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SCOTT'S RIGIDITY THEOREM FOR SEIFERT FIBERED SPACES; REVISITED

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ABSTRACT. We will present a new proof of the rigidity theorem for Seifert fibered spaces of infinite π_1 by Scott (1983) in the case when the base of the fibration is a hyperbolic triangle 2-orbifold. Our proof is based on arguments in the rigidity theorem for hyperbolic 3-manifolds by Gabai (1997).

It has been twenty years since Scott [11] proved the rigidity theorem for Seifert fibered spaces M with infinite fundamental groups. In fact, he proved the theorem in the case when the base of M is a triangle 2-orbifold $O(p,q,r)$ with $1/p+1/q+1/r \leq 1$. In all other cases, M is a Haken manifold and the proof of the theorem reduces to Waldhausen's rigidity theorem [14]. Scott's result was the first major step in the proof of the Seifert Fibered Space Theorem which was completed by Gabai [5] and Casson-Jungreis [3] independently. The theorem says that if the fundamental group $\pi_1(M)$ of a closed irreducible 3-manifold M contains a normal subgroup isomorphic to $\mathbf{Z}$, then M is a Seifert fibered space. On the other hand, the rigidity theorem for hyperbolic 3-manifolds was proved by Gabai [6] in the case when manifolds satisfy the insulator condition. After that, Gabai, Meyerhoff and N. Thurston [7] completed the hyperbolic rigidity theorem by showing that any closed hyperbolic 3-manifold satisfies this condition.

In this paper, we will present a new proof of Scott's Rigidity Theorem in the case when the base O of a Seifert fibered space M is a hyperbolic triangle 2-orbifold. Roughly speaking, our proof is the 2-dimensional (and hence a simpler) version of Gabai's argument in [6]. Such an approach is very natural and compatible with the widely recognized idea in three-manifold topology that any Seifert fibered structure with infinite π_1 is a degenerating limit of 3-dimensional hyperbolic structures. In fact, the covering $\hat{X}$ of M associated to an infinite cyclic normal subgroup of $\pi_1(M)$ is quasi-isometric to the hyperbolic plane $\mathbf{H}^2$. Our proof is completed by showing that the $\pi_1^{\mathrm{orb}}(O)$-orbit of some point $x \in \mathbf{H}^2$ satisfies the 2-dimensional insulator condition.

Theorem 0.1. *Let $f : M \longrightarrow N$ be a homotopy equivalence between closed orientable and irreducible 3-manifolds. If M admits a Seifert-fibration whose base is a hyperbolic triangle 2-orbifold, then f is homotopic to a homeomorphism.*

As a benefit of our proof of Theorem 0.1, we have a considerably shorter proof of the following, which is also the hyperbolic base case of the theorem given by Scott [12] (see also Boileau-Otal [2]).

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Theorem 0.2. *Let $f : M \longrightarrow M$ be a self-homeomorphism of a closed orientable Seifert fibered space with a hyperbolic base 2-orbifold. If f is homotopic to the identity Id_M of M, then f is isotopic to Id_M.*

1. 2- and 3-dimensional insulator theories; comparison

First of all, we will briefly review Gabai's insulator theory for hyperbolic 3-manifolds, which will be helpful to grasp our analogous approach for Seifert fibered spaces.

Hyperbolic case. Suppose that $f : M \longrightarrow N$ is a homotopy equivalence from a closed hyperbolic 3-manifold M to a closed irreducible 3-manifold N. Let ρ be the hyperbolic metric on M, and let ν be a fixed Riemannian metric on N. It is well known that there exist finite coverings $p_1 : X \longrightarrow M$, $q_1' : X' \longrightarrow N$ such that f is lifted to a map $f_1' : X \longrightarrow X'$ which is homotopic to a homeomorphism f_1. By identifying X with X' via f_1, we have the following homotopically commutative diagram, where $q_1 = q_1' \circ f_1$ and X_ρ (resp. X_ν) is supposed to have the Riemannian metric, still denoted by ρ (resp. ν), induced via p_1 (resp. q_1):

$$\begin{tikzcd} X_\rho \arrow[r, "\mathrm{Id}_X"] \arrow[d, "p_1"'] & X_\nu \arrow[d, "q_1"] \\ M_\rho \arrow[r, "f"] & N_\nu \end{tikzcd}$$

Let $p : \mathbf{H}^3_\rho \longrightarrow M_\rho$, $q : \mathbf{H}^3_\sigma \longrightarrow N_\nu$ be the universal coverings. Unfortunately, the definition of the insulator condition for hyperbolic 3-manifolds in [6] is not so simple, and it might be hard for us to take the notion at once. However, a striking theorem in [7] guarantees that any closed hyperbolic 3-manifold satisfies the condition. Roughly speaking, the insulator condition is used to designate an ambient isotopy class of $p_1^{-1}(c)$ in X, where c is a core of some open solid torus V in M. In fact, this condition gives a certain family $\{\lambda_{i:j}\}$ of simple loops in $\partial\mathbf{H}^3$ such that each component V_i of $p^{-1}(V)$ is the 'central' component of $\mathbf{H}^3 - \bigcup_{j\neq i} D_{i:j[\rho]}$, where $D_{i:j[\rho]}$ is a ρ-least area plane properly embedded in $\mathbf{H}^3_\rho$ with $\partial D_{i:j[\rho]} = \lambda_{i:j}$. In the original paper [6], $D_{i:j[\rho]}$ was a D^2-limit lamination in $\mathbf{H}^3_\rho$ spanned by $\lambda_{i:j}$. It was later shown that $D_{i:j[\rho]}$ can be supposed to be a properly embedded plane; see [13]. We note that intersections of properly embedded planes can be treated fairly easily compared with those of D^2-limit laminations. By studying the situation of $D_{i:j[\nu_t]}$ under a continuous deformation of Riemannian metrics ν_t $(0 \le t \le 1)$ on X with $\nu_0 = \rho$, $\nu_1 = \nu$, one can see that $p_1^{-1}(c)$ is ambient isotopic in X to $q_1^{-1}(c')$ for a core c' of some open solid torus V' in N. By using this fact, it is shown that f is homotopic to a map $g : M \longrightarrow N$ such that $g^{-1}(V') = V$ and $\psi = g|(M-V) : M-V \longrightarrow N-V'$ is a homotopy equivalence; see [6] for details. By Waldhausen's Rigidity Theorem [14], ψ is properly homotopic to a homeomorphism, which can be extended to a homeomorphism from M to N homotopic to f.

Next, suppose that $f : M \longrightarrow M$ is a self-homeomorphism of a closed hyperbolic 3-manifold homotopic to the identify Id_M. Let V be an open solid torus designated by the insulator condition for M with respect to the hyperbolic metric ρ on M. Then, $f(V)$ is a designated open solid torus with respect to the push-forward metric $f_*(\rho)$ on M. It follows that V is ambient isotopic to $f(V)$, and hence f is isotopic to a homeomorphism $f' : M \longrightarrow M$ with $f'|V = \mathrm{Id}_V$. By using this fact, one can show that f' and hence f are isotopic to Id_M.

Seifert fibered case. Let M be a closed Seifert fibered space with a hyperbolic base 2-orbifold O. For the proofs of Theorems 0.1 and 0.2, it suffices to consider the case when O is a triangle 2-orbifold.

Let $f : M \longrightarrow N$ be a homotopy equivalence from the manifold M to a closed irreducible 3-manifold N. There exist finite coverings $p_1 : X \longrightarrow M$, $q_1' : X' \longrightarrow N$ such that the induced fibration on X is an S^1-bundle over a closed hyperbolic surface, and f is lifted to a map f_1' homotopic to a homeomorphism $f_1 : X \longrightarrow X'$. Then, $q_1 = q_1' \circ f_1 : X \longrightarrow N$ is also a finite covering. Let $p : \hat{X} \longrightarrow M$, $q : \hat{X} \longrightarrow N$ be the coverings associated to $\langle\gamma\rangle \subset \pi_1(M)$, $f_*(\langle\gamma\rangle) \subset \pi_1(N)$, respectively. Both $G = \pi_1(M)/\langle\gamma\rangle$ and $G' = \pi_1(N)/f_*(\langle\gamma\rangle)$ can be regarded as covering transformation groups on $\hat{X}$. Let $\hat{\sigma} : \hat{X} \longrightarrow \mathbf{H}^2$ be the projection of the induced S^1-bundle structure. Note that the isometric action of $G = \pi_1^{\mathrm{orb}}(O)$ on $\mathbf{H}^2$ commutes the action of G on $\hat{X}$ via $\hat{\sigma}$. Suppose that $\hat{X}$ has the Riemannian metric ν induced from a fixed Riemannian metric on N, which is also denoted by ν. Then, G' acts on $\hat{X}$ isometrically. The definition of the 2-dimensional insulator condition is fairly simple compared with the 3-dimensional case, and it gives a certain family $\{\alpha_{i:j}\}$ of geodesic lines in $\mathbf{H}^2$ with $G\big(\bigcup_{i,j} \alpha_{i:j}\big) = \bigcup_{i,j} \alpha_{i:j}$. The proof of Lemma 2.3 which shows that any hyperbolic triangle group satisfies the insulator condition is short and elementary. By using the condition, we will prove that there exist G'-equivariant, mutually disjoint open solid tori $\hat{V}_i$ in $\hat{X}$. Here, each $\hat{V}_i$ is the 'central' component of $\hat{X} - \bigcup_{j \neq i} A_{i:j}$, where $A_{i:j}$ is a ν-least area open annulus properly homotopic in $\hat{X}$ to the open annulus $\hat{\sigma}^{-1}(\alpha_{i:j})$. Theorem 0.1 asserts that the map f is homotopic to a homeomorphism. In the proof, we do not need any deformation of Riemannian metrics on X. We just need the fact that $\hat{X} - \bigcup_i \mathrm{Int}\hat{W}_i$ is homeomorphic to $R \times S^1$ for some planar surface R, where $\hat{W}_i$ is a G'-equivariant, solid torus core of $\hat{V}_i$. This implies that N admits a Seifert fibration, and hence f is homotopic to a homeomorphism.

Theorem 0.2 asserts that a self-homeomorphism $f : M \longrightarrow M$ is isotopic to Id_M if f is homotopic to Id_M. It will be proved by arguments similar to those in the hyperbolic case. In particular, we need to study the situation of designated solid tori $\hat{V}_i$ in $\hat{X}$ under a continuous deformation of Riemannian metrics on M from ν to $f_*(\nu)$.

2. Insulator condition for Fuchsian orbits

We refer to Hempel [8] and Jaco [9] for the fundamental notation and definitions of 3-manifold topology. All 3-manifolds in the paper are assumed to be orientable.

Let M be a closed Seifert fibered space with the Seifert fibration $\sigma : M \longrightarrow O$, and let $f : M \longrightarrow N$ be a homotopy equivalence to a closed irreducible 3-manifold N. Suppose that the base O is the hyperbolic triangle orbifold $O(p, q, r)$, where p, q, r are integers with $2 \le p \le q \le r$ and $1/p + 1/q + 1/r < 1$. Let $\langle\gamma\rangle$ be the cyclic subgroup of $\pi_1(M)$ such that the generator γ is represented by a regular fiber of M. Since O is orientable, $\langle\gamma\rangle$ is contained in the center $Z(\pi_1(M))$ of $\pi_1(M)$. We set $G = \pi_1(M)/\langle\gamma\rangle$ and $G' = \pi_1(N)/\langle f_*(\gamma)\rangle$, and let $\overline{f}_* : G \longrightarrow G'$ be the isomorphism induced from $f_* : \pi_1(M) \longrightarrow \pi_1(N)$. The group G is naturally identified with the orbifold fundamental group $\pi_1^{\mathrm{orb}}(O)$. For any $g \in G$, the element

$\overline{f}_*(g) \in G'$ will be denoted by g'. Let $a : F \longrightarrow O$ be an orbifold covering such that F is a closed hyperbolic surface, and let $\hat{a} : \mathbf{H}^2 \longrightarrow F$ be the universal covering. Then, $\pi_1(F) = \pi_1^{\mathrm{orb}}(F)$ is regarded as a subgroup of $G = \pi_1^{\mathrm{orb}}(O)$. Consider the finite covering $p_1 : X \longrightarrow M$ associated to $\varphi^{-1}(\pi_1(F)) \subset \pi_1(M)$, where $\varphi : \pi_1(M) \longrightarrow G$ is the natural quotient epimorphism. If $q_1' : X' \longrightarrow N$ is the finite covering corresponding to p_1, then $f : M \longrightarrow N$ can be lifted to a homotopy equivalence $f_1' : X \longrightarrow X'$. Note that X has the S^1-bundle structure $\sigma_X : X \longrightarrow F$ induced from $\sigma : M \longrightarrow O$. Since X is a Haken manifold and X' is irreducible (see Meeks-Simon-Yau [10]), f_1' is homotopic to a homeomorphism f_1. Thus, we have the following commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{f_1' \,(\simeq f_1)} & X' \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle q_1'} \\ M & \xrightarrow{\quad f \quad} & N \end{array}$$

We will identify X with X' via the homeomorphism f_1. If $\hat{p} : \hat{X} \longrightarrow X$ is the covering associated to $\langle\gamma\rangle \subset \pi_1(X)$, then $p_1 \circ \hat{p} : \hat{X} \longrightarrow M$, $q_1 \circ \hat{p} : \hat{X} \longrightarrow N$ are also coverings, where $q_1 = q_1' \circ f_1$.

Fix a Riemannian metric ν on N, and also denote the induced metrics on X and $\hat{X}$ by ν. The manifold $\hat{X}$ has the S^1-bundle structure $\hat{\sigma} : \hat{X} \longrightarrow \mathbf{H}^2$ induced from σ_X. Then, we have the following diagram consisting of four blocks. The right lower block is homotopically commutative, and the other three are commutative in the usual sense.

$$\begin{array}{ccccc} \mathbf{H}^2 & \xleftarrow{\quad \hat{\sigma} \quad} & \hat{X} & \xrightarrow{\quad \mathrm{Id}_{\hat{X}} \quad} & \hat{X} \\ {\scriptstyle \hat{a}}\downarrow & & {\scriptstyle \hat{p}}\downarrow & & \downarrow{\scriptstyle \hat{p}} \\ F & \xleftarrow{\quad \sigma_X \quad} & X & \xrightarrow{\quad \mathrm{Id}_X \quad} & X \\ {\scriptstyle a}\downarrow & & {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle q_1} \\ O & \xleftarrow{\quad \sigma \quad} & M & \xrightarrow{\quad f \quad} & N \end{array}$$

The group G is regarded as an isometric and properly discontinuous transformation group on $\mathbf{H}^2$, i.e. G is a *Fuchsian group*, and also the covering tranformation group on $\hat{X}$ with respect to $p_1 \circ \hat{p}$. The group G' is regarded as the ν-isometric tranformation group on $\hat{X}$ with respect to $q_1 \circ \hat{p}$. Since the ν-lengths of S^1-fibers $\hat{\sigma}^{-1}(x)$ ($x \in \mathbf{H}^2$) are bounded, $\hat{\sigma} : \hat{X} \longrightarrow \mathbf{H}^2$ is a quasi-isometry. In particular, one can suppose that the compactification of $\hat{X}$ as a Gromov hyperbolic space has the boundary S^1_∞. The covering transformations of $\hat{X}$ corresponding to $g \in G$ and $g' \in G'$ are properly homotopic to each other such that the length of each trace of the homotopy is uniformly bounded. This means that the action of G on S^1_∞ is equal to that of G'. Thus, $\overline{f}_* : G \longrightarrow G'$ is the identity map as a transformation on S^1_∞.

The d-neighborhood $\{y \in \mathbf{H}^2; \mathrm{dist}(y, J) \le d\}$ of a closed subset J of $\mathbf{H}^2$ is denoted by $\mathcal{N}_d(J, \mathbf{H}^2)$. When J is a one point set $\{x\}$, $\mathcal{N}_d(\{x\}, \mathbf{H}^2)$ is also denoted by $B_d(x)$. For any geodesic line α in $\mathbf{H}^2$, $A^\natural_\alpha = \hat{\sigma}^{-1}(\alpha)$ is an open annulus properly embedded in $\hat{X}$. For $C > 0$, we set $L_C(\alpha) = \hat{\sigma}^{-1}(\mathcal{N}_C(\alpha, \mathbf{H}^2))$, which is a neighborhood of $A^\natural_\alpha$ in $\hat{X}$. An open annulus A properly embedded in $\hat{X}$ is said to be a *ν-least area open annulus associated to α* if A satisfies the following two conditions:

- There exists $C > 0$ with $A \subset L_C(\alpha)$ and such that A is properly homotopic to $A^\natural_\alpha$ in $L_C(\alpha)$. The number C is called a *deviation* of A.
- A is ν-least area. This means that any essential (compact) annulus A_0 in A has the least area among all immersed annuli A_1 in $\hat{X}$ with $\partial A_1 = \partial A_0$.

Lemma 2.1. *For any geodesic line $\alpha \subset \mathbf{H}^2$, there exists a ν-least area open annulus A_α associated to α. Moreover, a deviation $C > 0$ of A_α can be taken as a constant independent of α.*

Proof. The base orbifold $O(p, q, r)$ is divided by three geodesic segments u_1, u_2, u_3 into two hyperbolic triangles of interior angles π/p, π/q, π/r. The preimage $a^{-1}(u_1 \cup u_2 \cup u_3)$ is a union of finitely many closed geodesics $l_1, \cdots, l_n$ in F. Since $\pi_1(F)$ is residually finite, then replacing F by a suitable finite covering of F, if necessary, we may assume that $l_1, \cdots, l_n$ are simple geodesics. Note that $\hat{a}^{-1}(l_i)$ is a disjoint union of geodesic lines in $\mathbf{H}^2$. Figure 2.1 below illustrates the union $\bigcup_{i=1}^n \hat{a}^{-1}(l_i)$. Let $\mathcal{L}$ be the set of all geodesic lines contained in $\bigcup_{i=1}^n \hat{a}^{-1}(l_i)$. The preimage $T^\natural_i = \sigma_X^{-1}(l_i)$ is an incompressible torus in X. According to Freedman-Hass-Scott [4], there exists an embedded torus T_i in X which is ν-least area among all tori homotopic to $T^\natural_i$. Each component A_i of $\hat{p}^{-1}(T_i)$ is a ν-least area open annulus associated to a component of $\hat{a}^{-1}(l_i)$.

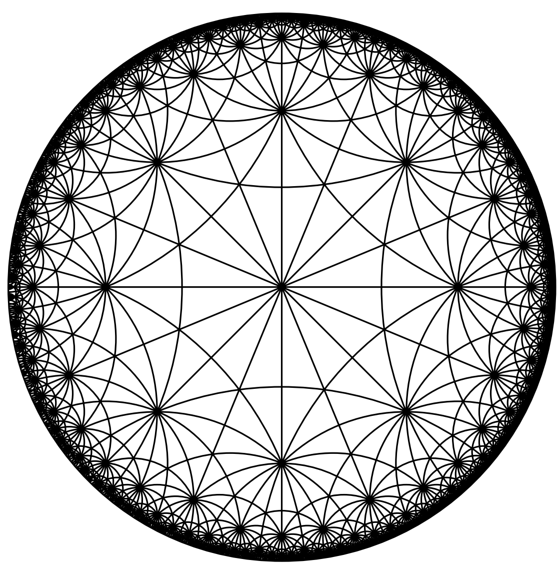

FIGURE 2.1. The case of $(p, q, r) = (2, 3, 8)$.

Since the ν-length of each trace of the homotopy between T_i and $T^\natural_i$ is uniformly bounded, that between $g'A_i$ and $g'A^\natural_i$ is uniformly bounded for any $g' \in G'$. Moreover, as is seen above, that between $g'A^\natural_i$ and $gA^\natural_i$ is uniformly bounded. It follows that $g'A_i$'s ($g' \in G', i = 1, \cdots, n$) have a common deviation $C_1 > 0$. We set $\mathcal{A} = \{g'A_i;\, g' \in G', i = 1, \cdots, n\}$. Consider a G-fundamental domain $D \subset \mathbf{H}^2$ of O consisting of two hyperbolic triangles. Then, there exist finitely many geodesic lines $\lambda_1, \cdots, \lambda_k \in \mathcal{L}$ in $\mathbf{H}^2 - D$ satisfying the following two conditions:

- For any $i \in \{1, \cdots, k\}$, λ_i is disjoint from λ_j's ($j \neq i-1, i, i+1$) and meets both λ_{i-1}, λ_{i+1} transversely in single points, where $\lambda_0 = \lambda_k$ and $\lambda_{k+1} = \lambda_1$.
- Let ζ, η be any geodesic lines in $\mathbf{H}^2$ meeting orthogonally in a single point of D. If η meets λ_i and λ_j respectively in the distinct components of $\mathbf{H}^2 - \zeta$, then $\mathcal{N}_{C_1}(\lambda_u, \mathbf{H}^2)$ is disjoint from ζ for any $u \in \{i-1, i, i+1, j-1, j, j+1\}$; see Figure 2.2(a).

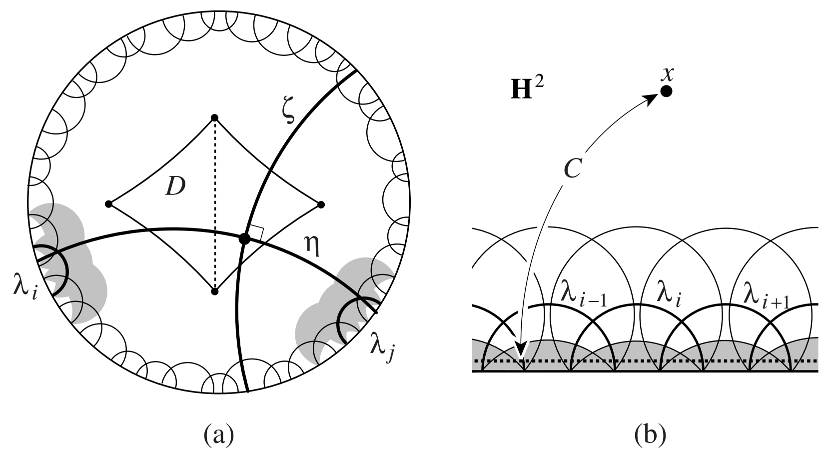

FIGURE 2.2. In (a), the two shaded regions represent respectively $\bigcup_{u=i-1}^{i+1} \mathcal{N}_{C_1}(\lambda_u, \mathbf{H}^2)$ and $\bigcup_{u=j-1}^{j+1} \mathcal{N}_{C_1}(\lambda_u, \mathbf{H}^2)$. In (b), the shaded region represents $Q_1 \cup \cdots \cup Q_k$, and the dotted line represents $\partial B_C(x)$.

For $i \in \{1, \cdots, k\}$, the component of $\mathbf{H}^2 - \mathrm{Int}\mathcal{N}_{C_1}(\lambda_i, \mathbf{H}^2)$ disjoint from D is denoted by Q_i. There exists a constant $C > 0$ with $\partial B_C(x) \subset \mathrm{Int}(Q_1 \cup \cdots \cup Q_k)$ for any $x \in D$; see Figure 2.2(b).

For any geodesic line α in $\mathbf{H}^2$, let $\mathcal{A}^\vee(\alpha)$ be the subset of $\mathcal{A}$ consisting of $A_i \in \mathcal{A}$ disjoint from the open annulus $A_\alpha^\natural = \hat{\sigma}^{-1}(\alpha)$. For $A_i \in \mathcal{A}^\vee(\alpha)$, the closure of the component of $\hat{X} - A_i$ disjoint from $A_\alpha^\natural$ is denoted by $E(A_i)$. Then, the union $E(\alpha) = \bigcup\{E(A_i);\, A_i \in \mathcal{A}^\vee(\alpha)\}$ consists of two components. For any geodesic line β meeting α orthogonally in a single point $x \in \mathbf{H}^2$, consider an element $g \in G$ with $g(x) \in D$. The points $y, z \in g(\beta)$ of distance C from $g(x)$ are contained respectively in $Q_{i-1} \cup Q_i \cup Q_{i+1}$ and $Q_{j-1} \cup Q_j \cup Q_{j+1}$, where i, j are the numbers satisfying the above condition if we set $g(\alpha) = \zeta$ and $g(\beta) = \eta$. Then, $\hat{\sigma}^{-1}(\{y, z\})$ is contained in $\mathrm{Int}E(g(\alpha))$. This implies that $L_C(\alpha)$ contains $\partial E(\alpha)$.

For convenience, we fix a direction of α. Consider a sequence $\{\alpha_0, \alpha_1, \alpha_2, \cdots\}$ of compact connected subsets (geodesic segments) of α satisfying $\alpha_n \subset \mathrm{Int}\alpha_{n+1}$ and $\bigcup_n \alpha_n = \alpha$. The annulus $\Lambda_n = \hat{\sigma}^{-1}(\alpha_n)$ is a deformation retract of $A_\alpha^\natural$.

For any $n \in \mathbf{N}$, let Λ_n^+, Λ_n^- be the components of $\Lambda_n - \mathrm{Int}\Lambda_{n-1}$, where Λ_n^+ is supposed to be ahead of Λ_n^- with respect to the direction of α. If necessary, replacing $\{\alpha_n\}$ by a suitable subsequence, one can have mutually disjoint elements $J_n^\epsilon \in \mathcal{A}$ ($\epsilon = \pm, n \in \mathbf{N}$) satisfying the following two conditions:

- J_n^ϵ meets Λ_n^ϵ non-trivially and $J_n^\epsilon \cap (A_\alpha^\natural - \mathrm{Int}\Lambda_n^\epsilon)$ is empty.
- The two end components of J_n^ϵ are contained in distinct components of $E(\alpha)$.

Let $\hat{X}_n$ be the closure of the middle component of $\hat{X} - J_n^+ \cup J_n^-$, and set $Y_n(\alpha) = \hat{X}_n - E(\alpha)$, $\partial_0 Y_n(\alpha) = \partial Y_n(\alpha) \cap (J_n^+ \cup J_n^-)$; see Figure 2.3. By the Annulus Theorem (for example, see Jaco [9, Theorem VIII.13]), there exists an essential annulus $(A_{n;\alpha}^\natural, \partial A_{n;\alpha}^\natural)$ properly embedded in $(Y_n(\alpha), \partial_0 Y_n(\alpha))$. Let $A_{n;\alpha}$ be a ν-least area annulus in $\hat{X}_n$ bounding $\partial A_{n;\alpha}^\natural$. Again by [4], $A_{n;\alpha}$ is an embedded annulus. Since all elements of $\mathcal{A}^\vee(\alpha)$ are ν-least area, $A_{n;\alpha}$ is contained in $Y_n(\alpha)$. For any integer $m \geq n$, set $A_{n;\alpha}^{(m)} = A_{m;\alpha} \cap Y_n(\alpha)$. Then, $\{(A_{n;\alpha}^{(m)}, \partial A_{n;\alpha}^{(m)})\}_{m \geq n}$ is a sequence of ν-least area essential annuli properly embedded in $(Y_n(\alpha), \partial_0 Y_n(\alpha))$ with $\sup_m\{\mathrm{Area}_\nu(A_{n;\alpha}^{(m)})\} \leq \mathrm{Area}_\nu(\partial Y_n(\alpha))/2$. By the Ascoli-Arzelà Theorem, the sequence has a subsequence which converges uniformly to a ν-least area annulus in $(Y_n(\alpha), \partial_0 Y_n(\alpha))$. By applying the diagonal argument to $\{A_{n,\alpha}\}$, one can show that the sequence has a subsequence converging uniformly on any compact set to a ν-least area open annulus A_α properly embedded in $\bigcup_{n=1}^\infty Y_n(\alpha) \subset L_C(\alpha)$. □

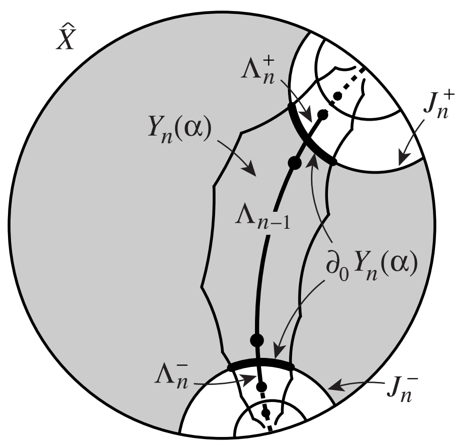

FIGURE 2.3. The shaded region represents $\hat{X}_n$.

Consider the G-orbit $Gx = \{x_i\}$ of $x \in \mathbf{H}^2$. For any ordered pair (x_i, x_j) of distinct points $x_i, x_j \in Gx$, let $\alpha_{i:j} = \alpha_{j:i}$ be the perpendicular bisector line of the geodesic segment connecting x_i with x_j, and let $H_{i:j}$ be the component of $\overline{\mathbf{H}^2} - \overline{\alpha}_{i:j}$ containing x_i. The closure $\delta_{i:j}$ of $S^1_\infty \cap H_{i:j}$ in S^1_∞ is an arc satisfying $\delta_{i:j} \cup \delta_{j:i} = S^1_\infty$ and $\delta_{i:j} \cap \delta_{j:i} = \partial\delta_{i:j} = \partial\delta_{j:i}$.

The three arcs $\delta_1, \delta_2, \delta_3$ in S^1_∞ satisfy the *tri-linking property* if $\delta_1 \cup \delta_2 \cup \delta_3 = S^1_\infty$.

Definition 2.2. Under the notation above, the orbit Gx satisfies the *insulator condition* if, for each $x_i \in Gx$, all triads $\delta_{j_1:i}, \delta_{j_2:i}, \delta_{j_3:i}$ do not have the tri-linking property.

Let Σ_O be the singular point set of the hyperbolic triangle 2-orbifold $O = O(p,q,r)$ with $2 \le p \le q \le r$. The point of Σ_O corresponding to an elliptic element of $G = \pi_1^{\mathrm{orb}}(O)$ with order r is denoted by $\overline{x}_0$.

The following lemma will play a crucial role in the proof of Lemma 3.1.

Lemma 2.3. *For any point $x_0 \in (a \circ \hat{a})^{-1}(\{\overline{x}_0\})$, the orbit Gx_0 satisfies the insulator condition.*

Proof. Let $\alpha_{0:j}$ be the perpendicular bisector line of the geodesic segment connecting x_0 with $x_j \in Gx_0 - \{x_0\}$, and let d_j be the distance between x_0 and $\alpha_{0:j}$, that is, $d_j = \mathrm{dist}(x_0, x_j)/2$. If $\delta_{j_1:0}, \delta_{j_2:0}, \delta_{j_3:0}$ satisfied the tri-linking property, then $\Delta_{j_1,j_2,j_3} = \overline{H_{0:j_1} \cap H_{0:j_2} \cap H_{0:j_3}}$ would be the hyperbolic triangle containing x_0. The distances between x_0 and the three edges of Δ_{j_1,j_2,j_3} are $d_{j_1}, d_{j_2}, d_{j_3}$. In particular, Δ_{j_1,j_2,j_3} contains the round disk D_{j_1,j_2,j_3} centered at x_0 of radius $d(x_0) = \min\{d_{j_1}, d_{j_2}, d_{j_3}\}$. Let Δ' be a maximal hyperbolic triangle containing Δ_{j_1,j_2,j_3}. Here, Δ' being *maximal* means that all vertices of Δ' are in S^1_∞. Thus, $d(x_0)$ would not be greater than the inscribed radius of Δ', which is $\log\sqrt{3}$ (for example see Beardon [1, §7.14]).

We will show that if $(p,q,r) \ne (2,3,7)$, then the inequality

$$d(x_0) > \log\sqrt{3} = 0.5493061\cdots \tag{2.1}$$

holds. As is suggested in Figure 2.4, if the inequality (2.1) holds for (p,q,r), then it also holds for any (p',q',r') with $p' \ge p$, $q' \ge q$, $r' \ge r$. Thus, it suffices to consider the cases of $(p,q,r) = (2,3,8), (2,4,5), (3,3,4)$. From the formula on hyperbolic right-angled triangles given in [1, Theorem 7.11.3(i)], if $(p,q,r) = (2,3,8)$ or $(3,3,4)$, then

$$d(x_0) = \operatorname{arc\,cosh}\left(\frac{\cos(\pi/3)}{\sin(\pi/8)}\right) = 0.7642854\cdots$$

holds; see Figure 2.5. Similarly, if $(p,q,r) = (2,4,5)$, then

$$d(x_0) = \operatorname{arc\,cosh}\left(\frac{\cos(\pi/4)}{\sin(\pi/5)}\right) = 0.6268696\cdots$$

holds. Since in either case the inequality (2.1) holds, Gx_0 satisfies the insulator condition whenever $(p,q,r) \ne (2,3,7)$.

On the other hand, if $(p,q,r) = (2,3,7)$, then

$$d(x_0) = \operatorname{arc\,cosh}\left(\frac{\cos(\pi/3)}{\sin(\pi/7)}\right) = 0.5452748\cdots$$

is *slightly* smaller than $\log\sqrt{3} = 0.5493061\cdots$. If $d_{j_1} = d_{j_2} = d_{j_3} = d(x_0)$, then $\alpha_{0:j_1}, \alpha_{0:j_2}, \alpha_{0:j_3}$ contain distinct edges of the regular heptagon H centered at x_0 as illustrated in Figure 2.6. It follows that $\delta_{j_1:0}, \delta_{j_2:0}, \delta_{j_3:0}$ do not satisfy the tri-linking property. Thus, at least one of d_{j_k} $(k = 1,2,3)$, say d_{j_1}, is greater than

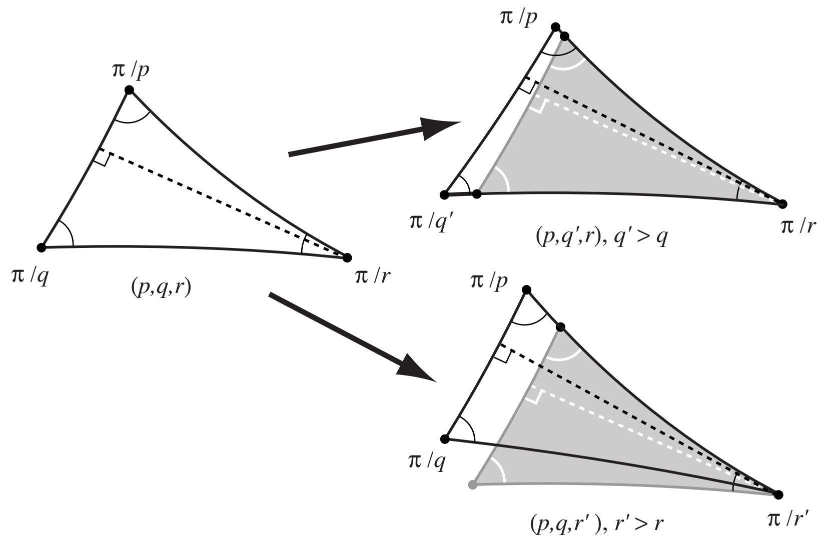

FIGURE 2.4. The cases of $q' > q$ and $r' > r$. The situation in the case of $p' > p$ is similar to that of $q' > q$.

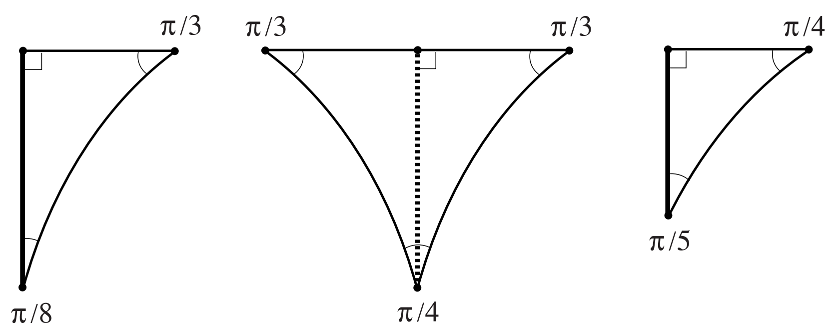

FIGURE 2.5.

$d(x_0)$. Then, we have

$$d_{j_1} \geq \frac{1}{2}\mathrm{dist}(x_0, x_8) = \mathrm{arc}\cosh\left(\frac{\cos(2\pi/7)}{\sin(\pi/7)}\right) = 0.9037998\cdots,$$

where x_8 is the point in $\mathbf{H}^2$ as shown in Figure 2.6. Suppose that $\delta_{j_1:0}, \delta_{j_2:0}, \delta_{j_3:0}$ satisfied the tri-linking property. Since the radius $d(x_0)$ of $D_{j_1,j_2,j_3} \subset \Delta_{j_1,j_2,j_3}$ has a value sufficiently near to $\log\sqrt{3}$, Δ_{j_1,j_2,j_3} would be geometrically well approximated by Δ'. From this, we know that Δ_{j_1,j_2,j_3} does not have any edge with $d_{j_k} > 0.9$, a contradiction. In fact, it is not hard to show that d_{j_k} is at most $0.55739\cdots$ by considering the extremal case when a round disk of radius $d(x_0)$ touches two edges of Δ'. Thus, also in the case of $(p, q, r) = (2, 3, 7)$, Gx_0 satisfies the insulator condition. $\square$

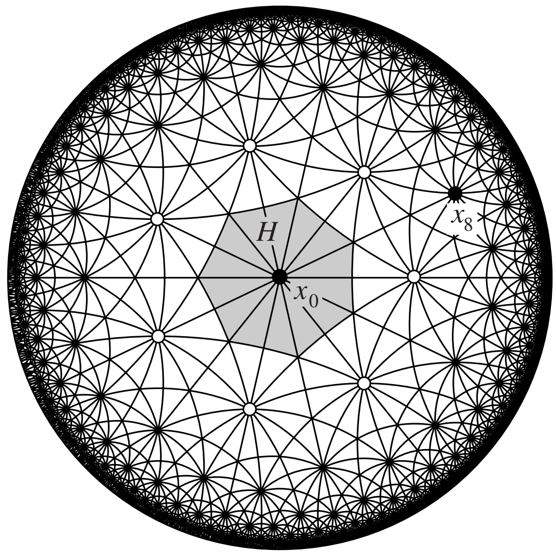

FIGURE 2.6. The seven white dots represent the points of Gx_0 nearest to x_0, and x_8 represents one of the points of Gx_0 next to these seven points.

3. Proofs of theorems

We will always assume that any two least area surfaces meet each other transversely, if necessary by deforming them slightly. Fix an $x_0 \in (a \circ \hat{a})^{-1}(\overline{x}_0)$ for the point $\overline{x}_0 \in \Sigma_O$ given in Lemma 2.3. Set $Gx_0 = \{x_i\}_{i\in\Gamma}$ for a countable set Γ. As in §2, for any ordered pair (i, j) of distinct elements of Γ, let $\alpha_{i:j} = \alpha_{j:i}$ be the perpendicular bisector line of the geodesic segment connecting x_i with x_j. Let $A_{i:j}$ be a ν-least area open annulus associated to $\alpha_{i:j}$. We will take them so that $g'A_{i:j} = A_{u:v}$ if $g(x_i) = x_u$, $g(x_j) = x_v$ for $g \in G$. Note that then $A_{i:j} \neq A_{j:i}$ may occur. If $\alpha_{i:j}$ and $\alpha_{k:l}$ do not have common end points, then by the ν-least area property for $A_{i:j}$ and $A_{k:l}$, the intersection $A_{i:j} \cap A_{k:l}$ is either empty or a single essential loop. If $\alpha_{i:j}$ and $\alpha_{k:l}$ has only one common end point, then $A_{i:j} \cap A_{k:l}$ has at most one compact component (an essential loop) and any other component ξ is a properly embedded arc in $A_{i:j}$ (resp. $A_{k:l}$) such that one of the two components of $A_{i:j} - \xi$ (resp. $A_{k:l} - \xi$) is an open disk. Note that the two end components of ξ are contained in a single end component of $A_{i:j}$. Similarly, if $\alpha_{i:j} = \alpha_{k:l}$, then either $A_{i:j} = A_{k:l}$ or $A_{i:j} \cap A_{k:l}$ consists of at most one essential loop and properly embedded arcs ξ each of which either excises from $A_{i:j}$ and $A_{k:l}$ open disks or cuts them into single open disks. The latter case occurs when ξ is a curve connecting the distinct end components of these annuli. The component of $\hat{X} - A_{i:j}$ adjacent to $\delta_{i:j} \subset \partial\hat{X} = S^1_\infty$ (resp. $\delta_{j:i}$) is denoted by $\hat{H}^+_{i:j}$ (resp. $\hat{H}^-_{i:j}$). Note that in the case when $A_{i:j} \neq A_{j:i}$, $\hat{H}^+_{i:j}$ is not equal to $\hat{H}^-_{j:i}$.

A simple loop c in an open solid torus V is called a *core* if $V - c$ is homeomorphic to $T^2 \times (0, 1)$.

Lemma 3.1. *For any $i \in \Gamma$, the intersection $\bigcap_{j\in\Gamma-\{i\}}(\hat{H}^+_{i:j} \cap \hat{H}^-_{j:i})$ has only one component $\hat{V}_i$ which is an open solid torus such that a core of $\hat{V}_i$ is also a core of $\hat{X}$. Moreover, any other component of $\bigcap_{j\in\Gamma-\{i\}}(\hat{H}^+_{i:j} \cap \hat{H}^-_{j:i})$ is an open 3-ball.*

Proof. By Lemma 2.1, one can have a common deviation for all $A_{i:j}$'s. Thus, there exist finitely many $\hat{H}_k$'s ($k = 1, \cdots, m$) with either $\hat{H}_k = \hat{H}^+_{i:j_k}$ or $\hat{H}_k = \hat{H}^-_{j_k:i}$ and such that

$$\bigcap_{j\in\Gamma-\{i\}} (\hat{H}^+_{i:j} \cap \hat{H}^-_{j:i}) = \hat{H}_1 \cap \cdots \cap \hat{H}_m.$$

We set $A_k = \partial\hat{H}_k$. Then, A_k is either $A_{i:j_k}$ or $A_{j_k:i}$. Suppose that $\mathcal{H}_{u-1} = \hat{H}_1 \cap \cdots \cap \hat{H}_{u-1}$ satisfies the assertion of this lemma; we will show that $\mathcal{H}_u$ also satisfies it. Since A_u is ν-least area, each component J of $A_u \cap \mathcal{H}_{u-1}$ is ether an open disk or an open annulus. Moreover, in the latter case, a simple essential loop in J is also essential in A_u. As is shown in Figure 3.1, $A_u \cap \mathcal{H}_{u-1}$ has at most one open annulus component J. The figure illustrates the case when $\partial_- J$ and $\partial_+ J$ meet respectively a compact component λ_- of $A_u \cap A_k$ and a non-compact component λ_+ of $A_u \cap A_l$ for some k, l with $1 \leq k, l \leq u-1$. The λ_- is an essential loop and λ_+ is a properly embedded curve in A_u. Note that the left side component of $A_u - \lambda_-$ is contained in $\text{Int}(\hat{X} - \hat{H}_k) \subset \text{Int}(\hat{X} - \mathcal{H}_{u-1})$.

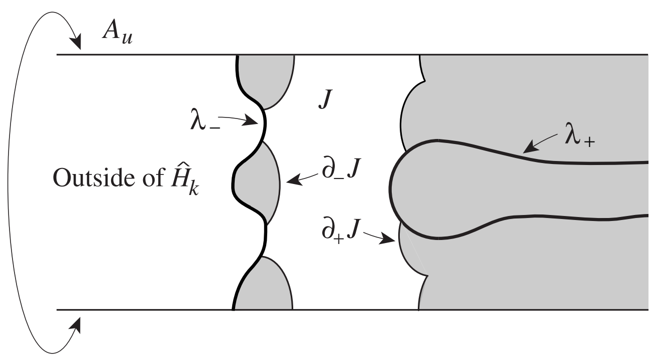

FIGURE 3.1. $A_u \cap \mathcal{H}_{u-1} - J$ is contained in the shaded part, which is a disjoint union of open disks. Thus, any component of $A_u \cap \mathcal{H}_{u-1}$ other than J is an open disk.

A core c of the open solid torus component V of $\mathcal{H}_{u-1}$ meets non-trivially only finitely many open disk components Δ_v of $A_u \cap \mathcal{H}_{u-1}$. If the intersection number $[\Delta_v] \cdot [c] \neq 0$, that is, Δ_v were a 'meridian disk' of V, then $V - \Delta_v$ would be an open ball. It follows that all components of $\mathcal{H}_u$ would be open 3-balls. As in [6, Lemma 4.3], one can show that some triad $\delta_{w_1:i}, \delta_{w_2:i}, \delta_{w_3:i}$ would have the tri-linking property. This contradicts the fact that Gx_0 satisfies the insulator condition by Lemma 2.3. Thus, each open disk Δ_v excises from V an open 3-ball B_v. Deforming c by an ambient isotopy with its support a small regular neighborhood of $\bigcup_v B_v$, one can cancel the intersection $c \cap (\bigcup_v \Delta_v)$. This shows that the union of open disk components of $A_u \cap \mathcal{H}_{u-1}$ divides V into one open solid torus V' and open 3-balls. Moreover, a core c' of V' is isotopic to c in V and hence c' is a core of $\hat{X}$. In the case when $A_u \cap \mathcal{H}_{u-1}$ contains an open annulus component J, J divides V' into two open solid tori V_1' and V_2'. One of them, say V_1', is contained in $\hat{H}_u$ and hence in $\mathcal{H}_u$, but the other is disjoint from $\hat{H}_u$. Thus, V_1' is a unique open solid torus component of $\mathcal{H}_u$. A simple essential loop c_J of J is a core of A_u and hence that of $\hat{X}$. A simple loop c_1' of V_1' isotopic to c_J in $\overline{V_1'}$ is a core of both V_1' and $\hat{X}$.

Repeating the same argument, one can show that $\mathcal{H}_m = \bigcap_{j\in\Gamma-\{i\}}(\hat{H}^+_{i:j} \cap \hat{H}^-_{j:i})$ satisfies the required conditions. □

Proof of Theorem 0.1. From the definition of open solid tori $\hat{V}_i$ $(i \in \Gamma)$ given in Lemma 3.1, they are mutually disjoint and G'-equivariant. In particular, for $g' \in G'$ corresponding to any element g of the stabilizer $\mathrm{Stab}_G(\overline{x}_0) \cong \mathbf{Z}_r$, we have $g'\hat{V}_0 = \hat{V}_0$. Then, there exists a G'-equivariant solid torus $\hat{W}_0 \subset \hat{V}_0$ such that a core of $\hat{W}_0$ is a core of $\hat{X}$. The image $W_0 = q_1 \circ \hat{p}(\hat{W}_0)$ is a solid torus in N. Set $\hat{W}_i = g'\hat{W}_0$ for $g \in G$ with $g(x_0) = x_i$. Since a core of any $\hat{W}_i$ is a core of $\hat{X}$ and since any two $\hat{W}_i$ and $\hat{W}_j$ are separated by the open annulus $A_{i:j}$, $\bigcup_i \hat{W}_i$ is an unknotted 'closed braid' in the open solid torus $\hat{X}$, that is, $\hat{Y} = \hat{X} - \bigcup_i \mathrm{Int}\hat{W}_i$ is homeomorphic to $R \times S^1$ for some planar surface R such that $\pi_1(R)$ is an infinitely generated free group. From $\pi_1(\hat{Y}) \cong \pi_1(R) \times \mathbf{Z}$, we know that $\pi_1(\hat{Y})$ has the center $Z = \langle h\rangle$ isomorphic to $\mathbf{Z}$. Since $q_1 \circ \hat{p}|\hat{Y} : \hat{Y} \longrightarrow N - \mathrm{Int}W_0$ is the restriction of the regular covering $q_1 \circ \hat{p}$, it is also a regular covering. Thus, $\pi_1(\hat{Y})$ can be regarded as a normal subgroup of $\pi_1(N - \mathrm{Int}W_0)$. In particular, for any $g \in \pi_1(N - \mathrm{Int}W_0)$, ghg^{-1} is an element of $\pi_1(\hat{Y})$. Moreover, for any $f \in \pi_1(\hat{Y})$, we have

$$\begin{aligned} f(ghg^{-1})f^{-1} &= g[(g^{-1}fg)h(g^{-1}fg)^{-1}]g^{-1} \\ &= g[h(g^{-1}fg)(g^{-1}fg)^{-1}]g^{-1} = ghg^{-1}. \end{aligned}$$

This implies that $ghg^{-1} \in Z$, and hence Z is a normal subgroup of $\pi_1(N - \mathrm{Int}W_0)$. Since $N - \mathrm{Int}W_0$ is a Haken manifold, this shows that $N - \mathrm{Int}W_0$ admits a Seifert fibration (see [9, Theorem VI.24]). Thus, N has a Seifert fibration extending that on $N - \mathrm{Int}W_0$. Since then $f : M \longrightarrow N$ is a homotopy equivalence between closed Seifert fibered spaces of infinite π_1, f is homotopic to a homeomorphism. □

Proof of Theorem 0.2. Let $f : M \longrightarrow M$ be a self-homeomorphism of a closed Seifert fibered space M with the hyperbolic base 2-orbifold O which is homotopic to the identity Id_M of M. If O is not a triangle orbifold, then M is a Haken manifold, and hence the proof of Theorem 0.2 is reduced to Waldhausen [14, Theorem 7.1]. So, we may assume that O is a triangle 2-orbifold $O(p, q, r)$ with $2 \le p \le q \le r$, $1/p + 1/q + 1/r < 1$. Moreover, one can suppose that f is a diffeomorphism. Fix a Riemannian metric ν on M. Then, $f_*(\nu)$ is another Riemannian metric on M. Consider a solid torus W_ν in M defined as $W_0 \subset N$ in the proof of Theorem 0.1 with respect to ν. Note that W_ν contains an exceptional fiber in M of order r. The other two exceptional fibers are contained in $M - \mathrm{Int}W_\nu$. Similarly, $f(W_\nu)$ can be regarded as a solid torus $W_{f_*(\nu)}$ in M with respect to $f_*(\nu)$.

First, we will show that Id_M is isotopic to a diffeomorphism $f' : M \longrightarrow M$ with $f'(W_\nu) = f(W_\nu)$. Let us consider a continuous deformation of Riemannian metrics on M from $\nu = \nu_0$ to $f_*(\nu) = \nu_1$ defined as follows. Let g_0, g_1 be the inner products on the tangent bundle $T(M)$ determining the Riemannian metrics ν_0, ν_1. Then, the Riemannian metric ν_t $(t \in [0, 1])$ on M is determined by the inner product g_t

on $T(M)$ such that

$$g_t(u,v) = (1-t)g_0(u,v) + tg_1(u,v) \tag{3.1}$$

for any $x \in M$ and any two vectors $u, v \in T_x(M)$. Let $\hat{V}_{i[t]}$ be a G-equivariant open solid torus in $\hat{X}$ defined as in Lemma 3.1 with respect to ν_t, and let W_t be an unknotted solid torus in the open solid torus $p_1 \circ \hat{p}(\hat{V}_{i[t]}) \subset M$. For the proof, it suffices to see that any $t_0 \in [0,1]$ has a neighborhood U such that, for each $t \in U$, W_t is ambient isotopic to W_{t_0} in M. If not, there would exist a sequence $\{t_n\}$ in $[0,1]$ converging to t_0 such that any W_{t_n} is not ambient isotopic to W_{t_0}. Let $l_1, \cdots, l_m$ be simple closed geodesics in F as in the proof of Lemma 2.1. Let $T_{k[t_n]}$ be a ν_{t_n}-least area torus homotopic to the embedded torus $T_k^\natural = \sigma_X^{-1}(l_k)$ in X. Note that the ν_{t_0}-areas of $T_{k[t_n]}$'s are bounded from above. In fact, from (3.1), we have a constant $K \geq 1$ independent of n such that

$$\mathrm{Area}_{t_0}(T_{k[t_n]}) \leq K\mathrm{Area}_{t_n}(T_{k[t_n]}) \leq K\mathrm{Area}_{t_n}(T_k^\natural) \leq K^2\mathrm{Area}_{t_0}(T_k^\natural).$$

By the Ascoli-Arzelà Theorem, if necessary passing to a subsequence of $\{t_n\}$, we may assume that each sequence $\{T_{k[t_n]}\}_{n=1}^\infty$ converges uniformly to a ν_{t_0}-least area torus $T^\bullet_{k[t_0]}$. Note that either $T_{k[t_0]} = T^\bullet_{k[t_0]}$ or $T_{k[t_0]} \cup T^\bullet_{k[t_0]}$ bounds a submanifold of X homeomorphic to $T^2 \times I$. From this convergence, one can take a constant $C_1 > 0$ independent of n as a deviation of any component $A_{k[t_n]}$ of $\hat{p}^{-1}(T_{k[t_n]})$. Then, as in the proof of Lemma 2.1, we have a constant $C > 0$ independent of n as a deviation of a ν_{t_n}-least area open annulus $A_{\alpha[t_n]}$ associated to any geodesic line α in $\mathbf{H}^2$. Again passing to a subsequence of $\{t_n\}$ if necessary, we may assume that the sequence $\{A_{i:j[t_n]}\}_{n=1}^\infty$ converges uniformly on all compact subsets to a ν_{t_0}-least area open annulus $A^\bullet_{i:j[t_0]}$ associated to $\alpha_{i:j}$. As in Lemma 3.1, $A^\bullet_{i:j[t_0]}$ and $A^\bullet_{j:i[t_0]}$ determine a G-equivariant open solid torus $\hat{V}^\bullet_{i[t_0]}$. Let $W^\bullet_{t_0}$ be an unknotted solid torus in $p_1 \circ \hat{p}(\hat{V}^\bullet_{i[t_0]})$. From the uniform convergence of $\{A_{i:j[t_n]}\} \longrightarrow A^\bullet_{i:j[t_0]}$ in compact subsets, we know that W_{t_n} is ambient isotopic to $W^\bullet_{t_0}$ in M for any $n \in \mathbf{N}$ greater than a sufficiently large n_0. Applying the argument in the proof of Lemma 3.1 to $A_{i:j[t_0]}, A_{j:i[t_0]}, A^\bullet_{i:j[t_0]}, A^\bullet_{j:i[t_0]}$ $(j \in \Gamma - \{i\})$, one can show that $\hat{V}_{i[t_0]} \cap \hat{V}^\bullet_{i[t_0]}$ has a unique open solid torus component $\hat{V}^\blacktriangle_{i[t_0]}$, a core of which is also a core of $\hat{X}$. This implies that a core of $\hat{V}_{i[t_0]}$ (resp. of $\hat{V}^\bullet_{i[t_0]}$) is isotopic to a core of $\hat{V}^\blacktriangle_{i[t_0]}$ in $\hat{V}_{i[t_0]}$ (resp. in $\hat{V}^\bullet_{i[t_0]}$). It follows that $W^\bullet_{t_0}$ and hence W_{t_n} $(n > n_0)$ are ambient isotopic to W_{t_0} in M. This contradiction shows the existence of our desired diffeomorphism f'.

Since the Seifert fibration on $M - \mathrm{Int}W_\nu$ has the base orbifold with a disk as its underlying space and with two exceptional fibers, $M - \mathrm{Int}W_\nu$ has a unique essential annulus A up to ambient isotopy. Thus, f' is isotopic to f'' with $f''(W_\nu) = f(W_\nu)$ and $f''(A) = f(A)$. Moreover, we may assume that, for some essential simple loop c in $\mathrm{Int}A$, $f''|c = f|c$. If $f^{-1} \circ f''|A : A \longrightarrow A$ were an orientation-reversing homeomorphism, then the homomorphism $\overline{f}_* : G \longrightarrow G$ induced from the identity $\mathrm{Id}_{\pi_1(M)} = f_* : \pi_1(M) \longrightarrow \pi_1(M)$ would exchange elliptic elements corresponding to the two exceptional fibers in $M - \mathrm{Int}W_\nu$, a contradiction. This shows that f'' is isotopic to f''' with $f'''|(T \cup A) = f|(T \cup A)$, where $T = \partial W_\nu$. Since the closure of each component of $M - T \cup A$ is a solid torus, f''' is isotopic to f. This completes the proof. $\square$

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