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AN ALTERNATIVE APPROACH TO HOMOTOPY OPERATIONS

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ABSTRACT. We give a particular choice of the higher Eilenberg-Mac Lane maps by a recursive formula. This choice leads to a simple description of the homotopy operations for simplicial $\mathbf{Z}/2$-algebras.

1. INTRODUCTION

This paper is about the ring of homotopy groups of a simplicial ring. This ring of homotopy classes has a huge amount of additional structure. The theory is best worked out for algebras over $\mathbb{F}_2$, and we will restrict ourselves to this case. §§2–3 in [3] contain a good survey with references to the original articles. We just recall the points which are most important to us.

The main observation is that the square of every element in positive degree is zero. Analyzing this fact gives rise to a divided power structure on the ring of homotopy groups. There is a refinement of this, which constructs a sequence of homotopy operations δ_i. These are defined by Dwyer in [2] and also by Bousfield.

The homotopy of a simplicial algebra $R_\bullet$ is isomorphic to the homology of the associated chain complex $C_*(R)$. We can represent an element in $\pi_n(R)$ by a cycle in the the chain complex, that is, by a class $z \in R_n = C_n(R)$, such that $\sum_{i=0}^n d_i(z) = 0$.

The problem which this article wants to solve is the following: Suppose that δ_i is defined on $\pi_n(R_\bullet)$. Can we give an explicit formula for an element in R_{n+i} that represents the element $\delta_i(z)$?

The reason that we care is that we are interested in explicit applications, as in [1], where we compute these operations.

We will focus on Dwyer's approach. In order to define the operations, he considers a sequence of natural transformations, defined for pairs of simplicial $\mathbb{F}_2$-vector spaces $V_\bullet$ and $W_\bullet$ as follows:

$$D^k\colon [C(V) \otimes C(W)]_m \to [C(V \otimes W)]_{m-k}.$$

The transformations satisfy recursive conditions, which we write down in the beginning of section 2.

These maps are considered to be higher analogues of the Eilenberg-Mac Lane map. Dwyer proves an existence and uniqueness result for these maps, but he does not give an explicit formula. He then uses them to define homotopy operations. We follow the reformulation of [3] at this point.

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Let $z \in R_n$ and define $\Theta_i(z)$ for $1 \le i \le n$ as the element

$$\Theta_i(z) = \mu D^{n-i}(z \otimes z) + \mu D^{n-i-1}(z \otimes \partial z).$$

Here μ is multiplication in the simplicial algebra. If z is a cycle, and $2 \le i \le n$, one sees that $\Theta_i(z)$ is also a cycle, and that the formula defines an operation $\delta_i \colon \pi_n(R_\bullet) \to \pi_{n+i}(R_\bullet)$.

We are going to give simpler formulas for D^i which satisfy the defining recursive relations. The derivation of these is where the hard work of this article is done. When we obtain the formulas, we can plug them into the definition of Θ_i, and obtain an explicit formula for $\delta_i(z)$. We now explain the formula we obtain in this fashion.

Write N_*R for the normalized chain complex with $N_qR = \bigcap_{1 \le i \le q} \ker(d_i)$ and differential d_0. Let $Z_qR \subseteq N_qR$ denote the cycles, i.e., the elements $z \in R_q$ with $d_iz = 0$ for $0 \le i \le q$.

To make the final formulas appear simpler, we assume that z is a normalized chain. This is no restriction, since the associated chain complex considered above contains the normalized chain complex as a quasi-isomorphic subcomplex.

Definition 1.1. For integers q, i with $1 \le i \le q$ we define $U(q,i)$ to be the set of pairs (μ, ν) of ordered sequences $\mu_1 < \cdots < \mu_i$, $\nu_1 < \cdots < \nu_i$ with disjoint union

$$\{\mu_1, \dots, \mu_i\} \sqcup \{\nu_1, \dots, \nu_i\} = \{q-i, q-i+1, \dots, q+i-1\}.$$

Let $V(q,i) \subseteq U(q,i)$ be the subset with $\mu_1 = q - i$.

Definition 1.2. For a cycle $z \in Z_qR$ we define $\delta_i(z) \in R_{q+i}$ by

$$\delta_i(z) = \sum_{(\mu,\nu) \in V(q,i)} s_{\nu_i} \dots s_{\nu_1}(z) s_{\mu_i} \dots s_{\mu_1}(z).$$

Note the close relationship to the Eilenberg-Mac Lane map D:

$$\mu D(z \otimes z) = \sum_{(\mu,\nu) \in U(q,q)} s_{\nu_i} \dots s_{\nu_1}(z) s_{\mu_i} \dots s_{\mu_1}(z).$$

Theorem 1.3. *When $2 \le i \le q$ the formula in Definition 1.2 defines a map $\delta_i : Z_qR \to Z_{q+i}R$. The induced map on homotopy $\delta_i : \pi_qR \to \pi_{q+i}R$ is the Bousfield-Dwyer homotopy operation.*

Proof. Let $z \in N_qR$ be a cycle. It is shown in Lemma 3.1 that $\delta_i(z)$ is a cycle in $N_{q+i}R$.

We use Definition 2.6 as our choice of higher Eilenberg-Mac Lane maps. The suspension operator $\mathcal{S}$ in the definition increases the simplicial degree by one. It preserves composition of simplicial maps and

$$\mathcal{S}(d_j) = d_{j+1}, \quad \mathcal{S}(s_j) = s_{j+1}, \quad \mathcal{S}(\mathrm{id}) = \mathrm{id}.$$

We have that $\Theta_i(z) = \mu D^{q-i}(z \otimes z)$. The statement we must show is that $\Theta_i(z) = \mu \mathcal{S}^{q-i}(D^0)(z \otimes z)$. So it suffices to see that $D^k(z \otimes z) = S^k(D^0)(z \otimes z)$, where $q - i = k$.

For $0 \leq j \leq k-1$ we have

$$\begin{aligned}\mathcal{S}^j(D^{k-j}) &= \mathcal{S}^{j+1}(D^{k-j-1}) + \begin{cases}\mathcal{S}^j(D^{k-j-1}(d_0 \otimes \mathrm{id})), & k-j \text{ even},\\ \mathcal{S}^j(D^{k-j-1}(\mathrm{id} \otimes d_0)), & k-j \text{ odd},\end{cases}\\ &= \mathcal{S}^{j+1}(D^{k-j-1}) + \begin{cases}\mathcal{S}^j(D^{k-j-1})(d_j \otimes \mathrm{id}), & k-j \text{ even},\\ \mathcal{S}^j(D^{k-j-1})(\mathrm{id} \otimes d_j), & k-j \text{ odd}.\end{cases}\end{aligned}$$

Thus $\mathcal{S}^j(D^{k-j})(z \otimes z) = \mathcal{S}^{j+1}(D^{k-j-1})(z \otimes z)$. We iterate this result and find that $D^k(z \otimes z) = \mathcal{S}^0(D^k)(z \otimes z) = \mathcal{S}^k(D^0)(z \otimes z)$. □

2. Higher Eilenberg-Mac Lane maps

Let $V_\bullet$, $W_\bullet$ be simplicial $\mathbb{F}_2$-vector spaces. The Eilenberg-Mac Lane map D can be considered as a set of linear maps

$$D_{i,j} \colon V_i \otimes W_j \to V_{i+j} \otimes W_{i+j}. \tag{2.1}$$

There is an explicit formula for these maps

$$D_{i,j} = \sum s_{\nu_j} \dots s_{\nu_1} \otimes s_{\mu_i} \dots s_{\mu_1},$$

where the sum is indexed by (i,j)-shuffles. An (i,j)-shuffle consists of two increasing sequences $\mu = (\mu_1, \mu_2, \dots, \mu_i)$ and $\nu = (\nu_1, \nu_2, \dots, \nu_j)$ such that each integer from $\{0, 1, \dots, i+j-1\}$ occurs exactly once as either a μ_r or a ν_s. There is an excellent discussion of this map in [4], chap. VIII, §8.

We will also consider the map ϕ_k with $(\phi_k)_{i,j} \colon V_i \otimes W_j \to V_{i+j-k} \otimes W_{i+j-k}$, defined to be the zero map, unless $i = j = k$. In this exceptional case, the map is the identity.

Let $T_{i,j} \colon V_i \otimes W_j \to W_j \otimes V_i$ be the map permuting the factors. According to [2] there is a sequence of maps D^k, $k = 0, 1, 2, \dots$, with

$$D^k_{i,j} \colon V_i \otimes W_j \to V_{i+j-k} \otimes W_{i+j-k},$$

only defined under the conditions that

$$0 \leq 2k \leq i+j, \tag{2.2}$$

which satisfy that

$$D^0 + TD^0T = D + \phi_0, \tag{2.3}$$

$$D^k + TD^kT = D^{k-1}\partial + \partial D^{k-1} + \phi_k. \tag{2.4}$$

There are many choices for the maps D^k, but Dwyer proves that the choices are equivalent up to homotopy (in a strong sense).

Before we give our definition of the maps D^k, we introduce some language, for the purpose of avoiding a nightmare of indices. We will write various formal sums of products of the simplicial generators d_r, s_r. Such an expression sometimes, but not always, defines a map $V_i \to V_j$ for all simplicial vector spaces $V_\bullet$. For instance, consider the simplicial relation $d_r s_r = \mathrm{id}$. The right-hand side of this relation defines a map (the identity) $V_i \to V_i$ for all i. The left-hand side does not define a map on V_i if $r > i$. But if $r \leq i$, both sides of the equation define maps, and in this case the relation says that the two maps $V_i \to V_i$ induced by the two sides agree.

We are going to consider tensor products of pairs of simplicial groups $V_\bullet \otimes W_\bullet$. For pairs of integers (i,j) and (k,l), we will write down natural transformations $V_i \otimes W_j \to V_k \otimes W_l$ and relations between such. We will specify them as sums of formal sequences of generators d_r, s_r. Every time we do this, we have to keep track of whether the formal sequences do indeed define natural transformations of the groups we write.

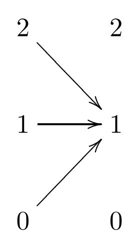
We start by introducing a "suspension operator" $\mathcal{S}$.

Definition 2.1. Let Δ denote the simplicial category. Define a functor $\mathcal{S}' : \Delta \to \Delta$ by $\mathcal{S}'([n]) = [n+1]$ on objects, and

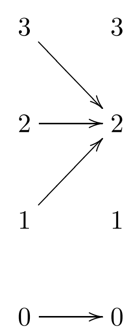
$$(\mathcal{S}'\alpha)(i) = \begin{cases} \alpha(i-1)+1 & \text{if } i \geq 1, \\ 0 & \text{if } i = 0, \end{cases}$$

on morphisms. Let $\mathcal{S} = (\mathcal{S}')^{op} : \Delta^{op} \to \Delta^{op}$ be the corresponding functor on the opposite category.

We can picture the suspension operator as follows:



$\alpha\colon [2] \to [2]$



$\mathcal{S}'\alpha\colon [3] \to [3]$

Note that d^0 defines a natural transformation $d^0 : \mathrm{Id} \to \mathcal{S}'$. So we have a natural transformation $d_0 : \mathcal{S} \to \mathrm{Id}$.

For a simplicial vector space $V\colon \Delta^{op} \to \{\mathbb{F}_2\text{-vector spaces}\}$, we define a new simplicial vector space $\mathcal{S}(V)_\bullet$ as the functor $V \circ \mathcal{S}$. There is a natural transformation of simplicial vector spaces $P\colon \mathcal{S}(V)_\bullet \to V_\bullet$, given degreewise as

$$P\colon \mathcal{S}(V)_i = V_{i+1} \xrightarrow{d_0} V_i.$$

Consider any natural transformation defined on the category of simplicial vector spaces of the form $\theta_V\colon V_i \to V_j$. For instance, this could be a map induced by a morphism $[j] \to [i]$ in Δ, or a linear combination of such maps.

Definition 2.2. The suspension of θ is the natural transformation

$$(\mathcal{S}\theta)_V\colon V_{i+1} = \mathcal{S}(V)_i \xrightarrow{\theta_{\mathcal{S}V}} \mathcal{S}(V)_j = V_{j+1}.$$

We obtain a commutative diagram:

$$\begin{tikzcd}
V_{i+1} \arrow[r, equal] \arrow[d, "(\mathcal{S}\theta)_V"'] & \mathcal{S}(V)_i \arrow[r, "P"] \arrow[d, "\theta_{\mathcal{S}V}"'] & V_i \arrow[d, "\theta_V"'] \\
V_{j+1} \arrow[r, equal] & \mathcal{S}(V)_j \arrow[r, "P"] & V_j
\end{tikzcd}$$

The composite of both rows is d_0, so it follows from this diagram and the naturality of θ that we have a relation

$$d_0(\mathcal{S}\theta) = \theta d_0 \colon \mathcal{S}(V)_i \to V_j. \tag{2.5}$$

We can also define a suspension on *formal* products of the simplicial generators d_i and s_i by $\mathcal{S}(d_i) = d_{i+1}$ and $\mathcal{S}(s_i) = s_{i+1}$. This suspension is compatible with the (formal) simplicial relations. In this way, we can formally define $\mathcal{S}$ on strings of simplicial generators by adding 1 to the index of every occurring generator s_i or d_j.

Lemma 2.3. *Let $V_\bullet$ be a simplicial $\mathbb{F}_2$-vector space.*

- *Let α be a formal products of generators d_r, s_r. Then α defines a natural transformation $\alpha_* \colon V_i \to V_j$ if and only if the corresponding natural transformation $\mathcal{S}(\alpha)_* : V_{i+1} \to V_{j+1}$ is also defined.*
- *Suspension commutes with passage to natural transformation, if the natural transformation is defined. That is,*
$$\mathcal{S}(\alpha_*) = (\mathcal{S}\alpha)_*.$$
- *If a formal sum $\sum \alpha_* \colon V_i \to V_j$ is defined and equals the zero map for i, then $\sum \mathcal{S}\alpha_* \colon V_{i+1} \to V_{j+1}$ is defined, and also equals the zero map.*

Proof. By induction on the number of factors in a product of generators, it is enough to check the first statement for the case of a generator s_j or d_j. For example, $(s_j)_*$ is defined on V_k if and only if $k \geq j$. But it is also true that $(\mathcal{S}(s_j))_* = (s_{j+1})_*$ is defined on V_{k+1} if and only if $k+1 \geq j+1$. The case d_j is treated in exactly the same way.

To prove the second statement, by induction it is enough to consider the case of a simplicial generator. That is, we have to check that $\mathcal{S}(s_i) = s_{i+1}$ and that $\mathcal{S}(d_i) = d_{i+1}$. But this follows directly from the definition of $\mathcal{S}$.

The third statement follows from the second, since

$$\left(\sum \mathcal{S}\alpha\right)_* = \mathcal{S}\left(\sum \alpha_*\right) = \mathcal{S}0 = 0.$$

□

All we have said can be generalized to tensor products of two simplicial vector spaces.

Lemma 2.4. *Let $V_\bullet$, $W_\bullet$ be simplicial $\mathbb{F}_2$-vector spaces. We can define the suspension of a natural transformation $\theta \colon V_i \otimes W_j \to V_k \otimes W_l$ as a natural transformation $\mathcal{S}\theta \colon V_{i+1} \otimes W_{j+1} \to V_{k+1} \otimes W_{l+1}$.*

- *Let α, β be formal products of generators d_i, s_i. If $\alpha_* \otimes \beta_*$ is defined for the index (i,j), then $\mathcal{S}(\alpha_* \otimes \beta_*)$ is defined for $(i+1, j+1)$.*
- *Suspension commutes with passage to natural transformation.*
- *If a formal sum $\sum \alpha_* \otimes \beta_*$ is defined and equals the zero map for (i,j), then $\mathcal{S}(\sum \alpha_* \otimes \beta_*)$ equals the zero map for $(i+1, j+1)$.*

We leave the proof to the dedicated reader.

Note that the map $(\phi_k)_{i,j}$ considered above is a natural transformation which is not defined as a formal combination of the simplicial generators d_i, s_i. But the suspension on it is still defined, and actually

$$(\phi_{k+1})_{i+1,j+1} = \mathcal{S}((\phi_k)_{i,j}) \colon V_{i+1} \otimes W_{j+1} \to V_{i+j+1-k} \otimes W_{i+j+1-k}.$$

However, ϕ_k is an example of a map with the following property.

Definition 2.5. An EM-type transformation F consists of the following data:

(1) An index function I_F, which to each pair of integers (i,j) associates a pair of integers $(k,l) = I_F(i,j)$.
(2) For each (i,j), we have a natural transformation on the category of simplicial vector spaces $V_i \otimes W_j \to V_k \otimes W_l$. Here, we conventionally define $V_i \otimes W_j = 0$ if either $i < 0$ or $j < 0$.

For instance, ϕ_k defines such an EM-type transformation. The index function for ϕ_k is $I_{\phi_k}(i,j) = (i+j-k, i+j-k)$.

The main example of such a map is the Eilenberg-Mac Lane map D. It is a collection of maps $D_{i,j}\colon V_i \otimes W_j \to V_{i+j} \otimes W_{i+j}$. The index function of D is $I_D(i,j) = (i+j, i+j)$.

We can always compose two EM-type transformations F, G. We can add them if the index function of F agrees with the index function of G. If F and G do not have the same index function, their sum is not defined. This is the price we pay for keeping easy control of the indices involved.

We can suspend an EM-type transformation. We define $I_{\mathcal{S}F}(i+1, j+1) = (1,1) + I_F(i,j)$ and $(\mathcal{S}F)_{i+1,j+1} = \mathcal{S}(F_{i,j})$. This is to be interpreted so that if either $i < 0$ or $j < 0$, then $(\mathcal{S}F)_{i+1,j+1} = 0$.

We can also twist an EM-type transformation by defining $\mathcal{T}(F)_{i,j} = TF_{j,i}T$. Suspension and twisting preserve sum and composition, and they commute.

Here are some examples of EM-type transformations, and relations between them. We insist that the relations are valid as relations between natural transformations with source $V_i \otimes W_j$ for all pairs of integers (i,j). When checking the formulas below, the main worry is to keep track of cases like $V_0 \otimes W_j$ and $V_i \otimes W_0$.

The sequences $d_0 \otimes \mathrm{id}$, $s_0 \otimes \mathrm{id}$ are EM-type transformations, with index functions $(i,j) \mapsto (i-1,j)$, respectively $(i,j) \mapsto (i+1,j)$. The thing to note is that the morphism $d_0 \otimes \mathrm{id}\colon V_0 \otimes W_j \to V_{-1} \otimes W_j$ makes sense (and equals the zero map), since we define $V_i = 0$ for $i = -1$, and $W_j = 0$ for $j = -1$.

The simplicial relation give relations between these functors:

$$(d_0 \otimes \mathrm{id})(s_0 \otimes \mathrm{id}) = \mathrm{id} \otimes \mathrm{id} \tag{2.6}$$

is true as a relation between EM-type transformations.

Another type of example is $\partial \otimes \mathrm{id}$, with index function $(i,j) \mapsto (i-1,j)$. It is defined by $(\partial \otimes \mathrm{id})_{i,j} = \sum_{0 \le r \le i} d_r \otimes \mathrm{id}$. Similarly, we can define $\mathrm{id} \otimes \partial$, with index function $(i,j) \mapsto (i, j-1)$.

When we apply suspension, we have to be careful. Here is an example of this: *unless* $i > 0$, $j = 0$, we have that $\mathcal{S}(\partial \otimes \mathrm{id})_{i,j} + (d_0 \otimes \mathrm{id})_{i,j} = (\partial \otimes \mathrm{id})_{i,j}$. If we postcompose with an EM-type transformation which vanishes on all groups $V_i \otimes W_0$, we get a genuine relation. For instance we have for any EM-type transformation F that

$$\mathcal{S}F\mathcal{S}(\partial \otimes \mathrm{id}) + \mathcal{S}F(d_0 \otimes \mathrm{id}) = \mathcal{S}F(\partial \otimes \mathrm{id}). \tag{2.7}$$

We can also use simplicial operations simultaneously in both factors. Using the index function $(i,j) \mapsto (i-1, j-1)$, we put

$$\delta = \sum_{0 \le r \le \min(i,j)} d_r \otimes d_r \colon V_i \otimes W_j \to V_{i-1} \otimes W_{j-1}.$$

$d_0 \otimes d_0$ is another EM-type transformation with the same index function, and

$$\mathcal{S}(\delta) = \delta + d_0 \otimes d_0. \tag{2.8}$$

Similarly (using the simplicial relations $d_0 s_0 = \mathrm{id} = d_1 s_0$, $d_i s_0 = s_0 d_{i-1}$ for $i \geq 2$) we get that

$$(\mathrm{id} \otimes \partial)(\mathrm{id} \otimes s_0) = (\mathrm{id} \otimes s_0)(\mathrm{id} \otimes \partial) + (\mathrm{id} \otimes s_0 d_0). \tag{2.9}$$

In the same way

$$(\mathrm{id} \otimes \partial)(\mathrm{id} \otimes d_0) = (\mathrm{id} \otimes d_0)(\mathrm{id} \otimes \partial) + (\mathrm{id} \otimes d_0 d_0). \tag{2.10}$$

For any EM-type transformation F, we have (because of (2.5))

$$(d_0 \otimes d_0)(\mathcal{S}F) = F(d_0 \otimes d_0). \tag{2.11}$$

We now give our definition of the higher Eilenberg-Mac Lane maps.

Definition 2.6. D^k is the EM-type transformation with index function $I_{D^k}(i,j) = (i+j-k, i+j-k)$, and defined by $D^0 = \mathcal{S}(D)(\mathrm{id} \otimes s_0)$ and inductively for $k \geq 1$ by the formula

$$D^k = \mathcal{S}(D^{k-1}) + \begin{cases} D^{k-1}(d_0 \otimes \mathrm{id}) & \text{if } k \text{ is even}, \\ D^{k-1}(\mathrm{id} \otimes d_0) & \text{if } k \text{ is odd}. \end{cases} \tag{2.12}$$

The sum on the right-hand side of (2.12) is defined because the EM-type transformations $\mathcal{S}(D^{k-1})$ and $D^{k-1}(\mathrm{id} \otimes d_0)$ have the same index function.

This equation (2.12) is an equation of EM-type transformations. Written out, it means that we have natural transformations $D^k_{i,j}\colon V_i \otimes W_j \to V_{i+j-k} \otimes W_{i+j-k}$ given inductively as

$$D^k_{i,j} = \begin{cases} \mathcal{S}(D^{k-1}_{i-1,j-1}) + D^{k-1}_{i-1,j}(d_0 \otimes \mathrm{id}) & \text{if } k \text{ is even, } i,j \geq 1, \\ \mathcal{S}(D^{k-1}_{i-1,j-1}) + D^{k-1}_{i,j-1}(\mathrm{id} \otimes d_0) & \text{if } k \text{ is odd, } i,j \geq 1, \\ D^{k-1}_{i-1,j}(d_0 \otimes \mathrm{id}) & \text{if } k \text{ is even, } i \geq 1, j = 0, \\ D^{k-1}_{i,j-1}(\mathrm{id} \otimes d_0) & \text{if } k \text{ is odd, } i = 0, j \geq 1, \\ 0 & \text{else.} \end{cases}$$

It is obvious that our D^0 satisfies (2.3). We have to prove that our choices of D^k, $k \geq 1$, satisfy (2.4). Our strategy for proving this is first to prove relations between EM-type transformations, that is, a relation between natural transformations valid for all pair of integers (i,j).

Definition 2.7. Let A^k be the EM-type transformation which has index function $I_{A^k}(i,j) = (i+j-k, i+j-k)$, and is defined by

$$A^0 = D^0 + \mathcal{T}D^0 + D,$$
$$A^k = D^k + \mathcal{T}D^k + \delta D^{k-1} + D^{k-1}(\partial \otimes \mathrm{id}) + D^{k-1}(\mathrm{id} \otimes \partial).$$

We have now defined all EM-type transformations that we will need. The main work of this article is done in proving the following recursion relation for A^k:

Lemma 2.8. *For $k \geq 1$ we have that*

$$A^k = \mathcal{S}(A^{k-1}) + \begin{cases} A^{k-1}(\mathrm{id} \otimes d_0) & \textit{if } k \textit{ is even}, \\ A^{k-1}(d_0 \otimes \mathrm{id}) & \textit{if } k \textit{ is odd}. \end{cases}$$

Proof. The proof is by direct computation. It is divided into three cases: $k = 1$; $k \geq 2$ and k even; $k \geq 3$ and k odd. The method used in the three cases is the same. We will write about a dozen relations, and then add all of them to give the desired recursion formula.

The case $k = 1$. We want to prove that

$$\begin{aligned} &D^1 + \mathcal{T}D^1 + D^0(\partial \otimes \mathrm{id}) + D^0(\mathrm{id} \otimes \partial) + \delta D^0 \\ &+ \mathcal{S}(D^0 + \mathcal{T}D^0 + D) + (D^0 + \mathcal{T}D^0 + D)(d_0 \otimes \mathrm{id}) = 0. \end{aligned} \tag{*}$$

We have that $D^0 + (\mathcal{S}D)(\mathrm{id} \otimes s_0) = 0$. From this follows immediately the relations

$$\delta D^0 + \delta(\mathcal{S}D)(\mathrm{id} \otimes s_0) = 0, \tag{2.13}$$

$$D^0(\mathrm{id} \otimes d_0) + \mathcal{S}D(\mathrm{id} \otimes s_0 d_0) = 0, \tag{2.14}$$

$$D^0(\partial \otimes \mathrm{id}) + \mathcal{S}D(\partial \otimes s_0) = 0, \tag{2.15}$$

$$D^0(\mathrm{id} \otimes \partial) + \mathcal{S}D(\mathrm{id} \otimes s_0)(\mathrm{id} \otimes \partial) = 0, \tag{2.16}$$

and

$$D^0(d_0 \otimes \mathrm{id}) + \mathcal{S}D(d_0 \otimes s_0) = 0. \tag{2.17}$$

There is also the defining relation for D^1, and we can apply the twist to it. This gives

$$D^1 + \mathcal{S}D^0 + D^0(\mathrm{id} \otimes d_0) = 0 \tag{2.18}$$

and

$$\mathcal{T}D^1 + \mathcal{S}\mathcal{T}D^0 + \mathcal{T}D^0(d_0 \otimes \mathrm{id}) = 0. \tag{2.19}$$

A fundamental property of the Eilenberg-Mac Lane map is that it is a chain map. In our notation, this says that $\delta D + D(\partial \otimes \mathrm{id}) + D(\mathrm{id} \otimes \partial) = 0$.

Suspending this, using (2.4), and right multiplying with $\mathrm{id} \otimes s_0$ we get

$$\mathcal{S}(\delta D)(\mathrm{id} \otimes s_0) + \mathcal{S}(D(\partial \otimes \mathrm{id}))(\mathrm{id} \otimes s_0) + \mathcal{S}(D(\mathrm{id} \otimes \partial))(\mathrm{id} \otimes s_0) = 0. \tag{2.20}$$

Now use that $\mathcal{S}$ is compatible with composition so $\mathcal{S}(D(\partial \otimes \mathrm{id})) = (\mathcal{S}D)\mathcal{S}(\partial \otimes \mathrm{id})$. The identitity (2.7) provides

$$\mathcal{S}(D(\partial \otimes \mathrm{id}))(\mathrm{id} \otimes s_0) + \mathcal{S}D(\partial \otimes s_0) + \mathcal{S}D(d_0 \otimes s_0) = 0. \tag{2.21}$$

Similarly, using twisted versions of (2.7) and (2.6) gives

$$\mathcal{S}(D(\mathrm{id} \otimes \partial))(\mathrm{id} \otimes s_0) + \mathcal{S}D(\mathrm{id} \otimes \partial)(\mathrm{id} \otimes s_0) + \mathcal{S}D = 0. \tag{2.22}$$

The relation (2.9) gives

$$\mathcal{S}D(\mathrm{id} \otimes \partial)(\mathrm{id} \otimes s_0) + \mathcal{S}D(\mathrm{id} \otimes s_0)(\mathrm{id} \otimes \partial) + \mathcal{S}D(\mathrm{id} \otimes s_0 d_0) = 0. \tag{2.23}$$

The relation (2.8) also gives a relation. We apply (2.11) for $F = D$ to it, and obtain

$$\delta(\mathcal{S}D)(\mathrm{id} \otimes s_0) + \mathcal{S}(\delta D)(\mathrm{id} \otimes s_0) + D(d_0 \otimes \mathrm{id}) = 0. \tag{2.24}$$

Adding the numbered relations (2.13)–(2.24) gives (*) and finishes the proof of case $k = 1$.

The case $k \geq 2$ and k even. The relation we want to prove is the following:

$$\begin{aligned} & D^k + \mathcal{T}D^k + D^{k-1}(\partial \otimes \mathrm{id}) + D^{k-1}(\mathrm{id} \otimes \partial) + \delta D^{k-1} \\ (**) \quad & + \mathcal{S}(D^{k-1} + \mathcal{T}D^{k-1} + D^{k-2}(\partial \otimes \mathrm{id}) + D^{k-2}(\mathrm{id} \otimes \partial) + \delta D^{k-2}) \\ & + (D^{k-1} + \mathcal{T}D^{k-1} + D^{k-2}(\partial \otimes \mathrm{id}) + D^{k-2}(\mathrm{id} \otimes \partial) + \delta D^{k-2})(\mathrm{id} \otimes d_0) = 0. \end{aligned}$$

The definition says that

$$D^k + \mathcal{S}D^{k-1} + D^{k-1}(d_0 \otimes \mathrm{id}) = 0. \tag{2.25}$$

Applying the twist to this, we get

$$\mathcal{T}D^k + \mathcal{S}\mathcal{T}D^{k-1} + \mathcal{T}D^{k-1}(\mathrm{id} \otimes d_0) = 0. \tag{2.26}$$

Since $k-1$ is odd, the definition says that $D^{k-1} = \mathcal{S}D^{k-2} + D^{k-2}(\mathrm{id} \otimes d_0)$. This gives us a sequence of relations:

$$D^{k-1}(d_0 \otimes \mathrm{id}) + \mathcal{S}D^{k-2}(d_0 \otimes \mathrm{id}) + D^{k-2}(d_0 \otimes d_0) = 0, \tag{2.27}$$

$$D^{k-1}(\mathrm{id} \otimes d_0) + \mathcal{S}D^{k-2}(\mathrm{id} \otimes d_0) + D^{k-2}(\mathrm{id} \otimes d_0 d_0) = 0, \tag{2.28}$$

$$\delta D^{k-1} + \delta D^{k-2}(\mathrm{id} \otimes d_0) + \delta\mathcal{S}D^{k-2} = 0, \tag{2.29}$$

$$D^{k-1}(\partial \otimes \mathrm{id}) + D^{k-2}(\partial \otimes \mathrm{id})(\mathrm{id} \otimes d_0) + \mathcal{S}D^{k-2}(\partial \otimes \mathrm{id}) = 0, \tag{2.30}$$

and

$$D^{k-1}(\mathrm{id} \otimes \partial) + \mathcal{S}D^{k-2}(\mathrm{id} \otimes \partial) + D^{k-2}(\mathrm{id} \otimes d_0\partial) = 0. \tag{2.31}$$

(2.10) gives us

$$D^{k-2}(\mathrm{id} \otimes \partial)(\mathrm{id} \otimes d_0) + D^{k-2}(\mathrm{id} \otimes d_0\partial) + D^{k-2}(\mathrm{id} \otimes d_0 d_0) = 0. \tag{2.32}$$

Using (2.11) for $F = D^{k-2}$ we get

$$(d_0 \otimes d_0)\mathcal{S}D^{k-2} + D^{k-2}(d_0 \otimes d_0) = 0. \tag{2.33}$$

Finally, the following equations follow from (2.7) and its twisted version, since in this case $k \geq 3$ and from (2.8), using the compatibility of suspension with products,

$$\mathcal{S}(D^{k-2}(\partial \otimes \mathrm{id})) + \mathcal{S}D^{k-2}(\partial \otimes \mathrm{id}) + \mathcal{S}D^{k-2}(d_0 \otimes \mathrm{id}), \tag{2.34}$$

$$\mathcal{S}(D^{k-2}(\mathrm{id} \otimes \partial)) + \mathcal{S}D^{k-2}(\mathrm{id} \otimes \partial) + \mathcal{S}D^{k-2}(\mathrm{id} \otimes d_0), \tag{2.35}$$

and

$$\mathcal{S}(\delta D^{k-2}) + \delta\mathcal{S}D^{k-2} + (d_0 \otimes d_0)\mathcal{S}D^{k-2}. \tag{2.36}$$

If we add all numbered equations (2.25)–(2.36), we get formula (**). This finishes the case $k \geq 2$, k even.

Case $k \geq 3$ and k odd. In this case, we want to prove that

(***)

$$\begin{aligned} & D^k + \mathcal{T}D^k + D^{k-1}(\partial \otimes \mathrm{id}) + D^{k-1}(\mathrm{id} \otimes \partial) + \delta D^{k-1} \\ & + \mathcal{S}(D^{k-1} + \mathcal{T}D^{k-1} + D^{k-2}(\partial \otimes \mathrm{id}) + D^{k-2}(\mathrm{id} \otimes \partial) + \delta D^{k-2}) \\ & + (D^{k-1} + \mathcal{T}D^{k-1} + D^{k-2}(\partial \otimes \mathrm{id}) + D^{k-2}(\mathrm{id} \otimes \partial) + \delta D^{k-2})(d_0 \otimes \mathrm{id}) = 0. \end{aligned}$$

The definition and its twist give

$$D^k + \mathcal{S}(D^{k-1}) + D^{k-1}(\mathrm{id} \otimes d_0) = 0, \tag{2.37}$$

$$\mathcal{T}D^k + \mathcal{S}\mathcal{T}(D^{k-1}) + \mathcal{T}D^{k-1}(d_0 \otimes \mathrm{id}) = 0. \tag{2.38}$$

Since $D^{k-1} = \mathcal{S}(D^{k-2}) + D^{k-2}(d_0 \otimes \mathrm{id})$, we have relations:

$$D^{k-1}(\mathrm{id} \otimes d_0) + \mathcal{S}D^{k-2}(\mathrm{id} \otimes d_0) + D^{k-2}(d_0 \otimes d_0) = 0, \tag{2.39}$$

$$D^{k-1}(d_0 \otimes \mathrm{id}) + \mathcal{S}D^{k-2}(d_0 \otimes \mathrm{id}) + D^{k-2}(d_0 d_0 \otimes \mathrm{id}) = 0, \tag{2.40}$$

$$\delta D^{k-1} + \delta D^{k-2}(d_0 \otimes \mathrm{id}) + \delta \mathcal{S} D^{k-2} = 0, \tag{2.41}$$

$$D^{k-1}(\partial \otimes \mathrm{id}) + D^{k-2}(d_0 \partial \otimes \mathrm{id}) + \mathcal{S}D^{k-2}(\partial \otimes \mathrm{id}) = 0, \tag{2.42}$$

and

$$D^{k-1}(\mathrm{id} \otimes \partial) + D^{k-2}(\mathrm{id} \otimes \partial)(d_0 \otimes \mathrm{id}) + \mathcal{S}D^{k-2}(\mathrm{id} \otimes \partial) = 0. \tag{2.43}$$

The twisted version of (2.10) gives us

$$D^{k-2}(\partial \otimes \mathrm{id})(d_0 \otimes \mathrm{id}) + D^{k-2}(\partial d_0 \otimes \mathrm{id}) + D^{k-2}(d_0 d_0 \otimes \mathrm{id}) = 0. \tag{2.44}$$

Adding the numbered equations (2.37)–(2.44) and the numbered equations (2.33)-(2.36), we get (***). ☐

Theorem 2.9. *Assume that* $i + j \geq 2k$. *Then*

$$D^0 + TD^0T = D + \phi_0,$$
$$D^k + TD^kT = D^{k-1}\partial + \partial D^{k-1} + \phi_k,$$

as natural transformations $V_i \otimes W_j \to V_{i+j-k} \otimes W_{i+j-k}$.

Proof. The statement we need to prove is that A^k and ϕ_k induce the same natural transformation if $i + j \geq 2k$. If $k = 0$, this is (2.3), which we have already verified. Now we assume inductively that the statement is true for $k - 1$, that is, that A^{k-1} and ϕ_{k-1} induce the same natural transformation if $i + j \geq 2k - 2$.

It follows that $A^{k-1}(d_0 \otimes \mathrm{id})$ and $\phi_{k-1}(d_0 \otimes \mathrm{id})$ induce the same transformation on $V_i \otimes W_j$ if $i + j \geq 2k - 1$. But $\phi_{k-1}(d_0 \otimes \mathrm{id})$ is only non-trivial if $i = k$ and $j = k - 1$, so $A^{k-1}(d_0 \otimes \mathrm{id})$ induces the trivial natural transformation on $V_i \otimes W_j$ if $i + j \geq 2k$. The same argument shows that $A^{k-1}(\mathrm{id} \otimes d_0)$ is trivial on $V_i \otimes W_j$ for $i + j \geq 2k$.

Because of of this and Lemma 2.8, we get that A^k induces the same transformation as $\mathcal{S}A^{k-1}$ on $V_i \otimes W_j$ for $i + j \geq 2k$. Now we use the induction assumption again, to see that this transformation agrees with $\mathcal{S}\phi_{k-1} = \phi_k$. ☐

3. Appendix

Lemma 3.1. *The formula in Definition* 1.2 *defines a map* $\delta_i : Z_q R \to Z_{q+i} R$ *when* $2 \leq i \leq q$.

Proof. Let $z \in Z_q R$. We must show that $\delta_i(z)$ is a cycle, i.e.,

$$d_j \delta_i(z) = \sum_{(\mu,\nu) \in V(q,i)} d_j s_{\nu_i} \dots s_{\nu_1}(z) d_j s_{\mu_i} \dots s_{\mu_1}(z) = 0, \quad 0 \leq j \leq q + i.$$

By one of the simplicial identities we have

$$d_i s_j = \begin{cases} s_{j-1} d_i, & i < j, \\ id, & i = j \text{ or } i = j+1, \\ s_j d_{i-1}, & i > j+1. \end{cases}$$

If $j \leq q - i$ we can commute the left d_j all the way through the degeneracy maps to z such that $d_j \delta_i(z) = 0$.

If $j = q - i + 1$ we find that all terms vanish except for those with $\nu_1 = q - i + 1$, $\mu_1 = q - i$. For such a term we have

$$d_j s_{\nu_i} \dots s_{\nu_1}(z) d_j s_{\mu_i} \dots s_{\mu_1}(z) = s_{\nu_i - 1} \dots s_{\nu_2 - 1}(z) s_{\mu_i - 1} \dots s_{\mu_2 - 1}(z),$$

so these cancel out in pairs.

If $j \geq q - i + 2$ the non-zero terms are those with $j \in \nu$, $j - 1 \in \mu$ or $j - 1 \in \nu$, $j \in \mu$. For such a term we can interchange the corresponding μ_r with the corresponding ν_s and get a new element in $V(q, i)$ since $j - 1 \geq q - i + 1$. So these terms cancel out in pairs. □

Remark 3.2. For $i = 1$ we have that $\delta_1(z) = s_q(z) s_{q-1}(z)$. Thus $d_j \delta_1(z) = 0$ for $j \neq q$ but $d_q \delta_1(z) = z^2$.

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