

ON A CLASS OF SPECIAL LINEAR SYSTEMS OF \mathbb{P}^3

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ABSTRACT. In this paper we deal with linear systems of \mathbb{P}^3 through fat points. We consider the behavior of these systems under a cubic Cremona transformation that allows us to produce a class of special systems which we conjecture to be the only ones.

INTRODUCTION

Let us take the projective space \mathbb{P}^n and let us consider the linear system of hypersurfaces of degree d having some points of fixed multiplicity. The virtual dimension of such systems is the dimension of the system of degree d polynomials minus the conditions imposed by the multiple points, and the expected dimension is the maximum between the virtual one and -1 . The systems whose dimension is bigger than the expected one are called *special systems*.

There exists a conjecture due to Harbourne (see [7]) and Hirschowitz (see [9]), characterizing special linear systems on \mathbb{P}^2 , which has been proved in some special cases (see [2, 3, 12, 11]).

In this paper we describe a class of special linear systems on \mathbb{P}^3 . The main tool will be the cubic Cremona transformation (2.1) which allows us to transform a linear system into another one. The dimension of the two systems is the same, while the virtual one may be different. This is a new phenomenon which does not occur in \mathbb{P}^2 . In Proposition 2.3 we give a formula expressing the difference between these virtual dimensions, and in particular we prove that if a transformation decreases the degree of a system, then it does not decrease its virtual dimension (Corollary 2.4). We will say that a system for which it is no longer possible to decrease the degree using one of these transformations is in *standard form*. Starting from a special system in standard form it is possible to construct infinitely many special systems by applying a sequence of cubic Cremona transformations. In this paper we describe two types of special systems in standard form, and we conjecture that they produce all the possible special cases.

The paper is organized as follows: in the first section we recall some definitions and notations. In Section 2 we give a description of a cubic Cremona transformation of \mathbb{P}^3 and its action on linear systems, while Section 3 deals with the resolution of the indeterminacy of this transformation. In Section 4 we state the conjecture and we give a procedure for evaluating the dimension of a system, and in the next section we give some motivations. Section 6 deals with special homogeneous linear

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systems according to the conjecture, and finally Section 7 provides some examples related to the procedure.

1. PRELIMINARIES

We start by fixing some definitions and notations.

Definition 1.1. Given a sheaf $\mathcal{F} = \mathcal{O}_{\mathbb{P}^3}(d) \otimes \mathcal{I}_V$, where \mathcal{I}_V is the ideal sheaf of a subscheme $V \subset \mathbb{P}^3$, we denote by $v(\mathcal{F})$ its *virtual dimension*, defined as

$$v(\mathcal{F}) = \chi(\mathcal{F}) - 1.$$

By $\mathcal{L} = \mathcal{L}_3(d, m_1, \dots, m_r)$ we will denote the linear system associated to the sheaf $\mathcal{O}_{\mathbb{P}^3}(d) \otimes \mathcal{I}_Z$, where $Z = \sum m_i p_i$ is a zero-dimensional scheme of fat points in general position. If no confusion arises, by abuse of notation we will use the same letter \mathcal{L} to also denote the sheaf. From the cohomology exact sequence associated to

$$0 \longrightarrow \mathcal{L} \longrightarrow \mathcal{O}_{\mathbb{P}^3}(d) \longrightarrow \mathcal{O}_Z \longrightarrow 0,$$

we obtain that $h^i(\mathcal{L}) = 0$ for $i = 2, 3$. Therefore we deduce that $v(\mathcal{L}) = h^0(\mathcal{O}_{\mathbb{P}^3}(d)) - h^0(\mathcal{O}_Z) - 1$, which may also be written as

$$v(\mathcal{L}) = \binom{d+3}{3} - \sum_{i=1}^r \binom{m_i+2}{3} - 1.$$

Let us denote by $e(\mathcal{L}) = \max(v(\mathcal{L}), -1)$ the *expected dimension* of \mathcal{L} .

Definition 1.2. A non-empty linear system \mathcal{L} is *special* if its expected dimension is strictly smaller than the effective one or, equivalently, if $h^1(\mathcal{L}) \neq 0$.

Let (X, π) be the blowing-up of \mathbb{P}^3 along $\{p_1, \dots, p_r\}$; by abuse of notation we will denote by \mathcal{L} the linear system associated to $L = dH - \sum m_i E_i$, where H is the pull-back of a hyperplane of \mathbb{P}^3 and $E_i = \pi^{-1}(p_i)$. Let $\langle H, E_1, \dots, E_r \rangle$ and $\langle h, e_1, \dots, e_r \rangle$ be two bases for the Chow groups $A^1(X)$ and $A^2(X)$, respectively, where h is the pull-back of a line and e_i is the class of a line in E_i . The intersection matrix, with respect to these two bases, is diagonal with the first element equal to 1 and the others equal to -1 (since $E_i e_i = -1$).

Given a curve $C \subset \mathbb{P}^3$, by abuse of notation we will denote by $\mathcal{L}C$ the intersection product of their strict transforms in X . We will write $C \in \ell_3(\delta, \mu_1, \dots, \mu_r)$ to denote a curve of degree δ with multiplicity μ_i at p_i . In this way, the intersection is given by the formula

$$(1.1) \quad \mathcal{L}C = d\delta - \sum_{i=1}^r \mu_i m_i.$$

We recall the Riemann-Roch formula for a divisor L on a smooth threefold X :

$$\chi(L) = \frac{L(L - K_X)(2L - K_X) + c_2(X)L}{12} + \chi(\mathcal{O}_X).$$

If the linear system associated to L can be written as $|L| = F + |M|$, where F is a fixed divisor of $|L|$ and $|M|$ is the residual system, then the above formula implies

$$(1.2) \quad v(L) = v(M) + v(F) + \frac{FM(L - K_X)}{2}.$$

2. CUBIC CREMONA TRANSFORMATIONS AND LINEAR SYSTEMS

In this section we focus our attention on a class of cubic Cremona transformations of \mathbb{P}^3 . Consider the system $\mathcal{L}_3(3, 2^4)$. By putting the four double points in the fundamental ones, the associated rational map is given by

$$(2.1) \quad \text{Cr} : (x_0 : x_1 : x_2 : x_3) \dashrightarrow (x_0^{-1} : x_1^{-1} : x_2^{-1} : x_3^{-1}).$$

The birational map (2.1) induces an action on the Picard group of X which can be described in the following way.

Proposition 2.1. *The action of transformation (2.1) on $\mathcal{L} = \mathcal{L}_3(d, m_1, \dots, m_r)$ is given by*

$$(2.2) \quad \text{Cr}(\mathcal{L}) = \mathcal{L}_3(d + k, m_1 + k, \dots, m_4 + k, m_5, \dots, m_r),$$

where $k = 2d - \sum_{i=1}^4 m_i$.

Proof. Since the transformation (2.1) is an isomorphism outside the base locus, we can reduce to the case $r = 4$. The exponents of monomials which generate \mathcal{L} (i.e. monomials of degree d in x_1, \dots, x_4 with multiplicity m_i at p_i for $i = 1, \dots, 4$) can be represented by the points of the truncated lattice tetrahedron

$$(2.3) \quad T := \{(a_1, \dots, a_4) \in \mathbb{Z}^4 \mid 0 \leq a_i \leq d - m_i \text{ and } \sum_{i=1}^4 a_i = d\}.$$

In the same way T' corresponds to $\mathcal{L}_3(d + k, m_1 + k, \dots, m_4 + k)$. Observe that

$$\text{Cr} \left(\prod_{i=1}^4 x_i^{a_i} \right) = \prod_{i=1}^4 x_i^{d-a_i},$$

where the right term may be written as

$$(x_1^{m_1} \dots x_4^{m_4}) \prod_{i=1}^4 x_i^{d-a_i-m_i}.$$

This may be summarized by saying that the transformation (2.1) induces the map $f(a_1, \dots, a_4) = (d - a_1 - m_1, \dots, d - a_4 - m_4)$ from \mathbb{Z}^4 to \mathbb{Z}^4 . We want to prove that f is a bijection between T and T' . First of all observe that if $(a_1, \dots, a_4) \in T$, then $0 \leq d - a_i - m_i \leq (d + k) - (m_i + k)$ and $\sum_{i=1}^4 (d - a_i - m_i) = d + k$, which implies that $f(T) \subseteq T'$. In the same way it is possible to prove that $f(T') \subseteq T$ and, since $f^2 = \mathbf{1}_{\mathbb{Z}^4}$, we get the claim. \square

Observe that $\dim \text{Cr}(\mathcal{L}) = \dim \mathcal{L}$, but in general the virtual dimensions of the two systems may be different.

Lemma 2.2. *Suppose that $2d - m_1 - m_2 - m_3 < 0$. Then $\text{Bs } \mathcal{L}_3(d, m_1, m_2, m_3)$ contains the plane through the three points.*

Proof. Consider T as before (with $m_4 = 0$). Then we have the inequality $a_1 + a_2 + a_3 \leq 3d - m_1 - m_2 - m_3 < d$. This implies that each monomial contains x_4 , which gives the fixed plane. \square

Proposition 2.3. *Let $\mathcal{L} = \mathcal{L}_3(d, m_1, \dots, m_r)$ be a linear system such that $2d \geq m_i + m_j + m_k$ for any choice of $\{i, j, k\} \subset \{1, 2, 3, 4\}$. Then*

$$(2.4) \quad v(\text{Cr}(\mathcal{L})) - v(\mathcal{L}) = \sum_{t_{ij} \geq 2} \binom{1+t_{ij}}{3} - \sum_{t_{ij} \leq -2} \binom{1-t_{ij}}{3},$$

where $t_{ij} = m_i + m_j - d$.

Proof. We can reduce to the case $\mathcal{L} = \mathcal{L}_3(d, m_1, \dots, m_4)$, since the contribution of the other points is 0 in equation (2.4). In this case, $\dim \mathcal{L} = \#T - 1$, where T is the set defined in (2.3). In order to evaluate this number, consider the sets

$$\begin{aligned} \Delta &:= \{(a_1, \dots, a_4) \in \mathbb{Z}_{\geq 0}^4 \mid \sum_{i=1}^4 a_i = d\}, \\ \Delta_i &:= \{(a_1, \dots, a_4) \in \Delta \mid a_i \geq d - m_i + 1\}, \end{aligned}$$

and observe that $T = \Delta \setminus \bigcup_{i=1}^4 \Delta_i$. By the inclusion-exclusion principle,

$$\#T = \#\Delta - \sum_i \#\Delta_i + \sum_{i < j} \#(\Delta_i \cap \Delta_j) - \sum_{i < j < k} \#(\Delta_i \cap \Delta_j \cap \Delta_k).$$

The virtual dimension $v(\mathcal{L})$ is one less than the difference of the first two terms on the right side of the equation above. A point (a_1, \dots, a_4) belongs to the intersection of the first three Δ_i 's if $a_i > d - m_i$ for $i = 1, 2, 3$; summing up these inequalities one obtains $d - a_4 > 3d - m_1 - m_2 - m_3$ which implies that $2d - m_1 - m_2 - m_3 < 0$, a contradiction to our assumption. The same argument holds for any other triple of points, so this gives the following formula:

$$\dim \mathcal{L} = v(\mathcal{L}) + \sum_{i < j} \#(\Delta_i \cap \Delta_j).$$

The set $\Delta_1 \cap \Delta_2$ is in one-to-one correspondence with the tetrahedron

$$\Delta_{12} := \{(b_1, \dots, b_4) \in \mathbb{Z}_{\geq 0}^4 \mid \sum_{i=1}^4 b_i = m_1 + m_2 - d - 2\},$$

where $b_i = a_i - d - 1 + m_i$ for $i = 1, 2$ and $b_i = a_i$ for $i = 3, 4$. The same holds for the other intersections and, since $m_i + m_j - d - 2 = t_{ij} - 2$, we have $\#(\Delta_i \cap \Delta_j) = \binom{t_{ij} + 1}{3}$. The equality $\dim \mathcal{L} = \dim \text{Cr}(\mathcal{L})$ implies that

$$v(\text{Cr}(\mathcal{L})) - v(\mathcal{L}) = \sum_{t_{ij} \geq 2} \binom{t_{ij} + 1}{3} - \sum_{t'_{ij} \geq 2} \binom{t'_{ij} + 1}{3},$$

where t'_{ij} is defined in the same way as t_{ij} for the system $\text{Cr}(\mathcal{L})$. This implies that $t'_{12} = (m_1 + k) + (m_2 + k) - (d + k) = t_{12} + k = -t_{34}$, and an analogous equality holds for each t'_{ij} . □

Corollary 2.4. *Under the assumptions of Proposition 2.3, if the degree of $\text{Cr}(\mathcal{L})$ is less than that of \mathcal{L} , then $v(\text{Cr}(\mathcal{L})) \geq v(\mathcal{L})$.*

Proof. The difference between the degree of $\text{Cr}(\mathcal{L})$ and that of \mathcal{L} is equal to $k = 2d - \sum_{i=1}^4 m_i$. From $2d < \sum_{i=1}^4 m_i$ we deduce that, if $t_{12} \geq 2$, then $d - m_3 - m_4 < m_1 + m_2 - d$ which is equivalent to $-t_{34} < t_{12}$. The same holds for each t_{ij} such that $t_{ij} \geq 2$, hence the right side of equation (2.4) is non-negative. □

3. THE ISOMORPHISMS OF $A^i(Y)$
INDUCED BY THE CUBIC TRANSFORMATION

The resolution of the indeterminacy of map (2.1) may be expressed by the following diagram:

$$(3.1) \quad \begin{array}{ccc} & Y & \\ p \swarrow & & \searrow p' \\ X & \overset{\text{---}}{\longrightarrow} & X \\ \pi \downarrow & & \downarrow \pi' \\ \mathbb{P}^3 & \overset{\text{---}}{\xrightarrow{\text{Cr}}} & \mathbb{P}^3 \end{array}$$

Here π is the blowing-up of \mathbb{P}^3 along the points p_1, \dots, p_4 , and p is the blowing-up of X along the strict transforms of the lines l_{ij} through p_i and p_j . The Picard group of X is generated by $\langle H, E_1, \dots, E_4 \rangle$, where E_i is the exceptional divisor corresponding to p_i . Let us denote $F_i = p^*E_i$ (this means that F_i is the blowing-up of a plane through three points), and let F_{ij} be the exceptional divisor corresponding to the blowing-up of the strict transform of l_{ij} . The divisor F_{ij} is a quadric, and we will denote by f_{ij}^v its “vertical” ruling, i.e. the one given by the lines of the blowing-up, and by f_{ij}^h the other one (the “horizontal”). The morphism p' contracts the quadrics F_{ij} along the horizontal rulings, while the divisors contracted by π' are the four fundamental planes. A base for $\text{Pic}(Y)$ is given by $\langle H, F_1, \dots, F_4, F_{12}, \dots, F_{34} \rangle$, where, with abuse of notation, we write H instead of p^*H . Let $H_1 = H - F_2 - F_3 - F_4 - F_{23} - F_{24} - F_{34}$ be the strict transform of the plane of \mathbb{P}^3 through the points p_2, p_3, p_4 and the three lines of the triangle. In the same way we define H_i for $i = 2, 3, 4$. The involution $\text{Cr} : \text{Pic}(Y) \rightarrow \text{Pic}(Y)$ induced by (2.1) is given by

$$(3.2) \quad \text{Cr}(F_i) = H_i, \quad \text{Cr}(F_{ij}) = F_{hk},$$

where $\{h, k\} = \{1, 2, 3, 4\} \setminus \{i, j\}$, which gives

$$(3.3) \quad \text{Cr}(H) = 3H - \sum_i 2F_i - \sum_{i < j} F_{ij}.$$

Proposition 3.1. *Let Z be the 0-dimensional scheme of four points p_1, \dots, p_4 with multiplicities m_1, \dots, m_4 and let W be the 1-dimensional scheme of the lines $\langle p_i, p_j \rangle$ with multiplicities n_{ij} . The map (2.1) transforms the sheaf $\mathcal{F} = \mathcal{O}_{\mathbb{P}^3}(d) \otimes \mathcal{I}_Z \otimes \mathcal{I}_W$ into $\mathcal{F}' = \mathcal{O}_{\mathbb{P}^3}(d + s) \otimes \mathcal{I}_{Z'} \otimes \mathcal{I}_{W'}$, where $s = 2d - \sum m_i$, $m'_i = m_i + s$ and $n'_{ij} = d - m_i - m_j + n_{hk}$.*

Proof. Let us take the pull-back of \mathcal{F} on Y :

$$p^* \pi^* \mathcal{F} = dH - \sum_i m_i F_i - \sum_{i < j} n_{ij} F_{ij}.$$

By (3.2) and (3.3) its image under the map Cr is

$$\text{Cr}(p^* \pi^* \mathcal{F}) = (d + s)H - \sum_i (m_i + s)F_i - \sum_{i < j} (d - m_i - m_j + n_{hk})F_{ij},$$

which finishes the proof. □

To each linear system $\mathcal{L} = \mathcal{L}_3(d, m_1, \dots, m_4)$ we associate a 1-cycle defined as

$$\Gamma(\mathcal{L}) := \sum_{t_{ij} \geq 1} t_{ij} l_{ij},$$

where $t_{ij} = m_i + m_j - d$ and l_{ij} is the line through p_i and p_j . Observe that by definition $H^0(\mathcal{L} \otimes \mathcal{I}_{\Gamma(\mathcal{L})}) = H^0(\mathcal{L})$, since each line $l_{ij} \in \Gamma(\mathcal{L})$ is contained in the base locus of \mathcal{L} with multiplicity at least t_{ij} .

Proposition 3.2. *With the preceding notations we have*

$$\chi(\mathcal{L} \otimes \mathcal{I}_{\Gamma(\mathcal{L})}) = \chi(\mathcal{L}) + \sum_{t_{ij} \geq 2} \binom{t_{ij} + 1}{3}.$$

Proof. We recall that by abuse of notation we denote by \mathcal{L} the linear system on X associated to $dH - \sum m_i E_i$. Consider the pull-back of $\mathcal{L} \otimes \mathcal{I}_{\Gamma(\mathcal{L})}$ on Y . In what follows we will concentrate on one of the six lines, and in order to simplify the notations we will omit the indices. Let $l \in \mathbf{A}^2(X)$ be the strict transform of the line, then $N_{l|X} \cong \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$. From the evaluation of the tautological line bundle associated to the blowing-up of l (see [5]), one obtains that $F|_F = -f^v - f^h$. The intersection $\mathcal{L}l = -t$ gives $p^*\mathcal{L}|_F = -tf^v$. The exact sequence

$$0 \longrightarrow p^*\mathcal{L} - (k + 1)F \longrightarrow p^*\mathcal{L} - kF \longrightarrow (p^*\mathcal{L} - kF)|_F \longrightarrow 0$$

and the preceding formulas imply that $h^0((p^*\mathcal{L} - kF)|_F) = h^0(kf^h + (k - t)f^v) = 0$ if and only if $k < t$. Therefore $tF \subseteq \text{Bs}(p^*\mathcal{L})$ and

$$\chi(p^*\mathcal{L} - tF) = \chi(p^*\mathcal{L}) - \sum_{k=0}^{t-1} \chi(kf^h + (k - t)f^v).$$

By the Riemann-Roch theorem on the quadric F we have $\chi(kf^h + (k - t)f^v) = (k + 1)(k + 1 - t)$. An easy calculation shows that the last sum of the above equation is equal to $-\binom{t+1}{3}$, hence applying this procedure to each of the F_{ij} one obtains

$$\chi(p^*\mathcal{L} - \sum_{t_{ij} \geq 2} t_{ij} F_{ij}) = \chi(p^*\mathcal{L}) + \sum_{t_{ij} \geq 2} \binom{t_{ij} + 1}{3}.$$

The proof follows from the projection formula $p_*p^*\mathcal{F} \cong \mathcal{F}$ and $R^i p_*p^*\mathcal{F} = 0$, with $i \geq 1$, for a blowing-up (see [8]). These properties imply that $H^i(Y, p^*\mathcal{F}) \cong H^i(X, \mathcal{F})$, where \mathcal{F} is \mathcal{L} or $\mathcal{L} \otimes \mathcal{I}_{\Gamma(\mathcal{L})}$. \square

Corollary 3.3. *Let $\mathcal{L} = \mathcal{L}_3(d, m_1, \dots, m_r)$ be a linear system and let $C_1, \dots, C_n \in \mathbf{A}^2(X)$ be a set of irreducible rational curves such that $N_{C_i|X} \cong \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ and $\mathcal{L}C_i = -t_i \leq -2$ for $i = 1, \dots, n$. Then*

$$\dim \mathcal{L} - v(\mathcal{L}) \geq \sum_{i=1}^n \binom{t_i + 1}{3} - h^2(\mathcal{L} \otimes \mathcal{I}_{\Gamma}),$$

where $\Gamma := \sum t_i C_i$ is the 1-dimensional scheme of the multiple curves.

Proof. Observe that in the proof of Proposition 3.2 the only assumption needed on the curve l is that its normal bundle in X is $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$. Since we are assuming the same for the C_i , we have that $\chi(\mathcal{L} \otimes \mathcal{I}_{\Gamma}) = \chi(\mathcal{L}) + \sum \binom{t_i + 1}{3}$. The equalities $h^0(\mathcal{L}) = h^0(\mathcal{L} \otimes \mathcal{I}_{\Gamma})$ and $h^2(\mathcal{L}) = 0$ finish the proof. \square

Remark 3.4. Let us take a system \mathcal{L} , choose four points and consider the associated sheaf $\mathcal{L} \otimes \mathcal{I}_{\Gamma(\mathcal{L})}$. Use Proposition 3.1 to transform this sheaf into $\mathcal{L}' \otimes \mathcal{I}_{\Gamma(\mathcal{L}')}$. The system \mathcal{L}' is obtained from \mathcal{L} by applying Proposition 2.1. Observe that $h^i(\mathcal{L} \otimes \mathcal{I}_{\Gamma(\mathcal{L})}) = h^i(\mathcal{L}' \otimes \mathcal{I}_{\Gamma(\mathcal{L}')})$, since the pull-back of the first system on Y is just the pull-back of the second one obtained by a base change on $\text{Pic}(Y)$. In particular this implies that the virtual dimensions of the two systems are the same. This, together with Proposition 3.2 gives another proof of Proposition 2.3.

Let us now consider the action of transformation (2.1) on the curves of Y . A basis for $\mathbf{A}^2(Y)$ may be given by $\langle h, f_1, \dots, f_4, f_{12}, \dots, f_{34} \rangle$, where h is the pull-back of the class of a line in \mathbb{P}^3 , f_i is the pull-back of a line of E_i and $f_{ij} = f_{ij}^v$ is the vertical ruling of F_{ij} .

Proposition 3.5. *The intersection matrix $M : \mathbf{A}^1(Y) \times \mathbf{A}^2(Y) \longrightarrow \mathbb{Z}$ with respect to the chosen bases of $\mathbf{A}^i(Y)$ is given by*

$$M = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & -I \end{array} \right).$$

Proof. By using the projection formula, since $(\pi \circ p)_* F_i = p_i$, F_i has a non-zero intersection only with f_i and $F_i f_i = F_i|_{F_i} f_i = -f_i^2 = -1$, where the second equation is due to $F_i^2 = p^* E_i^2 = p^* N_{E_i|X} = p^*(-e_i) = -f_i$. In the same way, since $(\pi \circ p)_* F_{ij} = l_{ij}$, F_{ij} has a non-vanishing intersection only with f_{ij} . Let F and f be F_{ij} and f_{ij} , respectively; then $Ff = F|_F f = (-f^v - f^h)f^v = -1$. Finally $Hh = 1$ since it is the intersection of a plane with a line. \square

Proposition 3.6. *With the same notation as before, we have that $HF_{ij} = f_{ij}$ and $F_{ij}^2 = -h + f_i + f_j - 2f_{ij}$.*

Proof. As before we will use F, f instead of F_{ij}, f_{ij} . The first equality is obvious, since $p_* H$ intersects $p_* F$ along a point and f is just the vertical fiber of the quadric over that point. Observe that since F^2 has a non-vanishing intersection only with H, F_i, F_j and F , then by Proposition 3.5 we have $F^2 = ah + m_i f_i + m_j f_j + bf$. The coefficient a is equal to $F^2 H = Ff = -1$. In the same way $m_i = -F^2 F_i = -Ff = 1$ and $b = -F^3 = -(-f^v - f^h)^2 = -2$. \square

Proposition 3.7. *The action of map (2.1) on $\mathbf{A}^2(Y)$ is given by*

$$\text{Cr}(h) = 3h - \sum_i f_i, \quad \text{Cr}(f_i) = 2h - \sum_{j \neq i} f_j, \quad \text{Cr}(f_{ij}) = h + f_{ks} - f_k - f_s,$$

where $\{k, s\} = \{1, 2, 3, 4\} \setminus \{i, j\}$.

Proof. Since $h = H^2$, $f_1 = -F_1^2$ and $f_{12} = HF_{12}$, we have

$$\begin{aligned} \text{Cr}(h) &= \text{Cr}(H)^2 \\ &= (3H - 2 \sum_i F_i - \sum_{i < j} F_{ij})^2 \\ &= 9h - 4 \sum_i f_i - \sum_{i < j} (h - f_i - f_j + 2f_{ij}) - 6 \sum_{i < j} f_{ij} + 8 \sum_{i < j} f_{ij} \\ &= 3h - \sum_i f_i, \end{aligned}$$

$$\begin{aligned}
 \text{Cr}(f_1) &= -\text{Cr}(F_1)^2 \\
 &= -(H - F_2 - F_3 - F_4 - F_{23} - F_{24} - F_{34})^2 \\
 &= -(h - f_2 - f_3 - f_4 - 3h + 2(f_2 + f_3 + f_4) - 2(f_{23} + f_{24} + f_{34}) \\
 &\quad - 2(f_{23} + f_{24} + f_{34}) + 4(f_{23} + f_{24} + f_{34})) \\
 &= 2h - f_2 - f_3 - f_4, \\
 \text{Cr}(f_{12}) &= \text{Cr}(H)\text{Cr}(F_{12}) \\
 &= (3H - 2 \sum_i F_i - \sum_{i < j} F_{ij})F_{34} \\
 &= 3f_{34} - 4f_{34} + h - f_3 - f_4 + 2f_{34} \\
 &= h + f_{34} - f_3 - f_4.
 \end{aligned}$$

□

Now, by $\ell_3(\delta, \mu_1, \dots, \mu_4, \beta_{12}, \dots, \beta_{34})$ we will denote the system of curves of \mathbb{P}^3 of degree δ with multiplicities μ_i in the four points and intersecting the line l_{ij} along β_{ij} points.

Proposition 3.8. *The image of the system $\ell = \ell_3(\delta, \mu_1, \dots, \mu_4, \beta_{12}, \dots, \beta_{34})$ by transformation (2.1) is $\text{Cr}(\ell) := \ell_3(\delta', \mu'_1, \dots, \mu'_4, \beta'_{12}, \dots, \beta'_{34})$, where*

$$(3.4) \quad \delta' = 3\delta - \sum_i 2\mu_i - \sum_{i < j} \beta_{ij}, \quad \mu'_r = \delta - \sum_{j \neq r} \mu_j - \sum_{\{i,j\} \neq r} \beta_{ij}, \quad \beta'_{ij} = \beta_{ks}.$$

Proof. The strict transform of a general element of ℓ may be written on Y as $\delta h - \sum \mu_i f_i - \sum \beta_{ij} f_{ij}$. The image of this element by the map Cr is

$$\delta(3h - \sum_i f_i) - \sum_i \mu_i(2h - \sum_{j \neq i} f_j) - \sum_{i < j} \beta_{ij}(h + f_{ks} - f_k - f_s).$$

This immediately gives the value of δ' and β'_{ij} . For μ'_r , observe that f_r appears in the first sum with coefficient $-\delta$, in the second with coefficient $\sum_{j \neq r} \mu_j$, and in the third sum it appears each time that the index $r \in \{k, s\}$ or, which is the same, each time that $r \notin \{i, j\}$. □

If the system ℓ has no intersection with any one of the six lines, then we have the following.

Corollary 3.9. *The action of transformation (2.1) on $\ell = \ell_3(\delta, \mu_1, \dots, \mu_r)$ is given by*

$$(3.5) \quad \text{Cr}(\ell) := \ell(\delta + 2\gamma, \mu_1 + \gamma, \dots, \mu_4 + \gamma, \mu_5, \dots, \mu_r),$$

where $\gamma = \delta - \sum_{i=1}^4 \mu_i$.

The action of Cr on the space of curves may be extended to a linear action on the space of polynomials $\mathbb{C}[\delta, \mu_1, \dots, \mu_r]$. An easy calculation gives

Corollary 3.10. *The following polynomials are invariant with respect to transformation (3.5):*

$$2\delta - \sum_{i=1}^s \mu_i, \quad \delta^2 - \sum_{i=1}^s 2\mu_i^2.$$

4. CONJECTURE

In this section we state a conjecture which allows us to give a procedure for computing the dimension of a linear system $\mathcal{L} = \mathcal{L}_3(d, m_1, \dots, m_r)$. First of all, since birational transformations do not change the effective dimension of \mathcal{L} , we can perform Cremona transformations until we get a system in standard form. If at some step we get a system \mathcal{L}^k with a negative multiplicity $-\alpha_i$ at p_i , then the system contains $\alpha_i E_i$ in its base locus. We remark that if this is the case, then there also exists a fixed component contained α_i times in the base locus of the starting system \mathcal{L} . This is the image of E_i by the sequence of Cremona transformations sending \mathcal{L}^k back to \mathcal{L} . This component can be removed without changing the dimension of \mathcal{L} . In particular we can remove $\alpha_i E_i$ from \mathcal{L}^k and keep performing Cremona if possible. That way we can reduce the problem to the study of the dimension of linear systems in standard form. In this direction let us state the following.

Conjecture 4.1. *A linear system $\mathcal{L} = \mathcal{L}_3(d, m_1, \dots, m_r)$ in standard form is special if and only if one of the following holds:*

- (i) *there exists a quadric $Q = \mathcal{L}_3(2, 1^9)$ such that $Q(\mathcal{L} - Q)(\mathcal{L} - K) < 0$;*
- (ii) *there exists a line $\ell = \ell_3(1, 1^2)$ such that $\mathcal{L}\ell \leq -2$.*

We remark that if condition (i) holds, then by (1.2) $v(\mathcal{L}) < v(\mathcal{L} - Q)$, while $\dim \mathcal{L} \geq \dim(\mathcal{L} - Q)$, which means that \mathcal{L} is special. In order to simplify the procedure we are going to prove that, under an extra assumption, condition (i) implies that the quadric Q is contained in the base locus of \mathcal{L} .

Lemma 4.2. *Let us suppose that the Harbourne-Hirschowitz Conjecture holds for linear systems on \mathbb{P}^2 with 10 fixed points. If $\mathcal{L} = \mathcal{L}_3(d, m_1, \dots, m_r)$ is a system in standard form and such that $Q(\mathcal{L} - Q)(\mathcal{L} - K) < 0$, then $Q \subset \text{Bs}(\mathcal{L})$.*

Proof. From the exact sequence

$$0 \longrightarrow \mathcal{L} - Q \longrightarrow \mathcal{L} \longrightarrow \mathcal{L}_{|Q} \longrightarrow 0,$$

we get $v(\mathcal{L}) = v(\mathcal{L} - Q) + v(\mathcal{L}_{|Q}) + 1$. If we compare with (1.2) we obtain that

$$v(\mathcal{L}_{|Q}) = \frac{Q(\mathcal{L} - Q)(\mathcal{L} - K)}{2} - 1 < 0.$$

The system $\mathcal{L}_{|Q} = \mathcal{L}_Q((d, d), m_1, \dots, m_9)$ is equivalent to the planar system $\mathcal{L}_2 = \mathcal{L}_2(2d - m_1, (d - m_1)^2, m_2, \dots, m_9)$ (see [10]). Therefore in order to prove the claim it is enough to prove that the system \mathcal{L}_2 is non-special or, since we assume that the Harbourne-Hirschowitz Conjecture holds for 10 multiple points, to prove that \mathcal{L}_2 is not (-1) -special. Let us compare the multiplicity $d - m_1$ to the m_i 's.

If $d - m_1 \geq m_3$, the system \mathcal{L}_2 is Cremona-stable (since $2d - m_1 \geq 2(d - m_1) + m_2$), and hence it is not special.

If $d - m_1 < m_4$ and \mathcal{L}_2 is not Cremona-stable, then $2d - m_1 < m_2 + m_3 + m_4$, which is not possible since we are assuming that \mathcal{L} is in standard form.

Finally, if $m_4 \leq d - m_1 < m_3$ and \mathcal{L}_2 is not Cremona-stable, then $d \leq m_2 + m_3 - 1$. If we write $d = m_2 + m_3 - t$, with $t \geq 1$, then

$$\mathcal{L}_2 = \mathcal{L}_2(2m_2 + 2m_3 - m_1 - 2t, m_2, m_3, (m_2 + m_3 - m_1 - t)^2, m_4, \dots, m_9).$$

Performing a Cremona transformation with the first three points we obtain a stable system (the degree equals the sum of the highest multiplicities). \square

Now we wonder: which system of quadrics may be contained in the base locus of a given linear system? The answer is given by the following.

Lemma 4.3. *If $|\sum_{i=1}^n r_i Q_i|$ is contained in the base locus of a linear system, then the Q_i must share 8 points, i.e. $Q_i \in \mathcal{L}_3(2, 1^8)$ for each i .*

Proof. Consider any two of the given quadrics, say Q_1, Q_2 , which share $s \leq 8$ points. Since $|Q_1 + Q_2| = \mathcal{L}_3(4, 2^s, 1^{18-2s})$ has virtual dimension $16 - 2s$, the system moves unless $s = 8$. Now consider any three quadrics, say Q_1, Q_2, Q_3 . If they have less than 8 common points, then $|Q_1 + Q_2 + Q_3| = \mathcal{L}_3(6, 3^7, 2^3)$ since, by the preceding observation, each pair must share 8 points. But this system has virtual dimension 1, which is not possible. If four or more quadrics share less than 8 points, then there exist three of them which do not share 8 points, and this is not possible as proved before.

On the other hand, let us prove that if $s = 8$, then the system $\mathcal{L} = |\sum_{i=1}^n r_i Q_i|$ cannot move. We can write $\mathcal{L} = \mathcal{L}_3(2r, r^8, r_1, \dots, r_n)$, where $r = \sum_{i=1}^n r_i$. From the exact sequence of Q_1 we get

$$0 \longrightarrow \mathcal{L} - Q_1 \longrightarrow \mathcal{L} \longrightarrow \mathcal{L}_{|Q_1} \longrightarrow 0.$$

Let us consider the restricted system $\mathcal{L}_{|Q_1} = \mathcal{L}_Q((2r, 2r), r^8, r_1)$. By [10] this is equivalent to the planar system $\mathcal{L}_2(3r, r^9, r_1)$, which is empty since $\mathcal{L}_2(3r, r^9)$ is the fixed cubic $\mathcal{L}_2(3, 1^9)$ counted r times. Therefore also $\mathcal{L}_{|Q_1} = \emptyset$, which implies that Q_1 is contained in the base locus of \mathcal{L} . This also implies that $r_1 Q_1$ is contained in the base locus of \mathcal{L} , and the same holds for $r_i Q_i$. Therefore $\mathcal{L} = \sum_{i=1}^n r_i Q_i$. \square

In order to conclude the part of the procedure concerning quadrics we need the following.

Lemma 4.4. *Let us suppose that the Harbourne-Hirschowitz Conjecture holds for linear systems on \mathbb{P}^2 with 10 points. If $\mathcal{L} = \mathcal{L}_3(d, m_1, \dots, m_r)$ is in standard form and $Q(\mathcal{L} - Q)(\mathcal{L} - K) < 0$, where Q is the quadric through the first 9 points, then $\mathcal{L} - Q$ is still in standard form.*

Proof. By hypothesis $2d \geq \sum_{i=1}^4 m_i$; moreover the degree of $\mathcal{L} - Q$ is $d - 2$ while the first 9 multiplicities are $m_i - 1$. Therefore $\mathcal{L} - Q$ is in standard form unless $m_{10} > m_4 - 1$. But this can happen only if $m_i = m$ for $i = 4, \dots, 10$. In this case \mathcal{L} would contain all the quadrics through p_1, p_2, p_3 and 6 of the points p_4, \dots, p_{10} , but by Lemma 4.3 this is not possible. \square

Therefore, assuming that the Harbourne-Hirschowitz Conjecture holds for up to 10 points, we can proceed by computing $Q(\mathcal{L} - Q)(\mathcal{L} - K)$. If it is negative we can remove the quadric Q and consider the system $\mathcal{L} - Q$, which is still in standard form. We reorder the multiplicities if necessary, and we keep removing the quadric as long as $Q(\mathcal{L} - Q)(\mathcal{L} - K) < 0$.

Let us consider part (ii) of Conjecture 4.1. We are going to prove that, in fact, such a system \mathcal{L} is special.

Proposition 4.5. *Let $\mathcal{L} = \mathcal{L}_3(d, m_1, \dots, m_r)$ be a non-empty linear system and let l be the line through p_1, p_2 . If $\mathcal{L}l = -t \leq -1$, then $h^2(\mathcal{L} \otimes \mathcal{I}_l) = 0$.*

Proof. Let $Z = Z' + Z''$, where $Z' = m_1 p_1 + m_2 p_2$ and $Z'' = m_3 p_3 + \dots + m_r p_r$, and let $\mathcal{L}' = \mathcal{O}_{\mathbb{P}^3}(d) \otimes \mathcal{I}_{Z'}$. The tensor product of the defining sequence of Z'' with

$\mathcal{L}' \otimes \mathcal{I}_{tl}$ gives

$$0 \longrightarrow \mathcal{L} \otimes \mathcal{I}_{tl} \longrightarrow \mathcal{L}' \otimes \mathcal{I}_{tl} \longrightarrow \mathcal{O}_{Z''} \longrightarrow 0.$$

Since $h^i(\mathcal{O}_{Z''}) = 0$ for $i \geq 1$, it is enough to prove that $h^2(\mathcal{L}' \otimes \mathcal{I}_{tl}) = 0$.

Let us take a plane $V \in \mathcal{L}(1, 1^2)$ and denote by W the corresponding element of $\mathcal{L}(1, 1^2) \otimes \mathcal{I}_l$. From the defining sequence of W we obtain

$$0 \longrightarrow (\mathcal{L}' - V) \otimes \mathcal{I}_{(t-1)l} \longrightarrow \mathcal{L}' \otimes \mathcal{I}_{tl} \longrightarrow (\mathcal{L}' \otimes \mathcal{I}_{tl})|_W \longrightarrow 0.$$

Observe that $h^2((\mathcal{L}' \otimes \mathcal{I}_{tl})|_W) = h^2(\mathcal{O}_{\mathbb{P}^2}(d-t) \otimes \mathcal{I}_{Z'}) = 0$, since $t = m_1 + m_2 - d \leq d$ (otherwise $m_1 > d$, and the system would be empty). This means that

$$h^2((\mathcal{L}' - V) \otimes \mathcal{I}_{(t-1)l}) \geq h^2(\mathcal{L}' \otimes \mathcal{I}_{tl}),$$

so we can proceed by induction on t until we obtain the system

$$\mathcal{L}_3(d - t, m_1 - t, m_2 - t)$$

whose h^2 vanishes. □

An easy consequence of Corollary 3.3 and Proposition 4.5 is the following.

Corollary 4.6. *Let \mathcal{L} and l be as before. If $\mathcal{L}l = -t \leq -2$, then \mathcal{L} is special and $\dim \mathcal{L} - v(\mathcal{L}) \geq \binom{t+1}{3}$.*

Observe that if there exists C_i with $i \geq 2$, such that $\mathcal{L}C_i = -t_i \leq -2$, then by the corollary above \mathcal{L} is special. It may happen that $h^i(\mathcal{L} \otimes \mathcal{I}_\Gamma) > 0$ for $i = 1, 2$, where $\Gamma = \sum t_i C_i$. This means that we are not able to compute the speciality of \mathcal{L} (see Example 7.6). For a system in standard form, we conjecture that $h^2(\mathcal{L} \otimes \mathcal{I}_\Gamma) = 0$ (this has been proved up to eight multiple points in [4]). Moreover, by assuming Conjecture 4.1, after removing the quadrics in $\text{Bs}(\mathcal{L})$, we have that $h^1(\mathcal{L}' \otimes \mathcal{I}_\Gamma) = 0$, where \mathcal{L}' is the residual system. This allows us to give the procedure shown in Table 1 for the evaluation of $\dim \mathcal{L}$.

TABLE 1.

<pre> While \mathcal{L} is not in standard form put $\mathcal{L} := \text{Cr}(\mathcal{L})$. If $m_i = -\alpha_i < 0$ then put $\mathcal{L} := \mathcal{L} - \alpha_i E_i$. While $Q(\mathcal{L} - Q)(\mathcal{L} - K) < 0$ put $\mathcal{L} := \mathcal{L} - Q$. Put $t_{ij} := m_i + m_j - d$. Return $\dim \mathcal{L} = v(\mathcal{L}) + \sum_{t_{ij} \geq 2} \binom{t_{ij} + 1}{3}$. </pre>

5. MOTIVATIONS

In [4] Conjecture 4.1 has been proved for systems $\mathcal{L}_3(d, m_1, \dots, m_r)$ with $r \leq 8$. With the help of a procedure written in the programming language of the computer algebra system SINGULAR [6], we verified the conjecture for systems $\mathcal{L}_3(d, m^r)$ with $m \leq 7$ and $r \leq 20$.

A consequence of the conjecture is that the only curves ℓ which can give speciality are the images of the line $\ell_3(1, 1^2)$ by a finite set of Cremona transformations. In one direction, observe that if there exists a curve ℓ which is the image of $\ell_3(1, 1^2)$ by a finite set of Cremona transformations and such that $\mathcal{L}\ell = -t \leq -2$, then \mathcal{L} is special. In fact, performing back Cremona, we get a system \mathcal{L}' such that $\mathcal{L}'\ell_3(1, 1^2) = -t$, since the intersection product is invariant under Cremona. Therefore, by Proposition 4.5, since cohomology groups are invariant under birational transformations, $h^2(\mathcal{L} \otimes \mathcal{I}_{t\ell}) = 0$, and \mathcal{L} is special. On the other hand, for other classes of curves one can have the following problem.

Example 5.1. Let $\ell = \ell_3(4, 1^8)$ be the system of quartics through 8 fixed points and suppose that $\mathcal{L}\ell = -t \leq -2$. The intersection product is $4d - \sum_{i=1}^8 m_i \leq -2$, and hence necessarily $2d - \sum_{i=1}^4 m_i \leq -1$, which implies that \mathcal{L} is not in standard form. Let us perform a Cremona transformation involving the first 4 points. Since the intersection product is invariant under this transformation, $\mathcal{L}'\ell' = \mathcal{L}\ell = -t$, where $\mathcal{L}' = \text{Cr}(\mathcal{L})$ and $\ell' = \text{Cr}(\ell)$. Moreover the degree d' of \mathcal{L}' is strictly smaller than d and, by Proposition 3.9, $\ell' = \ell$. Arguing as before we can say that \mathcal{L}' is not in standard form, and we can perform another transformation. Iterating this procedure we can transform \mathcal{L} until we get a system having negative degree, and hence we conclude that $\mathcal{L} = \emptyset$. Therefore the quartic cannot give speciality. The fact that $\ell_3(4, 1^8)$ cannot be obtained from $\ell_3(1, 1^2)$ by a finite set of Cremona transformations is also an easy consequence of Corollary 3.10, since the invariant $\delta^2 - \sum 2\mu_i^2$ is 0 for the quartic and -3 for the line.

Another consequence of Conjecture 4.1 is that if \mathcal{L} is in standard form, then we cannot find any curve $\ell \neq \ell_3(1, 1^2)$ which is the image of a line and such that $\mathcal{L}\ell \leq -2$. In order to give a motivation for this we prove the following.

Lemma 5.2. *Let us suppose that $\mathcal{L} = \mathcal{L}_3(d, m_1, \dots, m_r)$ and ℓ is obtained from $\ell_3(1, 1^2)$ by a finite set of Cremona transformations such that at each step the degree of the curve increases. Then we can write*

$$(5.1) \quad \mathcal{L}\ell = \beta_1(2d - \sum_{i=1}^4 m_i^{(1)}) + \dots + \beta_a(2d - \sum_{i=1}^4 m_i^{(a)}) + (d - m_h - m_k),$$

where $\beta_j \geq 1$, $m_i^{(j)}$ are chosen between m_1, \dots, m_r and $h, k \geq 5$.

Proof. We argue by induction on the number of Cremona transformations necessary to obtain ℓ from the line $\ell_3(1, 1^2)$. First of all, after one transformation the image of the line is the rational normal cubic $\ell_3(3, 1^6)$, having intersection product with \mathcal{L} equal to $3d - \sum_{i=1}^6 m_i = (2d - \sum_{i=1}^4 m_i) + (d - m_5 - m_6)$. Now we assume that the formula is true for $\ell = \ell_3(\delta, \mu_1, \dots, \mu_s)$, and we prove it for the curve $\ell' = \text{Cr}(\ell) = \ell_3(\delta', \mu'_1, \dots, \mu'_s)$ obtained from ℓ performing one more Cremona transformation increasing the degree. We can suppose that the transformation is based on the first 4 points. By formula (3.5), $\delta' = \delta + 2\gamma$ and $\mu'_i = \mu_i + \gamma$ for $i = 1, \dots, 4$, where $\gamma = \delta - \sum_{i=1}^4 \mu_i > 0$, and $\mu'_i = \mu_i$ for $i \geq 5$. Therefore $\mathcal{L}\ell' - \mathcal{L}\ell = \gamma(2d - \sum_{i=1}^4 m_i)$, which proves the claim. \square

Corollary 5.3. *If \mathcal{L} is not empty and in standard form and ℓ is obtained from $\ell_3(1, 1^2)$ by a finite set of Cremona transformations such that at each step the degree of the curve increases, then $\mathcal{L}\ell \geq 0$.*

Proof. Since \mathcal{L} is in standard form, $2d \geq \sum m_i^{(j)}$. Moreover $d - m_h - m_k \geq 0$, since otherwise $2d < m_1 + m_2 + m_h + m_k$. Hence all the terms on the right side of (5.1) are non-negative. \square

Therefore, if we assume that the following conjecture holds, we have that a system in standard form cannot have a negative intersection product with the image of a line.

Conjecture 5.4. *Let $\ell = \ell_3(\delta, \mu_1, \dots, \mu_r)$ be a curve that can be obtained from the line $\ell_3(1, 1^2)$ by a finite set of Cremona transformations. Then ℓ can be obtained by a finite set of Cremona transformations such that at each step the degree of the curve increases.*

6. HOMOGENEOUS LINEAR SYSTEMS

In this section we study special homogeneous systems $\mathcal{L} = \mathcal{L}_3(d, m^r)$.

Proposition 6.1. *The system \mathcal{L} is empty for $d \leq 2m - 1$ and $r \geq 8$.*

Proof. It is enough to show that the system $\mathcal{L}_3(2m - 1, m^8)$ is empty. We first prove by induction that performing Cremona on the first four points and the last four alternatively, at the i -th step we get the system

$$\mathcal{L}^i = \mathcal{L}_3(2m - 2i^2 - 1, (m - i^2 + i)^4, (m - i^2 - i)^4).$$

The basis of induction holds since applying Cremona to the starting system we get $\mathcal{L}^1 = \mathcal{L}_3(2m - 3, (m - 2)^4, m^4)$. If we now perform Cremona to \mathcal{L}^i taking the first four points we have that $k = 2(2m - 2i^2 - 1) - 4(m - i^2 + i) = -4i - 2$ and hence the transform system is

$$\text{Cr}(\mathcal{L}^i) = \mathcal{L}_3(2m - 2(i + 1)^2 - 1, (m - (i + 1)^2 - (i + 1))^4, (m - i^2 - i)^4).$$

The multiplicity $m - i^2 - i$ can be written as $m - (i + 1)^2 + (i + 1)$, and hence, reordering the points, we get that $\text{Cr}(\mathcal{L}^i) = \mathcal{L}^{i+1}$, which gives the inductive step.

We keep performing these transformations until $m - i^2 - i \leq 0$. At this step we have that $2m - 2i^2 - 1 < m - i^2 + i$, and hence the system \mathcal{L}^i is empty since it has some multiplicity bigger than the degree. \square

By assuming Conjecture 4.1 and the Harbourne-Hirschowitz Conjecture for linear systems on \mathbb{P}^2 with 10 points, we can prove the following.

Proposition 6.2. *If $d \geq 2m$ the system $\mathcal{L} = \mathcal{L}_3(d, m^r)$ is special if and only if $r = 9$ and $2m \leq d < [-1 + \frac{3}{2}\sqrt{2m^2 + 2m}]$.*

Proof. Our assumption on the degree implies that \mathcal{L} is in standard form. According to Conjecture 4.1, the system \mathcal{L} is special if and only if there exists a quadric $Q \in \mathcal{L}_3(2, 1^9)$ such that $Q(\mathcal{L} - Q)(\mathcal{L} - K) < 0$, or if there exists a line $\ell = \ell_3(1, 1^2)$ such that $\ell\mathcal{L} \leq -2$. But under our hypothesis on the degree, a line through 2 fixed points has non-negative intersection product with \mathcal{L} , and hence \mathcal{L} is special if and only if $Q(\mathcal{L} - Q)(\mathcal{L} - K) < 0$. But we can write $Q(\mathcal{L} - Q)(\mathcal{L} - K) = 2(d - 2)(d + 4) - 9(m - 1)(m + 2)$, which is negative if and only if $2m \leq d < [-1 + \frac{3}{2}\sqrt{2m^2 + 2m}]$. We finish the proof by showing that if \mathcal{L} is special, then the number r of fixed

points cannot be bigger than 9. If we suppose by contradiction that $r \geq 10$, then $Q_i(\mathcal{L} - Q_i)(\mathcal{L} - K) < 0$, where Q_i is the quadric through all the first 10 points but p_i . By Lemma 4.2 \mathcal{L} must contain the system of quadrics $\sum_{i=1}^{10} Q_i$, and this is not possible because of Lemma 4.3. \square

We can conclude that if the system \mathcal{L} has more than 9 fixed points (or exactly 8 points), then it is not special. If it has 9 fixed points, it is special if and only if $2m \leq d < [-1 + \frac{3}{2}\sqrt{2m^2 + 2m}]$. If $r \leq 7$ and $d \geq 2m$, the system cannot be special. Finally, if $r \leq 7$ and $d \leq 2m - 1$, we have to follow the procedure of Table 1.

7. EXAMPLES

We end the paper by presenting many examples of special systems and computing their dimension following the procedure of Table 1. We remark that for each example the dimension we found in this way agrees with the dimension computed with the help of a Singular program.

Notation 7.1. We will use the symbols $\overset{(i)}{\rightsquigarrow}$, $\overset{(ii)}{\rightsquigarrow}$ and $\overset{(iii)}{\rightsquigarrow}$ to signify that we are applying the first, the second and the third step of the procedure (in (i) we are applying a Cremona transformation, in (ii) we are removing a plane and in (iii) we are removing a quadric).

Example 7.2. Consider the system $\mathcal{L} := \mathcal{L}_3(7, 4^6)$ with $v(\mathcal{L}) = -1$. Following the procedure of Table 1 we can apply four Cremona transformations:

$$\mathcal{L}_3(7, 4^6) \overset{(i)}{\rightsquigarrow} \mathcal{L}_3(5, 4^2, 2^4) \overset{(i)}{\rightsquigarrow} \mathcal{L}_3(3, 2^4) \overset{(i)}{\rightsquigarrow} \mathcal{L}_3(1),$$

so that $\dim \mathcal{L} = 3$. Observe that since $\mathcal{L}_3(7, 4^6)\ell_3(3, 1^6) = -3$, the starting system contains the rational normal curve. After the first transformation $\ell_3(3, 1^6) \rightsquigarrow \ell_3(1, 1^2)$ and, in fact, $\mathcal{L}_3(5, 4^2, 2^4)\ell_3(1, 1^2) = -3$.

Example 7.3. Consider the system $\mathcal{L} := \mathcal{L}_3(12, 7^6)$ with $v(\mathcal{L}) = -50$. As before we can apply the following Cremona transformations:

$$\mathcal{L}_3(12, 7^6) \overset{(i)}{\rightsquigarrow} \mathcal{L}_3(8, 7^2, 3^4) \overset{(i)}{\rightsquigarrow} \mathcal{L}_3(4, 3^4, -1^2) \overset{(ii)}{\rightsquigarrow} \mathcal{L}_3(4, 3^4) \overset{(i)}{\rightsquigarrow} \mathcal{L}_3(0, -1^4) \overset{(ii)}{\rightsquigarrow} \mathcal{L}_3(0),$$

so that $\dim \mathcal{L} = 0$. From the procedure we may deduce that \mathcal{L} is given by the union of six surfaces of type $\mathcal{L}_3(2, 2, 1^5)$ (each surface can be obtained from a plane $\mathcal{L}_3(1, 1^3)$ applying one Cremona transformation).

In these two first examples, by performing some Cremona transformations and removing fixed planes we get a system in standard form which is not special. Let us give two examples where the system in standard form is still special.

Example 7.4. Consider the system $\mathcal{L} := \mathcal{L}_3(10, 6^5)$ with $v(\mathcal{L}) = 5$. Apply the Cremona transformation:

$$\mathcal{L}_3(10, 6^5) \overset{(i)}{\rightsquigarrow} \mathcal{L}_3(6, 6, 2^4).$$

The last system is in standard form and has virtual dimension 11 while its dimension is 15, as expected by our conjecture. In fact each of the four lines l_{1j} , $j = 2, \dots, 5$, has intersection product -2 with the system.

Example 7.5. Consider the system in standard form $\mathcal{L} := \mathcal{L}_3(16, 11, 7^8)$ with $v(\mathcal{L}) = 10$. Since $Q(\mathcal{L} - Q)(\mathcal{L} - K) = -2$ following the procedure, we may remove the quadric from the base locus of \mathcal{L} :

$$\mathcal{L}_3(16, 11, 7^8) \stackrel{\text{(iii)}}{\rightsquigarrow} \mathcal{L}_3(14, 10, 6^8).$$

The virtual dimension of the last system is 11, and its dimension is 19 (each of the 8 lines l_{1j} , $j = 2, \dots, 9$, has intersection product -2 with the system).

We remark that in the two examples above, when the linear system is in standard form and does not contain any quadric, then its speciality is given exactly by the sum of the binomials $\binom{t_{ij}+1}{3}$, where $-t_{ij}$ is the intersection product of the line l_{ij} with the system. In the following example we show that if a linear system \mathcal{L} is not in standard form, then its speciality can be strictly smaller than the sum of these binomials, or equivalently $h^2(\mathcal{L} \otimes \mathcal{I}_\Gamma) \neq 0$ (where Γ is the 1-dimensional scheme of the multiple lines).

Example 7.6. The system $\mathcal{L} = \mathcal{L}_3(3, 3^3)$ with $v(\mathcal{L}) = -11$ is not in standard form, hence

$$\mathcal{L}_3(3, 3^3) \stackrel{\text{(i)}}{\rightsquigarrow} \mathcal{L}_3(0, -3) \stackrel{\text{(ii)}}{\rightsquigarrow} \mathcal{L}_3(0).$$

This implies that $\dim \mathcal{L} = 0$, since it is three times the plane $\mathcal{L}_3(1, 1^3)$. For each line l_{ij} through two of the three points we have $\mathcal{L}l_{ij} = -3$. Let $\Gamma = 3(l_{12} + l_{13} + l_{23})$. By Corollary 3.3 $\dim \mathcal{L} - v(\mathcal{L}) \geq 12 - h^2(\mathcal{L} \otimes \mathcal{I}_\Gamma)$, which implies that $h^2(\mathcal{L} \otimes \mathcal{I}_\Gamma) \geq 1$.

In the next example we give a way to construct a class of systems whose speciality is due to multiple quadrics in the fixed locus.

Example 7.7. Let r_1, \dots, r_n be positive integers and let $r = \sum r_i$. As we proved in Lemma 4.3, the system $\mathcal{L} = \mathcal{L}_3(2r, r^8, r_1, \dots, r_n)$ has dimension 0, while its virtual dimension is $\sum(r_i - \binom{r_i+2}{3}) \leq 0$. Moreover this sum is 0 if and only if each $r_i = 1$, otherwise the system is special.

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