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SIGN-CHANGING CRITICAL POINTS FROM LINKING TYPE THEOREMS

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Abstract. In this paper, the relationships between sign-changing critical point theorems and the linking type theorems of M. Schechter and the saddle point theorems of P. Rabinowitz are established. The abstract results are applied to the study of the existence of sign-changing solutions for the nonlinear Schrödinger equation $-\Delta u+V(x)u=f(x,u), u\in H^1(\mathbf{R}^N)$, where $f(x,u)$ is a Carathéodory function. Problems of jumping or oscillating nonlinearities and of double resonance are considered.

1. Introduction

We first describe a general method of obtaining critical points of functionals. We do this because the solutions of problems in partial differential equations can often be found as critical points of functionals. Let E be a Hilbert space endowed with norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$. Define a class of contractions of E as follows:

$$\begin{aligned}\Phi := \Big\{&\Gamma(\cdot,\cdot)\in \mathbf{C}([0,1]\times E,E) : \Gamma(0,\cdot)=\mathbf{id}; \text{ for each } t\in[0,1), \Gamma(t,\cdot) \text{ is a}\\ &\text{homeomorphism of } E \text{ onto itself and } \Gamma^{-1}(\cdot,\cdot) \text{ is continuous on } [0,1)\times E;\\ &\text{there exists an } x_0\in E \text{ such that } \Gamma(1,x)=x_0 \text{ for each } x\in E \text{ and that}\\ &\Gamma(t,x)\to x_0 \text{ as } t\to 1 \text{ uniformly on bounded subsets of } E\Big\}.\end{aligned}$$

Obviously, $\Gamma(t,u)=(1-t)u\in\Phi$. A subset A of E links a subset B of E if $A\cap B=\emptyset$ and, for every $\Gamma\in\Phi$, there is a $t\in[0,1]$ such that $\Gamma(t,A)\cap B\neq\emptyset$. Particularly, if A and B are closed and bounded, and $E\backslash A$ is path connected, then A linking B implies that B links A, that is, in this case linking is symmetric (cf. [27, 33]). The following theorem can be found in [27, 33]. We refer the readers to [25, 34] for previous work concerning linking.

Theorem A. *Let $G\in\mathbf{C}^1(E,\mathbf{R})$ and let A,B be subsets of E such that A links B and*

$$a_0:=\sup_A G\le b_0:=\inf_B G.$$

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Assume that

$$a := \inf_{\Gamma\in\Phi} \sup_{s\in[0,1],u\in A} G(\Gamma(s,u))$$

is finite. Then there is a sequence $\{u_k\} \subset E$ *such that* $G(u_k) \to a, G'(u_k) \to 0.$

Such a sequence is called a Palais-Smale (PS) sequence. The existence of such a sequence does not guarantee the existence of a critical point. However, if it has a convergent subsequence, then existence is guaranteed. If $a = b_0$, then $\mathrm{dist}(u_k, B) \to 0$. By this theorem, one can obtain a critical point if the functional satisfies the usual Palais-Smale condition, i.e., if every PS sequence has a convergent subsequence.

An open problem is: when will the critical point be sign-changing, i.e., when will it take on both positive and negative values? This is a very delicate question, much more difficult than finding mere solutions. Many researchers have studied this problem, but very little progress has been obtained.

We approach the problem in the following way. We designate a positive (negative) cone P $(-P)$ of E. Members of the positive or negative cone are the functions that do not change sign. Thus, we are looking for solutions to our problems that are not contained in these cones.

In the present paper, we are going to answer this question. Although some technical conditions are needed and will be given in the next section, we would like to state the following theorems loosely. They will be proved in Section 2.

Theorem B. *Assume that a compact subset* A *of* E *links a closed subset* B *which includes only sign-changing elements of* E, $G' = \mathbf{id} - K_G$, *where* $K_G : E \to E$ *is a compact operator, and* G *satisfies a weak (PS) condition (to be explained). If*

$$a_0 := \sup_A G \leq b_0 := \inf_B G,$$

then there is a sign-changing critical point of G *with critical value in* $[b_0 - \varepsilon, \sup\limits_{(t,u)\in[0,1]\times A} G((1-t)u) + \varepsilon]$ *for all* ε *small.*

As a consequence, we present a variation of the Saddle Point Theorems of Rabinowitz [25] and Schechter [27], which leads to the existence of a sign-changing critical point.

Theorem C. *Assume that* $E = N \oplus M, 1 < \dim N < \infty$, $G \in \mathbf{C}^1(E, \mathbf{R})$, *satisfies the (PS) condition and* $G' = \mathbf{id} - K_G$, *where* $K_G : E \to E$ *is a compact operator. Suppose:*

(1) $G(v) \leq \delta$ *for all* $v \in N$; *where* $\delta > 0$ *is a constant.*
(2) $G(w) \geq \delta$ *for all* $w \in M$ *with* $\|w\| = \rho$; *where* $\rho > 0$ *is a constant.*
(3) $G(sw_0 + v) \leq C_0$ *for all* $s \geq 0, v \in N$; $w_0 \in M\backslash\{0\}$ *is a fixed element,* C_0 *is a constant.*

Then there exists a sequence $\{u_n\} \subset E\backslash(-P \cup P)$ *such that*

$$G'(u_n) \to 0, \quad G'(u_n) = \frac{C_n}{n} u_n, \quad G(u_n) \to c,$$

where $\{C_n\}$ *is a bounded sequence and* $c \in [\delta/2,\ 2C_0]$.

The novelty in Theorem C is only the sign-changing property of the (PS) sequence (and of the eventual critical point). We are able to accomplish this because of the special form of G'. Without this restriction, the existence of a (PS) sequence without the sign-changing property can be proved even under weaker assumptions than (3) (cf. [27, Theorem 2.7.3, p. 44]). However, it should be noted that in the original form of the saddle point theorem [25], it is required that

$$G(sw_0 + v) \leq 0 \text{ for all } s \geq 0, v \in N, \|sw_0 + v\| = R \tag{1.1}$$

holds for some $R \geq \rho$. In practice, one has to show that

$$\limsup_{R\to\infty}\{G(sw_0 + v) : s \geq 0, v \in N, \|sw_0 + v\| = R\} < 0$$

in order to get (1.1). This is much more demanding than (3) of Theorem C or the hypotheses of the theorems of [27, 29, 32] (which are weaker than (3)). For our special case, we are able to obtain a sign-changing sequence even when only (3) holds.

As applications, we consider sign-changing solutions to problems concerning the following stationary Schrödinger equation:

$$\begin{cases} -\Delta u + V(x)u = f(x,u), \\ \quad u \in H^1(\mathbf{R}^N), \end{cases} \tag{1.2}$$

where $f(x,t) : \mathbf{R}^N \times \mathbf{R} \to \mathbf{R}$ is a Carathéodory function and the potential $V(x)$ satisfies a certain geometrical condition. We will consider the following problems.

- Jumping nonlinearities:

$$\lim_{t\to+\infty}\frac{f(x,t)}{t} := b_+(x); \quad \lim_{t\to-\infty}\frac{f(x,t)}{t} := b_-(x).$$

Our results permit the jump to cross arbitrarily many eigenvalues and do not involve the classical Fučík spectrum. In fact, the Fučík spectrum corresponding to the Schrödinger equation (1.2) has not been studied.

- Jumping and oscillating linearities:

$$\liminf_{t\to\pm\infty}\frac{f(x,t)}{t} := f_\pm(x); \quad \limsup_{t\to\pm\infty}\frac{f(x,t)}{t} := g_\pm(x).$$

- Double resonance:

$$\lambda_k \leq L(x) := \liminf_{|t|\to\infty}\frac{f(x,t)}{t} \leq \limsup_{|t|\to\infty}\frac{f(x,t)}{t} := K(x) \leq \lambda_{k+1},$$

where $\{\lambda_k\}$ are the eigenvalues of $-\Delta + V(x)$.

We will strengthen the existence results due to Cac [11], Berestycki-de Figueiredo [3], Lazer-Mckenna [21], Schechter [27], etc. by showing there are solutions that are indeed sign-changing.

2. A linking theorem

We consider the following type of functional $G \in \mathbf{C}^1(E, \mathbf{R})$. Its gradient G' is of the form $G'(u) = i(u)u - K_G u$, where $i(u) : E \to [1/2, 1]$ is a locally Lipschitz continuous function; $K_G : E \to E$ is a compact operator. Let $K := \{u \in E : G'(u) = 0\}$ and $\tilde{E} := E \backslash K$.

A locally Lipschitz continuous map $V : \tilde{E} \to E$ is called a pseudo-gradient vector field for G if

- $\langle G'(u), V(u)\rangle \geq \frac{1}{2}\|G'(u)\|^2$ for all $u \in \tilde{E}$.
- $\|V(u)\| \leq 2\|G'(u)\|$ for all $u \in \tilde{E}$.

It is well known that the initial value problem

$$\frac{d\sigma(t,u)}{dt} = -V(\sigma(t,u)), \quad \sigma(0,u) = u,$$

has a unique solution (called flow or trajectory) $\sigma : [0, T(u)) \times \tilde{E} \to E$, where $T(u) \in (0, \infty]$ is the maximum time of the existence of the flow with initial value u.

Let P $(-P)$ denote the closed convex positive (negative) cone of E. For $\mu_0 > 0$, define

$$\pm\mathcal{D}_0 := \{u \in E : \operatorname{dist}(u, \pm P) < \mu_0\}, \quad \mathcal{D} := \mathcal{D}_0 \cup (-\mathcal{D}_0), \quad \mathcal{S} = E \backslash \mathcal{D},$$
$$\pm\mathcal{D}_1 := \{u \in E : \operatorname{dist}(u, \pm P) < \mu_0/2\}.$$

Then $\mathcal{D}_0$ and $\mathcal{D}_1$ are open convex, $\mathcal{D}$ is open, $\pm P \subset \pm\mathcal{D}_1 \subset \pm\mathcal{D}_0$, $\mathcal{S}$ is closed. We make the following assumption.

($\mathbf{a_1}$) $K_G(\pm\mathcal{D}_0) \subset \pm\mathcal{D}_1$.

Lemma 2.1. *Assume* (a_1). *Then there exists a locally Lipschitz continuous map $B_0 : \tilde{E} \to E$ such that $B_0(\pm\mathcal{D}_0 \cap \tilde{E}) \subset \pm\mathcal{D}_1$ and that $V(u) := i(u)u - B_0(u)$ is a pseudo-gradient vector field of G.*

This lemma improves the lemmas in [38] and [22]. But in [38], $\mathcal{D}$ itself is a convex set. In [22], $K_G(\partial\mathcal{D}_0) \subset \mathcal{D}_0, G' = \mathbf{id} - K_G$.

Proof. For any $w \in \tilde{E}, \|G'(w)\| \neq 0$. We define

$$U(w) := \{u \in \tilde{E} : \|K_G u - K_G w\| < \frac{1}{8}\|G'(w)\|, \|G'(u)\| > \frac{1}{2}\|G'(w)\|\}.$$

Then $\{U(w) : w \in \tilde{E}\}$ is an open covering of $\tilde{E}$ in the topology of E, and we can find a locally finite refinement open covering $\{\tilde{U}(\lambda) : \lambda \in \Lambda\}$ of $\tilde{E}$, where Λ is the index set. For any $\lambda \in \Lambda$, only one of the following cases occurs:

(1) $\tilde{U}(\lambda) \cap \mathcal{D}_0 = \emptyset, \quad \tilde{U}(\lambda) \cap (-\mathcal{D}_0) = \emptyset$;
(2) $\tilde{U}(\lambda) \cap \mathcal{D}_0 \neq \emptyset, \quad \tilde{U}(\lambda) \cap (-\mathcal{D}_0) = \emptyset$;
(3) $\tilde{U}(\lambda) \cap \mathcal{D}_0 = \emptyset, \quad \tilde{U}(\lambda) \cap (-\mathcal{D}_0) \neq \emptyset$;
(4) $\tilde{U}(\lambda) \cap \mathcal{D}_0 \cap (-\mathcal{D}_0) \neq \emptyset$;
(5) $\tilde{U}(\lambda) \cap \mathcal{D}_0 \neq \emptyset, \quad \tilde{U}(\lambda) \cap (-\mathcal{D}_0) \neq \emptyset$, but $\tilde{U}(\lambda) \cap \mathcal{D}_0 \cap (-\mathcal{D}_0) = \emptyset$.

If the last case happens, we remove $\tilde{U}(\lambda)$ from the covering and replace it with $\tilde{U}(\lambda) \backslash \bar{\mathcal{D}}_0$ and $\tilde{U}(\lambda) \backslash (-\bar{\mathcal{D}}_0)$. In this way, we arrange that the covering has only the properties (1)-(4). For each $\lambda \in \Lambda$, define

$$\alpha_\lambda(u) := \operatorname{dist}(u, \tilde{E} \backslash \tilde{U}_\lambda), \quad \phi_\lambda(u) := \frac{\alpha_\lambda(u)}{\sum_{\lambda \in \Lambda} \alpha_\lambda(u)}, \quad u \in \tilde{E};$$

then $0 \le \phi_\lambda(u) \le 1$ and $\phi_\lambda : \tilde{E} \to E$ is locally Lipschitz continuous. For each $\lambda \in \Lambda$, choose $a_\lambda \in \tilde{U}(\lambda)$ such that a_λ is arbitrary in case (1); $a_\lambda \in \tilde{U}(\lambda) \cap \mathcal{D}_0$ in case (2); $a_\lambda \in \tilde{U}(\lambda) \cap (-\mathcal{D}_0)$ in case (3); $a_\lambda \in \tilde{U}(\lambda) \cap \mathcal{D}_0 \cap (-\mathcal{D}_0)$ in case (4). Define $B_0(u) = \sum_{\lambda \in \Lambda} \phi_\lambda(u) K_G a_\lambda, \ u \in \tilde{E}$. Then $B_0 : \tilde{E} \to E$ is locally Lipschitz coninuous. Let $V(u) := i(u)u - B_0 u$. We shall prove that B_0 and V are what we want.

For any $u \in \tilde{E}$, there are only finitely many numbers $\lambda_1, \cdots, \lambda_s \in \Lambda$ such that $u \in \tilde{U}(\lambda_1) \cap \cdots \cap \tilde{U}(\lambda_s)$. Moreover, there are $w_1, \cdots, w_s \in \tilde{E}$ such that $\tilde{U}(\lambda_i) \subset U(w_i)$ for $i = 1, \cdots, s$. Then $B_0(u) = \sum_{i=1}^{s} \phi_{\lambda_i}(u) K_G a_{\lambda_i}$, where $a_{\lambda_i} \in \tilde{U}(\lambda_i)$ for $i = 1, \cdots, s$. Note that

$$\begin{aligned}\|K_G u - K_G a_{\lambda_i}\| &\le \|K_G u - K_G w_i\| + \|K_G w_i - K_G a_{\lambda_i}\| \\ &\le \frac{1}{4}\|G'(w_i)\| \\ &\le \frac{1}{2}\|G'(u)\|\end{aligned}$$

for $i = 1, \cdots, s$, and

$$\begin{aligned}\|K_G u - B_0 u\| &= \|K_G u - \sum_{i=1}^{s} \phi_{\lambda_i}(u) K_G a_{\lambda_i}\| \\ &= \|\sum_{i=1}^{s} \phi_{\lambda_i}(u)(K_G u - K_G a_{\lambda_i})\| \le \frac{1}{2}\|G'(u)\|,\end{aligned}$$

hence

$$\begin{aligned}\|V(u)\| &= \|i(u)u - B_0(u)\| \\ &\le \|i(u)u - K_G u\| + \|K_G u - B_0(u)\| \\ &\le \|G'(u)\| + \|\sum_{i=1}^{s} \phi_{\lambda_i}(u) K_G u - \sum_{i=1}^{s} \phi_{\lambda_i}(u) K_G a_{\lambda_i}\| \\ &\le \frac{3}{2}\|G'(u)\| \qquad (2.1)\end{aligned}$$

and $|\langle G'(u), K_G u - B_0 u\rangle| \le \frac{1}{2}\|G'(u)\|^2$. It follows that

$$\langle G'(u), V(u)\rangle = \|G'(u)\|^2 + \langle G'(u), K_G u - B_0 u\rangle \ge \frac{1}{2}\|G'(u)\|^2. \qquad (2.2)$$

Inequalities (2.1) and (2.2) imply that $V(u) := i(u)u - B_0 u$ is a pseudo-gradient vector field for G. Next, we show that $B_0(\pm\mathcal{D}_0 \cap \tilde{E}) \subset \pm\mathcal{D}_1$. In fact, for any $u \in \mathcal{D}_0 \cap \tilde{E}$, there are finitely many $\phi_\lambda(u)$, say $\phi_{\lambda_i}(u)$ $(i = 1, \cdots, s)$, which are nonzero. Then $B_0 u = \sum_{i=1}^{s} \phi_{\lambda_i}(u) K_G a_{\lambda_i}$ and $u \in \tilde{U}(\lambda_i) \cap \mathcal{D}_0$ for $i = 1, \cdots, s$. Hence, $a_{\lambda_i} \in \tilde{U}(\lambda_i) \cap \mathcal{D}_0$ by the definition of a_λ. It follows that $K_G a_{\lambda_i} \in \mathcal{D}_1$ by recalling the condition (a_1). It implies that $B_0(u) \in \mathcal{D}_1$, since $\mathcal{D}_1$ is also convex. This proves that $B_0(\mathcal{D}_0 \cap \tilde{E}) \subset \mathcal{D}_1$. Similarly, we have that $B_0(-\mathcal{D}_0 \cap \tilde{E}) \subset -\mathcal{D}_1$. □

Consider the following vector field:

$$W(u) := \frac{(1 + \|u\|)^2 V(u)}{(1 + \|u\|)^2 \|V(u)\|^2 + 1}.$$

Then W is a locally Lipschitz continuous vector field on $\tilde{E}$. Obviously, $\|W(u)\| \le \|u\| + 1$ for all $u \in \tilde{E}$. We need the following lemma which can be found in [17, Theorem 4.1] (see also Brezis [10, Theorem 1], K. C. Chang [12]).

Lemma 2.2. *Assume E is a Banach space, $\mathcal{M}$ is a closed convex subset of E, $H : \mathcal{M} \to E$ is locally Lipschitz continuous and*

$$\lim_{\lambda \to 0^+} \frac{\operatorname{dist}(u + \lambda H(u), \mathcal{M})}{\lambda} = 0, \quad \forall u \in \mathcal{M}.$$

Then for any given $u_0 \in \mathcal{M}$, there exists a $\delta > 0$ such that the initial value problem

$$\frac{d\eta(t,u_0)}{dt} = H(\eta(t,u_0)), \quad \eta(0,u_0) = u_0,$$

has a unique solution $\eta(t,u_0)$ defined on $[0,\delta)$. Moreover, $\eta(t,u_0) \in \mathcal{M}$ for all $t \in [0,\delta)$.

Recall the weak Palais-Smale condition at c ((w-PS)$_c$, for short): if for any sequence $\{u_n\}$ such that $G(u_n) \to c$ and $(1+\|u\|)G'(u_n) \to 0$, then $\{u_n\}$ has a convergent subsequence. This version of (w-PS)$_c$ was first introduced in [15] and used by [35, 39, 40] in some variants.

We denote $K[a,b] := \{u \in E : G'(u) = 0, a \le G(u) \le b\}, G^c := \{u \in E : G(u) \le c\}, B_R(0) := \{u \in E : \|u\| \le R\}$. Define

$$\Phi^* := \{\Gamma \in \Phi : \Gamma(t,\mathcal{D}) \subset \mathcal{D}\}.$$

Then $\Gamma(t,u) = (1-t)u \in \Phi^*$. The main results of this section are the following theorems.

Theorem 2.1. *Suppose that* (a$_1$) *holds. Assume that a compact subset A of E links a closed subset B of $\mathcal{S}$ and*

$$a_0 := \sup_A G \le b_0 := \inf_B G.$$

If G satisfies the (w-PS)$_c$ condition for any $c \in [b_0, \sup_{(t,u)\in[0,1]\times A} G((1-t)u)]$, then $K[a^*-\varepsilon, a^*+\varepsilon] \cap (E\backslash(-P\cup P)) \ne \emptyset$ for all ε small, where

$$a^* := \inf_{\Gamma\in\Phi^*} \sup_{\Gamma([0,1],A)\cap\mathcal{S}} G(u) \in [b_0, \sup_{(t,u)\in[0,1]\times A} G((1-t)u)].$$

Moreover, if $a^* = b_0$, then $K[a^*,a^*] \subset B$.

Theorem 2.2. *Suppose that* (a$_1$) *holds. Assume that $E = N \oplus M, 1 < \dim N < \infty$, and that*

1. *$G(v) \le \delta$ for all $v \in N$; where δ is a positive constant.*
2. *$G(w) \ge \delta$ for all $w \in \{w : w \in M, \|w\| = \rho\} \subset \mathcal{S}$; where ρ is a positive constant.*
3. *$G(sw_0 + v) \le C_0$ for all $s \ge 0, v \in N$; $w_0 \in M\backslash\{0\}$ is a fixed element, C_0 is a constant.*

If G satisfies the (w-PS)$_c$ condition for all $c > 0$, then there exists a sequence $\{u_n\} \subset E\backslash(-P\cup P)$ such that

$$G'(u_n) \to 0, \quad G'(u_n) = \frac{C_n}{n}u_n, \quad G(u_n) \to c,$$

where $\{C_n\}$ is a bounded sequence and $c \in [\delta/2,\ 2C_0]$.

The statement $G'(u_n) = \frac{C_n}{n}u_n$ in Theorem 2.2 is quite helpful for showing the sign-change of the limit of the (PS) sequence $\{u_n\}$.

Proof of Theorem 2.1. Evidently, a^* is well defined since A links B and $B \subset \mathcal{S}$. Moreover, $a^* \in [b_0, \sup_{(t,u)\in[0,1]\times A} G((1-t)u)]$.

We first consider the case of $a^* > b_0$. By contradiction, we assume that

$$K[a^*-\varepsilon_0, a^*+\varepsilon_0] \cap (E\backslash(-P\cup P)) = \emptyset$$

for some ε_0 small enough. Then $K[a^*-\varepsilon_0, a^*+\varepsilon_0] \subset (-P\cup P)$.

Case 1. Assume $K[a^* - \varepsilon_0, a^* + \varepsilon_0] \neq \emptyset$.

Since $K[a^* - \varepsilon_0, a^* + \varepsilon_0]$ is compact, we may assume that

$$\operatorname{dist}(K[a^* - \varepsilon_0, a^* + \varepsilon_0], \mathcal{S}) := \delta_0 > 0.$$

By the (w-PS) condition, there is an $\bar{\varepsilon} > 0$ such that

$$\frac{(1 + \|u\|)^2 \|G'(u)\|^2}{(1 + \|u\|)^2 \|G'(u)\|^2 + 1} \geq \bar{\varepsilon} \tag{2.3}$$

for $u \in G^{-1}[a^* - \bar{\varepsilon}, a^* + \bar{\varepsilon}] \backslash (K[a^* - \varepsilon_0, a^* + \varepsilon_0])_{\delta_0/2}$, where $(Z)_c := \{u \in E : \operatorname{dist}(u, Z) \leq c\}$. By decreasing $\bar{\varepsilon}$, we may assume that $\bar{\varepsilon} < a^* - b_0, \bar{\varepsilon} < \varepsilon_0/3$. Then $\langle G'(u), W(u) \rangle \geq \bar{\varepsilon}/8$ for any $u \in G^{-1}[a^* - \bar{\varepsilon}, a^* + \bar{\varepsilon}] \backslash (K[a^* - \varepsilon_0, a^* + \varepsilon_0])_{\delta_0/2}$. Let

$$Q_1 = \{u \in E : |G(u) - a^*| \geq 3\bar{\varepsilon}\}, \quad Q_2 = \{u \in E : |G(u) - a^*| \leq 2\bar{\varepsilon}\}$$

and

$$\eta(u) = \frac{\operatorname{dist}(u, Q_1)}{\operatorname{dist}(u, Q_1) + \operatorname{dist}(u, Q_2)}.$$

Let $\xi(u) : E \to [0, 1]$ be locally Lipschitz continuous such that $\xi(u) = 1$ for all $u \in E \backslash (K[a^* - \varepsilon_0, a^* + \varepsilon_0])_{\delta_0/2}$; $\xi(u) = 0$ for all $u \in (K[a^* - \varepsilon_0, a^* + \varepsilon_0])_{\delta_0/3}$. Let $\bar{W}(u) = \eta(u)\xi(u)W(u)$ for $u \in \tilde{E}$; $\bar{W}(u) = 0$ otherwise. Then $\bar{W}$ is a locally Lipschitz vector field on E. We consider the following Cauchy initial value problem:

$$\frac{d\sigma(t, u)}{dt} = -\bar{W}(\sigma(t, u)), \ \sigma(0, u) = u,$$

which has a unique continuous solution $\sigma(t, u)$ in E. Evidently,

$$\frac{dG(\sigma(t, u))}{dt} \leq 0.$$

By the definition of a^*, there exists a $\Gamma \in \Phi^*$ such that $\Gamma([0, 1], A) \cap \mathcal{S} \subset E^{a^* + \bar{\varepsilon}}$. Therefore, $\Gamma([0, 1], A)$ is a subset of $E^{a^* + \bar{\varepsilon}} \cup \mathcal{D}$. Denote $A_1 := \Gamma([0, 1], A)$. We claim that there exists a $T_1 > 0$ such that $\sigma(T_1, A_1) \subset E^{a^* - \bar{\varepsilon}/4} \cup \mathcal{D}$.

First, if $u \in \mathcal{D}$, we are going to show that $\sigma(t, u) \in \mathcal{D}$ for all $t \geq 0$. Without loss of generality, we may assume that $u \in \mathcal{D}_0$. Suppose there exists a $t_0 > 0$ such that $\sigma(t_0, u) \notin \mathcal{D}_0$. We may choose a neighborhood U_u of u such that $U_u \subset \mathcal{D}_0$, since $\mathcal{D}_0$ is open. By the theory of ordinary equations in Banach space, we can find a neighborhood U_{t_0} of $\sigma(t_0, u)$ such that $\sigma(t_0, \cdot) : U_u \to U_{t_0}$ is a homeomorphism. Since $\sigma(t_0, u) \notin \mathcal{D}_0$, we can take a $w \in U_{t_0} \backslash \bar{\mathcal{D}}_0$. Correspondingly, we find a $v \in U_u$ such that $\sigma(t_0, v) = w$. Hence, we may find a $t_1 \in (0, t_0)$ such that $\sigma(t_1, v) \in \partial\mathcal{D}_0$ and $\sigma(t, v) \notin \bar{\mathcal{D}}_0$ for all $t \in (t_1, t_0]$.

On the other hand, for any $z \in \bar{\mathcal{D}}_0 \cap K$, $\bar{W}(z) = 0$, hence

$$\operatorname{dist}(z + \lambda(-\bar{W}(z)), \bar{\mathcal{D}}_0) = 0, \quad \text{for all } \lambda > 0. \tag{2.4}$$

For any $z \in \bar{\mathcal{D}}_0 \cap \tilde{E}$, we have $B_0(z) \in \bar{\mathcal{D}}_1$ since $B_0(\mathcal{D}_0 \cap \tilde{E}) \subset \mathcal{D}_1$ by Lemma 2.1. Therefore, by a property of the cone P: $xP + yP \subset P$ for all $x, y \geq 0$, we have

$$\begin{aligned}
& \operatorname{dist}(z + \lambda(-\bar{W}(z)), P) \\
= \ & \operatorname{dist}(z - \lambda\eta(z)\xi(z)W(z), P) \\
= \ & \operatorname{dist}\Big((1 - \frac{\lambda\eta(z)\xi(z)(1+\|z\|)^2 i(z)}{(1+\|z\|)^2\|V(z)\|^2+1})z \\
+ \ & \frac{\lambda\eta(z)\xi(z)(1+\|z\|)^2}{(1+\|z\|)^2\|V(z)\|^2+1}B_0(z), P\Big) \\
\leq \ & \operatorname{dist}\Big((1 - \frac{\lambda\eta(z)\xi(z)(1+\|z\|)^2 i(z)}{(1+\|z\|)^2\|V(z)\|^2+1})z + \frac{\lambda\eta(z)\xi(z)(1+\|z\|)^2}{(1+\|z\|)^2\|V(z)\|^2+1}B_0(z), \\
& (1 - \frac{\lambda\eta(z)\xi(z)(1+\|z\|)^2 i(z)}{(1+\|z\|)^2\|V(z)\|^2+1})P + \frac{\lambda\eta(z)\xi(z)(1+\|z\|)^2}{(1+\|z\|)^2\|V(z)\|^2+1}P\Big) \\
\leq \ & (1 - \frac{\lambda\eta(z)\xi(z)(1+\|z\|)^2 i(z)}{(1+\|z\|)^2\|V(z)\|^2+1})\operatorname{dist}(z, P) \\
+ \ & \frac{\lambda\eta(z)\xi(z)(1+\|z\|)^2}{(1+\|z\|)^2\|V(z)\|^2+1}\operatorname{dist}(B_0(z), P) \\
\leq \ & (1 - \frac{\lambda\eta(z)\xi(z)(1+\|z\|)^2 i(z)}{(1+\|z\|)^2\|V(z)\|^2+1})\mu_0 + \frac{\lambda\eta(z)\xi(z)(1+\|z\|)^2}{(1+\|z\|)^2\|V(z)\|^2+1}\frac{\mu_0}{2} \\
\leq \ & \mu_0
\end{aligned}$$

for $\lambda > 0$ small enough since $i(z) \geq 1/2$. That is, $z + \lambda(-\bar{W}(z)) \in \bar{\mathcal{D}}_0$ for λ small. Once again, we get

$$\operatorname{dist}(z + \lambda(-\bar{W}(z)), \bar{\mathcal{D}}_0) = 0, \quad \text{for all } \lambda > 0 \quad \text{small enough.} \tag{2.5}$$

Combining (2.4) and (2.5), we obtain

$$\lim_{\lambda \to 0+} \frac{\operatorname{dist}(z + \lambda(-\bar{W}(z)), \bar{\mathcal{D}}_0)}{\lambda} = 0, \quad \forall z \in \bar{\mathcal{D}}_0.$$

Consider the following initial value problem:

$$\frac{d\sigma(t, \sigma(t_1, v))}{dt} = -\bar{W}(\sigma(t, \sigma(t_1, v))), \quad \sigma(0, \sigma(t_1, v)) = \sigma(t_1, v) \in \bar{\mathcal{D}}_0.$$

It has a unique solution $\sigma(t, \sigma(t_1, v))$. By Lemma 2.2, there is a $\delta > 0$ such that

$$\sigma(t, \sigma(t_1, v)) \in \bar{\mathcal{D}}_0 \quad \text{ for all } t \in [0, \delta).$$

Hence, by the semi-group property, $\sigma(t, v) \in \bar{\mathcal{D}}_0$ for all $t \in [t_1, t_1 + \delta)$, which contradicts the definition of t_1. Therefore, $\sigma(t, u) \in \mathcal{D}$ for all $t \geq 0$.

If $u \in A_1, u \notin \mathcal{D}$. Then we observe that $G(u) \leq a^* + \bar{\varepsilon}$. If $G(u) \leq a^* - \bar{\varepsilon}$, then $G(\sigma(t, u)) \leq G(u) \leq a^* - \bar{\varepsilon}$ for all $t \geq 0$. Assume $G(u) > a^* - \bar{\varepsilon}$. Then $u \in G^{-1}[a^* - \bar{\varepsilon}, a^* + \bar{\varepsilon}]$. If $\operatorname{dist}\Big(\sigma([0, \infty), u),\ K[a^* - \varepsilon_0, a^* + \varepsilon_0]\Big) \leq \delta_0/2$, then there exists a t_m such that $\operatorname{dist}(\sigma(t_m, u), \mathcal{S}) \geq \delta_0/4$, i.e., $\sigma(t_m, u) \in \mathcal{D}$. Assume that $\operatorname{dist}\Big(\sigma([0, \infty), u), K[a^* - \varepsilon_0, a^* + \varepsilon_0]\Big) > \delta_0/2 > 0$. Similarly, we assume that $G(\sigma(t, u)) > a^* - \bar{\varepsilon}$ for all $t \geq 0$ (otherwise, we are done). Then, by (2.3),

$$\frac{(1 + \|\sigma(t, u)\|)^2\|G'(\sigma(t, u))\|^2}{(1 + \|\sigma(t, u)\|)^2\|G'(\sigma(t, u))\|^2 + 1} \geq \bar{\varepsilon}, \quad \eta(\sigma(t, u)) = \xi(\eta(t, u)) = 1 \tag{2.6}$$

for all $t \geq 0$. Therefore,

$$G(\sigma(2,u)) = G(u) + \int_0^2 dG(\sigma(s,u)) \leq a^* - 2\bar{\varepsilon}. \tag{2.7}$$

By combining the above arguments, we see that for any $u \in A_1 \backslash \mathcal{D}$, there exists a $T_u > 0$ such that $\sigma(T_u, u) \in E^{a^*-\bar{\varepsilon}/2} \cup \mathcal{D}$. By continuity, there exists a neighborhood U_u such that $\sigma(T_u, U_u) \subset E^{a^*-\bar{\varepsilon}/3} \cup \mathcal{D}$. Since $A_1 \backslash \mathcal{D}$ is compact, we get a $T_1 > 0$ such that $\sigma(T_1, A_1 \backslash \mathcal{D}) \subset E^{a^*-\bar{\varepsilon}/4} \cup \mathcal{D}$. Then

$$\sigma(T_1, A_1) \subset E^{a^*-\bar{\varepsilon}/4} \cup \mathcal{D}. \tag{2.8}$$

Case 2. If $K[a^*-\varepsilon_0, a^*+\varepsilon_0] = \emptyset$, then (2.3) holds with $(K[a^*-\varepsilon_0, a^*+\varepsilon_0])_{\delta_0/2} = \emptyset$ and $\xi(u) \equiv 1$. Then, trivially, (2.6)-(2.8) are still true.

Now we define

$$\Gamma^*(s,u) = \begin{cases} \sigma(2T_1 s, u), & s \in [0,1/2], \\ \sigma(T_1, \Gamma(2s-1,u)), & s \in [1/2,1]. \end{cases}$$

Then, $\Gamma^* \in \Phi^*$. If $s \in [0,1/2]$, we have that $\Gamma^*(s,A) \cap \mathcal{S} \subset \sigma(2T_1 s, A) \cap \mathcal{S} \subset E^{a_0} \cap \mathcal{S} \subset E^{a^*-\bar{\varepsilon}/4}$. If $s \in [1/2,1]$, $\Gamma^*(s,A) \cap \mathcal{S} \subset \sigma(T_1, \Gamma(2s-1,A)) \cap \mathcal{S} \subset \sigma(T_1, A_1) \cap \mathcal{S} \subset (E^{a^*-\bar{\varepsilon}/4} \cup \mathcal{D}) \cap \mathcal{S} \subset E^{a^*-\bar{\varepsilon}/4} \cap \mathcal{S} \subset E^{a^*-\bar{\varepsilon}/4}$. It follows that $G(\Gamma^*([0,1],A) \cap \mathcal{S}) \leq a^* - \bar{\varepsilon}/4$, a contradiction.

Next we consider the case of $a^* = b_0$. The idea is similar to that in [41]. Here we have to construct a different vector field and need a careful analysis of the flow. We shall prove that $K[a^*,\ a^*] \cap B \neq \emptyset$. If it were not true, there would exist numbers $\varepsilon_1, \varepsilon_2, \varepsilon_3$ such that

$$\frac{(1+\|u\|)^2 \|G'(u)\|^2}{1+(1+\|u\|)^2\|G'(u)\|^2} \geq \varepsilon_1 \quad \text{for } |G(u)-a^*| < \varepsilon_2 \text{ and } \operatorname{dist}(u,B) \leq \varepsilon_3. \tag{2.9}$$

By decreasing ε_2, we may assume that $\varepsilon_2 < \varepsilon_1 \varepsilon_3/16$. Let

$$Q_3 := \{u \in E : \operatorname{dist}(u,B) \leq \varepsilon_3/2, |G(u)-a^*| \leq \varepsilon_2/2\},$$
$$Q_4 := \{u \in E : \operatorname{dist}(u,B) \leq \varepsilon_3/3, |G(u)-a^*| \leq \varepsilon_2/3\}.$$

Then $K \subset E \backslash Q_3$. Choose $\Gamma \in \Phi^*$ such that $\sup\limits_{\Gamma([0,1],A)\cap S} G(u) \leq a^* + \varepsilon_2/3$. We can find a $u_0 \in \Gamma([0,1],A) \cap B \cap \mathcal{S} \neq \emptyset$ since A links B and $B \subset \mathcal{S}$. This implies that $b_0 \leq G(u_0) \leq \sup\limits_{\Gamma([0,1],A)\cap S} G(u) \leq a^* + \varepsilon_2/3$, i.e., that $u_0 \in Q_4 \subset Q_3$. Let

$$\eta_1(u) = \frac{\operatorname{dist}(u, E\backslash Q_3)}{\operatorname{dist}(u, E\backslash Q_3) + \operatorname{dist}(u, Q_4)},$$

and consider the following Cauchy initial value problem:

$$\frac{d\sigma_1(t,u)}{dt} = -\eta_1(\sigma_1(t,u)) W(\sigma_1(t,u)),\ \sigma_1(0,u) = u \in E,$$

which has a unique continuous solution $\sigma_1(t,u)$ in E. Obviously, by (2.9),

$$\frac{dG(\sigma_1(t,u))}{dt} \leq -\frac{\varepsilon_1}{8} \eta_1(\sigma_1(t,u)). \tag{2.10}$$

If $u \in E^{a^*+\varepsilon_2/3}$, then $G(\sigma_1(t,u)) \le G(u) \le a^* + \varepsilon_2/3$ for all $t \ge 0$. If there is a $t_1 \le \varepsilon_3/4$ such that $\sigma_1(t_1,u) \notin Q_4$, then either $G(\sigma_1(t_1,u)) < a^* - \varepsilon_2/3$ or $\operatorname{dist}(\sigma_1(t_1,u),B) > \varepsilon_3/3$. For the latter case, we observe that $\operatorname{dist}(\sigma_1(t,u),B) \ge \varepsilon_3/12$, and hence, $\sigma_1(t,u) \notin B$ for all $t \in [0,\varepsilon_3/4]$. If $\sigma_1(t,u) \in Q_4$ for all $t \in [0,\varepsilon_3/4]$, then

$$G(\sigma_1(\frac{\varepsilon_3}{4},u)) = G(u) + \int_0^{\frac{\varepsilon_3}{4}} dG(\sigma_1(t,u)) \le a^* + \frac{\varepsilon_2}{3} - \frac{\varepsilon_3\varepsilon_1}{32} \le a^* - \frac{\varepsilon_2}{6}.$$

That is, either $G(\sigma_1(\varepsilon_3/4,u)) < a^* - \varepsilon_2/6 = b_0 - \varepsilon_2/6$ or $\sigma_1(t,u) \notin B$ for all $t \in [0,\varepsilon_3/4]$ and each $u \in E^{a^*+\varepsilon_2/3}$. It follows that $\sigma_1(\varepsilon_3/4,u) \notin B$ for any $u \in E^{a^*+\varepsilon_2/3}$. Next we prove that $\forall u \in A, t \in [0,\varepsilon_3/4]$, we have $\sigma_1(t,u) \notin B$. Note that, if $u \in A, u \notin \mathcal{S}$, then $u \in \mathcal{D}$. Following an argument similar to that of the proof of the first case, we see that $\sigma_1(t,u) \in \mathcal{D}$. Hence $\sigma_1(t,u) \notin B \subset \mathcal{S}$ for all $t \ge 0$. Therefore, we may just consider the case $u \in A \cap \mathcal{S}$. Evidently, $\sigma_1(\varepsilon_3/4,u) \notin B$. Furthermore, by (2.10),

$$G(\sigma_1(t,u)) \le G(u) - \frac{\varepsilon_1}{8}\int_0^t \eta_1(\sigma_1(t,u))dt \le a^* - \frac{\varepsilon_1}{8}\int_0^t \eta_1(\sigma_1(s,u))ds.$$

If $\sigma_1(t,u) \in B$, then $G(\sigma_1(t,u)) \ge b_0 = a^*$, and we must have $\eta_1(\sigma_1(s,u)) \equiv 0$ for $s \in [0,t]$. This implies that $\sigma_1(s,u) \notin Q_4$ and either $G(\sigma_1(s,u)) < a^* - \varepsilon_2/3$ or $\operatorname{dist}(\sigma_1(s,u),B) > \varepsilon_3/3$ for all $s \in [0,t]$. Both cases imply $\sigma_1(t,u) \notin B$. This proves that $\sigma_1([0,\varepsilon_3/4],A) \cap B = \emptyset$. Let

$$\Gamma_1(t,u) = \begin{cases} \sigma_1(2t\varepsilon_3/4,u), & 0 \le t \le 1/2,\\ \sigma_1(\varepsilon_3/4,\Gamma(2t-1,u)), & 1/2 \le t \le 1.\end{cases}$$

Then it is easy to check that $\Gamma_1 \in \Phi^*$. But by the above arguments, $\Gamma_1([0,1],A) \cap B = \emptyset$, which contradicts the fact that A links B. □

Proof of Theorem 2.2. Define $\xi \in \mathbf{C}^\infty(\mathbf{R})$ such that $\xi = 0$ in $(-\infty,1/2)$ and $\xi = 1$ in $(1,\infty), 0 \le \xi \le 1$. We may assume that $\|w_0\| = 1$. Write $u \in E$ as $u = v + w, v \in N, w \in M$. Let

$$G_n(u) = G(u) - (C_0 + \frac{1}{n})\xi(\frac{\|u\|^2}{n}), \quad n = 1,2,\cdots.$$

Then

$$G'(u) - G_n'(u) = 2(C_0 + \frac{1}{n})\xi'(\frac{\|u\|^2}{n})\frac{u}{n},$$
$$\|G'(u) - G_n'(u)\| \le C_1 n^{-1/2}.$$

We claim that G_n satisfies (w-PS) for each n sufficiently large if G does. In fact, assume that $\{u_k\}$ is a (w-PS) sequence: $G_n(u_k) \to c$ and $(1+\|u_k\|)G_n'(u_k) \to 0$ as $k \to \infty$. If, for a renamed subsequence, $\frac{\|u_k\|^2}{n} > 1$, then $\xi'(\frac{\|u_k\|^2}{n}) = 0$ and $(1+\|u_k\|)G_n'(u_k) = (1+\|u_k\|)G'(u_k) \to 0$. Then $\{u_k\}$ has a convergent subsequence. If $\frac{\|u_k\|^2}{n} \le 1$, then $\{u_k\}$ is bounded and (w-PS) follows immediately. To see this, note that

$$G'(u_k) - 2(C_0 + \frac{1}{n})\xi'(\frac{\|u_k\|^2}{n})\frac{u_k}{n} \to 0.$$

Take n so large that

$$b(u) = i(u) - [2(C_0 + \frac{1}{n})\xi'(\frac{\|u\|^2}{n})/n]$$

is bounded and bounded away from 0. Then

$$b(u_k)u_k - K_G u_k \to 0 \text{ as } k \to \infty.$$

Since the u_k are bounded, there is a renamed subsequence such that $b(u_k)$ and $K_G u_k$ converge. Hence, this subsequence converges as well. Thus, in both cases, the (w-PS) condition is satisfied. Moreover, $G_n(v) \le \delta$ for all $v \in N$. For any $w \in M$, if $\|w\| = \rho$, then $\xi(\frac{\|w\|^2}{n}) = 0$ for $n > 2\rho^2$ and consequently $G_n(w) = G(w) \ge \delta$. Choose $\|sw_0 + v\| := n^{1/2} := R_n$. Then $R_n > \rho$ if n large enough, and

$$G_n(sw_0 + v) = G(sw_0 + v) - (C_0 + 1/n)\xi(\frac{\|sw_0 + v\|^2}{n}) \le -\frac{1}{n}.$$

Let

$$B := \{w \in M : \|w\| = \rho\}$$

and

$$A_n := \{v \in N : \|v\| \le R_n\} \cup \{sw_0 + v : s \ge 0, v \in N, \|sw_0 + v\| = R_n\}.$$

Then A_n links B (cf. [27, page 38]), and G_n satisfies all the conditions of Theorem 2.1. Hence, there exists a $u_n \in E\backslash(-P \cup P)$ such that

$$G_n'(u_n) = 0, \quad G_n(u_n) \in [\delta/2, \sup_{(t,u)\in[0,1]\times A_n} G_n((1-t)u)].$$

Evidently,

$$\begin{gathered}\|G'(u_n) - G_n'(u_n)\| = \|G'(u_n)\| \le C_1 n^{-1/2} \to 0,\\ \delta/2 \le G_n(u_n) \le G(u_n) \le G_n(u_n) + C_0 + 1/n,\\ \sup_{(t,u)\in[0,1]\times A_n} G_n((1-t)u) \le C_0.\end{gathered}$$

Therefore, $G(u_n) \to c \in [\delta/2, 2C_0]$. Finally,

$$G'(u_n) = G'(u_n) - G_n'(u_n) = 2(C_0 + \frac{1}{n})\xi'(\frac{\|u_n\|}{n})\frac{u_n}{n} = \frac{C_n}{n}u_n,$$

where $\{C_n\}$ is a bounded sequence. □

3. Applications

Consider sign-changing solutions to the following stationary Schrödinger equation:

$$\left\{\begin{array}{c} -\Delta u + V(x)u = f(x,u), \\ u \in H^1(\mathbf{R}^N), \end{array}\right. \tag{3.1}$$

where $f(x,t) : \mathbf{R}^N \times \mathbf{R} \to \mathbf{R}$ is a Carathéodory function and the potential $V(x)$ satisfies the following geometric condition:

($\mathbf{e}_0$) $V(x) \in L^\infty_{loc}(\mathbf{R}^N), V_0 := \operatorname{essinf}_{\mathbf{R}^N} V(x) > 0$. For any $M > 0$ and any $r > 0$,

$$\operatorname{meas}(\{x \in B_r(y) : V(x) \le M\}) \to 0 \text{ as } |y| \to \infty, \tag{3.2}$$

where $B_r(y)$ denotes the ball centered at y with radius r.

The role of (e_0) ensures the compactness of certain embeddings. The limit (3.2) can be replaced by one of the following stronger conditions:

$(\mathbf{e}_0)'$ $\operatorname{meas}(\{x \in \mathbf{R}^N : V(x) \le M\}) < \infty$ for any $M > 0$ (cf. [5]).

$(\mathbf{e}_0)''$ $V(x) \to \infty$ as $|x| \to \infty$ (cf. [26]).

In recent years many existence results have been obtained for (3.1) under various conditions on $V(x)$ and $f(x,t)$. In [26] the author had obtained one positive and one negative solution of (1.1) under assumption $(\mathrm{e}_0)''$. A generalization of [26] can be found in [5]. In [7], existence and multiplicity results were obtained under the assumption $(\mathrm{e}_0)'$. One sign-changing solution was obtained in [9] for Dirichlet problems (see also [8]). A recent paper [4] studied (3.1) with superlinear $f(x,u)$. In the case $f(x,u)$ is odd in u, infinitely many sign-changing solutions were obtained in [4] by using genus. An estimate of the number of nodal domains was given there. It should be noted that all the papers mentioned above dealt with superlinear cases. In [36] (see also [37]), the double resonance of (3.1) was considered, but no information concerning the sign-changing solutions was obtained. To the best of our knowledge, the existence of sign-changing solutions of (3.1) with asymptotically linear and sublinear nonlinearities has not been studied before. In this section, we consider the asymptotically linear or sublinear case with either jumping (oscillating) nonlinearities or double resonance.

To study the sign-changing solutions, several authors established some abstract theories. In [1], the author established an abstract critical theory in partially ordered Hilbert spaces by virtue of critical groups and studied superlinear problems. In [23], a Ljusternik-Schnirelmann theory was established for studying the sign-changing solutions of an even functional. Some linking type theorems were also obtained in partially ordered Hilbert spaces. The methods and abstract critical point theory of [1, 6, 23, 24] involved the dense Banach space $C(\Omega)$ of continuous functions in the Hilbert space $H_0^1(\Omega)$, where the cone has nonempty interior. This plays a crucial role. To fit that framework, much stronger hypotheses (e.g., boundedness of the domain and stronger smoothness of the nonlinearities) are imposed. In [4], the method of dealing with superlinear nonodd f is based on [22] by using arguments of invariant sets and by a careful analysis of the descending flow. In the proof of [9], there was constructed a series of Dirichlet problems on the ball, and the ball was expanded to the whole space. Other papers on sign-changing solutions include [2, 8, 13, 14, 16, 18, 19, 41].

Let E be the Hilbert space

$$E := \{u \in H^1(\mathbf{R}^N) : \int_{\mathbf{R}^N} V(x)u^2 < \infty\}$$

endowed with the inner product $\langle u, v\rangle := \int_{\mathbf{R}^N} (\triangledown u \triangledown v + V(x)uv)dx$ for $u, v \in E$ and norm $\|u\| := \langle u,\ u\rangle^{1/2}$.

By [5], the hypothesis (e_0) implies that the eigenvalue problem

$$-\Delta u + V(x)u = \lambda u, \quad x \in \mathbf{R}^N,$$

possesses a sequence of positive eigenvalues: $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_k < \cdots \to \infty$, the principal eigenvalue λ_1 is simple with positive eigenfunction φ_1, and the eigenfunctions φ_k corresponding to λ_k $(k \ge 2)$ are sign-changing. Let N_k denote the eigenspace of λ_k. Then $\dim N_k < \infty$. We fix k and let $E_k := N_1 \oplus \cdots \oplus N_k$. Since we are going study the asymptotically linear or sublinear cases, we assume, throughout this section, that

$$|f(x,t)t| \le F_0 t^2 \quad \text{for all } x \in \mathbf{R}^N, t \in \mathbf{R}, \tag{3.3}$$

where $F_0 > \lambda_k$ is a constant.

3.1. **Jumping nonlinearities.** We assume

$$\lim_{t\to+\infty} \frac{f(x,t)}{t} = b_+(x); \quad \lim_{t\to-\infty} \frac{f(x,t)}{t} = b_-(x) \tag{3.4}$$

uniformly for $x \in \mathbf{R}^N$.

We introduce the following assumptions. The letter c will be indiscriminately used to denote various constants when the exact values are irrelevant.

(e$_1$) $f(x,t)t \ge 0$ for $(x,t) \in \mathbf{R}^N \times \mathbf{R}$; $\limsup_{t\to 0} \frac{f(x,t)}{t} \le V_0/3$ uniformly for $x \in \mathbf{R}^N$.

(e$_2$) $2F(x,t) \ge \lambda_{k-1}t^2 - W_0(x)$ for $(x,t)\in\mathbf{R}^N\times\mathbf{R}$, where $F(x,t)=\int_0^t f(x,\ s)ds$ and $0 < \int_{\mathbf{R}^N} W_0(x)dx < \infty$.

Choose l such that

$$\lambda_l \ge \frac{74\lambda_k^2}{\lambda_{k-1}(\lambda_k - \lambda_{k-1})} F_0. \tag{3.5}$$

Then there is a constant C_{l-1} such that $\|u\|_\infty = \sup_{\mathbf{R}^N} |u| \le C_{l-1}\|u\|$ for any $u \in E_{l-1}$. We need the following local condition around zero:

(e$_3$) $2F(x,t) \le \frac{\lambda_k + \lambda_{k-1}}{2} t^2$ for $x \in \mathbf{R}^N$ and $|t| \le r_0$, where

$$r_0 > C_{l-1}\Big(\frac{24\lambda_k}{\lambda_k - \lambda_{k-1}} \int_{\mathbf{R}^N} W_0(x)dx\Big)^{1/2}.$$

The first result deals with the case of a jump not crossing eigenvalues: $\lambda_k < b_\pm(x) \le \lambda_{k+1}$. Resonance may occur at λ_{k+1}.

Theorem 3.1. *Assume that* (e$_0$)-(e_3) *and* (3.4) *hold with $\lambda_k < b_\pm(x) \le \lambda_{k+1}$. If either $b_+(x) < \lambda_{k+1}$ for $x \in \mathbf{R}^N$ or $b_-(x) < \lambda_{k+1}$ for $x \in \mathbf{R}^N$, then equation* (3.1) *has a sign-changing solution.*

If we strengthen the condition on f, we have the following theorem where the jump is allowed to cross an arbitrarily finite number of eigenvalues.

Theorem 3.2. *Assume that* (e$_0$)-(e_3) *and* (3.4) *hold. If $\lambda_k < b_\pm(x)$ for $x \in \mathbf{R}^N$, and*

(e$_4$) *there exists a $C_0(x) \in L^1(\mathbf{R}^N)$ such that*
 (i) $f(x,t)t - 2F(x,t) \ge C_0(x), \quad$ *for* $(x,t) \in \mathbf{R}^N \times \mathbf{R}$,
 (ii) $\lim_{|t|\to\infty} (f(x,t)t - 2F(x,t)) = \infty$ *for $x \in \Omega$, where $\Omega := \{x \in \mathbf{R}^N : V(x) \le 3F_0\}$,*

then (3.1) *has a sign-changing solution.*

Remark 3.1. Theorem 3.2 permits $b_\pm(x)$ to be arbitrarily bounded functions greater than λ_k and to cross an arbitrarily finite number of eigenvalues of $-\Delta + V$. Therefore, the jump has much more freedom. A condition similar to (e$_4$) was introduced in [36, 37] with a different Ω. However, whether or not the solution is sign-changing was not decided there.

For the Dirichlet boundary value problem

$$(3.6)\qquad \begin{cases} -\Delta u = f(x,u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbf{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$ and $f(x,t)$ jumps at $\pm\infty$ in the sense

$$\begin{cases} f(x,t)/t \to a & \text{a.e. } x \in \Omega \quad \text{as } t \to -\infty, \\ f(x,t)/t \to b & \text{a.e. } x \in \Omega \quad \text{as } t \to \infty, \end{cases}$$

the existence of solutions of (3.6) is closely related to the equation

$$-\Delta u = bu^+ - au^-, \quad \text{where } u^\pm = \max\{\pm u, 0\}.$$

Conventionally, the set

$$\Sigma := \{(a,b) \in \mathbf{R}^2 : -\Delta u = bu^+ - au^- \text{ has nontrivial solutions}\}$$

is called the Fučík spectrum of $-\Delta$ (cf. [20, 29]). It plays a key role in most results of this aspect. However, so far **no** complete description of Σ has been found. The Fučík spectrum corresponding to the Schrödinger equation (3.1) has not been studied and seems interesting itself. Furtunately, the results of Theorems 3.1-3.2 and the following theorems do not need to involve the idea of the Fučík spectrum.

3.2. **Jumping and oscillating.** In this subsection, we will consider the following case:

$$(3.7)\qquad \liminf_{t\to\pm\infty} \frac{f(x,t)}{t} := f_\pm(x); \quad \limsup_{t\to\pm\infty} \frac{f(x,t)}{t} := g_\pm(x).$$

Assumption (3.7) implies that the nonlinearities are jumping and oscillating. Assume

($\mathbf{e}_5$) $2F(x,t) \geq \max\{\lambda_{k-1}t^2 - W_0(x), \quad f_+(x)(t^+)^2 + f_-(x)(t^-)^2 - W^*(x)\}$ for $(x,t) \in \mathbf{R}^N \times \mathbf{R}$, where $k > 1$ and

$$0 < \int_{\mathbf{R}^N} W_0(x)dx < \infty; \quad 0 < \int_{\mathbf{R}^N} W^*(x)dx < \infty.$$

Theorem 3.3. *Assume* $(\mathrm{e}_0), (\mathrm{e}_1), (\mathrm{e}_3), (\mathrm{e}_5)$. *For each pair of numbers* α_+, β_- *in the interval* $(\lambda_k, \lambda_{k+1})$ *there are numbers* $\alpha_- < \lambda_k$ *and* $\beta_+ > \lambda_{k+1}$ *such that*

$$\alpha_\pm \leq f_\pm(x) \leq g_\pm(x) \leq \beta_\pm, \quad x \in \mathbf{R}^N.$$

Then equation (3.1) *has a sign-changing solution.*

Theorem 3.4. *Assume* $(\mathrm{e}_0), (\mathrm{e}_1), (\mathrm{e}_3), (\mathrm{e}_5)$. *For each pair of numbers* α_-, β_+ *in the interval* $(\lambda_k, \lambda_{k+1})$ *there are numbers* $\alpha_+ < \lambda_k$ *and* $\beta_- > \lambda_{k+1}$ *such that*

$$\alpha_\pm \leq f_\pm(x) \leq g_\pm(x) \leq \beta_\pm, \quad x \in \mathbf{R}^N.$$

Then equation (3.1) *has a sign-changing solution.*

Theorem 3.5. *Assume* $(\mathrm{e}_0), (\mathrm{e}_1), (\mathrm{e}_3), (\mathrm{e}_5)$. *Suppose that*

$$\|v\|^2 \leq \int_{\mathbf{R}^N} (f_+(v^+)^2 + f_-(v^-)^2)dx, \quad \forall v \in E_k; \quad g_\pm(x) \leq \lambda_{k+1}, \ \ x \in \mathbf{R}^N,$$

and that no eigenfunction corresponding to λ_{k+1} *satisfies* $-\Delta u + V(x)u = g_+u^+ - g_-u^-$, *and no function in* $E_k\backslash\{0\}$ *satisfies* $-\Delta u + V(x)u = f_+u^+ - f_-u^-$. *Then equation* (3.1) *has a sign-changing solution.*

Theorem 3.6. *Assume* $(\mathrm{e}_0), (\mathrm{e}_1), (\mathrm{e}_3), (\mathrm{e}_5)$. *Suppose that*

$$\lambda_k \le f_\pm(x) \le g_\pm(x) \le \lambda_{k+1}, \quad x \in \mathbf{R}^N,$$

and that no eigenfunction corresponding to λ_k *satisfies* $-\Delta u + V(x)u = f_+u^+ - f_-u^-$ *and that no eigenfunction corresponding to* λ_{k+1} *satisfies* $-\Delta u + V(x)u = g_+u^+ - g_-u^-$. *Then* (3.1) *has a sign-changing solution.*

The existence results of Theorems 3.3-3.6 are essentially known (cf., e.g., [30]) (see also [11, 3]). But in those papers the signs of the solutions cannot be decided. Theorems 3.3-3.6 are neither consequences of the usual linking theorems nor consequences of the methods developed in [1, 23, 4], etc.

3.3. **Double resonance case.** We will consider the following case:

$$\lambda_k \le L(x) := \liminf_{|t|\to\infty} \frac{f(x,t)}{t} \le \limsup_{|t|\to\infty} \frac{f(x,t)}{t} := K(x) \le \lambda_{k+1} \tag{3.8}$$

uniformly for $x \in \mathbf{R}^N$, and the eigenfunctions of λ_k are $\neq 0$ a.e. We have

Theorem 3.7. *Assume that* (e_0)-(e_4) *and* (3.8) *hold with* $L(x) \not\equiv \lambda_k$. *Then equation* (3.1) *has a sign-changing solution.*

Next we proceed to prove the above theorems. Define

$$G(u) = \frac{1}{2}\|u\|^2 - \int_{\mathbf{R}^N} F(x,u)dx, \quad u \in E.$$

Then $G \in \mathbf{C}^1(E, \mathbf{R})$.

Lemma 3.1. *Under the assumptions of Theorems* 3.1-3.2, $G(u) \to -\infty$ *for* $u \in E_k$ *as* $\|u\| \to \infty$.

Proof. Rewrite G as

$$G(u) = \frac{1}{2}\|u\|^2 - \int_{\mathbf{R}^N} \Big(\frac{1}{2}b_+(x)(u^+)^2 + \frac{1}{2}b_-(x)(u^-)^2 + P(x,u)\Big)dx, \quad u \in E,$$

where $P(x,u) := \int_0^u p(x,t)dt$; $p(x,t) = f(x,t) - (b_+(x)t^+ - b_-(x)t^-)$; $t^\pm = \max\{\pm t, 0\}$. Note that $\min\{b_+(x), b_-(x)\} > \lambda_k$ and recall the variational characterization of eigenvalues $\{\lambda_k\}$ (cf., e.g., [36]), we have the following estimates for any $u \in E_k$:

$$\begin{aligned}
&G(u)\\
&= \frac{1}{2}\|u\|^2 - \frac{1}{2}\Big(\int_{b_-(x)\geq b_+(x)} + \int_{b_-(x)<b_+(x)}\Big)(b_+(x)(u^+)^2 + b_-(x)(u^-)^2)dx\\
&\quad - \int_{\mathbf{R}^N} P(x,u)dx\\
&= \frac{1}{2}\|u\|^2 - \frac{1}{2}\int_{b_-(x)\geq b_+(x)} b_+(x)u^2dx\\
&\quad -\frac{1}{2}\int_{b_-(x)\geq b_+(x)}\Big(b_-(x)-b_+(x)\Big)(u^-)^2dx - \frac{1}{2}\int_{b_-(x)<b_+(x)} b_-(x)u^2dx\\
&\quad -\frac{1}{2}\int_{b_-(x)<b_+(x)}\Big(b_+(x)-b_-(x)\Big)(u^+)^2dx - \int_{\mathbf{R}^N} P(x,u)dx\\
&\leq \frac{1}{2}\|u\|^2 - \quad \frac{1}{2}\int_{b_-(x)\geq b_+(x)} b_+(x)u^2dx\\
&\quad -\frac{1}{2}\int_{b_-(x)<b_+(x)} b_-(x)u^2dx - \int_{\mathbf{R}^N} P(x,u)dx\\
&\leq \frac{1}{2}\|u\|^2 - \frac{1}{2}\int_{\mathbf{R}^N} \min\{b_+(x), b_-(x)\}u^2dx - \int_{\mathbf{R}^N} P(x,u)dx\\
&\leq -\delta\|u\|^2 - \int_{\mathbf{R}^N} P(x,u)dx,
\end{aligned}$$

where $\delta > 0$ is a constant. Therefore,

$$\lim_{\|u\|\to\infty, u\in E_k} \frac{G(u)}{\|u\|^2} \leq -\delta$$

since

$$\lim_{t\to\infty} \frac{p(x,t)}{t} = 0$$

and $\dim E_k < \infty$. □

Lemma 3.2. *Under the assumptions of Theorem* 3.7*, $G(u) \to -\infty$ for $u \in E_k$ as $\|u\| \to \infty$.*

Proof. Since $L(x) \geq \lambda_k, L(x) \not\equiv \lambda_k$ and $\dim E_k < \infty$, by the variational characterization (cf., e.g., [36]) of the eigenvalues $\{\lambda_k\}$, there is a $\delta > 0$ such that

$$\|u\|^2 - \int_{\mathbf{R}^N} L(x)u^2dx \leq -\delta\|u\|^2 \quad \text{for all } u \in E_k. \tag{3.9}$$

In fact, the left-hand side of (3.9) is clearly ≤ 0. The only way it can vanish is when $u(x)$ is an eigenfunction of λ_k and $L(x) = \lambda_k$ on the support of $u(x)$. But the support of $u(x)$ has measure 0, contradicting the hypothesis on $L(x)$. This implies (3.9) (cf., e.g., [27]). On the other hand, since $L(x) \in L^\infty(\mathbf{R}^N)$, we may find an $R_0 > 0$ such that

$$\int_{\mathbf{R}^N\setminus B_R(0)} \frac{L(x)u^2}{\|u\|^2}dx \leq \frac{\delta}{4}, \quad \int_{\mathbf{R}^N\setminus B_R(0)} |F(x,u)|dx \leq \frac{\delta}{8}\|u\|^2 \tag{3.10}$$

for all $R \geq R_0$ and $u \in E_k$. It follows that

$$\|u\|^2 - \int_{B_R(0)} L(x)u^2 dx \leq -\frac{3\delta}{4}\|u\|^2 \quad \text{for all } u \in E_k, R \geq R_0. \tag{3.11}$$

Furthermore, by (3.8), for $\varepsilon < V_0\delta/10$, there exists a $C_\varepsilon > 0$ such that $\frac{1}{2}L(x)t^2 - F(x,t) \leq \frac{1}{2}\varepsilon t^2 + C_\varepsilon$ for all $x \in B_R(0), t \in \mathbf{R}$. Therefore, combining (3.9)-(3.11),

$$\begin{aligned} G(u) &= \frac{1}{2}\|u\|^2 - \frac{1}{2}\int_{B_R(0)} L(x)u^2 dx + \int_{B_R(0)} (\frac{1}{2}L(x)u^2 - F(x,u))dx \\ &\quad - \int_{\mathbf{R}^N \setminus B_R(0)} F(x,u)dx \\ &\leq -\frac{3\delta}{8}\|u\|^2 + \frac{\delta}{8}\|u\|^2 + \int_{B_R(0)} (\frac{1}{2}\varepsilon u^2 + C_\varepsilon)dx \\ &\leq -\frac{\delta}{5}\|u\|^2 + \int_{B_R(0)} C_\varepsilon dx. \end{aligned}$$

The lemma follows immediately. □

Lemma 3.3. *Assume* (e_2) *or* (e_5). *Then* $G(u) \leq \frac{1}{2}\int_{\mathbf{R^N}} W_0(x)dx$ *for all* $u \in E_{k-1}$.

Proof. This is an immediate consequence of conditions (e_2) or (e_5). □

Lemma 3.4. *Assume* (e_3). *There exists a* $\rho_0 > 0$ *such that*

$$G(u) \geq \frac{1}{2}\int_{\mathbf{R}^N} W_0(x)dx \text{ for } \ u \in E_{k-1}^\perp, \quad \|u\| = \rho_0.$$

Proof. By a simple computation,

$$2F(x,t) \leq 2F_0t^2 - F_0r_0^2 \quad \text{for } |t| \geq r_0, x \in \mathbf{R}^N, \tag{3.12}$$

where r_0 comes from (e_3). For any $u \in E_{k-1}^\perp$, we write $u = v + w$ with $v \in N_k \oplus N_{k+1} \oplus \cdots \oplus N_{l-1}$ and $w \in E_{l-1}^\perp$, where l is given in (3.5). Let $\beta_0 = \frac{\lambda_k+\lambda_{k-1}}{2}$ and

$$\xi_1 := \frac{(2F_0 + \lambda_l)}{4}w^2 + \frac{(\lambda_k + \beta_0)}{4}v^2 - F(x, v+w). \tag{3.13}$$

If $|v+w| \leq r_0$, then by condition (e_3) and the choice of λ_l, we see that

$$\begin{aligned} \xi_1 &\geq \frac{(2F_0 + \lambda_l)}{4}w^2 + \frac{(\lambda_k + \beta_0)}{4}v^2 - \frac{1}{2}\beta_0(v+w)^2 \\ &\geq \frac{(2F_0 + \lambda_l) - 2\beta_0}{4}w^2 + \frac{(\lambda_k + \beta_0) - 2\beta_0}{4}v^2 - \beta_0|vw| \\ &\geq \Big(\frac{((2F_0 + \lambda_l - 2\beta_0)(\lambda_k - \beta_0))^{1/2}}{2} - \beta_0\Big)|vw| \\ &\geq 0. \end{aligned} \tag{3.14}$$

If $|v+w| > r_0$, then by (3.13), we conclude that

$$\begin{aligned} xi_1 &\geq \Big(\frac{(\lambda_l + 2F_0) - 4F_0}{4}w^2 + \frac{(\lambda_k + \beta_0) - 4F_0}{4}v^2 - 2F_0vw + \frac{F_0r_0^2}{2}\Big) \\ &:= \xi_2 + \xi_3, \end{aligned} \tag{3.15}$$

where

$$\xi_2 := \frac{(\lambda_l - 2F_0)}{8} w^2 + \frac{(\lambda_k - \beta_0)}{4} v^2 - \beta_0 vw,$$

$$\xi_3 := \frac{\lambda_l - 2F_0}{8} w^2 - \frac{2F_0 - \beta_0}{2} v^2 - (2F_0 - \beta_0) vw + \frac{F_0 r_0^2}{2}. \tag{3.16}$$

Next, we estimate ξ_2 and ξ_3. If

$$\frac{(\lambda_k - \beta_0)}{4} |v| - \beta_0 |w| \geq 0,$$

then

$$\xi_2 \geq \frac{\lambda_l - 2F_0}{8} w^2 + (\frac{\lambda_k - \beta_0}{4} |v| - \beta_0 |w|) |v| \geq 0. \tag{3.17}$$

If

$$\frac{(\lambda_k - \beta_0)}{4} |v| - \beta_0 |w| \leq 0,$$

by the choice of λ_l, we deduce that

$$\xi_2 \geq (\frac{\lambda_l - 2F_0}{8} - \frac{4\beta_0^2}{\lambda_k - \beta_0}) w^2 + \frac{\lambda_k - \beta_0}{4} v^2 \geq 0. \tag{3.18}$$

On the other hand,

$$\xi_3 \geq \frac{(\lambda_l + 2F_0) - 4F_0}{8} w^2 - (2F_0 - \beta_0)(|v| + |w|)|v| + \frac{F_0 r_0^2}{2} := \xi_3'. \tag{3.19}$$

Thus

$$\begin{aligned} \xi_3' &\geq \frac{(\lambda_l - 10F_0 + 4\beta_0)}{8} w^2 - \frac{3(2F_0 - \beta_0)}{2} v^2 + \frac{F_0 r_0^2}{2} \\ &\geq -\frac{3(2F_0 - \beta_0)}{2} v^2 + \frac{F_0 r_0^2}{2}. \end{aligned} \tag{3.20}$$

Choose $\rho_0 := \frac{1}{C_{l-1}} (\frac{F_0 r_0^2}{3(2F_0 - \beta_0)})^{1/2}$. If $\|u\| = \rho_0$, then $\|v\|_\infty \leq C_{l-1} \|v\| \leq C_{l-1} \|u\| \leq C_{l-1} \rho_0$. Hence, $\xi_3' \geq 0$. Therefore, by (3.14)-(3.20), $\xi_1 \geq 0$. Finally,

$$\begin{aligned} G(u) &= G(v + w) \\ &= \frac{1}{2} (\|v\|^2 + \|w\|^2 - 2 \int_{\mathbf{R}^N} F(x, v + w) dx) \\ &\geq \frac{1}{4} \|v\|^2 + \frac{1}{4} \|w\|^2 + \frac{1}{4} \lambda_k \|v\|_2^2 + \frac{1}{4} \lambda_l \|w\|_2^2 - \int_{\mathbf{R}^N} F(x, u) dx \\ &\geq \frac{1}{4} (1 - \frac{\beta_0}{\lambda_k}) \|v\|^2 + \frac{1}{4} (1 - \frac{2F_0}{\lambda_l}) \|w\|^2 + \int_{\mathbf{R}^N} \xi_1 dx \\ &\geq \frac{1}{4} \min\{(1 - \frac{\beta_0}{\lambda_k}), (1 - \frac{2F_0}{\lambda_l})\} \|u\|^2 \\ &\geq \frac{1}{4} (1 - \frac{\beta_0}{\lambda_k}) \rho_0^2 \\ &\geq \frac{1}{2} \int_{\mathbf{R}^N} W_0(x) dx. \end{aligned}$$

□

Lemma 3.5. *Under the assumptions of Theorem* 3.1*, G satisfies the (w-PS) condition.*

Proof. Assume that $\{u_n\}$ is a (w-PS) sequence:

$$G(u_n) \to c, \quad (1 + \|u_n\|)G'(u_n) \to 0.$$

By negation, we assume that $\|u_n\| \to \infty$ as $n \to \infty$. Let $w_n = u_n/\|u_n\|$. Then $\|w_n\| = 1$ and there is a renamed subsequence such that $w_n \to w$ weakly in E, strongly in $L^2(\mathbf{R}^N)$ and *a.e.* in $\mathbf{R}^N$. Moreover,

$$\langle G'(u_n), v\rangle = \langle u_n, v\rangle - \int_{\mathbf{R}^N} f(x, u_n)v dx \to 0$$

and

$$\langle w_n, v\rangle - \int_{\mathbf{R}^N} \frac{f(x, u_n)v}{\|u_n\|} dx \to 0.$$

By (3.4), we see that $-\Delta w + V(x)w = b_+ w^+ - b_- w^-$. Since $G(u_n)/\|u_n\|^2 = 1/2 - \int_{\mathbf{R}^N} F(x, u_n)dx/\|u_n\|^2 \to 0$, we see that $\int_{\mathbf{R}^N}(b_+(w^+)^2 + b_-(w^-)^2)dx = 1$. It implies that $w \neq 0$. Let $w := w_- + w_+$ with $w_- \in E_k$, $w_+ \in E_k^\perp$, $\tilde{w} := w_+ - w_-$. Let $q(x) = b_+(x)$ when $w(x) \geq 0$; $q(x) = b_-(x)$ when $w(x) < 0$. Then we have that $-\Delta w + V(x)w = q(x)w$ and hence

$$\|w_+\|^2 - \|w_-\|^2 = \int_{\mathbf{R}^N} q(x)(w_+)^2 dx - \int_{\mathbf{R}^N} q(x)(w_-)^2 dx.$$

It follows that

$$\begin{aligned} 0 &\leq \|w_+\|^2 - \lambda_{k+1}\|w_+\|_2^2 \leq \|w_+\|^2 - \int_{\mathbf{R}^N} q(x)(w_+)^2 dx \\ &= \|w_-\|^2 - \int_{\mathbf{R}^N} q(x)(w_-)^2 dx \leq \|w_-\|^2 - \lambda_k \int_{\mathbf{R}^N} (w_-)^2 dx \leq 0. \end{aligned}$$

That is, $\|w_\pm\|^2 = \int_{\mathbf{R}^N} q(x)(w_\pm)^2$. The only way this can happen is $q(x) = \lambda_k$ when $w_-(x) \neq 0$ and $q(x) = \lambda_{k+1}$ when $w_+(x) \neq 0$, and therefore, either w_- is an eigenfunction of λ_k or w_+ is an eigenfunction of λ_{k+1}. But the first case cannot occur since $b_\pm > \lambda_k$. If w_+ is an eigenfunction of λ_{k+1}, then w_+ is sign-changing. Since $-\Delta w_+ + V(x)w_+ = b_+(x)w_+^+ - b_-(x)w_+^-$, we have $b_- = \lambda_{k+1}$ on a subset of $\mathbf{R}^N$ of positive measure and $b_+ = \lambda_{k+1}$ on another subset of $\mathbf{R}^N$ of positive measure. This contradicts the assumption of the theorem. □

Lemma 3.6. *Under the assumptions of Theorems* 3.2 *and* 3.7*, G satisfies the (w-PS) condition.*

Proof. Assume that $\{u_n\}$ is a (w-PS) sequence: $(1 + \|u_n\|)\|G'(u_n)\| \to 0$ and $\{G(u_n)\}$ is bounded. Then

$$G(u_n) - \frac{1}{2}\langle G'(u_n), u_n\rangle = \int_{\mathbf{R}^N} (\frac{1}{2} f(x, u_n)u_n - F(x, u_n))dx < c \tag{3.21}$$

and

$$\begin{aligned} \frac{1}{2}\|u_n\|^2 &\le c + \int_{\mathbf{R}^N} F(x,u_n)dx \\ &\le c + \int_{\Omega} F(x,u_n)dx + \int_{\mathbf{R}^N\setminus\Omega} F(x,u_n)dx \\ &\le c + \int_{\Omega} F(x,u_n)dx + \int_{\mathbf{R}^N\setminus\Omega} \frac{1}{6}V_0(x)u_n^2dx \\ &\le c + \int_{\Omega} F(x,u_n) + \frac{1}{6}\|u_n\|^2. \end{aligned}$$

Therefore, $\frac{1}{3}\|u\|^2 \le c + \frac{1}{2}\int_\Omega F(x,u_n)dx \le c + \int_\Omega F_0 u_n^2 dx$. If $\{\|u_n\|\}$ is unbounded, then, for a renamed subsequence, $1 \le 3F_0 \lim_{n\to\infty}\int_\Omega \frac{u_n^2}{\|u_n\|^2}dx$. It follows that $\lim_{n\to\infty}|u_n|^2 = \infty$ on a subset of Ω with positive measure. Combining this with (e_4), we have $\int_{\mathbf{R}^N} \frac{1}{2}f(x,u_n)u_n - F(x,u_n))dx \to \infty$, which contradicts (3.21). $\square$

Lemma 3.7. *Under the assumptions of Theorem* 3.3*, G satisfies the (PS) condition.*

Proof. First of all, we claim that, for each pair of numbers $\alpha_+, \beta_- \in (\lambda_k, \lambda_{k+1})$, there are numbers $\alpha_- < \lambda_k, \beta_+ > \lambda_{k+1}$ such that

$$\|u\|^2 < \int_{\mathbf{R}^N} (\alpha_+(u^+)^2 + \alpha_-(u^-)^2)dx, \quad \forall u \in E_k\setminus\{0\}; \tag{3.22}$$

$$\|u\|^2 > \int_{\mathbf{R}^N} (\beta_+(u^+)^2 + \beta_-(u^-)^2)dx, \quad \forall u \in E_k^\perp\setminus\{0\}. \tag{3.23}$$

A similar result for Dirichlet boundary value problems can be found in [11, 21] (see also [30]). By a slight modification, their proofs work perfectly for this claim. We omit the details. Then the conditions of Theorem 3.3 and (3.22)-(3.23) imply that

$$\|u\|^2 < \int_{\mathbf{R}^N} (f_+(u^+)^2 + f_-(u^-)^2)dx, \quad \forall u \in E_k\setminus\{0\}; \tag{3.24}$$

$$\|u\|^2 > \int_{\mathbf{R}^N} (g_+(u^+)^2 + g_-(u^-)^2)dx, \quad \forall u \in E_k^\perp\setminus\{0\}. \tag{3.25}$$

Now let $\{u_n\}$ be a (PS) sequence: $\|G'(u_n)\| \to 0$ and $\{G(u_n)\}$ is bounded. We just have to show that $\{u_n\}$ is bounded. To show this, assume that $\|u_n\| \to \infty$. Let $\bar{u}_n = u_n/\|u_n\|$. Then $\bar{u}_n \to \bar{u}$ weakly in E, strongly in $L^2(\mathbf{R}^N)$, and a.e. in $\mathbf{R}^N$. Since $\frac{|f(x,u_n)|}{\|u_n\|} \le F_0|\bar{u}_n|$, we may assume that $\frac{f(x,u_n)}{\|u_n\|}$ converges strongly in $L^2(\mathbf{R}^N)$ to a function $h(x)$. Observe that

$$\liminf_{n\to\infty} \frac{f(x,u_n)}{\|u_n\|} \ge \bar{u}(x) \liminf_{t\to\infty} \frac{f(x,t)}{t} = \bar{u}(x)f_+(x), \quad \text{if } \bar{u}(x) > 0.$$

In a similar way, we can show that

$$\begin{aligned} \bar{u}(x)f_+(x) \le \liminf_{n\to\infty} \frac{f(x,u_n)}{\|u_n\|} \le \limsup_{n\to\infty} \frac{f(x,u_n)}{\|u_n\|} \le \bar{u}(x)g_+(x), \text{ if } \bar{u}(x) > 0; \\ \bar{u}(x)g_-(x) \le \liminf_{n\to\infty} \frac{f(x,u_n)}{\|u_n\|} \le \limsup_{n\to\infty} \frac{f(x,u_n)}{\|u_n\|} \le \bar{u}(x)f_-(x), \text{ if } \bar{u}(x) < 0. \end{aligned}$$

This gives

$$\bar{u}(x)f_+(x) \le h(x) \le \bar{u}(x)g_+(x), \text{ if } \bar{u}(x) > 0;$$
$$\bar{u}(x)g_-(x) \le h(x) \le \bar{u}(x)f_-(x), \text{ if } \bar{u}(x) < 0.$$

Let $q(x) = h(x)/\bar{u}(x)$ if $\bar{u}(x) \neq 0$, otherwise, $q(x) = 0$. Then

(3.26)
$$f_+(x) \le q(x) \le g_+(x), \text{ if } \bar{u}(x) > 0; \qquad f_-(x) \le q(x) \le g_-(x), \text{ if } \bar{u}(x) < 0.$$

On the other hand, $G'(u_n) \to 0$ implies that

$$\langle \bar{u}(x), v\rangle - \int_{\mathbf{R}^N} h(x)v dx = \langle \bar{u}(x), v\rangle - \int_{\mathbf{R}^N} q(x)\bar{u}v dx = 0. \tag{3.27}$$

Let $\bar{u} = \bar{v} + \bar{w}$ with $\bar{v} \in E_k,\ \bar{w} \in E_k^{\perp},\ \tilde{u} = \bar{w} - \bar{v}$. Therefore, by (3.27),

$$\|\bar{w}\|^2 - \|\bar{v}\|^2 = \int_{\mathbf{R}^N} q(x)(\bar{w})^2 dx - \int_{\mathbf{R}^N} q(x)(\bar{v})^2 dx. \tag{3.28}$$

Recalling (3.24)-(3.26) and (3.28), we have

$$\begin{aligned} 0 &\le \int_{\mathbf{R}^N} (f_+(\bar{v}^+)^2 + f_-(\bar{v}^-)^2)dx - \|\bar{v}\|^2 \le \int_{\mathbf{R}^N} q(x)(\bar{v})^2 dx - \|\bar{v}\|^2 \\ &= \int_{\mathbf{R}^N} q(x)(\bar{w})^2 dx - \|\bar{w}\|^2 \le \int_{\mathbf{R}^N} (g_+(\bar{w}^+)^2 + g_-(\bar{w}^-)^2)dx - \|\bar{w}\|^2 \le 0. \end{aligned} \tag{3.29}$$

It follows that

$$\int_{\mathbf{R}^N} (f_+(\bar{v}^+)^2 + f_-(\bar{v}^-)^2)dx = \|\bar{v}\|^2; \quad \int_{\mathbf{R}^N} (g_+(\bar{w}^+)^2 + g_-(\bar{w}^-)^2)dx = \|\bar{w}\|^2. \tag{3.30}$$

Using (3.24)-(3.25) once again, we see that $\bar{v} = \bar{w} = \bar{u} = 0$. Hence,

$$\langle G'(u_n), \frac{u_n}{\|u_n\|^2}\rangle = 1 - \int_{\mathbf{R}^N} \frac{f(x, u_n)}{\|u_n\|} \bar{u}_n(x) dx \to 1,$$

providing a contradiction. □

By a similar argument, we can prove

Lemma 3.8. *Under the assumptions of Theorem* 3.4*, G satisfies the (PS) condition.*

Lemma 3.9. *Under the assumptions of Theorem* 3.5*, G satisfies the (PS) condition.*

Proof. By the assumptions of the theorem, we have

$$\|u\|^2 \le \int_{\mathbf{R}^N} (f_+(u^+)^2 + f_-(u^-)^2)dx, \quad u \in E_k;$$
$$\|u\|^2 \ge \lambda_{k+1}\|u\|_2^2 \ge \int_{\mathbf{R}^N} (g_+(u^+)^2 + g_-(u^-)^2)dx, \quad u \in E_k^{\perp}.$$

Then (3.30) still holds. Hence

$$\int_{\mathbf{R}^N} (\lambda_{k+1} - g_+)(\bar{w}^+)^2 dx + \int_{\mathbf{R}^N} (\lambda_{k+1} - g_-)(\bar{w}^-)^2 dx = 0.$$

It follows that $g_+ = \lambda_{k+1}$ if $\bar{w} > 0$, $g_- = \lambda_{k+1}$ if $\bar{w} < 0$ and $\bar{w}$ is an eigenfunction of λ_{k+1}. Therefore, $-\Delta\bar{w} + V(x)\bar{w} = \lambda_{k+1}\bar{w} = g_+\bar{w}^+ - g_-\bar{w}^-$, which implies that $\bar{w} = 0$. Furthermore,

$$\int_{\mathbf{R}^N} (f_+(\bar{v}^+)^2 + f_-(\bar{v}^-)^2)dx = \int_{\mathbf{R}^N} q(x)(\bar{v})^2 dx.$$

Thus, $q(x) = f_+(x)$ if $\bar{u} > 0$; $q(x) = f_-(x)$ if $\bar{u} < 0$ and $-\Delta\bar{u} + V(x)\bar{u} = f_+\bar{u}^+ - f_-\bar{u}^-$. It follows that $\bar{u} = \bar{v} = 0$. Using an argument similar to that used in proving Lemma 3.7, we get a contradiction if the (PS) sequence is unbounded. $\square$

Similarly, we have

Lemma 3.10. *Under the assumptions of Theorem* 3.6, *G satisfies the (PS) condition.*

To prove Theorems 3.1-3.7, we apply Theorem 2.1. First, we let

$$P := \{u \in E : u(x) \geq 0 \text{ for a.e. } x \in \mathbf{R}^N\}.$$

Then P $(-P)$ is the positive (negative) cone of E, and $\pm P$ has empty interior.

Consider E_m with $m > k+2$. Define

$$B_m := (N_k \oplus N_{k+1} \oplus \cdots \oplus N_m) \cap B_{\rho_0}(0),$$

where ρ_0 comes from Lemma 3.4;

$$A := \{u = v + sy_0 : v \in E_{k-1}, s \geq 0, \|u\| = R\} \cup (E_{k-1} \cap B_R(0)),$$
$$y_0 \in N_k, \|y_0\| = 1.$$

Then A and B_m link each other (cf. [27, 28]), and each u of B_m is sign-changing. Let $P_m = P \cap E_m$ be the positive cone in E_m. Then it is easy to check that $\text{dist}(B_m, -P_m \cup P_m) = \delta_m > 0$ since B_m is compact.

We define

$$\mathcal{D}_0(m, \mu_0) := \{u \in E_m : \text{dist}(u, P_m) < \mu_0 \},$$
$$\mathcal{D}_1(m, \mu_0) := \{u \in E_m : \text{dist}(u, P_m) < \mu_0/2 \}.$$

Now we consider $G_m := G|_{E_m}$. Then

$$G_m'(u) = u - \text{Proj}_m K_G u, \quad u \in E_m,$$

where Proj_m denotes the projection of E onto E_m. We make use of

Lemma 3.11. *Under the assumptions of* (e_1), *there exists a $\mu_0 \in (0, \delta_m)$ such that* $\text{Proj}_m K_G(\mathcal{D}_0(m, \mu_0)) \subset \mathcal{D}_1(m, \mu_0)$.

Proof. The proof is essentially due to [4]. Write $u^\pm = \max\{\pm u, 0\}$. For any $u \in E_m$,

$$\|u^+\|_2 = \min_{w \in (-P_m)} \|u - w\|_2 \leq \frac{1}{V_0^{1/2}} \min_{w \in (-P_m)} \|u - w\| = \frac{1}{V_0^{1/2}} \text{dist}(u, -P_m) \tag{3.31}$$

and, for each $s \in (2, 2^*]$, there exists a $C_s > 0$ such that

$$\|u^\pm\|_s \leq C_s \text{dist}(u, \mp P_m).$$

By assumption (e_1), for each $\varepsilon' > 0$, there exists a $C_{\varepsilon'} > 0$ such that

$$f(x,t)t \leq (V_0/3 + \varepsilon')t^2 + C_{\varepsilon'}|t|^p, \quad x \in \mathbf{R}^N, t \in \mathbf{R}, \tag{3.32}$$

where $p > 2$ is a constant. Let $v = \mathrm{Proj}_m K_E(u)$. Then by (3.31) and (3.32),

$$\begin{aligned}
\mathrm{dist}(v, -P_m)\|v^+\| &\leq \|v^+\|^2 \\
&= \langle v, v^+\rangle \\
&= \int_{\mathbf{R}^N} f(x, u^+)v^+ dx \\
&\leq \int_{\mathbf{R}^N} ((V_0/3 + \varepsilon')|u^+| + C_{\varepsilon'}|u^+|^{p-1})|v^+| \\
&\leq \Big(\frac{2}{5}\mathrm{dist}(u, -P_m) + C\mathrm{dist}(u, -P_m)^{p-1}\Big)\|v^+\|.
\end{aligned}$$

That is, $\mathrm{dist}\Big(\mathrm{Proj_m}K_G(u), -P_m\Big) \leq (\frac{2}{5})\mathrm{dist}(u, -P_m) + C\mathrm{dist}(u, -P_m)^{p-1}$. So there exists a $\mu_0 < \delta_m$ such that $\mathrm{dist}(\mathrm{Proj_m}K_E(u), -P_m) \leq \frac{1}{2}\mu_0$ for every $u \in -\mathcal{D}_0(m, \mu_0)$. Similarly, $\mathrm{dist}(\mathrm{Proj}_m K_G(u), P_m) \leq \frac{1}{2}\mu_0$ for every $u \in \mathcal{D}_0(m, \mu_0)$. The conclusion follows. □

Proofs of Theorems 3.1-3.2 *and* 3.7. Let $\mathcal{D}(m) := -\mathcal{D}_0(m, \mu_0) \cup \mathcal{D}_0(m, \mu_0), \mathcal{S}_m := E_m \backslash \mathcal{D}_m$. Then $B_m \subset \mathcal{S}_m$, and Lemma 3.11 says that condition (a_1) of Theorem 2.1 is satisfied.

Therefore, by Theorem 2.1, there exists a $u_m \in E_m \backslash(-P_m \cup P_m)$ (sign-changing critical point) such that

$$G'_m(u_m) = 0, G_m(u_m) \in [b_0 - \bar{\varepsilon}, \sup_{(t,u)\in[0,1]\times A} G((1-t)u) + \bar{\varepsilon}],$$

where $b_0 = \frac{1}{2}\int_{\mathbf{R}^N} W_0(x)dx > 0$ and $\sup_{(t,u)\in[0,1]\times A} G((1-t)u)$ are independent of m, $\bar{\varepsilon}$ is small enough.

To prove G has a sign-changing critical point, we just have to prove that $\{u_m\}$ has a convergent subsequence whose limit is still sign-changing. We first need to prove that $\{u_m\}$ is bounded. Once this is known, the existence of a convergent subsequence follows, since the equation is of subcritical growth. However, the proof of the boundedness of $\{u_m\}$ is the same as the proof of the (w-PS) condition of Lemmas 3.5-3.7. To prove that the limit of the subsequence is sign-changing, we adopt the ideas of [9]. Let $u_m^{\pm} := \max\{\pm u_m, 0\}$. Then

$$\|u_m^{\pm}\|^2 = \int_{\mathbf{R}^N} f(x, u_m^{\pm})u_m^{\pm} dx.$$

By (e_1), there exists a $C > 0$ such that

$$f(x, u)u \leq \frac{V_0}{3}|u|^2 + C|u|^p, \quad x \in \mathbf{R}^N, u \in \mathbf{R},$$

where $p > 2$ is a constant. It follows that

$$\|u_m^{\pm}\|^2 \leq \frac{1}{3}\|u_m^{\pm}\|^2 + C\|u_m^{\pm}\|_p^p.$$

Hence, $\|u_m^{\pm}\| \geq s_0 > 0$, where s_0 is a constant independent of m. This implies that the limit of the subsequence is sign-changing. □

Proofs of Theorems 3.3-3.6. We intend to use Theorem 2.2. By Lemmas 3.3-3.4, we see that $G(u) \leq \frac{1}{2}\int_{\mathbf{R}^N} W_0(x)dx$ for all $u \in E_{k-1} := N$ and $G(u) \geq \frac{1}{2}\int_{\mathbf{R}^N} W_0(x)dx$ for all $u \in B_m$. By (e_5) and (3.24), we have $G(u) \leq \frac{1}{2}\int_{\mathbf{R}^N} W^*(x)dx$ for all $u \in E_k$.

Then, G_m satisfies all the conditions of Theorem 2.2. Therefore, there exists a sequence $\{u_k\} \in E_m \backslash (-P_m \cup P_m)$ such that

$$G'_m(u_k) \to 0, G'_m(u_k) = C_k u_k/k, G_m(u_k) \in [\frac{1}{4}\int_{\mathbf{R}^N} W_0(x)dx, \int_{\mathbf{R}^N} W^*(x)dx],$$

as $k \to \infty$, where the sequence $\{C_k\}$ is bounded. By Lemmas 3.7-3.10, $u_k \to u(m)$, where $u(m)$ satisfies

$$G'_m(u(m)) = 0, G_m(u(m)) \in [\frac{1}{4}\int_{\mathbf{R}^N} W_0(x)dx, \int_{\mathbf{R}^N} W^*(x)dx].$$

We now show that $u(m)$ is sign-changing. In fact, since $G'_m(u_k) - C_k u_k/k = 0$, we have

$$\|u_k^{\pm}\|^2 - \frac{C_k}{k}\|u_k^{\pm}\|^2 = \int_{\mathbf{R}^N} f(x, u_k^{\pm})u_k^{\pm}dx \le \frac{1}{3}\|u_k^{\pm}\|^2 + C\|u_k^{\pm}\|_p^p.$$

It follows that $\|u_k^{\pm}\| \ge s_0 > 0$, where s_0 is a constant independent of k, m. This implies that the limit $u(m)$ is sign-changing. By a similar argument, $u(m) \to u^*$ as $m \to \infty$, where u^* is sign-changing and

$$G'(u^*) = 0, G(u^*) \in [\frac{1}{4}\int_{\mathbf{R}^N} W_0(x)dx, \int_{\mathbf{R}^N} W^*(x)dx].$$

$\square$

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