x_0 \in L$ and $\lambda T x_0 + C x_0 = p^*$, and the proof is finished. □

It should be noted that the assumption that (E) has only the zero solution for any $\mu > 0$ cannot be omitted, in its entirety, in Theorem 5. In fact, let $Tx = ax^m$, $Cx = b|x|^m$, $\lambda = 2$, $\gamma = m$, $b > 2a > 0$. Here, m is an odd positive integer. The equation

$$\lambda T x + C x + \mu \widetilde{J}_m x = 2ax^m + b|x|^m + \mu x^m = (2a + \mu)x^m + b|x|^m = 0$$

has every negative number x as a solution for $\mu = b - 2a$ or $2a + \mu = b$. Also, the operator $\lambda T x + C x = b|x|^m + 2ax^m \ge 2a(|x|^m + x^m) \ge 0$ is not surjective. We note that $\lambda \notin \Lambda(T, C)$ because $\lambda T x + C x = 0$ implies $x = 0$. Here, we have used $\widetilde{J}_m x = \operatorname{grad} |x|^{m+1}/(m+1) = |x|^m \operatorname{grad} |x| = |x|^m \operatorname{sgn} x = x^m$, for all $x \neq 0$.

6. Operators C defined on $\overline{D(T)}$

In this section we are not assuming everywhere that, for the Banach space X, X, X^* are locally uniformly convex. The following result was given by Guan and Kartsatos in [9].

Theorem A. *Assume that $T : X \supset \overline{D} \to X$ is accretive, bounded and $C : \overline{D} \to X$ is compact, where D is an open, bounded subset of X with $0 \in D$. Assume that there exists a constant $c > 0$ such that the equation $Tx - Cx + cx = 0$ has no solution $x \in \overline{D}$ and let one of the following conditions be satisfied:*

(i) *X^* is uniformly convex and T is demicontinuous.*
(ii) *T is continuous.*

Then there exists $(\lambda_0, x_0) \in (0,1) \times \partial D$ such that $Tx_0 - \lambda_0 C x_0 + c x_0 = 0$. If, moreover, $0 \notin T(\partial D)$, T is ϕ-expansive on ∂D and $Tx - Cx + cx = 0$ has no solution $x \in \overline{D}$ for all small $c > 0$, there exists $(\lambda_0, x_0) \in (0,1) \times \partial D$ such that $Tx_0 - \lambda_0 C x_0 = 0$.

Li and Huang gave two eigenvalue results in [16] extending Theorem A, where T is maximal monotone and C is compact or completely continuous. As was mentioned by Guan and Kartsatos in [9], as well as other authors before, a considerable amount of eigenvalue existence theory is based on a result which is a simple but fundamental consequence of the Leray-Schauder theory in combination with situations like theorem A above. According to this result, if the compact operator $C : \overline{G} \to X$ has no fixed points in $\overline{G}$, then there exists $(\lambda_0, x_0) \in (0,1) \times \partial G$ such that $(I - \lambda_0 C)x_0 = 0$. It is easy to see that in this particular case we have

$d(I-C,G,0)=0$. Such considerations, with substantial extensions and refinements, were used by the authors in [12].

This section provides an eigenvalue result along these lines for operators $T,\ C$, where T is maximal monotone with compact resolvents and C is defined on $\overline{D(T)}$ and is continuous and bounded there. This result complements the two results of Li and Huang in [16].

The resolvents $(T+\varepsilon J)^{-1}$ of the maximal monotone operator T are strongly continuous mappings for all $\varepsilon > 0$. Also, if one of them is compact, then they all are (cf., e.g., Kartsatos [11]).

Theorem 6. *Let $T : X \supset D(T) \to 2^{X^*}$ be maximal monotone with compact resolvents. Assume that $C : X \supset \overline{D(T)} \to X^*$ is continuous and bounded. Let $G \subset X$ be open and bounded and such that $0 \in D(T) \cap G$ and $0 \in T(0)$. Let $\varepsilon,\ \varepsilon_0,\ \Lambda$ be given positive numbers. Assume that*

($\mathcal{P}$) *there exists $\lambda \in (0,\Lambda]$ such that the inclusion*

$$Tx+\lambda Cx+\varepsilon Jx \ni 0 \tag{89}$$

has no solution in $D(T)\cap G$. Then

(i) *there exists $(\lambda_0,x_0) \in (0,\Lambda]\times(D(T)\cap\partial G)$ such that*

$$Tx_0+\lambda_0 Cx_0+\varepsilon Jx_0 \ni 0; \tag{90}$$

(ii) *if $0 \notin T(D(T)\cap\partial G)$ and property ($\mathcal{P}$) is satisfied for every $\varepsilon \in (0,\varepsilon_0]$, then there exists $(\lambda_0,x_0) \in (0,\Lambda]\times(D(T)\cap\partial G)$ such that $Tx_0+\lambda_0 Cx_0 \ni 0$.*

Proof. (i) We now consider the homotopy equation

$$H(\lambda,u) \equiv u+\lambda C(T+\varepsilon J)^{-1}u = 0, \tag{91}$$

for $\lambda \in (0,\Lambda],\ u \in D \equiv (T+\varepsilon J)(D(T)\cap\overline{G}) \subset X^*$. We notice that when $u \in D$ we have

$$C(T+\varepsilon J)^{-1}u \in C(D(T)\cap\overline{G}),$$

where the set on the right is bounded. Consequently, every solution $u\in D$ of (91) satisfies

$$\|u\| \le \Lambda K, \quad \text{where } K = \sup_{x\in D(T)\cap\overline{G}}\{\|Cx\|\}. \tag{92}$$

We fix $s > \Lambda K$ and let $Q \equiv T+\varepsilon J$. We note that Q is injective and surjective with a continuous inverse $Q^{-1} : X^* \to X$. This implies that $Q(D(T)\cap G)$ is open and $Q(D(T)\cap\overline{G})$ is closed in X^*. In addition,

$$Q(D(T)\cap\overline{G}) = Q(D(T)\cap G)\cup Q(D(T)\cap\partial G)$$

and

$$\overline{Q(D(T)\cap G)} = Q(D(T)\cap G)\cup \partial Q(D(T)\cap G)$$

imply

$$Q(D(T)\cap\overline{G}) \supset \overline{Q(D(T)\cap G)} \tag{93}$$

and

$$Q(D(T)\cap\partial G) \supset \partial Q(D(T)\cap G). \tag{94}$$

It follows that in order to solve (91) in $(T+\varepsilon J)(D(T)\cap\overline{G})$, via Leray-Schauder degree theory, it suffices to consider it only for $u \in \overline{U\cap B_s(0)}$, where $U \equiv (T+\varepsilon J)(D(T)\cap G)$.

We note that the set $U \cap B_s(0)$ is open and bounded. We also note that the set $C(T+\varepsilon J)^{-1}(\overline{U \cap B_s(0)})$ is compact by the compactness of the resolvent $(T+\varepsilon J)^{-1}$ and the continuity of C. Thus, the Leray-Schauder degree $d(H(\lambda,\cdot),U,0)$ will be well defined for all $\lambda \in [0,\Lambda]$ if

$$0 \notin (I+\lambda F)(\partial(U \cap B_s(0))), \tag{95}$$

where $F \equiv C(T+\varepsilon J)^{-1} : \overline{U \cap B_s(0)} \to X^*$. Note that $0 \in U \cap B_s(0)$. Since $\partial(U \cap B_s(0)) \subset \partial U \cup \partial B_s(0)$, we know that (95) is true for $\lambda = 0$. We assume that $\lambda > 0$ and that (90) is not true. Then a) for every $\lambda \in (0,\Lambda]$ equation (89) has no solution $x \in D(T) \cap \partial G$.

Let us assume that b) there is $\lambda \in (0,\Lambda]$ such that $(I+\lambda F)u = 0$ has a solution $u \in \partial(U \cap B_s(0))$. We know that $\|u\| \le \Lambda K < s$, so that $u \notin \partial B_s(0)$. Actually, $u \notin \partial U$ either. In fact, if $u \in \partial U$, then (94) implies $u \in Q(D(T) \cap \partial G) = (T+\varepsilon J)(D(T) \cap \partial G)$. Letting $u = y^* + \varepsilon Jx$, with $x \in D(T) \cap \partial G$ and $y^* \in Tx$, we see from (91) that

$$y^* + \varepsilon Jx + \lambda Cx = 0$$

or

$$Tx + \lambda Cx + \varepsilon Jx \ni 0.$$

Since $x \in D(T) \cap \partial G$, we have a contradiction to our assumption a).

It follows that

$$d(H(\lambda,\cdot),U,0) = d(H(0,\cdot),U,0) = 1.$$

Consequently, the equation $H(\lambda,u) \equiv (I+\lambda F)u = 0$ is solvable with $u \in ((T+\varepsilon J)(D(T) \cap G)) \cap B_s(0)$ for every $\lambda \in (0,\Lambda]$. This implies that equation (89) is solvable in $D(T) \cap G$ for every $\lambda \in (0,\Lambda]$, i.e. a contradiction to $(\mathcal{P})$. Consequently, (90) holds and (i) is true.

(ii) We may assume that there exists a sequence $\{(\lambda_n, x_n)\} \subset (0,\Lambda] \times (D(T) \cap \partial G)$ such that

$$Tx_n + \lambda_n Cx_n + (1/n)Jx_n \ni 0, \quad n = 1,2,\ldots. \tag{96}$$

Then

$$x_n = (T+J)^{-1}\left[-\lambda_n Cx_n + (1-(1/n))Jx_n\right].$$

Since $\{x_n\}$ and $\{\lambda_n Cx_n\}$ are bounded, the compactness of $(T+J)^{-1}$ implies that $\{x_n\}$ lies in a compact set. Thus, we may assume that $x_n \to x_0 \in \overline{D(T)} \cap \partial G$. We may also assume that $\lambda_n \to \lambda_0 \in [0,\Lambda]$. If $\lambda_0 = 0$, then the closedness of T implies $x_0 \in D(T)$ and $0 \in Tx_0$. This however is a contradiction to our assumption $0 \notin T(D(T) \cap \partial G)$. Consequently, $\lambda_0 > 0$ and, again by the closedness of T, $x_0 \in D(T)$ and $Tx_0 + \lambda_0 Cx_0 \ni 0$. The proof is finished. □

If the operator C in Theorem 6 is assumed to be defined just on the open and bounded set $\overline{G}$, then it can be extended to an operator $\widetilde{C}$ on all of X by Dugundji's theorem. The operator $\widetilde{C}$ is continuous and its range lies in the convex hull of the range of C. It is thus bounded. The proof of Theorem 6 goes through in this case, with C replaced by $\widetilde{C}$, without any further modifications. Other versions of Theorem 6 include the case where the resolvents of T are completely continuous and C is bounded and demicontinuous.

7. Continuous branches of eigenvectors

It is easy to see that condition $(\mathcal{P})$ does not allow $C(\lambda,0)=0$ in Theorem 1 , or $C(0)=0$ in Theorems 2-4, where λ is as in condition $(\mathcal{P})$. As it was easily shown in [9, Lemma 4.2] for accretive operators T and $J_\psi = J$, if $T : X \supset D(T) \to 2^{X^*}$, $C : X \supset D(C) \to X^*$ are such that $|(T+\lambda C)x| \geq \alpha > 0$ for $x \in D(T+C)\cap G$, then there exists $\varepsilon_0 > 0$ such that the inclusion $Tx + \lambda Cx + \varepsilon J_\psi x \ni 0$ has no solution x in $D(T+C)\cap G$ for any $\varepsilon \in (0,\varepsilon_0)$. Here, λ is a fixed positive number, and for a set A, $|A| = inf\{\|x\| \; ; \; x \in A\}$. In fact, if the constant α is as above,

$$M = \sup\{\psi(\|x\|) \; : \; x \in G\},$$

$\varepsilon_0 \in (0, \alpha/M),\ \varepsilon \in (0,\varepsilon_0),\ x \in D(T+C)\cap G$ and $y^* \in Tx$, then

$$\|y^* + \lambda Cx + \varepsilon J_\psi x\| \geq |Tx + \lambda Cx| - \varepsilon_0 \psi(\|x\|) \geq \alpha - \varepsilon_0 M > \alpha - \alpha = 0.$$

Thus, the assumption $(\mathcal{P})$ in several theorems above may be replaced by an assumption like

$(\mathcal{P}_1)$ there exists $\lambda \in (0,\Lambda]$ and $\alpha > 0$ such that

$$|Tx + \lambda Cx| \geq \alpha, \quad x \in D(T+C)\cap G.$$

Condition $(\mathcal{P}_1)$ implies that $C(0) \neq 0$. The conclusion in this case is obvious.

Analogous remarks are valid for the implicit case $C = C(\lambda, x)$.

We are now going to show that the results of this paper allow for the existence of continuous branches of eigenvectors. We need the following definition.

Definition 1. Let $T : X \supset D(T) \to 2^{X^*},\ C : \mathcal{R} \times X \supset D(C) \to X^*$, be given and consider the problem

$$Tx + C(\lambda, x) \ni 0. \tag{97}$$

An "eigenvector" x is a solution of (97) for some "eigenvalue" λ with $x \in D(T)$ and $(\lambda, x) \in D(C)$. We say that the nonzero eigenvectors of the problem (97) form a "continuous branch of infinite length" if there exists $r_0 > 0$ such that, for every $r \geq r_0$, the sphere $\partial B_r(0)$ contains at least one nonzero eigenvector of (97).

We give below a result according to which the problem

$$Tx + \lambda Cx \ni 0 \tag{98}$$

has nonzero eigenvectors forming a continuous branch of infinite length. This is done in the setting of Theorem 4. Analogous results hold for the other eigenvalue problems studied above.

Theorem 7. *Assume that the operator $T : X \supset D(T) \to 2^{X^*}$ is maximal monotone with $0 \in D(T)$ and $0 \in T(0)$. Assume that the operator $C : X \supset D(C) \to X^*$ is quasibounded, with $L \subset D(C)$, and satisfies $(\widetilde{S}_+)$ and c_3). Let Λ be a positive number. Assume that $Tx \ni 0$ implies $x = 0$, T satisfies (S_q) and*

$(\mathcal{P}_1)$ there exist $\alpha > 0$ and $\lambda \in (0,\Lambda]$ such that

$$|Tx + \lambda Cx| \geq \alpha, \quad x \in D(T+C). \tag{99}$$

Then the nonzero eigenvectors of the problem (98) form a continuous branch of infinite length with corresponding eigenvalues $\lambda \in (0,\Lambda]$.

Proof. Let $r_0 > 0$ be given. Let $\varepsilon_0 > 0$ be so small that $\varepsilon_0 r_0 < \alpha$. Then

$$|Tx + \lambda Cx + \varepsilon Jx| \geq \alpha - \varepsilon\|x\| \geq \alpha - \varepsilon_0 r_0 > 0, \quad x \in D(T+C),$$

implies that the inclusion

$$Tx + \lambda Cx + \varepsilon Jx \ni 0 \tag{100}$$

has no solution $x \in D(T+C) \cap B_{r_0}(0)$ for any $\varepsilon \in (0, \varepsilon_0]$. Since $0 \notin T(\partial B_{r_0}(0))$ and T is of type (S_q), Theorem 4 implies the existence of a solution $x_{\lambda_0} \in D(T+C) \cap \partial B_{r_0}(0)$, for some $\lambda_0 \in (0, \Lambda]$. The same argument can be repeated for any number $r > r_0$ instead of r_0 itself. The proof is complete. □

In the following result we assume that the operator T is defined and bounded on all of X. We do this in order to demonstrate the fact that the assumption $|Tx + Cx| \geq \alpha$ may be replaced in this case by the assumption $\|Cx\| \geq \alpha$ on $D(C)$. An operator $T : X \supset D(T) \to 2^{X^*}$ is called "bounded" if for every bounded set $M \subset X$ the set $\bigcup\{Tx \ : \ x \in D(T) \cap M\}$ is bounded.

Theorem 8. *Assume that the operator $T : D(T) = X \to 2^{X^*}$ is maximal monotone and bounded with $0 \in T(0)$. Assume that the operator $C : X \supset D(C) \to X^*$ is quasibounded, with $L \subset D(C)$, and satisfies $(\widetilde{S}_+)$ and c_3). Assume that $Tx \ni 0$ implies $x = 0$, T satisfies (S_q) and there exists $\alpha > 0$ such that*

$$\|Cx\| \geq \alpha, \quad x \in D(C). \tag{101}$$

Then the nonzero eigenvectors of the problem (98) *form a continuous branch of infinite length.*

Proof. We show that the problem (98) possesses eigenvectors on the set $\partial B_r(0)$ for every $r > 0$. To this end, we fix $r > 0$, $\varepsilon > 0$ and show first that there exists $\widetilde{\lambda} > 0$ such that

$$d(T + \widetilde{\lambda} C + \varepsilon J, B_r(0), 0) = 0. \tag{103}$$

If this is not true, then there exists a sequence $\{\lambda_n\} \subset (0, \infty)$ such that $\lambda_n \to \infty$ and one of the following holds:

(i) the degree $d(T + \lambda_n C + \varepsilon J, B_r(0), 0)$ is not well defined;
(ii) $d(T + \lambda_n C + \varepsilon J, B_r(0), 0) \neq 0$.

In case (i) there exist eigenvectors $x_n \in \partial B_r(0)$ such that

$$Tx_n + \lambda_n Cx_n + \varepsilon Jx_n \ni 0. \tag{104}$$

In case (ii) there exist eigenvectors $x_n \in B_r(0)$ such that (104) holds. Thus, in either case, there exists a sequence $\{x_n\} \subset \overline{B_r(0)}$ such that (104) holds. However, this leads to a contradiction because $\|\lambda_n Cx_n + \varepsilon J\| \geq \alpha\lambda_n - \varepsilon r \to \infty$, while the sets Tx_n lie in a fixed bounded set. Thus, (103) is true for some $\widetilde{\lambda} > 0$.

We consider the homotopy

$$H(t, x) \equiv Tx + t\widetilde{\lambda} Cx + \varepsilon Jx, \quad t \in [0,1], \ x \in D(H(t, \cdot)). \tag{105}$$

Either there exist $t_0 \in [0,1]$ and $x_{t_0} \in \partial B_r(0)$ such that

$$Tx_{t_0} + t_0\widetilde{\lambda} Cx_{t_0} + \varepsilon Jx_{t_0} \ni 0, \tag{106}$$

or

$$d(H(t, \cdot) = d(H(1, \cdot) = 0 = d(H(0, \cdot) = 1, \quad t \in [0,1], \tag{107}$$

i.e. a contradiction. The last equality in (107) follows from Theorem 3, (i), in [15]. It follows that (106) is true. Naturally, we must have $t_0 \neq 0$ in (106) because otherwise $0 \in (T + \varepsilon J)(\partial B_r(0))$. This cannot happen because we already have $0 \in (T + \varepsilon J)(0)$ and $T + \varepsilon J$ is one-to-one.

From (106) we obtain sequences $\lambda_n \in (0, \infty)$, $\{x_n\} \subset \partial B_r(0)$ such that

$$Tx_n + \lambda_n Cx_n + (1/n)Jx_n \ni 0. \tag{108}$$

We may assume that $x_n \rightharpoonup x_0$. Again, $\{\lambda_n\}$ cannot contain a subsequence $\{\lambda_{n_k}\}$ such that $\lambda_{n_k} \to \infty$ as $k \to \infty$ because the sequence $\{Tx_{n_k} + (1/n_k)Jx_{n_k}\}$ lies in a bounded set and $\lambda_{n_k}\|Cx_{n_k}\| \to \infty$ as $k \to \infty$.

From the quasiboundedness of C and

$$\langle Cx_n, x_n\rangle = -(1/\lambda_n)\langle y_n^* + (1/n)Jx_n, x_n\rangle \leq 0,$$

we obtain that $\{Cx_n\}$ is bounded. Thus, we may assume that $Cx_n \rightharpoonup c^* \in X^*$. Since the sequence $\{\lambda_n\}$ is bounded, we may assume that $\lambda_n \to \lambda_0$. Again, $\lambda_0 \neq 0$ otherwise $\lambda_n Cx_n + (1/n)Jx_n \to 0$ and the (S_q)-property of T would imply that $x_n \to x_0 \in \partial G$. Since T is demiclosed, Lemma A would imply $Tx_0 \ni 0$. This is a contradiction to our assumption that $Tx \ni 0$ implies $x = 0$. It follows that $\lambda_0 > 0$ and we may also assume that $\lambda_n > 0$ for all n.

It is now easy to see that

$$\limsup_{n\to\infty}\langle Cx_n, x_n - x_0\rangle \leq 0.$$

Arguments like this were used in the proofs of Theorems 3 and 4. From the $(\widetilde{S}_+)$-property of C we conclude that $x_n \to x_0$, $x_0 \in D(C)$ and $Cx_0 = c^*$. Thus, since $y_n^* \rightharpoonup -\lambda_0 Cx_0$, $x_n \to x_0$ and T is demiclosed (see Lemma A), we obtain $x_0 \in D(C) \cap \partial B_r(0)$ and $Tx_0 + \lambda_0 Cx_0 \ni 0$.

Since $r > 0$ is arbitrary, the nonzero eigenvectors of problem (98) form a continuous branch of infinite length. □

The following result is a variant of Theorem 8.

Theorem 9. *Assume that the operator $T : D(T) = X \to 2^{X^*}$ is maximal monotone and bounded. Assume that the operator $C : X \supset D(C) \to X^*$ is bounded, with $L \subset D(C)$, and satisfies $(\widetilde{S}_+)$ and c_3). Assume that $Tx \ni 0$ implies $x = 0$, T satisfies (S_q) and there exists $r_0 > 0$ such that*

$$\inf\{\|Cx\| \ :\ x \in D(C) \setminus B_{r_0}(0)\} \equiv \alpha > 0 \tag{109}$$

and, for each $r \geq r_0$,

$$\overline{\left\{\frac{Cx}{\|Cx\|}\ :\ x \in D(C) \cap \partial B_r(0)\right\}} \neq \partial B_1(0). \tag{110}$$

Then the nonzero eigenvectors of the problem (98) form a continuous branch of infinite length.

Proof. Fix $r \geq r_0$. Then, by (110), there exists $y_0^* \in \partial B_1(0)$ such that

$$y_0^* \notin \overline{\left\{\frac{Cx}{\|Cx\|}\ :\ x \in D(C) \cap \partial B_r(0)\right\}}. \tag{110_a}$$

We fix $\varepsilon > 0$ and show that there exists $\widetilde{\lambda} > 0$ such that

$$(T + \widetilde{\lambda}C + \varepsilon J)x \not\ni \eta y_0^*, \quad (\eta, x) \in (0, \infty) \times (D(T + C) \cap \partial B_r(0)). \tag{111}$$

Assume that this is not true. Then there exist sequences $\{\lambda_n\} \subset (0,\infty),\ \eta_n \subset (0,\infty),\ \{x_n\} \subset D(T+C)\cap\partial B_r(0),\ y_n^* \in Tx_n$ such that $\lambda_n \to \infty$ and

$$y_n^* + \lambda_n C x_n + \varepsilon J x_n = \eta_n y_0^*. \tag{112}$$

Dividing above by λ_n and taking into consideration that $\{y_n^* + \varepsilon J x_n\}$ is bounded, we obtain

$$\lim_{n\to\infty}\left(Cx_n - \frac{\eta_n}{\lambda_n} y_0^*\right) = 0. \tag{113}$$

By (109), $\|Cx_n\| \geq \alpha$. Thus, (113) implies that the sequence $\{\eta_n/\lambda_n\}$ is bounded. We may assume that $\eta_n/\lambda_n \to \mu \in (0,\infty)$. Obviously, this implies $\eta_n \to \infty$. Consequently, from (112) we obtain

$$\frac{\lambda_n}{\eta_n}\|Cx_n\| \to \|y_0^*\| = 1.$$

This, along with

$$\frac{\lambda_n}{\eta_n} C x_n \to y_0^*,$$

says that

$$\frac{Cx_n}{\|Cx_n\|} \to y_0^*,$$

which contradicts (110_a). It follows that (111) is true.

Now assume that $(T+\widetilde{\lambda}C+\varepsilon J)x \not\ni 0,\ x \in D(T+C)\cap\partial B_r(0)$. Then (111) holds for every $\eta \geq 0$. We claim that the degree $d(T+\widetilde{\lambda}C+\varepsilon J, B_r(0), 0)$, which is well defined by (111), equals 0. In fact, assume that the contrary holds and consider the homotopy function

$$H_n(t,x) \equiv (T+\widetilde{\lambda}C+\varepsilon J)x - nty_0^*, \quad (t,x) \in [0,1]\times(D(T+C)\cap\overline{B_r(0)}).$$

We note that this is an admissible homotopy for our degree in [15] (see [15, Theorem 3, (iv)]). Because of this, we have

$$d(H_n(1,\cdot), B_r(0), 0) = d(H_n(0,\cdot), B_r(0), 0) = d(T+\lambda C+\varepsilon J, B_r(0), 0) \neq 0.$$

Consequently, for every n, the inclusion

$$Tx_n + \widetilde{\lambda}Cx_n + \varepsilon J x_n \ni ny_0^*$$

is solvable with solution $x_n \in B_r(0)$. This, however, is a contradiction to the boundedness of the operator $T+\widetilde{\lambda}C+\varepsilon J$. It follows that $d(T+\widetilde{\lambda}C+\varepsilon J, B_r(0), 0) = 0$. Since

$$d(T+\varepsilon J, B_r(0), 0) = 1,$$

by Theorem 3, (i), of [15], there must exist $\lambda_\varepsilon > 0$ such that $(T+\lambda_\varepsilon C+\varepsilon J)x_\varepsilon \ni 0$ for some $x_\varepsilon \in \partial B_r(0)$.

Let $x_n \in \partial B_r(0)$ solve

$$Tx_n + \lambda_n C x_n + (1/n)Jx_n \ni 0.$$

Then, since T and J are bounded and C is bounded below, we cannot have a subsequence of $\{\lambda_n\}$ converging to ∞ as $n \to \infty$. Thus, we may assume that $\lambda_n \to \lambda_0 \in [0,\infty)$. We may also assume that $x_n \rightharpoonup x_0 \in \overline{B_r(0)}$ and $Cx_n \rightharpoonup h^*$. If $\lambda_0 = 0$, then the (S_q)-property of T implies that $x_n \to x_0 \in \partial B_r(0)$, while its demiclosedness says that $x_0 \in D(T)\cap\partial B_r(0)$ and $Tx_0 = 0$, i.e. a contradiction. It follows that $\lambda_0 > 0$.

We can now work as in the proof of Theorem 8 to show that

$$\limsup_{n\to\infty}\langle Cx_n, x_n - x_0\rangle \le 0.$$

This implies that $x_n \to x_0 \in D(C)$ and $Cx_0 = h^*$. Again, the demiclosedness of T says that $x_0 \in D(T)$ and $Tx_0 + \lambda_0 Cx_0 \ni 0$. Since $r \ge r_0$ is arbitrary, we have that the nonzero eigenvectors of problem (98) form a continuous branch of infinite length. □

8. Applications

Application 1. This application is connected with Theorem 3. We shall study the existence of eigenvectors of second order nonlinear elliptic equations normalized by their norms in $L^2(\Omega)$.

We assume that Ω is a bounded open set in $\mathcal{R}^n$ with boundary $\partial\Omega$ belonging to $C^{2,\alpha}$, for some $\alpha > 0$.

Assume that the functions $a_i(x,u),\ i = 0,1,\dots,n$, are defined for $x \in \Omega,\ u \in \mathcal{R}$, measurable w.r.t. x for all u, and continuous w.r.t. u for almost all x. We also assume the inequalities

$$\begin{aligned} &|a_i(x,u)| \le \nu_1,\ i = 1,\dots,n, \\ &|a_0(x,u)| \le \nu_1|u| + a(x) \end{aligned} \tag{114}$$

with a positive constant ν_1 and $a \in L^2(\Omega)$.

We consider the eigenvalue problem

$$\Delta u + \lambda\left\{\sum_{i=1}^{n} a_i(x,u)\frac{\partial u}{\partial x_i} + a_0(x,u)\right\} = 0, \quad x \in \Omega, \tag{115}$$

$$u(x) = 0, \quad x \in \partial\Omega, \tag{116}$$

with normalized condition

$$\|u\|_{L^2(\Omega)} = 1. \tag{117}$$

We also consider the auxiliary equation

$$\Delta u + \varepsilon u + \lambda_\varepsilon\left\{\sum_{i=1}^{n} a_i(x,u)\frac{\partial u}{\partial x_i} + a_0(x,u)\right\} = 0, \quad x \in \Omega. \tag{118}$$

We study the solvability of the eigenvalue problem (115)-(117) in $W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$.

Theorem 10. *Assume that the above conditions on Ω and $a_i(x,u),\ i = 0,1,\dots,n$, are satisfied. Assume that for some positive number ε_0 and arbitrary $\varepsilon \in (0,\varepsilon_0)$ there exists a number $\lambda_\varepsilon \in (0,1]$ such that the problem ((118), (116)) has no solution $u(x)$ satisfying the conditions*

$$u \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega), \quad \|u\|_{L^2(\Omega)} < 1. \tag{119}$$

Then there exists a solution (λ_0, u_0) of the problem (115)-(117) such that $\lambda_0 \in (0,1]$.

Proof. We shall apply Theorem 3 with $X = L^2(\Omega)$. We define the operators $T,\ C$ as follows:

$$(120)\qquad \begin{aligned} &Tu = \Delta u, \quad D(T) = W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega), \\ &Cu = \left\{ \sum_{i=1}^n a_i(x,u) \frac{\partial u}{\partial x_i} + a_0(x,u) \right\}, \quad \mathcal{D}(C) = W^{1,2}(\Omega). \end{aligned}$$

The solvability of the equation

$$\Delta u + \tau u = f(x)$$

in $W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ is well known with boundary condition (117), $\tau > 0$ and $f \in L^2(\Omega)$. Consequently, the operator T is maximal monotone.

Now, we verify conditions $c_1),\ c_2)$ for the operator C. To see that condition $c_1)$ is satisfied, let $u \in D(T)$ and let the inequalities

$$(121)\qquad \langle Tu + Cu, u \rangle \le 0, \quad \|u\| \le S$$

hold, where $\|\cdot\|$ is the norm in $L^2(\Omega)$. The first inequality in (121) implies

$$(122)\qquad \int_\Omega \left\{ \sum_{i=1}^n \left| \frac{\partial u}{\partial x_i} \right|^2 + \left[\sum_{i=1}^n a_i(x,u) \frac{\partial u}{\partial x_i} + a_0(x,u) \right] u \right\} dx \le 0.$$

Using the inequalities (114) and the second inequality in (121), we obtain immediately from (122) the estimate

$$(123)\qquad \|u\|_{W^{1,2}(\Omega)} \le K_1(S).$$

From (123) we have $\|Cu\| \le K_2(S)$ with some number $K_2(S)$ depending only on known parameters and S. Therefore the quasiboundedness of C with respect to T is established.

To check condition $c_2)$, let $\{u_n\} \subset D(T)$ be such that $u_n \rightharpoonup u_0$, $Cu_n \rightharpoonup h_0$ and

$$(124)\qquad \limsup_{n\to\infty} \langle Cu_n, u_n - u_0 \rangle \le 0, \quad \langle Tu_n + Cu_n, u_n \rangle \le 0.$$

As in the case of (123), we have from (124)

$$\|u_n\|_{W^{1,2}(\Omega)} \le K_3,$$

which guarantees that $u_n \to u_0,\ u_0 \in D(C)$. Using the weak convergence of $\frac{\partial u_n}{\partial x_i}$ in $L^2(\Omega)$ and the strong convergence of $u_n(x)$ in $L^2(\Omega)$, we obtain $Cu_0 = h_0$.

It is easy to show that all the other conditions of Theorem 3 are satisfied. This completes the proof. □

Application 2. Consider the elliptic problem

$$(125)\qquad \begin{aligned} &-(1+\mu) \sum_{i=1}^n \frac{\partial}{\partial x_i} \left[|\nabla u|^{m-2} \frac{\partial u}{\partial x_i} \right] + \frac{c(x)}{|x|^r} |u|^{m-1} + \lambda \frac{f(x)}{|x|^q} |u|^{m-2} u \\ &= -\sum_{i=1}^n \frac{\partial}{\partial x_i} h_i(x), \quad x \in \Omega, \end{aligned}$$

$$(126)\qquad u(x) = 0, \quad x \in \partial\Omega,$$

where $\mu > 0$, $\lambda \ge 1$, $0 < r\frac{mn}{mn-n+m} < q < n$, $h_i \in L^{\frac{m}{m-1}}(\Omega),\ c,\ f \in L^\infty(\Omega)$, $f(x) \ge 1,\ 1 < m < n,\ \gamma > 0$.

We shall consider solutions $u(x)$ of the problem ((125), (126)) such that

$$u(x) \in \overset{\circ}{W}^{1,m}(\Omega), \quad \frac{1}{|x|^q}|u(x)|^{m-1} \in L^1(\Omega), \tag{127}$$

and equation (125) is satisfied in the sense of distributions.

Remark 2. We shall study the problem ((125), (126)) using a variant of Theorem 5. Namely, we assume conditions $t_1) - t_4)$ for the operator T without the restriction $L = D(T)$. The proof of such a variant remains the same.

Theorem 11. *Assume that $0 \in \Omega$ and let λ be such that the homogeneous problem ((125), (126)) (with $h_i(x) \equiv 0$) has only the zero solution for any $\mu < 0$. Then the problem ((125), (126)), for $\mu = 0$, has a solution for any functions $h_i \in L^{\frac{m}{m-1}}(\Omega)$, $i = 1, \ldots, n$.*

Proof. We shall apply Theorem 5 with $X = \overset{\circ}{W}^{1,m}(\Omega)$ and operators T, C, $\tilde{J}_m$ defined as follows:

$$\begin{aligned} \langle Tu, \varphi \rangle &= \int_\Omega \frac{f(x)}{|x|^q}|u|^{m-2}u\,\varphi dx, \\ \langle Cu, \varphi \rangle &= \int_\Omega \left\{ \sum_{i=1}^n |\nabla u|^{m-2}\frac{\partial u}{\partial x_i}\frac{\partial \varphi}{\partial x_i} + \frac{c(x)}{|x|^r}|u|^{m-1}\varphi \right\} dx, \\ \langle \tilde{J}_m u, \varphi \rangle &= \sum_{i=1}^n \int_\Omega |\nabla u|^{m-2}\frac{\partial u}{\partial x_i}\frac{\partial \varphi}{\partial x_i} dx. \end{aligned} \tag{128}$$

We let $D(T)$ be the set of all functions $u(x)$ satisfying conditions (127) and the inequality

$$\left| \int_\Omega \frac{f(x)}{|x|^q}|u|^{m-2}u\varphi dx \right| \le C_1 \|\varphi\|_{\overset{\circ}{W}^{1,m}(\Omega)}, \tag{129}$$

for all $\varphi \in C_0^\infty(\Omega)$, with some constant C_1 depending on u.

We let $D(C)$ be the set of all functions $u(x) \in \overset{\circ}{W}^{1,m}(\Omega)$ satisfying the inequality

$$\left| \int_\Omega \frac{c(x)}{|x|^r}|u|^{m-2}u\varphi dx \right| \le C_2 \|\varphi\|_{\overset{\circ}{W}^{1,m}(\Omega)} \tag{130}$$

for $\varphi \in C_0^\infty(\Omega)$ with C_2 depending on u.

The proof of the needed properties of the operator T was established in the paper [14]. It is therefore omitted.

We shall check the properties $c_1)$, $c_2)$ of the operator C.

To show $c_1)$, let us assume that $u \in D(T) \cap D(C)$

$$\langle Tu + Cu, u \rangle \le 0, \quad \|u\| \le S.$$

Then we have, immediately,

$$\int_\Omega \left\{ |\nabla u|^m + \frac{1}{|x|^q}|u|^m \right\} dx \le C_3. \tag{131}$$

The estimate for the norm $\|Cu\|$ follows from the inequality

$$\tag{132} \begin{aligned} \left|\int\limits_\Omega \frac{c(x)}{|x|^r}\ |u|^{m-1}|\varphi|dx\right| &\le C_4\left\{\int\limits_\Omega\left(\frac{1}{|x|^r}|u|^{m-1}\right)^{\frac{mn}{mn-n+m}}dx\right\}^{\frac{mn-n+m}{mn}}\cdot\|\varphi\| \\ &\le C_5\left\{\int\limits_\Omega\frac{1}{|x|^q}|u|^m dx\right\}^{\frac{r}{q}}\cdot\left\{\int\limits_\Omega|u|^{\frac{mn}{n-m}}dx\right\}^{(m-1-m\frac{r}{q})\frac{n-m}{mn}}\cdot\|\varphi\|, \end{aligned}$$

and shows that the operator C is quasibounded.

For the proof of c_2), let us consider a sequence $\{u_n\}\subset D(T)\cap D(C)$ such that $u_n\rightharpoonup u_0,\ Cu_n\rightharpoonup h_0$ and

$$\tag{133} \limsup_{n\to\infty}\langle Cu_n,u_n-u_0\rangle\le 0,\quad \langle Tu_n+Cu_n,u_n\rangle\le S.$$

Working as in (132), we obtain

$$\lim_{n\to\infty}\int\limits_\Omega\frac{c(x)}{|x|^r}|u_n|^{m-1}(u_n-u_0)dx=0,$$

and the first inequality of (133) implies $u_n\to u_0$. It is easy to show that $u_0\in\mathcal{D}(C)$, $Cu_0=h$.

The assertion of Theorem 11 follows now from Theorem 5 and Remark 2. □

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