TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 358, Number 9, September 2006, Pages 3997–4019
S 0002-9947(05)03782-7
Article electronically published on August 1, 2005

MANIFOLDS WITH AN $SU(2)$-ACTION ON THE TANGENT BUNDLE

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ABSTRACT. We study manifolds arising as spaces of sections of complex manifolds fibering over $\mathbb{C}P^1$ with the normal bundle of each section isomorphic to $\mathcal{O}(k) \otimes \mathbb{C}^n$.

Any hypercomplex manifold can be constructed as a space of sections of a complex manifold Z fibering over $\mathbb{C}P^1$. The normal bundle of each section must be the sum of $\mathcal{O}(1)$'s, and this suggests that interesting geometric structures can be obtained if we replace $\mathcal{O}(1)$ with other line bundles. Such structures have been introduced by many authors [1, 2, 3, 7, 8, 11, 13] and are variously known as conic, Grassman, paraconformal or $\mathcal{P}$-structures.

Spaces of sections with normal bundle $\mathcal{O}(n) \oplus \mathcal{O}(n)$ have been studied recently, in detail, by Maciej Dunajski and Lionel Mason [7, 8], and much of the present paper can be viewed as a translation of their work from spinor language.

We adopt the point of view that spaces of sections of complex manifolds fibering over $\mathbb{C}P^1$ are *manifolds with a fibrewise linear action of* $SU(2)$ *on the tangent bundle*. The integrability condition says that the ideal in $(\Omega^* M)^{\mathbb{C}}$ generated by the highest weight 1-forms is closed, for any Borel subgroup. We call such manifolds *generalised hypercomplex manifolds* (or k*-hypercomplex manifolds* if the representation of $SU(2)$ splits into copies of the k-th symmetric power of the standard representation). These manifolds seem to be interesting from several points of view, quite apart from "intrinsic worth". On the one hand, they provide a natural setting for certain integrable systems, "monopoles", as we discuss in sections 4 and 5. These comprise the Bogomolny hierarchy of Mason and Sparling [14] and the self-dual hierarchy. On the other hand, even if one is interested primarily in hypercomplex or hyperkähler manifolds, one may wish to consider these generalised hypercomplex manifolds, since (for odd k) they are foliated by hypercomplex submanifolds (see section 7; this observation goes back to Gindikin [10]; see also [7, 8]).

The paper is organised as follows: in the next section we recall some basic facts about foliations and distributions. In section 2 we define the generalised hypercomplex manifolds and their twistor spaces. Section 3 is devoted to an interpretation of a construction of D. Quillen [16], which we need later on. In sections 4 and 5 we discuss the Ward correspondence and monopoles on k-hypercomplex manifolds. In the following section we show that a k-hypercomplex manifold M has, analogously to hypercomplex manifolds, an S^2-worth of certain integrable structures. For odd k, M is always a complex manifold, but the S^2-worth of complex structures are

Received by the editors September 29, 2003 and, in revised form, June 16, 2004.
2000 *Mathematics Subject Classification.* Primary 53C26, 53C28.

combined quite differently from hypercomplex structures. In section 7 we show that M is foliated by $(k-2i)$-hypercomplex submanifolds, for any $i < k/2$. In particular, as mentioned above, for odd k we obtain a foliation by hypercomplex submanifolds. In section 8 we construct a "hypercomplex extension" of a k-hypercomplex manifold M, i.e. a hypercomplex manifold $\tilde{M}$ fibering over M with the property that a monopole on M is equivalent to a solution of self-duality equations on $\tilde{M}$. In the next section we discuss maps between generalised hypercomplex manifolds, and in section 10 symplectic structures and symplectic quotients. In the last section we give some examples—the k-hypercomplex manifolds arising as moduli spaces of solutions to analogues of Nahm's equations seem to be particularly interesting.

Finally, we should ask whether we could replace the action of $SU(2)$ on TM with other compact semisimple Lie groups. Thus we ask for manifolds with a fibrewise G-action on TM such that the ideal in $(\Omega^* M)^{\mathbb{C}}$ generated by the highest weight 1-forms is closed for any Borel subgroup of $G^{\mathbb{C}}$. An example of such a manifold is the group G itself. Much of the theory goes through for such "*Borel-Weil manifolds*", but there is dearth of examples. In any case, twistor considerations show that for $G \neq SU(2)$, the canonical linear connection on M (see Remark 5.7) is *flat* (not necessarily torsion-free).

Remark on notation. If X is a complex manifold, then $TX, T^*X, \Omega^p(X)$ all denote holomorphic objects, i.e. the $(1,0)$-tangent and -cotangent bundle and the sheaf of holomorphic p-forms, respectively. If E is a holomorphic bundle on X, we also denote by E the sheaf of its sections.

1. Foliations and involutive structures

Here, we recall some basic facts about foliations and involutive structures (i.e. complex distributions). The basic references are [15] and [18].

Definition 1.1. A *foliated manifold* is a manifold M of dimension n modelled on the fibration $\mathbb{R}^q \times \mathbb{R}^{n-q} \to \mathbb{R}^q$. This means that there is a smooth atlas $\{U_i, \phi_i\}$, $\phi_i : U_i \to \mathbb{R}^q \times \mathbb{R}^{n-q}$, such that the transition functions $\phi_i \circ \phi_j^{-1} : U_i \cap U_j \to \mathbb{R}^q \times \mathbb{R}^{n-q}$ are of the form

$$\mathbb{R}^q \times \mathbb{R}^{n-q} \ni (x,y) \longmapsto (h_{ij}(x), p_{ij}(x,y)) \in \mathbb{R}^q \times \mathbb{R}^{n-q}. \tag{1.1}$$

Equivalently, a foliation is given by an integrable distribution $\mathcal{P} \subset TM$. The integral submanifolds (leaves) of $\mathcal{P}$ determine a partition F of M, and, following [15], we avoid the constant reference to the foliated atlas through the expediency of denoting a foliated manifold by (M, F). We shall also refer to F as a foliation (of codimension q).

A foliation is *simple* if it is given by a submersion with connected fibres, i.e. the space of leaves (with the quotient topology) is a manifold. In particular, the foliated atlas of a foliated manifold consists of simple open sets.

Now let P be a property of manifolds which is determined by a reduction of the pseudogroup of diffeomorphisms of $\mathbb{R}^m$ to a subpseudogroup $\Pi = \Pi_m$ (e.g. complex, affine, etc.). We say that a foliation F is a *P-foliation* if every leaf has property P, i.e. if for every $x \in \mathbb{R}^q$, the functions $p_{ij}(x, \cdot)$ in (1.1) belong to Π_{n-q}. We shall say that a foliation F is a *transversely P-foliation* if the functions h_{ij} in (1.1) belong to Π_q.

We also need the definition of bundles and forms in this setting.

Definition 1.2. Let (M, F) be a foliated manifold and let E be a vector (or principal) bundle on M. E is said to be a *foliated bundle* if the restriction of E to any simple open subset U of M is a pullback of a bundle on the local quotient manifold $\bar{U}$.

Definition 1.3. An r-form ω on a foliated manifold (M, F) is called *basic*, if it is locally a pullback of a form on a local quotient manifold. In other words ω can be represented locally using only the "transverse" coordinates x in (1.1).

We now turn to complex distributions.

Definition 1.4. An *involutive* structure on a smooth manifold M is an involutive vector subbundle $\mathcal{V}$ of $T^{\mathbb{C}}M$, i.e. $\mathcal{V}$ satisfies $[\mathcal{V}, \mathcal{V}] \subset \mathcal{V}$.

The dual description gives us a subbundle T' of complex 1-form which vanishes on $\mathcal{V}$. If $\mathcal{V}$ is involutive, then T' is *closed*, i.e. for any local smooth section ϕ of T' there are sections $\psi_1, \dots, \psi_m$ of T' and smooth differential 1-forms $\omega_1, \dots, \omega_m$ such that

$$d\phi = \psi_1 \wedge \omega_1 + \cdots + \psi_m \wedge \psi_m.$$

Definition 1.5. An involutive structure on a smooth manifold M given by the vector bundle $\mathcal{V}$ is called *elliptic* if $T^{\mathbb{C}}M = \mathcal{V} + \bar{\mathcal{V}}$ (i.e. $T' \cap \bar{T}' = 0$).

The importance of elliptic structures follows from the following generalisation of the Newlander-Nirenberg Theorem (cf. [18], Theorem VI.7.1):

Theorem 1.6. *Let $\mathcal{V} \subset T^{\mathbb{C}}M$ be an elliptic involutive structure. Then $\mathcal{V}$ is integrable, i.e. every point $m \in M$ has a neighbourhood with local coordinates $z_1, \dots, z_r, t_1, \dots, t_k$, where $z_i \in \mathbb{C}$ such that*

$$T' = \mathit{span}\{dz_1, \dots, dz_m\}.$$

Equivalently

$$\mathcal{V} = \mathit{span}\{\partial/\partial\bar{z}_j, \partial/\partial t_k\}.$$

Let $\mathcal{F} = \mathcal{V} \cap \bar{\mathcal{V}} \cap TM$, where $\bar{\mathcal{V}}$ is the complex conjugate of $\mathcal{V}$. Then $\mathcal{F}$ is a (real) integrable distribution on M and the above theorem says the corresponding foliation is a transversely holomorphic foliation. In particular, if $\mathcal{F}$ is simple, then the space of leaves $M/\mathcal{F}$ is a complex manifold.

Remark 1.7. If M and the involutive structure $\mathcal{V}$ are real-analytic, we can extend $\mathcal{V}$ to an involutive subbundle of $T^{1,0}M^{\mathbb{C}}$, where $M^{\mathbb{C}}$ is a complexification of M. Thus an involutive structure on M corresponds to a holomorphic foliation of $M^{\mathbb{C}}$.

2. Generalised hypercomplex manifolds

Almost complex manifolds can be thought of as manifolds whose tangent bundle admits a fibrewise action of $U(1)$ such that each tangent space is the direct sum of the standard 2-dimensional representations of $U(1)$. Similarly, an almost hypercomplex manifold is a manifold whose tangent bundle admits a fibrewise action of $SU(2)$ such that each tangent space is the direct sum of the standard 4-dimensional representations of $SU(2)$. A less well-known example is that of an f-structure [21, 17], which amounts to giving a fibrewise S^1-action on TM such that each tangent space decomposes into standard or trivial S^1-representations.

2.1. **Generalised almost hypercomplex structures.** Let M be a smooth manifold. A *generalised almost hypercomplex structure* on M is a smooth fibrewise action of $SU(2)$ on TM such that each tangent space is isomorphic to $V\otimes\mathbb{R}^n$, where V is a fixed non-trivial irreducible representation of $SU(2)$ (on a real vector space). The complexified representation $V^{\mathbb{C}}$ is then one or two copies of the k-th symmetric power of the standard 2-dimensional unitary representation of $SU(2)$, and we shall also call M an *almost k-hypercomplex manifold*.

An almost k-hypercomplex manifold has dimension $m(k+1)$, where m is even if k is odd. The structure group of such a manifold reduces to the centraliser of $SU(2)$ in $GL\big(m(k+1),\mathbb{R}\big)$, i.e. to $GL(m,\mathbb{R})$ if k is even and to $GL(m/2,\mathbb{H})$ if k is odd. Let $E=E_M$ be the complex vector bundle on M associated to the standard representation of $GL(n/2,\mathbb{H})$ or $GL(m,\mathbb{R})$ on $\mathbb{C}^m$ (in the second case, the representation is the complexification of the standard real representation). Let H be the trivial bundle with fibre $S^k\mathbb{C}^2$. We then have a canonical isomorphism:

$$TM^{\mathbb{C}}\simeq E_M\otimes H. \tag{2.1}$$

For even k there is a corresponding splitting of the real tangent bundle, but we prefer to treat both even and odd k uniformly.

Remark 2.1. The splitting as in (2.1) with trivial bundle H is called in literature a *right-flat almost Grassmann structure* [13, 2]. Obviously, if we have such a splitting and the bundles E,H are equipped with either quaternionic or real structures (depending on the parity of $\dim M$), we can define an almost k-hypercomplex structure on M. Thus there is no difference between generalised almost hypercomplex structures and right-flat almost Grassman structures. The difference occurs when we consider integrability conditions.

We now turn to constructing almost k-hypercomplex manifolds. One way of realising the irreducible representations of $SL(2,\mathbb{C})$ is as sections of line bundles over $\mathbb{C}P^1$. Similarly an irreducible representation of $SU(2)$ on a real vector space can be realised as the space of *real* sections of an irreducible σ-bundle on $\mathbb{C}P^1$ [16]. Here σ-bundle means a holomorphic bundle equipped with an anti-holomorphic involution τ covering the antipodal map σ on $\mathbb{C}P^1$. An irreducible σ-bundle is isomorphic to either $\mathcal{O}(2k)$ or $\mathcal{O}(2k+1)\oplus\mathcal{O}(2k+1)$. Therefore each tangent space of a generalised almost hypercomplex manifold can be realised as the space of real sections of a σ-bundle E on $\mathbb{C}P^1$, and one way of obtaining manifolds with a generalised almost hypercomplex structure is as the space of sections of a complex manifold fibering over $\mathbb{C}P^1$. Indeed, we have

Proposition 2.2. *Let Z be a complex manifold fibering over $\mathbb{C}P^1$ and equipped with an antiholomorphic involution τ covering the antipodal map σ on $\mathbb{C}P^1$. Suppose that there exists a holomorphic and τ-invariant section of $Z\to\mathbb{C}P^1$ whose normal bundle is isomorphic to $\mathcal{O}(k)\otimes\mathbb{C}^n$, $k>0$. Then the space of such sections is a manifold of dimension $n(k+1)$, equipped with a canonical almost k-hypercomplex structure.*

Proof. The fact that the space M of real sections with normal bundle $\mathcal{O}(k)\otimes\mathbb{C}^n$ is a manifold of dimension $n(k+1)$ follows from the Kodaira deformation theory, given that $h^1(\mathcal{O}(k))=0$. To show that there is a canonical almost k-hypercomplex structure on M, consider the vertical bundle $N\subset TZ$, i.e. the kernel of $d\pi:TZ\to T\mathbb{C}P^1$. For any $m\in M$, i.e. a section $m:\mathbb{C}P^1\to Z$, the normal bundle to m is just the restriction of N to m. The tangent space $T^{\mathbb{C}}_m M$ is identified with $H^0(m, N_{|_m})$. Let $L = \mathcal{O}(k)$. We have a canonical identification of bundles on Z:

$$N \simeq (N \otimes \pi^*(L^*)) \otimes \pi^*(L)$$

which, given the fact that $N_{|_m} \simeq L \otimes \mathbb{C}^n$, leads to the decomposition

$$T_m M^{\mathbb{C}} \simeq H^0(m, N \otimes \pi^*(L^*)) \otimes H^0(m, \pi^*(L)).$$

In other words the tangent bundle $T^{\mathbb{C}}M$ decomposes as the tensor product of two bundles on M, $H^0(\cdot, N \otimes \pi^*(L^*)$ and $H^0(\cdot, \pi^*(L))$. This second bundle is trivial, and so the action of $SU(2)$ on $H^0(\mathbb{C}P^1, L)$ induces a canonical fibrewise action of $SU(2)$ on $T^{\mathbb{C}}M$. This action restricts to the real tangent bundle, as the existence of the real structure τ implies that the representation of $SU(2)$ on $H^0(\mathbb{C}P^1, L \otimes \mathbb{C}^n)$ is real. $\square$

Example 2.3. The 3-sphere carries a canonical 2-hypercomplex structure, defined by identifying $TS^3 \simeq SU(2) \times \mathfrak{su}(2)$ and considering the adjoint action of $SU(2)$ on the second factor. The integrability can be checked directly, or we can view S^3 as the space of real sections of $\mathbb{P}\big(\mathcal{O}(1) \oplus \mathcal{O}(1)\big) \simeq \mathbb{C}P^1 \times \mathbb{C}P^1$.

2.2. **Integrability of GHC-structures.**

Definition 2.4. We shall say that a generalised almost hypercomplex structure is integrable if it arises locally as in the previous proposition, i.e. M can be realised (at least locally) as the space of real sections of a complex manifold Z fibering over $\mathbb{C}P^1$. We shall call an integrable generalised almost hypercomplex structure (resp. integrable almost k-hypercomplex structure) a generalised hypercomplex structure (resp. a k-hypercomplex structure). We shall often abbreviate the words "generalised hypercomplex" to GHC.

Before proceeding, it will be useful to describe the map

$$(\textit{an irreducible representation of } SL(2,\mathbb{C})) \mapsto (\textit{a line bundle on } \mathbb{C}P^1).$$

If H is such a representation, then $SL(2,\mathbb{C})$ acts irreducibly on H^*. Fix an isomorphism $\mathbb{C}P^1 \simeq SL(2,\mathbb{C})/B$, where B is a Borel subgroup. For every Borel subgroup B_q corresponding to a point $q \in \mathbb{C}P^1$, let l_q be the line of highest weight vectors for B_q. This determines a holomorphic line bundle $\tilde{L}$ on $\mathbb{C}P^1$. In fact, we obtain an imbedding $\mathbb{C}P^1 \to \mathbb{P}(H^*)$ and $\tilde{L}$ is the pullback of the tautological bundle on $\mathbb{P}(H^*)$. It follows that $L = \tilde{L}^*$ is positive and $H \simeq H^0(\mathbb{C}P^1, L)$ as representations of $SL(2,\mathbb{C})$.

We now describe the condition on integrability of generalised almost hypercomplex structures. Consider the action of $SL(2,\mathbb{C})$ on the complexified cotangent bundle $(T^*M)^{\mathbb{C}}$. For for any point $q \in \mathbb{C}P^1$ which corresponds to a Borel subgroup B of $SL(2,\mathbb{C})$, we define the following subbundles of $T^{\mathbb{C}}M$:

- $\mathcal{U}_q$ - the subbundle of $(T^*M)^{\mathbb{C}}$ corresponding to the highest weight;
- $\mathcal{K}_q$ - the subbundle of $T^{\mathbb{C}}M$ annihilated by $\mathcal{U}_q$;
- $\mathcal{F}_q = \mathcal{K}_q \cap \overline{\mathcal{K}}_q \cap TM$ - a distribution on M.

Thus $\mathcal{K}_q$ consists of all but the lowest weight tangent vectors. We have:

Theorem 2.5. *An almost k-hypercomplex structure on a manifold M is integrable if and only if for every $q \in \mathbb{C}P^1$ the corresponding subbundle $\mathcal{K}_q$ of $T^{\mathbb{C}}M$ is involutive, i.e. $[\mathcal{K}_q, \mathcal{K}_q] \subset \mathcal{K}_q$.*

Proof. Suppose that the almost generalised hypercomplex structure is integrable, i.e. M is given (locally) as the parameter space of τ-invariant sections of a holomorphic manifold Z fibering over $\mathbb{C}P^1$. Then M has the natural complexification $M^{\mathbb{C}}$ defined as the parameter space of all sections of Z with the normal bundle $\mathcal{O}(k) \otimes \mathbb{C}^n$. For $q \in \mathbb{C}P^1$ consider the natural holomorphic map p_q from $M^{\mathbb{C}}$ to the fiber Z_q of Z over q given by intersecting a section with the fibre. The map dp_q is given by highest-weight 1-forms (this follows from the proof of Proposition 2.2). The kernel of dp_q is a subbundle $\mathcal{W}$ of $T^{1,0}M^{\mathbb{C}}$. At points of M we have a canonical identification $T_m^{\mathbb{C}}M \simeq T_m^{1,0}M^{\mathbb{C}}$ under which $\mathcal{W} \simeq \mathcal{K}_q$.

Conversely, suppose that the subbundle $\mathcal{K}_q$ is involutive for any point q of $\mathbb{C}P^1$. We define a subbundle $\mathcal{V}$ of the complexified tangent bundle to $M \times \mathbb{C}P^1$ as follows:

$$\mathcal{V}_{(m,q)} = (\mathcal{K}_q)_m \oplus T_q^{0,1}\mathbb{C}P^1, \tag{2.2}$$

where $\mathbb{C}P^1$ is equipped with the canonical complex structure. We claim that this subbundle is involutive. Let us choose a point q in $\mathbb{C}P^1$ and let v be a local section of $\mathcal{K}_q$. Let $u(\zeta)$ be a local holomorphic section of $SL(2,\mathbb{C}) \to \mathbb{C}P^1$ (in a neighbourhood of q). Then, because $\mathcal{K}$ is a homogeneous bundle, $u(\zeta)v$ is a section of $\mathcal{K}_{u(\zeta)q}$ for each ζ. The vector fields $u(\zeta)v$ and $\partial/\partial\bar{\zeta}_i$, where ζ_i are local holomorphic coordinates on $\mathbb{C}P^1$, generate the bundle $\mathcal{V}$. Therefore $\mathcal{V}$ is involutive. This involutive structure is elliptic and hence (replacing M by an open subset, if Hausdorffness is a problem) $(M \times \mathbb{C}P^1)/\mathcal{F}$, $\mathcal{F} = \mathcal{V} \cap \bar{\mathcal{V}} \cap TM$, is a complex manifold. This is our twistor space Z. It is clear that Z is a complex fibration over $\mathbb{C}P^1$ with the fiber at q equal to $M/\mathcal{F}_q$. Points of m give rise to sections of $M \times \mathbb{C}P^1 \to \mathbb{C}P^1$, which then descend to sections of Z. That these sections have correct normal bundle follows from the remarks following Definition 2.4. $\square$

Remark 2.6. One can now generalise the notion of GHC-structures by allowing group actions on TM such that each tangent space splits into *nonequivalent* irreducible representations and by using Theorem 2.5 as the definition. These are the $\mathcal{P}$-structures of Simon Gindikin [11].

2.3. **The twistor foliation of a GHC-manifold.** The proof of the above theorem shows how to construct the twistor space Z of a GHC-manifold. Unfortunately, it also shows that Z usually will not be Hausdorff. Therefore we need, in general, to view the twistor space Z as a foliation, given by the distribution (2.2). We adopt the following definition:

Definition 2.7. The *twistor foliation* Z of a generalised hypercomplex manifold M is the foliation of $M \times \mathbb{C}P^1$ determined by the integrable distribution $\mathcal{Z}$, where $\mathcal{Z}_{(m,q)} = (\mathcal{F}_q)_m$.

Definition 2.8. A GHC-manifold is *regular* if its twistor foliation is simple.

In the case of a regular GHC-manifold, we have a genuine twistor space which is a complex manifold fibering over $\mathbb{C}P^1$. In general, the twistor foliation Z is a transversely holomorphic foliation. In other words, if we consider a small open subset of U (where the foliation is simple), then the space of leaves of $U \times \mathbb{C}P^1$ is a complex manifold. Moreover it fibers over $\mathbb{C}P^1$. Globally, this means (directly from the definition) that the distribution $\mathcal{Z}$ is contained in the kernel of $d\pi$, where $\pi : M \times \mathbb{C}P^1 \to \mathbb{C}P^1$ is the projection. We shall often abuse notation and write $\pi : Z \to \mathbb{C}P^1$. We shall also refer to a leaf of Z as a point of Z, and write $z \in Z$ (this is justified if we identify the space of leaves with a non-Hausdorff manifold).

We shall try as much as possible to limit ourselves to regular GHC-manifolds and just indicate how the results can be generalised.

2.4. **Complex GHC-manifolds.** If M is a GHC-manifold arising as the space of real sections of a complex manifold $Z \to \mathbb{C}P^1$, then M has a natural complexification $M^{\mathbb{C}}$ given as the space of all sections (not just real) with the correct normal bundle. For a general GHC-manifold we still have complexifications, not necessarily realised as a space of sections. We adopt the following definition:

Definition 2.9. A *complex* GHC-manifold is a complex manifold X with a holomorphic action of $SL(2,\mathbb{C})$ on TX such that each tangent space is isomorphic to $S^k\mathbb{C}^2 \otimes \mathbb{C}^n$, and such that the differential ideal generated by highest weight forms is closed for each Borel subgroup of $SL(2,\mathbb{C})$.

As in the case of real GHC-manifolds we obtain, for each $q \in \mathbb{C}P^1$, a holomorphic subbundle $\mathcal{K}_q$ of $TX = T^{1,0}X$ and, putting these together, a holomorphic foliation Z of $X \times \mathbb{C}P^1$. Again, we call Z the twistor foliation. Following the tradition, we shall call its leaves *α-surfaces*.

As before, we have the isomorphism:

$$TX \simeq E_X \otimes H.$$

Consider the projection $\tau : X \times \mathbb{C}P^1 \to X$ and the bundle τ^*E_X. A simple open subset U of $X \times \mathbb{C}P^1$ has a quotient manifold $U/\mathcal{K}$ which is a local twistor space. The proof of Proposition 2.2 shows that τ^*E_X restricted to U arises as a pullback of a bundle on $U/\mathcal{K}$. Therefore:

Proposition 2.10. *The bundle τ^*E_X is a foliated bundle on $(X \times \mathbb{C}P^1, Z)$.* □

We observe that if X is equipped with an antiholomorphic involution σ compatible with the $SL(2,\mathbb{C})$-action, then the fixed point set X^σ is a GHC-manifold, providing that $\dim_{\mathbb{R}} X^\sigma = \dim_{\mathbb{C}} X$. In this case, the twistor foliations on X and on $M = X^\sigma$ are *transversely equivalent* ([15], section 2.7), and so we can think of them as the same "manifold of leaves".

It will often be convenient to replace a GHC-manifold M with a complex GHC-manifold X such that $M = X^\sigma$. We shall then write $M^{\mathbb{C}}$ for X.

3. On Quillen's resolution

D. Quillen [16] has defined a canonical resolution of sheaves on $\mathbb{C}P^1$. In this section we shall view this resolution and its splitting in terms of homogeneous vector bundles on $\mathbb{C}P^1$.

Let H be an irreducible representation of $SL(2,\mathbb{C})$ arising as the space of sections of an ample line bundle $L = \mathcal{O}(k)$ on $\mathbb{C}P^1$. In this section we shall write G for $SL(2,\mathbb{C})$ and B for the standard Borel subgroup of upper-triangular matrices. Thus L is the homogeneous line bundle $G \times_B \mathbb{C}_k$, where $\mathbb{C}_k$ is the 1-dimensional representation of B: $[b_{ij}] \cdot z = b_{11}^{-k} z$.

We consider the homogeneous vector bundle $G \times_B H$. Since the action of B on H extends to an action of G, $G \times_B H$ is trivial as a vector bundle. We have the canonical equivariant map

$$G \times_B H \to L$$

induced by the B-equivariant map $H \to \mathbb{C}_k$. If we identify $G \times_B H$ with the trivial bundle $\underline{H} = H \times \mathbb{C}P^1$, then this map simply sends (h, q), $h \in H = \Gamma(L)$, to $h(q)$. We obtain an exact sequence of homogeneous vector bundles on $\mathbb{C}P^1$:

$$0 \to K \to G \times_B H \to L \to 0. \tag{3.1}$$

The cohomology sequence of the dual to (3.1) implies, given that $H^0(L^*) = H^1(\underline{H}^*) = 0$,

$$0 \to H^0(\underline{H}^*) \to H^0(K^*) \to H^1(L^*) \to 0. \tag{3.2}$$

This is an exact sequence of representations of $SL(2, \mathbb{C})$. The representation $H' = H^1(\mathbb{C}P^1, L^*)$ is isomorphic to $S^{k-2}\mathbb{C}^2$ and so irreducible. Therefore

$$\hat{H} = H^0(\mathbb{C}P^1, K^*)$$

is simply the direct sum of H^* and H', and the following sequence of representations splits:

$$0 \to H^* \xrightarrow{i} \hat{H} \xrightarrow{j} H' \to 0. \tag{3.3}$$

We observe:

Lemma 3.1. *The vector bundle K^* splits as a direct sum of line bundles $\mathcal{O}(1)$.*

Proof. The long exact sequence of (3.1) implies that all cohomology of K vanishes (given that the second map induces an isomorphism on H^0), and hence K is a sum of $\mathcal{O}(-1)$'s. □

Therefore $\hat{H}$ can be given structure of a complex quaternionic vector space.

We shall now consider the sequence (3.3) in greater detail and construct a canonical equivariant isomorphism of $H^1(\mathbb{C}P^1, L^*)$ with the space of sections of another homogeneous bundle. Let, for every $q \in \mathbb{C}P^1$, S_q denote the subspace of highest weight vectors in H (for the Borel corresponding to q).

Lemma 3.2. *The bundle S is a homogeneous subbundle of K.*

Proof. It is clear that S is a homogeneous subbundle of $G \times_B H$, since the map of fibers $S_1 \to H$ over [1] is B-equivariant. On the other hand we also have $S_1 \subset K_1$, since $K_1 = \{(v_1, \ldots, v_k, 0)^T\}$ and $S_1 = \{(v_1, 0, \ldots, 0)^T\}$. □

We consider the short exact sequence of homogeneous vector bundles:

$$0 \to (K/S)^* \to K^* \to S^* \to 0. \tag{3.4}$$

The long exact sequence of cohomology of this sequence begins as

$$0 \to H^0\big((K/S)^*\big) \to \hat{H} \to H^0(S^*).$$

The construction of a line bundle from representation, recalled after Definition 2.4, shows that there is a canonical isomorphism (of representations) $H^* \to H^0(S^*)$. Thus we obtain a map $p : \hat{H} \to H^*$.

Lemma 3.3. *The map p is a left inverse of the map i in* (3.3).

Proof. This is a matter of going through various identifications. We start with a section s of the trivial bundle $\underline{H}^*$. The map i means that we evaluate s on K. Following this by p means that we evaluate s on S. This however is exactly how we obtain the element s of H^*: it is a section of $H^0(S^*)$. □

Finally:

Lemma 3.4. *As homogeneous vector bundles*

$$(K/S)^* \simeq G \times_B H'.$$

In particular, K/S is a trivial vector bundle.

Proof. As both sides are homogeneous vector bundles, it is enough to show that their fibers over [1] coincide as B-modules. $H \simeq \mathbb{C}^{k+1}$ was the k-th symmetric power $S^k\mathbb{C}^2$ of the standard representation of $SL(2,\mathbb{C})$. The fiber K_1 of K consists of vectors in H of the form $(v_1, \ldots, v_k, 0)^T$ and the fiber of S_1 of vectors $(v_1, 0, \ldots, 0)^T$. Now the mapping $K_1/S_1 \to \mathbb{C}^{k-1}$ induced by

$$(v_1, v_2, \ldots, v_k, 0)^T \mapsto (v_2, \ldots, v_k)^T$$

provides an isomorphism between B-modules K_1/S_1 and $S^{k-2}\mathbb{C}^2$. □

4. Some natural bundles on generalised hypercomplex manifolds

Let M be a regular GHC-manifold and let $Z \to \mathbb{C}P^1$ be its twistor space. We have the double fibration

$$Z \xleftarrow{\eta} Y = M^{\mathbb{C}} \times \mathbb{C}P^1 \xrightarrow{\tau} M^{\mathbb{C}}. \tag{4.1}$$

The kernel of $d\eta$ is the bundle $\mathcal{K}$ (defined in section 2).

We shall consider several natural (locally free) sheaves over $M^{\mathbb{C}}$, which arise as direct images of sheaves on Y.

We shall write $\tau_*^i(F)$ for the i-th direct image of the sheaf of sections of F, and τ_* for τ_*^0. The first bundle on $M^{\mathbb{C}}$ we wish to consider is the tangent bundle $TM^{\mathbb{C}} = T^{1,0}M^{\mathbb{C}}$. This, by the proof of Proposition 2.2, can be written as $\tau_*\eta^*(T_\pi Z)$, where $T_\pi Z$ is the vertical tangent bundle. We recall the decomposition (2.1):

$$TM^{\mathbb{C}} \simeq E_M \otimes H. \tag{4.2}$$

The proof of Proposition 2.2 shows that $E_M = \tau_*\eta^*\big(T_\pi Z \otimes \pi^* L^*\big)$ and H is the trivial bundle with fiber equal to the representation $H^0(\mathbb{C}P^1, L) = S^k\mathbb{C}^2$ of $SL(2,\mathbb{C})$. If k is odd, so that L has a quaternionic structure, then both E_M and H have quaternionic structures covering the real structure on $M^{\mathbb{C}}$. Similarly, if k is even, then both E_M and H have real structures. In particular, if k is even, the bundle E has a real form $E^{\mathbb{R}}$ on M, and $TM \simeq E^{\mathbb{R}} \otimes H^{\mathbb{R}}$, where $H^{\mathbb{R}}$ is the underlying representation of $SU(2)$ on $\mathbb{R}^{k+1}$.

Next we consider direct images of $\eta^*(T_\pi^* Z)$.

Proposition 4.1. *We have:*

$$\tau_*^0\eta^*\left(T_\pi^* Z\right) = 0,$$
$$\tau_*^1\eta^*\left(T_\pi^* Z\right) \simeq E^* \otimes H',$$

where $E = E_M$ and H' is the trivial bundle over $M^{\mathbb{C}}$ with fiber given by the representation $S^{k-2}\mathbb{C}^2 \simeq H^1(\mathbb{C}P^1, L^*)$ of $SL(2,\mathbb{C})$.

Proof. We can write $T_\pi^* Z \simeq \big(T_\pi^* Z \otimes \pi^*(L)\big) \otimes \pi^*(L^*)$. Since the first bundle in the tensor product is trivial on each twistor line and L is positive, the result follows. □

Now we consider the sheaf Ω^*_η of η-vertical holomorphic forms on Y, i.e. the exterior algebra of $\Omega^1(Y)/\eta^*\big(\Omega^1(Z)\big)$. We are particularly interested in the direct image $\tau_*\Omega^1_\eta$ (it is well known [20, 13, 4] that the Ward correspondence gives us differential operators $F \to \tau_*\Omega^1_\eta \otimes F$ on bundles over M).

We have

Proposition 4.2. *The sheaf $\tau_*(\Omega^1_\eta)$ is isomorphic to $E^* \otimes \hat{H}$, where $\hat{H} = H^* \oplus H'$ is the representation defined in the previous section.*

Proof. Let $x \in M^{\mathbb{C}}$ and let $\mathbb{C}P^1_x$ be the fibre of τ over x (i.e. a section of Z corresponding to x). The η-normal bundle of $\mathbb{C}P^1_x$ inside Y is the bundle $\mathcal{K}$, i.e. the subbundle of $T_x \times \mathbb{C}P^1$ given by choosing the subspace $\mathcal{K}_q$ of T_x for each q. Therefore

$$\big(\tau_*\Omega^1_\eta\big)_x = \Gamma(\mathbb{C}P^1_x, \mathcal{K}^*). \tag{4.3}$$

However, as T_x decomposes as $\mathbb{C}^n \otimes H$, where $\mathbb{C}^n$ is the fiber of E at x, the bundle $\mathcal{K}$ decomposes as $\mathbb{C}^n \otimes K$, where K is defined in (3.1). The result follows. □

We observe that Ω^1_η fits into the exact sequence

$$0 \to \eta^*(T^*_\pi Z) \to \tau^*\left(\Omega^1 M^{\mathbb{C}}\right) \to \Omega^1_\eta \to 0. \tag{4.4}$$

The corresponding long exact sequence of direct images gives, using Proposition 4.1:

$$0 \to \Omega^1 M^{\mathbb{C}} \to \tau_*\Omega^1_\eta \to E^* \otimes H' \to 0. \tag{4.5}$$

All the sheaves in this sequence are tensor products of E^* with (sheaves of sections of) trivial bundles, and we have the key:

Proposition 4.3. *On a regular GHC-manifold the sequence* (4.5) *splits canonically:*

$$\tau_*\Omega^1_\eta = \Omega^1 \oplus (E^* \otimes H')\,. \tag{4.6}$$

The splitting is induced by that of Lemma 3.3*.*

Proof. The only thing to prove is that the maps in (4.5) are identity on E^*. This follows from the identification of the bundles involved at each point x of $M^{\mathbb{C}}$, as in the proof of Proposition 4.2. □

It is now clear how to compute direct images of Ω^i_η for any i. For example, the same arguments as in the proof of Proposition 4.2 yield:

Proposition 4.4. *The sheaf $\tau_*(\Omega^2_\eta)$ is isomorphic to $\big(S^2E^* \otimes H_-\big) \oplus \big(\Lambda^2 E^* \otimes H_+\big)$, where $H_- = H^0(\mathbb{C}P^1, \Lambda^2 K^*)$ and $H_+ = H^0(\mathbb{C}P^1, S^2K^*)$.* □.

Remark 4.5. Although we have limited ourselves to regular GHC-manifolds, everything in this section remains valid for arbitrary GHC-manifolds, as the definitions of sheaves Ω^*_η and of $\eta^*(T^*_\pi Z)$ ("sheaf of vertical basic 1-forms") are unchanged.

5. Monopoles

We shall consider geometric structures arising from holomorphic vector bundles F on the twistor space Z of a regular generalised hypercomplex manifold M, under the condition that F is trivial on twistor sections. We use the notation of the previous section, in particular the double fibration

$$Z \xleftarrow{\eta} Y = M^{\mathbb{C}} \times \mathbb{C}P^1 \xrightarrow{\tau} M^{\mathbb{C}},$$

and the sheaf Ω^*_η of η-vertical holomorphic forms. By composing the exterior derivative on Y with the projection onto Ω^1_η we obtain a first-order differential operator

$$d_\eta : \Omega^0 \to \Omega^1_\eta$$

which annuls $\eta^*\Omega^0_Z$. If F is a holomorphic bundle on Z, then it is well known (see, e.g., [13] or [4]) that d_η extends to a flat relative connection

$$\nabla_\eta : \eta^* F \to \Omega^1_\eta(F)$$

on the pullback η^*F.

If the bundle F is trivial on each section of Z, then η^*F is trivial on each fibre of τ and we consider the direct image

$$\hat{F} = \tau_*\eta^* F$$

which is a vector bundle on $M^{\mathbb{C}}$ of the same rank as F. By pushing down ∇_η we obtain a first-order differential operator

$$D := \tau_*\nabla_\eta : \hat{F} \to \tau_*\Omega^1_\eta \otimes \hat{F}. \tag{5.1}$$

This operator satisfies

$$D(fs) = fD(s) + \partial f \otimes s,$$

where $\partial = \tau_* d_\eta$.

We recall from the previous section that we have a canonical isomorphism

$$\big(\tau_*\Omega^1_\eta\big)_x = H^0(\mathbb{C}P^1_x, \mathcal{K}^*), \tag{5.2}$$

where the bundle $\mathcal{K}$ is the subbundle of $T_x \times \mathbb{C}P^1$ annihilated by the highest weight forms for each $q \in \mathbb{C}P^1$. Therefore we have the canonical map

$$\imath^*_q : \tau_*\Omega^1_\eta \to \mathcal{K}^*_q \tag{5.3}$$

for each $q \in \mathbb{C}P^1$. In particular, if we restrict both $\hat{F}$ and D to the leaf of the foliation $\mathcal{K}_q$ passing through x, i.e. to the submanifold $\tau\big(\eta^{-1}(z)\big)$ where $z = \eta(\tau^{-1})(x)$, then we recover ∇_η, i.e. D induces a flat connection on $\hat{F}$ restricted to this submanifold.

All of the above is well known and works for arbitrary twistor spaces [13, 4]. In the case of generalised hypercomplex manifolds we can use, however, the results of the previous section, in particular the splitting

$$\tau_*\Omega^1_\eta = \Omega^1 \oplus (E^* \otimes H')\,. \tag{5.4}$$

The operator ∂ becomes simply the exterior derivative:

Lemma 5.1. *Under the isomorphism* (5.4)*, the operator* $\partial : \Omega^0 \to \tau_*\Omega^1_\eta$ *becomes* $\partial = d \oplus 0$.

Proof. From the equation (4.4) we have the commutative diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Omega^0 & = & \Omega^0 & \longrightarrow & 0 \\ & & {\scriptstyle d}\big\downarrow & & {\scriptstyle \partial}\big\downarrow & & \\ 0 & \longrightarrow & \Omega^1 & \longrightarrow & \tau_*\Omega^1_\eta & \longrightarrow & E^*\otimes H'. \end{array}$$

□

Therefore on a generalised hypercomplex manifold the operator D is given by a connection ∇ and a Higgs field, i.e. a section Φ of $\mathrm{End}(\hat{F})\otimes(E^*\otimes H')$.

Moreover $\nabla+\Phi$ is flat on each α-subspace, i.e. on each submanifold S_z of $M^{\mathbb{C}}$ of the form $\tau\big(\eta^{-1}(z)\big)$, $z\in Z$.

We can formulate the results as follows.

Theorem 5.2. *Let M be a regular generalised hypercomplex manifold and let Z be its twistor space. There exists a* 1-1 *correspondence between*

(a) *holomorphic vector bundles on Z trivial on each twistor line, and*
(b) *monopoles on $M^{\mathbb{C}}$, i.e. a connection ∇ on a holomorphic vector bundle $\hat{F}$ on $M^{\mathbb{C}}$ and a section Φ of* $\mathrm{End}(\hat{F})\otimes(E^*\otimes H')$ *such that $\nabla\oplus\Phi$ is flat on each α-subspace, i.e. on each submanifold of $M^{\mathbb{C}}$ of the form $\tau\big(\eta^{-1}(z)\big)$, $z\in Z$.*

This correspondence remains valid in the presence of a real structure, giving monopoles on M. □

Here, the connection $\nabla\oplus\Phi$ on an α-subspace S_z corresponding to a point $z\in Z$ over $q\in\mathbb{C}P^1$ is given by

$$\hat{F}\xrightarrow{\nabla\oplus\Phi}\Big(\hat{F}\otimes E^*\otimes H^*\Big)\oplus\Big(\hat{F}\otimes E^*\otimes H'\Big)=\hat{F}\otimes E^*\otimes\hat{H}\xrightarrow{\imath_q^*}\hat{F}\otimes\Omega^1 S_z,$$

where $\imath_q^*$ is given by (5.3).

Remark 5.3. The above theorem describes monopoles for the group $GL(m,\mathbb{C})$ (or $GL(m,\mathbb{R})$), m being the rank of F. By considering bundles whose structure group reduces, we obtain a 1-1 correspondence between bundles on Z and monopoles for other groups G.

Example 5.4 (**Monopoles on $\mathbb{R}^5$**)**.** Let M be $\mathbb{R}^5$ viewed as the real form of the fourth symmetric power of the defining representation of $\mathfrak{su}(2)$. Then M is a 4-hypercomplex manifold, and its twistor space is the total space of the line bundle $\mathcal{O}(4)$. Thus $M^{\mathbb{C}}$ is identified with polynomials

$$z_0+z_1\zeta+z_2\zeta^2+z_3\zeta^3+z_4\zeta^4, \tag{5.5}$$

and M with polynomials invariant under

$$z_i\mapsto(-1)^i\overline{z}_{4-i}. \tag{5.6}$$

We wish to discuss the monopoles on a (trivial) vector bundle F over $\mathbb{R}^5$ (cf. [14]). We first discuss monopoles on $\mathbb{C}^5$. In this case E_M is the trivial 1-dimensional bundle and, according to the above theorem, a monopole is given by a connection and a section Φ of $\mathrm{End}(F)\otimes H' \simeq \mathrm{End}(F)\otimes \mathbb{C}^3$, i.e. a triple Φ_1, Φ_2, Φ_3 of sections of $\mathrm{End}(F)$. Let the 1-form of the connection be

$$\sum_{i=0}^{4} A_i dz_i.$$

In order to find the monopole equation we have to consider the map $\imath_q^* : \mathbb{C}^5 \oplus \mathbb{C}^3 \to K_q^*$ given by (5.3). Its image consists of polynomials (5.5) vanishing at $\zeta = q$, and $\imath_q^*$ on $\mathbb{C}^5$ is simply the projection. On the other hand $\imath_q^*$ on $\mathbb{C}^3$ is described by Lemma 3.4 and the equation (3.4). It is equivariant for the Borel subgroup corresponding to q.

Consider $q = 0$. Then the connection ∇^0 on each α-surface S_z obtained by freezing z_0 in (5.5) has the 1-form:

$$(A_1 + \Phi_1)dz_1 + (A_2 + \Phi_2)dz_2 + (A_3 + \Phi_3)dz_3 + A_4 dz_4.$$

Thus this connection must be flat and similarly for every q. We would obtain 18 equations due to Mason and Sparling [14]. In practice it is simpler to obtain the equations as a reduction of self-duality equations on the connection

$$\sum_{i=1}^{3} \Phi_i dt_i + \sum_{i=0}^{4} A_i dz_i$$

on $\mathbb{R}^8$. This will be discussed in detail in section 8.

Monopoles on $\mathbb{R}^5$ will arise when we impose the reality condition (5.6) on the A_i and a similar one on the Φ_i.

We briefly discuss the (well-known) case of 1-hypercomplex manifolds. In this case there is no Φ and a connection ∇ obtained as in Theorem 5.2 is called *self-dual* [6] or *hyperholomorphic* [19]. It is easy to describe this condition. Since, for $k = 1$, $\tau_*\Omega^1_\eta = \Omega^1$, we have the natural operator

$$\Omega^2 = \tau_*\Omega^1_\eta \wedge \tau_*\Omega^1_\eta \longrightarrow \tau_*\Omega^2_\eta. \tag{5.7}$$

The curvature of the relative connection ∇_η lies in Ω^2_η, and it vanishes. Therefore a connection ∇ will be self-dual if and only if its curvature lies in the kernel of (5.7) [4]. Proposition 4.4 describes $\tau_*\Omega^2_\eta$ and, since in this case $K^* = \mathcal{O}(1)$ and $H = \mathbb{C}^2$, we conclude:

Proposition 5.5. *A connection ∇ on a holomorphic vector bundle over the complexified hypercomplex manifold $M^{\mathbb{C}}$ is self-dual if and only if its curvature lies in the component $S^2E^* \otimes \Lambda^2 H$ of $\Lambda^2 T^*M \simeq \big(S^2E^*\otimes\Lambda^2H\big) \oplus \big(\Lambda^2E^*\otimes S^2H\big)$.* □

An equivalent way of expressing this condition is that the curvature is $SL(2,\mathbb{C})$-invariant [19].

Remark 5.6. Again, the results of this section hold for non-regular GHC-manifolds, providing we replace holomorphic bundles on Z with foliated holomorphic bundles on (Y, Z) (cf. Definition 1.2). In particular, there is a monopole on the bundle E_M over $M^{\mathbb{C}}$.

Remark 5.7. Let M be a k-hypercomplex manifold of dimension $n(k+1)$ (n is even if k is odd). Since the bundle E has a natural connection ∇^E and $T^{\mathbb{C}}M = E \otimes H$, we obtain a canonical connection ∇_M on M by tensoring ∇^E with the flat connection on H. Since the action of $SU(2)$ on TM is parallel for this connection, the holonomy of ∇_M is contained in the centraliser of $SU(2)$ in $GL(n(k+1),\mathbb{R})$, i.e. in $GL(n,\mathbb{R})$, if k is even and in $GL(n/2,\mathbb{H})$ if k is odd. Apart from $k=1$, the connection ∇_M is not torsion-free. It is perhaps of interest to know more about ∇_M, e.g. how its torsion is related to the Higgs field arising from E.

6. Structure of generalised hypercomplex manifolds

From the very definition, a hypercomplex manifold has the structure of a complex manifold for every $q \in \mathbb{C}P^1$. For generalised hypercomplex manifolds, we have shown so far (in section 2) that a GHC-manifold M has, for every $q \in \mathbb{C}P^1$, the structure of a transverse holomorphic foliation (M, Z_q), where leaves of Z_q are the integral manifolds of the distribution $\mathcal{F}_q$. In this section we shall prove a much more precise structure theorem.

6.1. **Weight subbundles.** Let M be an almost k-hypercomplex manifold. Then, for every $q \in \mathbb{C}P^1$, we can decompose $T^{\mathbb{C}}M$ into eigenspaces of the circle normalising the Borel subgroup corresponding to q:

$$T^{\mathbb{C}}M = \bigoplus_{i=0}^{k} \mathcal{S}_q(k-2i), \tag{6.1}$$

where we adopt the convention that $\mathcal{S}_q(-k)$ is the bundle of lowest weight vectors and $\mathcal{S}_q(i+1)$ is obtained from $\mathcal{S}_q(i)$ by the action of the unipotent radical of B_q. In particular the subbundle $\mathcal{K}_q$ defined in section 2 is the bundle $\bigoplus_{j\neq -k} \mathcal{S}_q(j)$.

On a k-hypercomplex manifold these distributions are defined on all of $M^{\mathbb{C}}$. We have:

Theorem 6.1. *The subbundles $\mathcal{S}_q(j)$ of $TM^{\mathbb{C}}$ are involutive for every j and q.*

This follows from the following result:

Proposition 6.2. *For every $q \in \mathbb{C}P^1$ and every $l \geq -k$, the subbundle*

$$\mathcal{K}_q(l) = \bigoplus_{j\geq l} \mathcal{S}_q(j)$$

of $TM^{\mathbb{C}}$ is involutive.

We observe that $\mathcal{K}_q(-k) = TM^{\mathbb{C}}$ and $\mathcal{K}_q(-k+1) = \mathcal{K}_q$, where $\mathcal{K}_q$ is the leaf of the twistor foliation over q. Theorem 6.1 follows immediately from this proposition, as $\mathcal{S}_q(j) = \mathcal{K}_q(j) \cap \mathcal{K}_{\sigma(q)}(-j)$, where σ is the antipodal map on $\mathbb{C}P^1$.

To prove the above proposition, consider a regular neighbourhood U of a point $m \in M^{\mathbb{C}}$ with the corresponding twistor space Z_U. Then the integral submanifold of $\mathcal{K}_q(l)$ through m is the space of all sections of $Z_U \to \mathbb{C}P^1$ which coincide with the section $m : \mathbb{C}P^1 \to Z_U$ up to order $l-1$ at q (equivalently these are sections of Z_U blown up $l-1$ times lying in the same connected component as m).

6.2. **Structure theorems.** We shall now show that, for every $q \in \mathbb{C}P^1$, a generalised hypercomplex manifold M *locally* looks like the following sequence of submersions:

$$M \to Z_q(p) \to \cdots \to Z_q(1) = Z_q,$$

where $p = (k+1)/2$ if k is odd and $p = k/2$ if k is even. Each $Z_q(i)$ is a complex manifold of dimension $2ni$ if k is odd and ni if k is even. Moreover the projections $Z_q(i) \to Z_q(i-1)$ are holomorphic.

We first define the relevant pseudogroups of transition functions. For k odd, let $\Gamma_{k,n}$ be the pseudogroup of local holomorphic diffeomorphisms of $\mathbb{C}^{2n} \otimes \mathbb{C}^{(k+1)/2}$ of the form

$$(x_1, x_2, \dots, x_{\frac{k+1}{2}}) \longmapsto \big(\phi_1(x_1), \phi_2(x_1, x_2), \phi_3(x_1, x_2, x_3), \dots, \phi_{\frac{k+1}{2}}(x_1, \dots, x_{\frac{k+1}{2}})\big),$$

where for every $i > 0$, ϕ_i is a local holomorphic transformation of $\mathbb{C}^{2n}$. For k even, we define $\Gamma_{k,n}$ similarly as the pseudogroup of local diffeomorphisms of $\big(\mathbb{C}^n \otimes \mathbb{C}^{k/2}\big) \oplus \mathbb{R}^n$, where each ϕ_i for $i = 1, \dots, \frac{k}{2}$ is as before and $\phi_{\frac{k}{2}+1}(x_1, \dots, x_{\frac{k}{2}}, x_{\frac{k}{2}+1})$ is real in the "last" variable.

We now have the following structure theorems.

Theorem 6.3. *Let k be odd and let M be a k-hypercomplex manifold of dimension $2n(k+1)$. Then, for every $q \in \mathbb{C}P^1$, M has a complex atlas with transition functions belonging to $\Gamma_{k,n}$.*

Theorem 6.4. *Let k be even and let M be a k-hypercomplex manifold of dimension $n(k+1)$. Then, for every $q \in \mathbb{C}P^1$, M has an atlas of charts in $\mathbb{C}^{nk} \oplus \mathbb{R}^n$ with transition functions belonging to $\Gamma_{k,n}$.*

Proof. These are a simple consequence of Proposition 6.2 and the Newlander-Nirenberg Theorem 1.6. □

Remark 6.5. For k odd, M becomes a complex manifold. The proof shows how to describe the complex structure: on each weight subbundle $F_i = \big(\mathcal{S}_q(i) \oplus \mathcal{S}_q(-i)\big) \cap TM$ of TM choose an element I_i of the circle normalising B_q such that $I_i^2 = -1$ on F_i. Then $I_q = \bigoplus I_i|F_i$ is a complex structure on M.

7. From k-hypercomplex to $(k-2i)$-hypercomplex

We again consider the decomposition (6.1) of $T^{\mathbb{C}}M$ into weight subbundles. For $q \in \mathbb{C}P^1$ and any $l > 0$, $l < k/2$, we consider the subbundle of TM given by

$$\mathcal{F}_q(l) = TM \cap \bigoplus_{i=0}^{k-2l} \mathcal{S}_q(k-2l-2i). \tag{7.1}$$

Thus $\mathcal{F}_q(0) = TM$ and $\mathcal{F}_q(1) = \mathcal{F}_q$, where $\mathcal{F}_q$ is the leaf of the twistor foliation over q.

We are going to sketch a proof (cf. [10, 7, 8]) of:

Theorem 7.1. *Let M be a k-hypercomplex manifold. Then, for any $q \in \mathbb{C}P^1$ and any $0 \le l \le k/2$, every leaf L of $\mathcal{F}_q(l)$ carries a canonical $(k-2l)$-hypercomplex structure. In particular, if k is odd, a $2n(k+1)$-dimensional k-hypercomplex manifold M is foliated by $4n$-dimensional hypercomplex manifolds.*

Proof. We first construct an almost $(k-2l)$-structure on each leaf L. The complexified tangent space at m to L is canonically the subspace of $T^{\mathbb{C}}_m M$ consisting of all but l highest and l lowest weight subspaces for the Borel B_q. This subspace has canonically the structure of an $SL(2,\mathbb{C})$-module with highest weight $k-2l$ (just set the actions of B_q and of the opposite Borel to be zero on $S_q(k-2l)$ and on $S_q(-k+2l)$, respectively). As everything is compatible with the real structure, we obtain an almost $(k-2l)$-hypercomplex structure on L. To prove its integrability, consider a small neighbourhood U of m, where the twistor foliation is simple. Let Z_U be the twistor space of U. First consider the case $l=1$. A leaf of $\mathcal{F}_q(1)$ is simply a fiber $\eta^{-1}(z)$ of $\eta: U \to Z_U$, where $q=\pi(z)$. Its complexification is the intersection of the fibers $\eta^{-1}(z)$ and $\eta^{-1}(\tau(z))$ of $\eta: U^{\mathbb{C}} \to Z_U$, where $\tau: Z_U \to Z_U$ is the real structure. One can check that this intersection is the space of sections passing through the exceptional divisors of the manifold $\tilde{Z}_U$ which is Z_U blown up at z and $\tau(z)$. The almost $(k-2)$-hypercomplex structure on L is the one obtained from Proposition 2.2 applied to $\tilde{Z}_U$ and hence it is integrable. Blowing up successively proves the result. $\square$

8. A hypercomplex extension of a k-hypercomplex manifold

If M is a flat k-hypercomplex manifold, i.e. a vector space of the form $H^{\mathbb{R}} \otimes \mathbb{R}^n$, where $H^{\mathbb{R}}$ is the real form of $S^k\mathbb{C}^2$, then the sequence dual to (3.3) can be interpreted as saying that there is a 1-hypercomplex manifold $\tilde{M}$ fibering over M such that $T\tilde{M}$ can be canonically identified with $\tau_* \Omega^1_\eta$. In particular monopoles on M will correspond to self-dual connections on $\tilde{M}$. In this section we shall show that such an $\tilde{M}$ always exists. Let M be a k-hypercomplex manifold. Then $TM^{\mathbb{C}} \simeq E_M \otimes H$ and both E_M and H are equipped with real or with quaternionic structures. The same is true then about E_M and $H' \simeq S^{k-2}\mathbb{C}^2$, and we have a real vector bundle $(E_M \otimes H')^{\mathbb{R}}$ over M.

The total space of $(E_M \otimes H')^{\mathbb{R}}$ is an obvious candidate for $\tilde{M}$. One way of defining an almost hypercomplex structure is to use the connection on E_M to define a connection on $\tilde{M}^{\mathbb{C}} = E_M \otimes H'$ and obtain a splitting

$$T\tilde{M}^{\mathbb{C}} = p^*(E_M \otimes H') \oplus p^*(E_M \otimes H) \simeq p^*(E_M \otimes \hat{H}), \tag{8.1}$$

where $p: \tilde{M}^{\mathbb{C}} \to M^{\mathbb{C}}$ is the projection. Since $\hat{H} = H^0(K^*)$, Lemma 3.1 provides an almost hypercomplex structure on $\tilde{M}$. This will not, however, usually be integrable. We have to twist the first term of the middle expression in (8.1) by the monopole on E_M in order to get an integrable hypercomplex structure. Such constructions are given in [5]. The hypercomplex structure exists only on an open subset where the monopole is non-degenerate. Here we shall give a twistorial and local proof that such an integrable structure exists, providing that certain topological conditions are fulfilled. To define these, consider the bundles $\mathcal{S}_q(k)$ of highest weight vectors on $M^{\mathbb{C}}$. These bundles combine to give us a bundle $\mathcal{S}$ on $Y = M^{\mathbb{C}} \times \mathbb{C}P^1$ which is a subbundle of $\mathcal{Z}$, i.e. of the twistor distribution. From the previous section we know that the bundles $\mathcal{S}_q(k)$ are involutive, and, hence, so is $\mathcal{S}$.

We can state the theorem:

Theorem 8.1. *Let M be a regular k-hypercomplex manifold such that the space of leaves of the distribution $\mathcal{S}$ is a manifold $\tilde{Z}$. Then there exists a hypercomplex manifold $\tilde{M}$ such that:*

- *There is a projection $p: \tilde{M} \to M$ and a section $M \to \tilde{M}$ of this projection, whose tubular neighbourhood can be identified with a neighbourhood of the zero section of the bundle $(E_M \otimes H')^{\mathbb{R}}$ over M.*

- *There is a canonical identification of* $\Omega^1_{\tilde\eta}$ *with* $p^*\Omega^1_\eta$ *of sheaves on* $\tilde Y = \tilde M^{\mathbb{C}} \times \mathbb{C}P^1$*, which makes the following diagram commute:*

$$\begin{array}{ccc} \Omega^1_{\tilde Y} & \longrightarrow & \Omega^1_{\tilde\eta} \\ \uparrow & & \| \\ p^*\Omega^1_Y & \longrightarrow & p^*\Omega^1_\eta. \end{array} \tag{8.2}$$

Corollary 8.2. *In the above situation, there is a canonical isomorphism*

$$\Omega^1 \tilde M^{\mathbb{C}} \simeq p^*\tau_*\Omega^1_\eta. \tag{8.3}$$

Proof. Under the assumptions we have a well-defined twistor space Z of M and the Hausdorff manifold $\tilde Z$ of leaves of $\mathcal{S}$. We shall show that $\tilde Z$ is a twistor space of a hypercomplex manifold. Consider the double fibration $Z \xleftarrow{\eta} Y = M^{\mathbb{C}} \times \mathbb{C}P^1 \xrightarrow{\tau} M^{\mathbb{C}}$. Since $\mathcal{S} \subset \operatorname{Ker} d\eta$, $\tilde Z$ fibers over Z and has a canonical real structure given by the real structures on Y and on $\mathcal{S}$. A point of $M^{\mathbb{C}}$ gives a section of $Y \to \mathbb{C}P^1$ and hence a section of $\tilde Z$. We claim the normal bundle of such a section in $\tilde Z$ splits into the direct sum of $\mathcal{O}(1)$'s. Indeed, the vertical tangent bundle of the fibration $Y \to \mathbb{C}P^1$ is simply $\tau^*E_M \otimes \underline{H}$, where $\underline{H}$ is the trivial bundle over $\mathbb{C}P^1$ whose fiber is $H = S^k\mathbb{C}^2$. Therefore the vertical tangent bundle of $\tilde Z$ is

$$T_{\tilde\pi}\tilde Z = \big(\tau^*E_M \otimes \underline{H}\big)/\mathcal{S} \simeq \tau^*E_M \otimes (\underline{H}/S),$$

where S is the bundle of highest weight vectors in $\underline{H}$. The sequences (3.1) and (3.4) imply that $\underline{H}/S \simeq K^*$, and by Lemma 3.1 this splits into the direct sum of $\mathcal{O}(1)$'s. Since τ^*E_M is trivial on sections, we conclude that the space of real sections of $\tilde Z$ is a hypercomplex manifold, at least in a neighbourhood $\tilde M$ of M.

As any section of $\tilde Z$ projects to a section of Z, we obtain a projection $p : \tilde M \to M$. To identify a tubular neighbourhood of M in $\tilde M$ observe that $T_{\tilde\pi}\tilde Z$ fits into the diagram:

$$\begin{array}{ccccccc} 0 \longrightarrow (T_\eta Y)/\mathcal{S} & \longrightarrow & T_{\tilde\pi}\tilde Z & \longrightarrow & T_\pi Z & \longrightarrow & 0 \\ \| & & \| & & \| & & \\ 0 \longrightarrow \tau^*E_M \otimes (K/S) & \longrightarrow & \tau^*E_M \otimes (\underline{H}/S) & \longrightarrow & (T_\pi Z \otimes L^*) \otimes L & \longrightarrow & 0. \end{array}$$

This allows us, using Lemma 3.4, to identify a tubular neighbourhood of $M^{\mathbb{C}}$ in $\tilde M^{\mathbb{C}}$ with a neighbourhood of the zero section in the bundle $E_M \otimes H'$ over $M^{\mathbb{C}}$.

We now prove the second statement. From the above diagram, $T\tilde Y$ is identified with $(p^*\tau^*E_M) \otimes \hat H^*$ and $\tilde\eta^*\Omega^1_{\tilde\pi}$ with $(p^*\tau^*E_M^*) \otimes (H/S)^*$. Therefore $\Omega^1_{\tilde\eta}$ is $p^*\tau^*E_M^*$ tensored with the cokernel of the map $(H/S)^* \to \hat H$. Homogeneous bundle arguments, similar to those in section 3, show that this cokernel is equal to K^*. Hence $\Omega^1_{\tilde\eta}$ is isomorphic to $(p^*\tau^*E_M^*) \otimes K^*$ which is $p^*\Omega^1_\eta$ (see section 4). The diagram (8.2) commutes, as the maps are identity on $p^*\tau^*E_M$ and the horizontal maps are given by appropriate maps on sheaves on $\mathbb{C}P^1$, as in section 3. □

The above theorem allows us to view monopoles on M as a reduction of self-dual connections (i.e. monopoles without Φ) on $\tilde M$. Indeed, we recall from section 5 that a monopole is equivalent to a first-order operator $D : F \to \tau_*\Omega^1_\eta \otimes F$ on a bundle F. Pulling D back to $\tilde M^{\mathbb{C}}$ we obtain a connection $\tilde\nabla$ on p^*F (note that

(8.2) guarantees that the isomorphism $\Omega^1_{\tilde\eta} \simeq p^*\Omega^1_\eta$ commutes with differentials $d_{\tilde\eta}$ and d_η). We have:

Theorem 8.3. *Let M and $\tilde M$ be as in the previous theorem. Then a pair (∇, Φ) on a bundle F over $M^{\mathbb{C}}$ is a monopole if and only if the connection $\tilde\nabla$ on p^*F over $\tilde M^{\mathbb{C}}$ is self-dual.*

Proof. This is now automatic, since both (∇, Φ) and $\tilde\nabla$ arise from flat relative connections on Ω^1_η and $\Omega^1_{\tilde\eta}$, respectively. □

9. Maps between generalised hypercomplex manifolds

We wish to consider maps between GHC-manifolds. What we clearly need are maps which, for regular GHC-manifolds, give rise to fibrewise mappings of the twistor spaces. The following definition is the translation of this condition.

Definition 9.1. A morphism between two GHC-manifolds is a smooth map $f : M \to M'$ such that, for any $q \in \mathbb{C}P^1$, the differential df satisfies the following two conditions:

- df maps the subbundle $\mathcal{F}_q$ of TM to the subbundle $\mathcal{F}_q$ of TM';
- the induced map $df : TM/\mathcal{F}_q \to TM'/\mathcal{F}_q$ commutes with the action of the maximal torus T corresponding to the point q.

Example 9.2. Let M be a hypercomplex manifold equipped with a tri-holomorphic action of a Lie group H for which a moment map exists. Such a moment map $\mu : M \to \mathfrak{h}^* \otimes \mathbb{R}^3$ can be viewed as a morphism of two GHC-manifolds, M and $\mathfrak{h}^* \otimes \mathbb{R}^3$, where the latter is equipped with the flat 2-hypercomplex structure given by the action of $SU(2)$ on the second factor.

The following fact follows directly from the definition of morphisms.

Proposition 9.3. *A map $f : M \to M'$ of two regular GHC-manifolds is a morphism if and only if there is a holomorphic bundle map F between the twistor spaces Z, Z' of M and M' such that $\widetilde{f(m)} = F(\tilde m)$, where $\tilde p$ denotes a section of the twistor space corresponding to a point p of the manifold.*

We also have:

Proposition 9.4. *Let M and M' be two k-hypercomplex manifolds. Then any GHC-morphism f between M and M' respects the generalised hypercomplex structure, i.e. df commutes with the action of $SU(2)$.*

Proof. This is a local statement, so we can assume that M and M' are regular. Let F be a fibrewise mapping of the twistor spaces of M and M' given by the previous proposition. Thus dF at a section is a linear mapping between $L \otimes \mathbb{C}^n$ and $L \otimes \mathbb{C}^m$, where L is the line bundle corresponding to the representation V. Such a mapping is given by a constant linear map from $\mathbb{C}^n$ to $\mathbb{C}^m$ and hence df commutes with the action of $SU(2)$. □

The following proposition generates a large number of generalised hypercomplex structures. It is an analogue of the fact [9] that 4-dimensional hypercomplex manifolds with circle symmetry fiber over hyper-CR-manifolds.

Proposition 9.5. *Let M be a regular k-hypercomplex manifold of dimension $(k+1)^2$ equipped with a free circle action which respects the k-hypercomplex structure and such that the vector field V generated by the action satisfies the following two conditions:*

- *for any $q \in \mathbb{C}P^1$ and any $m \in M$, $V_m \notin (\mathcal{F}_q)_m$;*
- *for any m in M, $SU(2)V_m$ linearly generates all of T_mM.*

Then M/S^1 is a $(k+1)$-hypercomplex manifold and the projection $M \to M/S^1$ is a morphism of generalised hypercomplex structures.

Proof. Let Z be the twistor space of M. The first condition on V implies that there is a free local action of $\mathbb{C}^*$ on fibers of Z, and we can form a complex manifold Z_{red} by taking fibrewise quotients (at least locally). Sections of Z descend to sections of Z_{red} and we have to show that they have correct normal bundle. This normal bundle N fits in the exact sequence

$$0 \to \mathcal{O} \to \mathcal{O}(k)^{k+1} \to N \to 0.$$

Thus N is a vector bundle of rank k and the first Chern class $k(k+1)$. In addition, the second condition on V implies that $\mathcal{O}$ does not embed into any proper subbundle of $\mathcal{O}(k)^{k+1}$ of the form $\mathcal{O}(k)^s$. Therefore any line bundle which is a direct summand of N has the first Chern class at least $k+1$. It follows that $N \simeq \mathcal{O}(k+1)^k$. □

10. Symplectic k-hypercomplex structures

Let M be a generalised hypercomplex manifold. Let V be the irreducible representation of $SU(2)$ such that $T_mM \simeq V \otimes \mathbb{R}^n$ for any $m \in M$. If $V^{\mathbb{C}} \simeq S^k\mathbb{C}^2$, then we denote by $V^{[2]}$ the irreducible representation of $SU(2)$ on a real vector space such that $(V^{[2]})^{\mathbb{C}} \simeq S^{2k}\mathbb{C}^2$.

Definition 10.1. A k-symplectic structure on M is a closed non-degenerate $SU(2)$-invariant 2-form ω with values in $V^{[2]}$, i.e. $\omega : \Lambda^2TM \to V^{[2]}$.

To explain this notion, recall the decomposition $T^{\mathbb{C}}M \simeq E \otimes H$, where H is the trivial bundle with fiber $S^k\mathbb{C}^2$. The space of 2-vectors decomposes as

$$\Lambda^2TM^{\mathbb{C}} = \left(\Lambda^2E \otimes S^2H\right) \oplus \left(S^2E \otimes \Lambda^2H\right).$$

Observe that this expression contains exactly one component $S^{2k}\mathbb{C}^2$, which lies in the first term. Therefore a k-symplectic form ω is equivalent to a non-degenerate 2-form ω_E on E, compatible with the quaternionic or real structure, such that the canonical map

$$\Lambda^2\left(TM^{\mathbb{C}}\right) \longrightarrow \Lambda^2E \otimes S^{2k}\mathbb{C}^2 \xrightarrow{\omega_E} S^{2k}\mathbb{C}^2$$

defines a closed $H^{[2]}$-valued 2-form.

For regular GHC-manifolds we can also give an interpretation in terms of the twistor space Z. We recall that Z fibers over $\mathbb{C}P^1$ and M can be identified with the space of real sections with normal bundle $L \otimes \mathbb{C}^n$, where L is a given ample line bundle on $\mathbb{C}P^1$. The k-symplectic form is equivalent to a fibrewise complex-symplectic form ω_Z on Z respecting the real structure of Z. We note that ω_Z takes values in the line bundle L^2, whose space of real sections is $V^{[2]}$. Such a definition of a k-symplectic structure is given in [7].

Remark 10.2. Suppose that the form ω_E is positive on $(e, \tilde{e})$, where $e \mapsto \tilde{e}$ is the quaternionic or real structure of E. If M is a symplectic k-hypercomplex manifold with k odd, then M admits a canonical Riemannian metric. Indeed, $S^2TM^{\mathbb{C}}$ splits into the direct sum of $S^2E \otimes S^2H$ and $\Lambda^2 E \otimes \Lambda^2 H$. Since, for k odd, the decomposition of $\Lambda^2 S^k \mathbb{C}^2$ into irreducibles contains a trivial representation, $\Lambda^2 H$ has a canonical symplectic 2-form ω_H, also compatible with the quaternionic structure. Then $\omega_E \otimes \omega_H$ is a non-degenerate symmetric tensor on $TM^{\mathbb{C}}$, compatible with the real structure and giving a metric on M.

If k is even, then it is S^2H which contains a trivial representation, and so we obtain a canonical symplectic form on M, via the decomposition $\Lambda^2 TM^{\mathbb{C}} = \big(\Lambda^2 E \otimes S^2 H\big) \oplus \big(S^2 E \otimes \Lambda^2 H\big)$.

Now suppose that a symplectic GHC-manifold admits a proper and free action of a Lie group G which respects the GHC-structure and the symplectic form. Suppose, in addition, that the action is Hamiltonian, i.e. there is a G-equivariant map

$$\mu : M \to V^{[2]} \otimes \mathfrak{g}^*$$

having the usual property that its differential evaluated on an element ρ of $\mathfrak{g}$ is equal to the contraction of ω with the vector field generated by the action of $\exp(t\rho)$.

We can now construct GHC-manifolds using the symplectic quotient construction, i.e. defining the reduced manifold as $\mu^{-1}(s)/G$, where $s \in V^{[2]} \otimes \mathfrak{g}^*$ is invariant under G-action. For regular manifolds this is equivalent to taking the complex-symplectic quotient along the fibers of Z to obtain a new twistor space Z_{red}. The generic section of $Z_{\rm red}$ which descended from a section of Z will have the correct normal bundle. Globality is however hard to come by, as even the linear version of the symplectic quotient works only generically. Thus, if T is a trivial isotropic σ-subbundle of a symplectic σ-bundle $E = \mathcal{O}(k) \otimes \mathbb{C}^{2n}$, then it is not always true that $T^{\perp}/T$ splits into the sum of line bundles of degree k.

11. Examples

The symplectic quotient described in the previous section provides many examples of generalised hypercomplex structures. Thus we start with the flat k-hypercomplex structure on $V = \mathbb{R}^{k+1} \otimes \mathbb{R}^{2n}$. The twistor space of this GHC-manifold is the total space of the vector bundle $\mathcal{O}(k) \otimes \mathbb{C}^{2n}$. It is naturally a symplectic bundle, and we consider the induced k-symplectic structure on V. Now we choose a group H preserving the symplectic GHC-structure. H must be a subgroup of $Sp(n, \mathbb{C})$ and its representation on $\mathbb{C}^{2n}$ is quaternionic if k is odd and real if k is even. Thus H is a subgroup of $Sp(n)$ if n is odd and of $Sp(2n, \mathbb{R})$ if n is even. The k-symplectic quotient M of V by H will have a (symplectic) k-hypercomplex manifold on an open dense subset. We observe that the fibers of the twistor space of M look generically the same as for the twistor space of the hypercomplex manifold obtained by the same quotient construction with k replaced by 1. Thus we can think of resulting manifolds as k-hypercomplex analogues of ALE-spaces, complex coadjoint orbits, toric hyperkähler manifolds, etc.

We shall consider two examples in greater detail.

11.1. ***k*-Eguchi-Hanson manifold.** We begin with $\mathbb{C}^{k+1} \otimes \mathbb{C}^4$ which we view as a complexified flat k-hypercomplex manifold. We have the $\mathbb{C}^*$-action on the second factor $t \cdot (z_1, z_2, w_1, w_2) = (tz_1, tz_2, t^{-1}w_1, t^{-1}w_2)$ which induces an action of S^1

on the underlying k-hypercomplex space V which is $\mathbb{R}^{2k+2}\otimes\mathbb{R}^2$ for odd k, and $\mathbb{R}^{k+1}\otimes\mathbb{R}^4$ for even k. We consider the k-symplectic quotient of this space by the S^1-action. According to the discussion in the previous section, this is equivalent to taking the fibrewise symplectic quotient of $\mathcal{O}(k)\otimes\mathbb{C}^4$ by $\mathbb{C}^*$. In other words, we identify $V^{\mathbb{C}}$ with the space of sections of $\mathcal{O}(k)\otimes\mathbb{C}^4$, i.e. with quadruples of polynomials $\big(z_1(\zeta),z_2(\zeta),w_1(\zeta),w_2(\zeta)\big)$ of degree k. The moment map $\mu: V^{\mathbb{C}}\to H^0(\mathcal{O}(2k))$ is simply

$$\mu\big(z_1(\zeta),z_2(\zeta),w_1(\zeta),w_2(\zeta)\big)=z_1(\zeta)w_1(\zeta)+z_2(\zeta)w_2(\zeta).$$

It is clear that a section s of $\mathcal{O}(2k)$ will be a regular value for μ as soon as s does not have double zeros. This way we obtain a manifold $\mu^{-1}(s)/\mathbb{C}^*$. If we begin with real sections of $\mathcal{O}(k)\otimes\mathbb{C}^4$ and a real section s and quotient by S^1, we shall obtain a manifold M_k of dimension $2k+2$. This manifold carries a k-hypercomplex structure on an open dense subset, and one can call it the *k-Eguchi-Hanson manifold*.

We shall describe M_{k-1} or rather its complexification $M_{k-1}^{\mathbb{C}}$. Since the group (circle) is abelian, the moment map is invariant and we can take first the GIT-quotient of $\mathbb{C}^{4k}$ by $\mathbb{C}^*$ (i.e. the affine variety defined by $\mathbb{C}^*$-invariants) and then its intersection with a level set of the moment map, which becomes linear. In the present case the GIT quotient of $\mathbb{C}^{4k}$ by $\mathbb{C}^*$ is simply the variety of $2k\times 2k$-matrices of rank 1. Let us choose coordinates so that the polynomials $z_p(\zeta),w_p(\zeta)$, $p=1,2$, are of the form

$$z_1(\zeta)=\sum_{i=1}^{k} z_i\zeta^{i-1},\qquad z_2(\zeta)=\sum_{i=k+1}^{2k+2} z_i\zeta^{i-1},$$
$$w_1(\zeta)=\sum_{i=1}^{k} w_i\zeta^{k-i},\qquad w_2(\zeta)=\sum_{i=1}^{k} w_{k+i}\zeta^{2k-i}.$$

Then $M_{k-1}^{\mathbb{C}}$ is the intersection of the variety of rank 1 matrices $[a_{ij}]$ with the affine subspace described by equations

$$\sum_{i=1}^{k-p} a_{i,p+i}+\sum_{i=1}^{k-p} a_{k+i,k+p+i}=\tau_i,\ \ p=0,\dots,k-1,$$
$$\sum_{i=1}^{k-p} a_{p+i,i}+\sum_{i=1}^{k-p} a_{k+p+i,k+i}=\nu_i,\ \ p=0,\dots,k-1.$$

These equations fix the sum of entries of the matrix lying in the diagonal $(k\times k)$-blocks and parallel to the main diagonal. To obtain M we impose the reality condition $\bar{w}_i=z_i$. The k-hypercomplex structure on M should be equivalent to the one obtained in [8] via a solution to the Plebański heavenly equation.

11.2. **Infinite-dimensional quotients.** Just as in the case of hypercomplex manifolds, we can consider infinite-dimensional k-symplectic quotients. We shall discuss the case of $k=2$. Let G be a compact semisimple Lie group with Lie algebra $\mathfrak{g}$. Consider the space $\mathcal{A}$ of $\mathfrak{g}$-valued smooth functions $T_0,\dots T_5$ on an interval $[0,1]$. These can be viewed as real sections of the bundle $C^\infty\big([0,1],\mathfrak{g}^{\mathbb{C}}\big)\otimes(\mathcal{O}(2)\oplus\mathcal{O}(2))$ on $\mathbb{C}P^1$, and so we have a flat infinite-dimensional symplectic 2-hypercomplex manifold. The gauge group of transformations $g:[0,1]\to\mathfrak{g}$ acts via

$$T_0\mapsto \mathrm{ad}(g)T_0-\dot{g}g^1,\quad T_i\mapsto\mathrm{ad}(g)T_i,\ \ i=1,\dots,5.$$

The 2-symplectic quotient by the gauge subgroup of g with $g(0) = g(1) = 1$ will be described by an analogue of Nahm's equations. The simplest way to describe the equations is again to perform the quotient along the fibers of the twistor space. Let us write

$$\beta = T_4 + iT_5, \quad \gamma = T_2 + iT_3, \quad \alpha = T_0 + iT_1,$$

and put

$$B(\zeta) = \beta + \gamma\zeta + (\alpha + \alpha^*)\zeta^2 - \gamma^*\zeta^3 + \beta^*\zeta^4,$$
$$A(\zeta) = \alpha - \gamma^*\zeta + \beta^*\zeta^2.$$

Here A and $-A + B/\zeta^2$ are sections of $C^\infty\big([0,1], \mathfrak{g}^{\mathbb{C}}\big) \otimes \mathcal{O}(2)$ and provide coordinates on $\mathcal{A}$. Alternatively, we can choose B and T_0 as coordinates on $\mathcal{A}$, which corresponds to writing $\mathcal{O}(2) \oplus \mathcal{O}(2)$ as the extension

$$0 \to \mathcal{O} \to \mathcal{O}(2) \oplus \mathcal{O}(2) \to \mathcal{O}(4) \to 0.$$

The 2-symplectic quotient M is then described as the moduli space of solutions to

$$\dot{B}(\zeta) = [B(\zeta), A(\zeta)], \tag{11.1}$$

which is simply the space of sections of the fibrewise symplectic quotient of the twistor space $C^\infty\big([0,1], \mathfrak{g}^{\mathbb{C}}\big) \otimes (\mathcal{O}(2) \oplus \mathcal{O}(2))$.

The equations (11.1) are equivalent to the following:

$$\frac{d}{dt}\beta = [\beta, \alpha], \tag{11.2}$$

$$\frac{d}{dt}\gamma = [\gamma, \alpha] - [\beta, \gamma^*], \tag{11.3}$$

$$\frac{d}{dt}(\alpha + \alpha^*) = [\alpha^*, \alpha] + [\beta, \beta^*] - [\gamma, \gamma^*]. \tag{11.4}$$

The gauge group acts via

$$\alpha \mapsto \mathrm{ad}(g)\alpha - \dot{g}g^{-1}, \quad \beta \mapsto \mathrm{ad}(g)\beta, \quad \gamma \mapsto \mathrm{ad}(g)\gamma.$$

To identify M, use a gauge transformation g, with $g(0) = 1$, to make α hermitian. Then the map

$$[\alpha, \beta, \gamma] \longmapsto \big(g(1), \alpha(0) + \alpha^*(0), \beta(0), \gamma(0)\big)$$

is a diffeomorphism from M into $G \times \mathfrak{g} \times \mathfrak{g}^{\mathbb{C}} \times \mathfrak{g}^{\mathbb{C}}$. We observe that M fibers over a complex manifold Z_0 consisting of solutions to $\dot{\beta} = [\beta, \alpha]$ modulo *complex* gauge transformations. This space is easily identified with $G^{\mathbb{C}} \times g^{\mathbb{C}} \simeq T^*G^{\mathbb{C}}$ (one can make $\alpha = 0$ via a complex gauge transformation g with $g(0) = 1$).

The 2-hypercomplex structure is G-invariant and symplectic. It would be interesting to identify the hypercomplex extension of M, given by Theorem 8.1. An obvious candidate is the hypercomplex structure on a neighbourhood of the zero section of $T^*H^{\mathbb{C}}$, where H is the tangent group of G, i.e. the semidirect product of G and $\mathfrak{g}$.

Even more interesting would be to consider the 3-hypercomplex manifold given by a similar construction and identify the hypercomplex structures, described by Theorem 7.1, on real α-surfaces.

Finally, one can impose different boundary conditions (poles at $t = 0$, conjugacy classes at infinity, etc.) on equations (11.2)-(11.4) and their k-hypercomplex, $k > 2$, analogues. This yields not only many k-hypercomplex structures, but also, in view of Theorems 7.1 and 8.1, many hypercomplex structures.

Acknowledgment

This work was supported by an Advanced Research Fellowship from the Engineering and Physical Sciences Research Council of Great Britain.

I thank Charles Boyer, David Calderbank and Misha Verbitsky for reading the manuscript and for comments.

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