THE ARONSSON EQUATION FOR ABSOLUTE MINIMIZERS OF \(L^\infty\)-FUNCTIONALS ASSOCIATED WITH VECTOR FIELDS SATISFYING HÖRMANDEr’S CONDITION

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Abstract. Given a Carnot-Carathéodory metric space \((\mathbb{R}^n, d_X)\) generated by vector fields \(\{X_i\}_{i=1}^m\) satisfying Hörmander’s condition, we prove in Theorem A that any absolute minimizer \(u \in W^{1, \infty}_X(\Omega)\) to \(F(v, \Omega) = \text{ess sup}_{x \in \Omega} f(x, Xv(x))\) is a viscosity solution to the Aronsson equation

\[- \sum_{i=1}^m X_i(f(x, Xu(x)))f_{p_i}(x, Xu(x)) = 0, \quad \text{in } \Omega,\]

under suitable conditions on \(f\). In particular, any AMLE is a viscosity solution to the subelliptic \(\infty\)-Laplacian equation

\[\Delta_{\infty}^{(X)} u := - \sum_{i,j=1}^m X_i u X_j u X_i X_j u = 0, \quad \text{in } \Omega.\]

If the Carnot-Carathéodory space is a Carnot group \(G\) and \(f\) is independent of the \(x\)-variable, we establish in Theorem C the uniqueness of viscosity solutions to the Aronsson equation

\[A(Xu, (D^2 u)^+) := - \sum_{i,j=1}^m f_{p_i}(Xu) f_{p_j}(Xu) X_i X_j u = 0, \quad \text{in } \Omega,\]

\[u = \phi, \quad \text{on } \partial \Omega,\]

under suitable conditions on \(f\). As a consequence, the uniqueness of both AMLE and viscosity solutions to the subelliptic \(\infty\)-Laplacian equation is established on any Carnot group \(G\).

§1. Introduction

Variational problems in \(L^\infty\) are very important because of both their analytic difficulties and their frequent appearance in applications. (See the survey article by Barron [3]). The study began with Aronsson [A1, A2]. The simplest model is to consider minimal Lipschitz extensions (or MLE): for a bounded, Lipschitz domain \(\Omega \subset \mathbb{R}^n\) and \(g \in \text{Lip}(\Omega)\), find \(u \in W^{1, \infty}(\Omega)\), with \(u|_{\partial \Omega} = g\), such that

\[\|Du\|_{L^\infty(\Omega)} \leq \| Dw\|_{L^\infty(\Omega)}, \quad \forall w \in W^{1, \infty}_0(\Omega) \text{ with } w|_{\partial \Omega} = g.\]  

(1.1)

Since MLE may be neither unique nor smooth, Aronsson [A1] introduced the notation of absolutely minimizing Lipschitz extensions (or AMLE for short), and proved

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that any $C^2$ AMLE solves the $\infty$-Laplacian equation

$$\Delta_\infty u := -\sum_{i,j=1}^n \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \frac{\partial^2 u}{\partial x_i \partial x_j} = 0, \text{ in } \Omega.$$  

However, (1.2) is a fully nonlinear, degenerate PDE and may not have $C^2$ solutions. The issue was finally settled by Jensen [J2], who not only established the equivalence between the AMLE and the viscosity solution to (1.2) (a notion of weak solutions first introduced by Crandall-Lions [CL]; see also Crandall-Ishii-Lions [CIL]), but also proved the uniqueness of viscosity solutions to (1.2) with Dirichlet boundary data. The remarkable analysis of [J2] involves approximation by $p$-Laplacian and Jensen’s maximum principle for semiconvex functions [J1]. The reader can consult with Evans [E] and Lindqvist-Manfredi [LM] for qualitative estimates on $\infty$-harmonic functions. Crandall-Evans-Gariepy [CEG] developed the comparison principle for viscosity solutions of (1.2) with cones, which are of the forms $a + b|x - x_0|$, and gave an alternative proof of the equivalence between AMLE and viscosity solution to (1.2). Furthermore, Crandall-Evans [CE] have utilized this property in their study on the regularity issue of $\infty$-harmonic functions. Recently, Barron-Jensen-Wang [BJW] considered general $L^\infty$-functionals

$$F(u, \Omega) := \text{ess sup}_{x \in \Omega} f(x, u(x), Du(x)), \forall u \in W^{1, \infty}(\Omega),$$  

and proved, under suitable conditions on $f$, that any absolute minimizer of $F(\cdot, \Omega)$ is a viscosity solution to the Aronsson equation

$$-\sum_{i=1}^n f_p(x, u(x), Du(x)) \frac{\partial}{\partial x_i} (f(x, u(x), Du(x))) = 0, \text{ in } \Omega.$$  

Shortly after [BJW], Crandall [C] gave an elegant proof of an improved version of [BJW]. Through [BJW] and [C], it becomes clear that the classical solution to the Hamilton-Jacobi equation $f(x, \phi(x), D\phi(x)) - c = 0$ plays an important role in the analysis. We would like to mention that there is a very interesting survey paper available on this subject by Aronsson-Crandall-Juntinen [ACJ].

Since the notion of AMLE can easily be formulated in any metric space, it is a very natural and interesting problem to study AMLE in Carnot-Carathéodory metric spaces, which include Riemannian manifolds and Subriemannian manifolds arising from Hörmander’s vector fields (e.g. Heisenberg groups, Carnot groups). There have been several works in this direction. For example, Juutinen [J] extended the main theorems of [J2] into Riemannian manifolds. Bieske [B1, B2] proved the uniqueness of viscosity solutions to the subelliptic $\infty$-Laplacian equation on the Heisenberg group $H^n$ and the Grushin-type space. Inspired by [C], Bieske-Capogna [BC] proved that any AMLE is a viscosity solution to the subelliptic $\infty$-Laplacian equation for any Carnot group $G$. In general, it was proved by [BC] that any horizontally $C^1$ AMLE is a viscosity solution to the subelliptic $\infty$-Laplacian equation associated with free systems of Hörmander’s vector fields. Bieske [B3] recently extended [BC] to the Grushin-type space.

In this paper, we are mainly interested in the Aronsson equation of AMLE’s and its uniqueness issue for any Carnot-Carathéodory metric space generated by vector fields satisfying Hörmander’s condition. In this direction, we prove that any AMLE is a viscosity solution to the subelliptic $\infty$-Laplacian equation. Moreover, if the vector fields are horizontal vector fields of a Carnot group $G$, then both AMLE
and viscosity solution to the Aronsson equation are unique. In fact some more general results are provided by Theorems A and C below.

In order to state our results, we first recall some preliminary facts.

For a bounded domain $\Omega \subset \mathbb{R}^n$ let $X_i \in C^{0,1}(\Omega, \mathbb{R}^n)$, $1 \leq i \leq m$, be given Lipschitz continuous vector fields on $\Omega$. We say that an absolutely continuous curve $r : [0,\delta] \to \Omega$ is admissible if there are measurable functions $a_j(t) : [0,\delta] \to \mathbb{R}$, $1 \leq j \leq m$, such that

$$\sum_{j=1}^{m} a_j^2(t) \leq 1, \quad r'(t) = \sum_{j=1}^{m} a_j(t) X_j(r(t)), \quad \text{for a.e. } t \in [0,\delta].$$

For any $x, y \in \Omega$, we define the Carnot-Carathéodory distance by

$$d_X(x, y) = \inf \{T \mid \exists \text{ an admissible curve } r : [0, T] \to \Omega, \text{ with } r(0) = x, r(T) = y \}.$$  

**Definition 1.1** (Carnot-Carathéodory space). For a bounded domain $\Omega \subset \mathbb{R}^n$ and a family of Lipschitz continuous vector fields, $\{X_i\}_{i=1}^{m}$, on $\Omega$, if the above-defined $d_X(\cdot, \cdot)$ satisfies the three properties for the distance function, then we call $(\Omega, d_X)$ a Carnot-Carathéodory space, i.e. there exists at least one admissible curve joining any two points in $\Omega$. Furthermore, we assume throughout this paper that $d_X$ is uniformly continuous with respect to the euclidean norm $\| \cdot \|$, namely

$$\omega(\delta) = \sup \{d_X(x, y) : x, y \in \Omega, \text{with } \|x - y\| \leq \delta \} \to 0, \text{ as } \delta \to 0.$$  

We will focus on two classes of Carnot-Carathéodory spaces: (i) Hörmander’s vector fields, and (ii) Carnot groups.

**Example 1.2** (Hörmander’s vector fields). Suppose that $\{X_i\}_{i=1}^{m} \subset C^\infty(\Omega, \mathbb{R}^n)$ satisfy Hörmander’s finite rank condition, i.e. there exists a step $r \geq 1$ such that $\{X_i(x)\}_{i=1}^{m}$ and their commutators up to at most order $r$ span $\mathbb{R}^n$ for any $x \in \Omega$. Then it follows from Nagel-Stein-Wainger [NSW] that $d_X$, defined by (1.4), is a distance. Moreover, for each compact set $K \subset \subset \Omega$ there exists a $C_K > 0$ so that

$$C_K^{-1} \|x - y\| \leq d_X(x, y) \leq C_K \|x - y\|^\frac{1}{r}, \quad \forall x, y \in K,$$

where $\| \cdot \|$ is the Euclidean norm on $\mathbb{R}^n$.

The description of Carnot groups will be given later in this section. We now recall the definition of horizontal Sobolev spaces and Lipschitz spaces.

For $u : \Omega \to \mathbb{R}$, denote $Xu := (X_1u, \cdots, X_m u)$ as the horizontal gradient of $u$. For $1 \leq p \leq \infty$, the horizontal Sobolev space is defined by

$$W^{1,p}_X(\Omega) \equiv \{u : \Omega \to \mathbb{R} \mid \|u\|_{W^{1,p}_X(\Omega)} \equiv \|u\|_{L^p(\Omega)} + \|Xu\|_{L^p(\Omega)} < \infty \}.$$

The Lipschitz space is defined by

$$\text{Lip}_X(\Omega) := \{u : \Omega \to \mathbb{R} \mid \|u\|_{\text{Lip}_X(\Omega)} \equiv \sup_{x,y \in \Omega, x \neq y} \frac{|u(x) - u(y)|}{d_X(x, y)} < \infty \}.$$  

It was proved by Garofalo-Nieu [GN] and Franchi-Serapioni-Serra [FSS] that for any Carnot-Carathéodory space $(\Omega, d_X)$, $u \in \text{Lip}_X(\Omega)$ iff $u \in W^{1,\infty}_X(\Omega)$.

We now give the definition of absolute minimizers of $\mathring{L}^\infty$-functionals over $W^{1,\infty}_X(\Omega)$. 

**Definition 1.3** (absolute minimizers). For any nonnegative, continuous function $f \in C(\Omega \times \mathbb{R}^m)$, let
\[ F(v, \Omega) = \text{esssup}_{x \in \Omega} f(x, Xu(x)), \quad \forall v \in W^{1, \infty}_X(\Omega). \]
A function $u \in W^{1, \infty}_X(\Omega)$ is an absolute minimizer of $F(\cdot, \Omega)$, if for any open subset $\Omega' \subset \Omega$ and $w \in W^{1, \infty}_X(\Omega')$, with $w = u$ on $\partial \Omega'$, we have
\[ F(u, \Omega') \leq F(w, \Omega'). \]
$u \in W^{1, \infty}_X(\Omega)$ is called an absolutely minimizing Lipschitz extension (or AMLE) if $u$ is an absolute minimizer of $F(\cdot, \Omega)$ when $f(x, p) = |p|^2$, for $(x, p) \in \Omega \times \mathbb{R}^m$.

Formal calculations yield that an absolute minimizer $u \in W^{1, \infty}_X(\Omega)$ to $F(\cdot, \Omega)$ satisfies the Aronsson equation
\[ -\sum_{i=1}^{m} X_i(f(x, Xu(x)))f_{p_i}(x, Xu(x)) = 0, \quad \text{in } \Omega. \tag{1.7} \]
In particular, the Aronsson equation of an AMLE is the (subelliptic) $\infty$-Laplace equation
\[ \Delta^{(X)}_\infty u := -\sum_{i,j=1}^{m} X_i u X_j u X_j u = 0, \quad \text{in } \Omega. \tag{1.8} \]

In order to show that an absolute minimizer is a solution to (1.7), we need to recall the notion of viscosity solutions to (1.7), which is a natural extension of that developed by Crandall-Lions [CL] (see also [CIL]) of second order (degenerate) elliptic PDEs.

Let $S^m$ denote the set of symmetric $m \times m$ matrices, equipped with the usual order. A function $A \in C(R^n \times R^m \times S^m)$ is called horizontally elliptic, if, for any $(x, p) \in R^n \times R^m$,
\[ A(x, p, M) \leq A(x, p, N), \quad \forall M, N \in S^m, \quad \text{with } N \leq M. \tag{1.9} \]
Let $(D^2u)^* \in S^m$ denote the horizontal hessian of $u$, defined by
\[ (D^2u)^*_{ij} = \frac{1}{2}(X_i X_j + X_j X_i)u, \quad \forall 1 \leq i, j \leq m. \]

We now have

**Definition 1.4.** Consider a horizontally elliptic equation:
\[ A(x, Xu(x), (D^2u)^*(x)) = 0, \quad \text{in } \Omega. \tag{1.10} \]
A function $u \in C(\Omega)$ is called a viscosity subsolution to (1.10), if, for any pair $(x_0, \phi) \in \Omega \times C^2(\Omega)$ such that $x_0$ is a local maximum point of $(u - \phi)$, then we have
\[ A(x_0, X\phi(x_0), (D^2\phi)^*(x_0)) \leq 0. \tag{1.11} \]
A function $u \in C(\Omega)$ is called a viscosity supersolution to (1.10) if $-u$ is a viscosity subsolution to (1.10). Finally, a function $u \in C(\Omega)$ is a viscosity solution to (1.10) if it is both a viscosity subsolution and a viscosity supersolution to (1.10).

**Remark 1.5.** If $\{X_i\}_{i=1}^m \in C^2(\Omega, \mathbb{R}^m)$, then we can define the horizontal $C^2$ space, $\Gamma^2(\Omega)$, by
\[ \Gamma^2(\Omega) = \{ u : \Omega \to \mathbb{R} \mid X_i u, X_i X_j u \in C(\Omega), 1 \leq i, j \leq m \}. \]
In [B1] [B2], [BC], Manfredi [FM], the authors introduced a slightly different notion of viscosity solution to (1.10) by requiring the test functions $\phi \in \Gamma^2(\Omega)$ in Definition 1.4. Since $C^2(\Omega) \subset \Gamma^2(\Omega)$, the notion of viscosity solution introduced here is weaker than that of [B1] [B2], [BC], and [FM]. However, when the underlying Carnot-Carathéodory space is induced by a Carnot group, then these two notions are equivalent (see [B1]).

It is easy to check that both (1.7) and (1.8) are horizontally elliptic equations. Now we are ready to state our first theorem.

**Theorem A.** Assume that \( \{X_i\}_{i=1}^m \subset C^2(\Omega, \mathbb{R}^n) \) and \((\Omega, d_X)\) is a Carnot-Carathéodory space. Suppose that \( u \in W^{1,\infty}_X(\Omega) \) is an absolute minimizer of

\[
F(v, \Omega) = \text{ess sup}_{x \in \Omega} f(x, Xv(x)), \quad v \in W^{1,\infty}_X(\Omega),
\]

where \( f \in C^2(\Omega \times \mathbb{R}^m, \mathbb{R}_+) \) satisfies

\begin{enumerate}[(f1)]
  \item for any \( x \in \Omega \), \( f(x, \cdot) \) is quasiconvex:
  \[
  (1.12) \quad f(x, tp_1 + (1-t)p_2) \leq \max\{f(x, p_1), f(x, p_2)\}, \quad \forall p_1, p_2 \in \mathbb{R}^m, \ 0 \leq t \leq 1.
  \]
  \item \( f \) is homogeneous of degree \( \alpha \geq 1 \) and \( f_p(0,0) = 0 \).
\end{enumerate}

Then \( u \) is a viscosity solution to the Aronsson equation (1.7).

The ideas to prove Theorem A are based on: (1) the observation of rewriting (1.7) into a euclidean form which enables us to adopt Crandall’s construction [C] of solutions to the Hamilton-Jacobi equation as test functions (see also [BJW]), and (2) the comparison principle of viscosity solutions to Hamilton-Jacobi equations without \( u \)-dependence (see [CIL] or [BJW]).

Since \( f(x, p) \equiv |p|^2 : \Omega \times \mathbb{R}^m \to \mathbb{R}_+ \) satisfies both (f1) and (f2), we have, as a consequence of Theorem A,

**Corollary B.** Assume that \( \{X_i\}_{i=1}^m \subset C^2(\Omega, \mathbb{R}^n) \) and \((\Omega, d_X)\) is a Carnot-Carathéodory space. Suppose that \( u \in W^{1,\infty}_X(\Omega) \) is an AMLE. Then \( u \) is a viscosity solution to the (subelliptic) \( \infty \)-Laplacian equation (1.8).

Now we turn to the discussion on the uniqueness problem of absolute minimizers of \( F(\cdot, \Omega) \) and viscosity solutions to (1.7). For this purpose, we restrict our attention to the case where the vector fields generating the Carnot-Carathéodory metric are the horizontal vector fields associated with a Carnot group \( G \).

Recall that a Carnot group of step \( r \geq 1 \) is a simply connected Lie group \( G \) whose Lie algebra \( g \) admits a vector space decomposition in \( r \) layers \( g = V^1 + V^2 + \cdots + V^r \) having two properties: (i) \( g \) is stratified, i.e., \([V^1, V^j] = V^{j+1}, j = 1, \cdots, r-1\); (ii) \( g \) is \( r \)-nilpotent, i.e. \([V^j, V^r] = 0, j = 1, \cdots, r\). \( V^1 \) is called the horizontal layer and \( V^j, j = 2, \cdots, r \), are vertical layers. It is well known (cf. Folland-Stein [FS]) that the exponential map, \( \exp : g \to G \), is a global diffeomorphism so that we can identify \( G \) with \( \mathbb{R}^n \) via \( \exp \) and \( G \) has an exponential coordinate system; here \( n = \dim(G) \) is the topological dimension of \( G \). More precisely, let \( X_{i,j}, 1 \leq i \leq m_j = \dim(V^j) \), be a basis of \( V^j \), for \( 1 \leq j \leq r \), which is orthonormal with respect to an arbitrarily chosen euclidean norm \( \| \cdot \| \) on \( g \), with respect to which the \( V^j \)’s are mutually orthogonal. Then \( p \in G \) has coordinate \((p_{ij})_{1 \leq i \leq m_j, 1 \leq j \leq r}\) if \( p = \exp(\sum_{j=1}^r \sum_{i=1}^{m_j} (p_{ij} X_{i,j})' \cdot) \). Let \( \cdot \) denote the group multiplication on \( G \). Then it is known (cf. [FS]) that the group law \((x, y) \to x \cdot y\) is a polynomial map with respect to the exponential map. From now on, we set \( m = m_1 = \dim(V^1) \) and denote \( X_i = X_{i,1} \) for \( 1 \leq i \leq m \). Two bi-Lipschitz equivalent metrics, on \( G \), we need are:
(1) the Carnot-Carathéodory metric $d_X$ on $G$ generated by $\{X_i\}_{i=1}^m$; (2) the gauge metric $d$ on $G$ given as follows. For $p = (p_{ij})_{1 \leq m, 1 \leq j \leq r}$, 
\[
    \|p\|^2_{G} = \sum_{j=1}^{r} \left( \sum_{i=1}^{m_j} |p_{ij}|^2 \right)^{\frac{m_j}{2}},
\]
with the induced gauge distance 
\[
    d(x, y) = \|x^{-1}y\|_G, \quad \forall x, y \in G,
\]
satisfying the invariant property: 
\[
    d(z \cdot x, z \cdot y) = d(x, y), \quad \forall x, y, z \in G.
\]

Now we mention the Heisenberg group $H^n$, the simplest Carnot group of step two, $H^n \equiv \mathbb{C}^n \times \mathbb{R}$ endowed with the group law: for $(z_1, \cdots, z_n, t), (z'_1, \cdots, z'_n, t') \in \mathbb{C}^n \times \mathbb{R}$ 
\[
    (z_1, \cdots, z_n, t) \cdot (z'_1, \cdots, z'_n, t') = (z_1 + z'_1, \cdots, z_n + z'_n, t + t' + 2\text{Im} \left( \sum_{i=1}^{n} z_i z'_i \right)),
\]
whose Lie algebra $h = V_1 + V_2$ with $V_1 = \text{span}\{X_i, Y_i\}_{1 \leq i \leq n}$ and $V_2 = \text{span}\{T\}$, where 
\[
    T = 4 \frac{\partial}{\partial t}, \quad X_i = \frac{\partial}{\partial x_i} - 2y_i \frac{\partial}{\partial t}, \quad Y_i = \frac{\partial}{\partial y_i} + 2x_i \frac{\partial}{\partial t}, \quad 1 \leq i \leq n.
\]

Recall that a function $f \in C^2(R^m)$ is strictly convex if there is a $C_0 > 0$ such that $D^2 f(x) \geq C_0$ for any $x \in R^m$. Now we are ready to state the uniqueness theorem.

**Theorem C.** Let $G$ be a Carnot group and let $\Omega \subset G$ be a bounded domain. Assume that $f \in C^2(R^m, R_+)$ is strictly convex, homogeneous of degree $\alpha \geq 1$, and $f(p) > 0$ for $p \neq 0$. Then, for any $\phi \in W^{1,\infty}_X(\Omega)$, the Dirichlet problem 
\[
    A(Xu, (D^2 u)^\alpha) := -\sum_{i,j=1}^{m} f_{p_i}(Xu)f_{p_j}(Xu)X_i X_j u = 0, \quad \text{in} \ \Omega,
\]
\[
    u = \phi, \quad \text{on} \ \partial \Omega,
\]
has at most one viscosity solution in $C(\bar{\Omega})$.
under small perturbations. (iv) Finally, we apply the comparison principle for the
two auxiliary equations to prove the uniqueness of (1.13).

As a consequence of Theorems A and C, we have

**Corollary D.** Let $G$ be a Carnot group and let $\Omega \subset G$ be a bounded domain.
Assume that $f \in C^2(R^m, R_+)$ is strictly convex, homogeneous of degree $\alpha \geq 1,$
and $f(p) > 0$ for $p \neq 0.$ Then, for any $\phi \in W^{1,\infty}_X(\Omega),$ there is a unique ab-
solute minimizer $u \in W^{1,\infty}_X(\Omega),$ with $u|_{\partial \Omega} = \phi,$ to the functional $F(v, \Omega) = \text{ess sup}_{x \in \Omega} f(Xv(x)),$ and (1.13) has a unique viscosity solution in $C(\overline{\Omega}).$ In partic-
ular, $\phi$ has a unique AMLE in $W^{1,\infty}_X(\Omega)$ and the subelliptic $\infty$-Laplacian equation
(1.8) has a unique viscosity solution.

We would like to remark that our method is of euclidean nature, which is con-
siderably different from that by [B1, B2] and [FM], where a subelliptic approach
was employed.

The paper is written as follows. In §2, we outline the proof of Theorem A. In
§3, we discuss the sup/inf convolution construction on any Carnot group $G.$ In
§4, we discuss the comparison principle between semiconvex subsolutions and
strict semiconcave supersolutions to any horizontally elliptic equations. In §5, we study
two auxiliary equations to (1.13), with horizontal gradient constraints. In §6, we
prove Theorem C.

### §2. Proof of Theorem A

This section is devoted to the proof of Theorem A. It consists of two steps: (i)
the construction of test functions by solving the Hamilton-Jacobi equation, moti-
vated by [BJW] and [C], and (ii) the comparison between viscosity subsolution and
classical strict supersolution of the Hamilton-Jacobi equation.

**Proof of Theorem A.** It suffices to prove that if $u$ is not a viscosity subsolution of
(1.12) at the point $x = 0 \in \Omega,$ then $u$ is not an absolute minimizer of $F(\cdot, \Omega).$ This
assumption implies that there exists $r_0 > 0$ and $\phi \in C^2(\Omega)$ such that $B_{r_0}(0) \subset \subset \Omega,$ and

\begin{equation}
0 = u(0) - \phi(0) \geq u(x) - \phi(x), \quad \forall x \in \Omega,
\end{equation}

but

\begin{equation}
- \sum_{i=1}^m X_i (f(x, X\phi(x))) f_{pi}(x, X\phi(x)) |_{x=0} = C_0 > 0.
\end{equation}

Now we have

**Lemma 2.1.** There exist a neighborhood $V$ of 0 and a function $\Phi \in C^2(V)$ such that

\begin{equation}
\Phi(0) = \phi(0), \quad D\Phi(0) = D\phi(0), \quad D^2\Phi(0) > D^2\phi(0),
\end{equation}

and

\begin{equation}
f(x, X\Phi(x)) = f(0, X\phi(0)) > 0, \quad \forall x \in V.
\end{equation}
Proof: Since \( \{X_i\}_{i=1}^m \subset C^2(\Omega, R^n) \), there is a matrix-valued function \((a_{ij}) \in C^2(\Omega, R^{mn})\) such that for \(1 \leq i \leq m\),
\[
X_i(x) = \sum_{j=1}^n a_{ij}(x) \frac{\partial}{\partial x_j}, \quad \forall x \in \Omega.
\]

Define \(\bar{f} : \Omega \times R^n \to R\) by
\[
\bar{f}(x, q_1, \ldots, q_n) = f(x, \sum_{j=1}^n a_{1j}(x)q_j, \ldots, \sum_{j=1}^n a_{mj}(x)q_j), \quad \forall (x, q_1, \ldots, q_n) \in \Omega \times R^n.
\]
Since \(f \in C^2(\Omega \times R^n)\) and \((a_{ij}) \in C^2(\Omega, R^{mn})\), it is easy to see that \(\bar{f} \in C^2(\Omega \times R^n)\).

Moreover, for any \((x, q) \in \Omega \times R^n\) and \(1 \leq i \leq n\), we have
\[
\frac{\partial \bar{f}}{\partial q_i}(x, q) = \sum_{k=1}^m a_{ki}(x) \frac{\partial f}{\partial p_k}(x, \sum_{j=1}^n a_{1j}(x)q_j, \ldots, \sum_{j=1}^n a_{mj}(x)q_j)
\]
and
\[
(2.5) \quad \bar{f}(x, D\phi(x)) = f(x, X_1\phi(x), \ldots, X_m\phi(x)) = f(x, X\phi(x)), \quad \forall x \in \Omega.
\]
Therefore, for any \(x \in \Omega\), we have
\[
(2.6) \quad \sum_{j=1}^m X_j(f(x, X\phi(x)))\bar{f}_{p_j}(x, X\phi(x)) = \sum_{i=1}^n \frac{\partial}{\partial x_i}(\bar{f}(x, D\phi(x))) \frac{\partial f}{\partial q_i}(x, D\phi(x)).
\]
This, combined with (2.2), implies
\[
(2.7) \quad A(0, D\phi(0), D^2\phi(0)) := -\sum_{i=1}^n \frac{\partial}{\partial x_i}(\bar{f}(x, D\phi)) \frac{\partial f}{\partial q_i}(x, D\phi)(0) = C_0 > 0.
\]

Now the argument in [C], pages 275-276, implies that there exist a neighborhood \(V\) of 0 and a function \(\Phi \in C^2(V)\) such that
\[
(2.8) \quad \Phi(0) = \phi(0), \quad D\Phi(0) = D\phi(0), \quad D^2\Phi(0) > D^2\phi(0),
\]
\[
(2.9) \quad \bar{f}(x, D\Phi(x)) = \bar{f}(0, D\phi(0)), \quad \forall x \in V.
\]

Now (2.8) and (2.9), combined with (2.5), give (2.4). For the convenience of the readers, we briefly sketch Crandall’s argument as follows. Let \(E = D_x\bar{f}(0, D\phi(0))\), \(F = -D_{\bar{f}}\bar{f}(0, D\phi(0))\), and \(G = E - D^2\phi(0)F\). Then (2.7) becomes \(G \cdot F > 0\). It is easy to check that \(X = P + D^2\phi(0)\), where \(P\) is given by
\[
PZ = \frac{Z \cdot G}{F \cdot G} + Z - \frac{Z \cdot F}{|F|^2} F, \quad \forall Z \in R^n,
\]
has the property that \(X > D^2\phi(0)\) and \(E - XF = 0\). We now define
\[
\psi(x) = \phi(0) + x \cdot D\phi(0) + \frac{1}{2} (XX \cdot x)
\]
and the hyperplane
\[
\Sigma = \{x \in R^n | x \cdot \{D_x\bar{f}(0, D\phi(0)) + D^2\phi(0)D_\eta\bar{f}(0, D\phi(0))\} = 0\},
\]
and solve the Cauchy problem
\[
\bar{f}(x, D\Phi(x)) = \bar{f}(0, D\phi(0)) \quad \text{and} \quad \Phi = \psi \text{ on } \Sigma
\]
such that
\[
D\Phi(0) = D\phi(0).
\]
in a neighborhood of 0. Since $\Sigma$ is not characteristic near 0, we have a local $C^2$ solution $\Phi$ to the Cauchy problem. The classical theory also implies that $D^2\Phi(0) = X > D^2\phi(0)$. This proves (2.8) and (2.9).

To see $f(0, X\phi(0)) > 0$, we observe that (2.2) implies

$$f_p(0, X\phi(0)) \equiv (\frac{\partial f}{\partial p_1}(0, X\phi(0)), \ldots, \frac{\partial f}{\partial p_m}(0, X\phi(0))) \neq 0.$$ 

This implies $X\phi(0) \neq 0$, for $f_p(0, 0) = 0$. Note that the homogeneity of $f$ implies that if $f(0, p) = 0$, then $p = 0$. Therefore, $f(0, X\phi(0)) > 0$. This finishes the proof of Lemma 2.1.

It follows from Lemma 2.1 that there exists an open neighborhood $V_1 \subset V$ of 0 such that $\Phi(0) = \phi(0) = u(0)$, and $\Phi(x) > \phi(x) \geq u(x)$ for any $0 \neq x \in V_1$. Therefore, for any small $\epsilon > 0$, there exists another neighborhood $V_\epsilon \subset V_1$ of 0 such that

$$\Phi(x) - \epsilon < u(x), \quad \forall x \in V_\epsilon; \quad \Phi(x) - \epsilon = u(x), \quad \forall x \in \partial V_\epsilon.$$ 

It follows from the absolute minimality of $u$ to $F(\cdot, \Omega)$ that

$$F(u, V_\epsilon) \leq F(\Phi - \epsilon, V_\epsilon) = \text{ess sup}_{x \in V_\epsilon} f(x, X\Phi(x)) = f(0, X\phi(0)).$$ 

Now we want to show that $u$ is a viscosity subsolution of the Hamilton-Jacobi equation (2.4) on $V_\epsilon$. For the reader’s convenience, we first recall from [CIL] the definition of viscosity solution to (2.4).

**Definition 2.2.** A function $v \in C(V_\epsilon)$ is a viscosity subsolution to the Hamilton-Jacobi equation

$$\bar{f}(x, Dv(x)) = K, \quad x \in V_\epsilon,$$

where $K > 0$ is a constant, if for any $(x, \phi) \in V_\epsilon \times C^1(V_\epsilon)$ such that $\phi$ touches $v$ at $x$ from above, we have

$$f(x, D\phi(x)) \leq K.$$ 

We remark that if $v \in C^1(V_\epsilon)$ is a classical subsolution to (2.12), then $v$ is also a viscosity subsolution to (2.12).

We also recall the following compactness theorem and a comparison principle for (2.12).

**Proposition 2.3 (CIL).** (1) Assume that $\{v_k\} \subset C(V_\epsilon)$ are a family of viscosity subsolutions to (2.12) with $K = C_k$. If $v_k \to v$ uniformly on $V_\epsilon$ and $C_k \to C > 0$, then $v \in C(V_\epsilon)$ is a viscosity subsolution to (2.12) with $K = C$.

(2) If $v \in C(V_\epsilon)$ is a viscosity subsolution to (2.12) and $w \in C^1(V_\epsilon)$ is a strict, classical supersolution to (2.12), i.e.

$$\bar{f}(x, Dw(x)) > K, \quad \forall x \in V_\epsilon,$$

then we have

$$\sup_{V_\epsilon} (v - w) = \sup_{\partial V_\epsilon} (v - w).$$

Now we have

**Lemma 2.4.** Under the same notations as above, $u \in W^{1,\infty}_X(V_\epsilon)$ is a viscosity subsolution to the Hamilton-Jacobi equation (2.4).
Proof: For any subdomain $U \subset \subset V$, and $0 < \delta < \text{dist}(U, \partial V)$, let $g_\delta : U \to R$ be the standard $\delta$-mollifier of $g$ for any function $g$ on $V$. Since $u \in W^{1,\infty}_x(V)$, $u_\delta$ converges uniformly to $u$ on $U$ as $\delta \to 0$. Since (f1) implies $f(x, \cdot)$ is quasiconvex, it follows from the Jensen inequality for quasiconvex functions (cf. [BJW], Theorem 1.1) that for any $x \in U$, we have

$$f(x, (Xu)_\delta(x)) \leq \text{ess sup}_{y \in B_\delta(x)} f(x, Xu(y)) \leq \text{ess sup}_{y \in B_\delta(x)} f(x, Xu(y)) \leq F(u, V_e) + C\delta,$$

where $C = \text{sup}\{|D_x f(x, p) : x \in U, |p| \leq 2\|Xu\|_{L^\infty(U)}\} < \infty$. Therefore we have

$$\max_{x \in U} f(x, (Xu)_\delta(x)) \leq f(0, X\phi(0)) + C\delta.$$

On the other hand, for any $1 \leq i \leq m$ and $x \in U$, we estimate $(X_iu)_\delta(x) - X_i(u_\delta)(x)$ as follows:

$$(X_iu)_\delta(x) - X_i(u_\delta)(x)$$

$$= \int_{R^n} \eta_\delta(x - y) \left( \sum_{j=1}^n a_{ij}(y) \frac{\partial}{\partial y_j} \right) (u(y) - u(x)) \, dy$$

$$- \int_{R^n} \sum_{j=1}^n a_{ij}(x) \frac{\partial \eta_\delta(x - y)}{\partial x_j} (u(y) - u(x)) \, dy$$

$$= \sum_{j=1}^n \int_{R^n} \left\{ - \frac{\partial}{\partial y_j} (a_{ij}(y) \eta_\delta(x - y)) - a_{ij}(x) \frac{\partial (\eta_\delta(x - y))}{\partial x_j} \right\} (u(y) - u(x)) \, dy$$

$$= \sum_{j=1}^n \int_{R^n} \left\{ a_{ij}(x) \frac{\partial \eta_\delta(x - y)}{\partial x_j} (u(y) - u(x)) \, dy$$

$$+ \sum_{j=1}^n \int_{R^n} \frac{\partial a_{ij}(y)}{\partial y_j} \eta_\delta(x - y) (u(y) - u(x)) \, dy.$$

Therefore we have, for any $x \in U$,

$$|(X_iu)_\delta(x) - X_i(u_\delta)(x)| \leq C \max_{1 \leq j \leq n} \|D a_{ij}\|_{L^\infty(U)} \int_{R^n} \{ \eta_\delta(x - y) \} |u(y) - u(x)|$$

$$+ |y - x| \|D \eta_\delta(x - y)\|_{L^\infty(U)} \|u(y) - u(x)\| \, dy$$

$$\leq C \|X_i\|_{C^1(U)} \sup_{y - x \leq \delta} |u(y) - u(x)|$$

$$\leq C \|X_i\|_{C^1(U)} \|u\|_{W^{1,\infty}(U)} \omega(\delta),$$

where $\omega(\delta) = \sup\{d_X(x, y) : \|y - x\| \leq \delta\}$ is the modular of continuity of $d_X$ with respect to the euclidean distance $\| \cdot \|$. This implies

$$f(x, (Xu)_\delta(x)) \leq \text{ess sup}_{x \in U} f(x, (Xu)_\delta(x)) \leq f(0, X\phi(0)) + \|f\|_{L^\infty(U)} \|Xu_\delta\|_{L^\infty(U)}$$

$$\leq f(0, X\phi(0)) + C(\delta + \omega(\delta)), \quad \forall x \in U.$$

Hence $u_\delta$ is a classical subsolution to (2.12), with

$$K = K_\delta \equiv f(0, X\phi(0)) + C(\delta + \omega(\delta)),$$

on $U$. Since (1.4) implies $\lim_{\delta \to 0} \omega(\delta) = 0$, we have that $K_\delta \to f(0, X\phi(0))$ as $\delta \to 0$. Note also that $u_\delta \to u$ uniformly on $U$, and Proposition 2.3(1) implies
that $u$ is a viscosity subsolution to (2.4) in $U$. Since $U$ exhausts $V_\epsilon$ as $\delta \to 0$, we have that $u$ is a viscosity subsolution of (2.4) in $V_\epsilon$. The proof of Lemma 2.4 is complete. \hfill \Box

Now we return to the proof of Theorem A. It follows from (f2) that

$$f(x, (1 + t)p) = (1 + t)^{\alpha} f(x, p) = (1 + g(t)) f(x, p), \forall t > 0, \forall (x, p) \in \Omega \times \mathbb{R}^n,$$

where $g(t) \equiv (1 + t)^{\alpha} - 1 > 0$ for $t > 0$ and $\alpha \geq 1$. This, combined with (2.8), implies that for any $t > 0$ we have

$$(2.17) \quad f(x, X((1 + t)\Phi_\epsilon)(x)) = (1 + g(t))f(0, X\phi(0)) = f(0, X\phi(0)) + \delta(t), \forall x \in V_\epsilon,$$

where $\Phi_\epsilon \equiv \Phi - \epsilon$ and $\delta(t) = g(t)f(0, X\phi(0)) > 0$. Therefore, for any $t > 0$, $(1 + t)\Phi_\epsilon$ is a strict, classical supersolution of (2.4). Therefore, Proposition 2.3(2) implies

$$(2.18) \quad \sup_{V_\epsilon}(u - (1 + t)\Phi_\epsilon) \leq \sup_{\partial V_\epsilon}(u - (1 + t)\Phi_\epsilon), \forall t > 0.$$

Taking $t$ into zero, we have

$$\sup_{V_\epsilon}(u - \Phi_\epsilon) \leq \sup_{\partial V_\epsilon}(u - \Phi_\epsilon) = 0.$$

This implies

$$u(x) \leq \Phi_\epsilon(x), \forall x \in V_\epsilon.$$

This clearly contradicts with (2.10). Therefore the proof of Theorem A is complete. \hfill \Box

\section{3. The construction of sup/inf convolutions on G}

This section is devoted to the construction of sup/inf convolutions on any Carnot group $G$, which is the necessary extension of Jensen-Lions-Souganidis [JLS] and plays an important role in the proof of Theorem C.

Let $\Omega \subset G$ be a bounded domain and let $d : G \times G \to R_+$ be the gauge distance defined in §1. For any $\epsilon > 0$, define

$$\Omega_\epsilon = \{x \in \Omega \mid \inf_{y \in G \setminus \Omega} d(x^{-1}, y^{-1})^{2r} \geq \epsilon\}.$$

\textbf{Definition 3.1}. For any $u \in C(\Omega)$ and $\epsilon > 0$, the sup convolution $u_\epsilon$ of $u$ is defined by

$$(3.1) \quad u_\epsilon(x) = \sup_{y \in \Omega} \{u(y) - \frac{1}{2\epsilon} d(x^{-1}, y^{-1})^{2r}\}, \forall x \in \Omega.$$

Similarly, the inf convolution $v_\epsilon$ of $v \in C(\bar{\Omega})$ is defined by

$$(3.2) \quad v_\epsilon(x) = \inf_{y \in \Omega} \{v(y) + \frac{1}{2\epsilon} d(x^{-1}, y^{-1})^{2r}\}, \forall x \in \Omega.$$

For $x \in G$, let $\|x\| := (\sum_{j=1}^{r} \sum_{i=1}^{m_j} x_{ij}^2)^{\frac{1}{2}}$ be the euclidean norm. We recall the notion of semiconvexity in the euclidean sense.

\textbf{Definition 3.2}. A function $u \in C(\Omega)$ is semiconvex if there is a constant $C > 0$ such that $u(x) + C\|x\|^2$ is convex; $u$ is semiconcave if $-u$ is semiconvex. Note that if $u \in C^2(\Omega)$ satisfies $D^2u(x) \geq -C$ for $x \in \Omega$, then $u$ is semiconvex. Here $D^2u = (\frac{\partial^2 u}{\partial x_i \partial x_j})_{1 \leq i, j \leq n}$ is the euclidean hessian of $u$.

We now have the generalized version of [JLS].
Proof. Since the proof for \( \epsilon \) w.r.t. \( x \) for any \( \Omega \) functions is convex, we have that

\[
\Omega \subset C(\bar{\Omega}),
\]

Then, for any \( \epsilon > 0 \), \( u^\epsilon, v^\epsilon \in W^{1,\infty}_x(\Omega) \) satisfy:

1. \( u^\epsilon \) is semiconvex and \( v^\epsilon \) is semiconcave.
2. \( u^\epsilon \) (or \( v^\epsilon \), resp.) is monotonically nondecreasing (or nonincreasing, resp.) w.r.t. \( \epsilon \) and converges uniformly to \( u \) on \( \Omega_{(1+4R_0)\epsilon} \).
3. If \( u \) (or \( v \), resp.) is a viscosity subsolution (or supersolution, resp.) to the horizontally elliptic equation

\[
B(Xu, (D^2u)^*) = 0 \text{ in } \Omega,
\]

then \( u^\epsilon \) (or \( v^\epsilon \), resp.) is a viscosity subsolution (or supersolution, resp.) to (3.3) in \( \Omega_{(1+4R_0)\epsilon} \).

Proof. Since the proof for \( v^\epsilon \) is parallel to that for \( u^\epsilon \), we only consider \( u^\epsilon \).

1. Since \( \Omega \subset \mathcal{G} \) is bounded, it is easy to see from the formula of \( d \) that

\[
C_d(\Omega) \equiv \|D^2_x\{d(x^{-1}, y^{-1})^{2r}\}\|_{L^\infty(\bar{\Omega} \times \Omega)} < \infty.
\]

Therefore, for any \( y \in \bar{\Omega} \), the function

\[
\tilde{u}^\epsilon_y(x) := u(y) - \frac{1}{2\epsilon}d(x^{-1}, y^{-1})^{2r} + \frac{C_d(\Omega)}{2\epsilon}|x|^2, \forall x \in \Omega,
\]

has nonnegative hessian and is convex. Since the supremum of a family of convex functions is convex, we have that

\[
u^\epsilon(x) + \frac{C_d(\Omega)}{2\epsilon}|x|^2 = \sup_{y \in \Omega} \tilde{u}^\epsilon_y(x)
\]

is convex so that \( \nu^\epsilon \) is semiconvex. Since any semiconvex function is Lipschitz with respect to the euclidean metric, we have \( u^\epsilon \in W^{1,\infty}_x(\Omega) \).

2. It is easy to see that for any \( \epsilon_1 \leq \epsilon_2 \), \( u^\epsilon_1(x) \leq u^\epsilon_2(x) \) and \( u(x) \leq u^\epsilon(x) \leq R_0 \) for any \( x \in \Omega \). Observe that for any \( x \in \Omega \) we have

\[
u^\epsilon(x) = \sup_{\Omega \cap \{d(x^{-1}, y^{-1})^{2r} \leq 4R_0\epsilon\}} (u(y) - \frac{1}{2\epsilon}d(x^{-1}, y^{-1})^{2r}).
\]

Therefore, for any \( x \in \Omega_{(1+4R_0)\epsilon} \), \( u^\epsilon(x) \) is attained at a point \( y \in \Omega \). To see \( u^\epsilon \to u \) uniformly on \( \Omega_{(1+4R_0)\epsilon} \), we observe that if \( u^\epsilon(x) \) is attained by \( x_\epsilon \), then

\[
u^\epsilon(x) \geq u(x_\epsilon) - \frac{1}{\epsilon}d(x^{-1}, x_\epsilon^{-1})^{2r} = u_\epsilon(x) - \frac{1}{2\epsilon}d(x^{-1}, x_\epsilon^{-1})^{2r}.
\]

This implies

\[
\lim_{\epsilon \to 0} \frac{1}{\epsilon}d(x^{-1}, x_\epsilon^{-1})^{2r} = 0,
\]

so that \( x_\epsilon \to x \) and \( \lim_{\epsilon \to 0} u^\epsilon(x) = u(x) \). Since

\[
|u^\epsilon(x_1) - u^\epsilon(x_2)| \leq |u(x_1) - u(x_2)|, \forall x_1, x_2 \in \Omega,
\]

\( u^\epsilon \to u \) uniformly on \( \Omega_{(1+4R_0)\epsilon} \).

3. For any \( x_0 \in \Omega_{(1+4R_0)\epsilon} \), let \( \phi \in C^2(\Omega_{(1+4R_0)\epsilon}) \) be such that

\[
u^\epsilon(x_0) - \phi(x_0) \geq u^\epsilon(x) - \phi(x), \forall x \in \Omega_{(1+4R_0)\epsilon}.
\]

It follows from the proof of (2) above that there exists a \( y_0 \in \Omega \) such that

\[
u^\epsilon(x_0) = u(y_0) - \frac{1}{2\epsilon}d(x_0^{-1}, y_0^{-1})^{2r}.
\]
Therefore we have
\[ u(y_0) - \frac{1}{2\epsilon}d(x_0^{-1}, y_0^{-1})^{2r_1} - \phi(x_0) \geq u(y) - \frac{1}{2\epsilon}d(x^{-1}, y^{-1})^{2r_1} - \phi(x), \forall x, y \in \Omega_{(1+4R_0)\epsilon}. \]
For \( y \) near \( y_0 \), we have \( x = x_0 \cdot y_0^{-1} \cdot y \in \Omega_{(1+4R_0)\epsilon} \) and hence
\[ u(y_0) - \phi(x_0 \cdot y_0^{-1} \cdot y_0) \geq u(y) - \phi(x_0 \cdot y_0^{-1} \cdot y). \]
Set \( \tilde{\phi}(y) = \phi(x_0 \cdot y_0^{-1} \cdot y) \) for \( y \in \Omega_{(1+4R_0)\epsilon} \) near \( y_0 \). Then \( \tilde{\phi} \) touches \( u \) from above at \( y = y_0 \) so that we have
\[ (3.4) \quad B(X\tilde{\phi}, (D^2\tilde{\phi})^*)(y_0) \leq 0. \]

It follows from the left invariance of \( X \) that
\[ X\tilde{\phi}(y) = (X\phi)(x_0 \cdot y_0^{-1} \cdot y), \quad (D^2(\tilde{\phi}))^*(y) = (D^2\phi)^*(x_0 \cdot y_0^{-1} \cdot y). \]
This implies
\[ B(X\phi(x_0), (D^2\phi)^*(x_0)) \leq 0. \]
Hence \( u^\epsilon \) is a viscosity subsolution of (3.3) on \( \Omega_{(1+4R_0)\epsilon} \). The proof is complete. \( \Box \)

§4. COMPARISON PRINCIPLE BETWEEN SEMICONVEX SUBSOLUTIONS AND SEMICONCAVE SUPERSOLUTIONS

In this section, we establish the comparison principle between semiconvex subsolutions and semiconcave strict supersolutions for any second order horizontally elliptic PDE on the Carnot-Carathéodory space. The argument is inspired by Jensen’s maximum principle for semiconvex functions (see [11, 12]) on second order elliptic PDEs. Here we assume that \( \{X_i = \sum_{j=1}^m a_{ij}(x) \frac{\partial}{\partial x_j}\}_{i=1}^m \) are \( C^1 \) vector fields on \( \mathbb{R}^n \). The main proposition of this section is

**Proposition 4.1.** For any bounded domain \( \Omega \subset \mathbb{R}^n \), suppose that \( B \in C(\Omega \times S^m) \) is horizontally elliptic. Assume that \( u \in C(\Omega) \) is a semiconvex subsolution to
\[ (4.1) \quad B(Xw, (D^2w)^*) = 0 \quad \text{in} \quad \Omega, \]
and \( v \in C(\Omega) \) is a semiconcave supersolution to
\[ (4.2) \quad B(Xw, (D^2w)^*) - \mu = 0 \quad \text{in} \quad \Omega \]
for some \( \mu > 0 \). Then
\[ (4.3) \quad \sup_{\Omega}(u - v) \leq \sup_{\partial\Omega}(u - v). \]

**Proof.** Suppose that (4.3) were false. Then
\[ \sup_{\Omega}(u - v) > \sup_{\partial\Omega}(u - v). \]
Hence \( u - v \) achieves its maximum on \( \partial\Omega \) at a point \( x_0 \in \Omega \). Since \( u - v \) is semiconvex, it is well known (cf. [12], page 67) that
\[ Du(x_0), Dv(x_0) \, \text{both exist and are equal}, \]
\[ u(x) - u(x_0) - \langle Du(x_0), x - x_0 \rangle = O(||x - x_0||^2), \]
\[ v(x) - v(x_0) - \langle Dv(x_0), x - x_0 \rangle = O(||x - x_0||^2), \]
where \( \langle \cdot, \cdot \rangle \) and \( || \cdot || \) denote the Euclidean inner product and norm. Denote \( R_0 = \inf\{||x_0 - y|| : y \in \partial\Omega\} \) and let \( R_1 > 0 \) be such that both (4.4) and (4.5) hold with
\|x - x_0\| < R_1. Set R_2 = \min\{R_0, R_1\} > 0 and define, for any \( \rho > 0 \), the rescaled maps \( u^\rho, v^\rho : B_{R_2\rho^{-1}} = \{ x \in \mathbb{R}^n ||x|| \leq R_2\rho^{-1} \} \rightarrow \mathbb{R} \) by

\[
\begin{align*}
u^\rho(x) &= \frac{1}{\rho^2}(u(x_0 + \rho x) - u(x_0)) - \rho(Du(x_0), x), \\
u^\rho(x) &= \frac{1}{\rho^2}(v(x_0 + \rho x) - v(x_0)) - \rho(Dv(x_0), x),
\end{align*}
\]

where the addition and scalar multiplication are the euclidean ones. Then it is easy to see that

\[
0 = (u^\rho - v^\rho)(0) \geq (u^\rho - v^\rho)(x), \quad \forall x \in B_{R_2\rho^{-1}}.
\]

It follows from (4.4) and (4.5) that, for any \( R > 0 \), there exists a \( \rho_0 = \rho_0(R) > 0 \) such that (i) \( \{u^\rho\}_{0 < \rho \leq \rho_0} \) are uniformly bounded, uniformly semiconcave, and uniformly Lipschitz continuous in \( B_R \); (ii) \( \{v^\rho\}_{0 < \rho \leq \rho_0} \) are uniformly bounded, uniformly semiconcave, and uniformly Lipschitz continuous in \( B_R \). Therefore, by the Cauchy diagonal process, we find that there is \( \rho_i \downarrow 0 \) such that \( u^{\rho_i} \rightarrow u^\ast, v^{\rho_i} \rightarrow v^\ast \) locally uniformly in \( \mathbb{R}^n \). Moreover, it is not difficult to see that \( u^\ast \) is locally bounded, semiconvex in \( \mathbb{R}^n \), \( v^\ast \) is locally bounded, semiconcave in \( \mathbb{R}^n \), and

\[
0 = (u^\ast - v^\ast)(0) \geq (u^\ast - v^\ast)(x), \quad \forall x \in \mathbb{R}^n.
\]

Now we need

**Claim 4.2.** \( u^\ast \) is a viscosity subsolution to

\[
B_1(D^2u) = 0 \quad \text{in} \quad \mathbb{R}^n,
\]

and \( v^\ast \) is a viscosity supersolution to

\[
B_2(D^2v) + \mu = 0 \quad \text{in} \quad \mathbb{R}^n,
\]

where \( B_1, B_2 : \mathcal{S}^n \rightarrow \mathbb{R} \) are defined by

\[
\begin{align*}
B_1(M) &= B(Xu(x_0), \{ \sum_{k,l=1}^n (a_{ik}(x_0)a_{jl}(x_0)M_{kl} + a_{ik}(x_0)\partial a_{jl}(x_0)\partial u(x_0)) \}_{1 \leq i,j \leq m}), \\
B_2(M) &= B(Xv(x_0), \{ \sum_{k,l=1}^n (a_{ik}(x_0)a_{jl}(x_0)M_{kl} + a_{ik}(x_0)\partial a_{jl}(x_0)\partial v(x_0)) \}_{1 \leq i,j \leq m}).
\end{align*}
\]

Before we prove this claim, we would like to remark that the horizontal ellipticity (1.9) of \( B \) implies that \( B_1, B_2 \in C(\mathcal{S}^n) \) is elliptic in the sense that

\[
B_i(M) \leq B_i(N), \quad \forall M, N \in \mathcal{S}^n, \quad \text{with} \quad M \geq N
\]

for \( i = 1, 2 \). This is also a key observation we need to use in the following proof.

To prove it, we first observe that \( u^\rho \) is a viscosity subsolution to

\[
B(Xu(x_0) + \rho X^\rho w, \{ \sum_{k,l=1}^n a_{ik}^\rho a_{jl}^\rho \partial^2 w_{\partial x_k \partial x_l} + a_{ik}^\rho (\partial a_{jl}^\rho)\rho(\partial u(x_0) + \rho \partial w(x_0)) \}_{1 \leq i,j \leq m}) = 0
\]

and \( v^\rho \) is a viscosity supersolution to

\[
B(Xv(x_0) + \rho X^\rho w, \{ \sum_{k,l=1}^n a_{ik}^\rho a_{jl}^\rho \partial^2 w_{\partial x_k \partial x_l} + a_{ik}^\rho (\partial a_{jl}^\rho)\rho(\partial v(x_0) + \rho \partial w(x_0)) \}_{1 \leq i,j \leq m}) = 0
\]
where $X^\rho = (X^\rho_1, \ldots, X^\rho_n)$, $X^\rho_i(x) = X_i(x_0 + \rho x)$, $a^\rho_{ik}(x) = a_{ik}(x_0 + \rho x)$, and $(\frac{\partial u}{\partial x_k})^\rho(x) = \frac{\partial a_{ik}}{\partial x_k}(x_0 + \rho x)$. For simplicity, we only indicate (4.8). Let $x_1 \in B_{R_2\rho^{-1}}$ and $\phi \in C^2(B_{R_2\rho^{-1}})$ be such that

$$0 = u^\rho(x_1) - \phi(x_1) \geq u^\rho(x) - \phi(x), \ \forall x \in B_{R_2\rho^{-1}}.$$ 

It is straightforward to see that

$$\phi^\rho(x) \equiv u(x_0) + \langle Du(x_0), x - x_0 \rangle + \rho^2 \phi\left(\frac{x - x_0}{\rho}\right)$$

satisfies

$$0 = u(x_0 + \rho x_1) - \phi^\rho(x_0 + \rho x_1) \geq u(x) - \phi^\rho(x), \ \forall x \in B_{R_0}(x_0).$$

This, combined with the fact that $u$ is a viscosity subsolution to (4.6), implies (4.10)

$$B(X\phi^\rho, (D^2\phi^\rho)^\ast)(x_0 + \rho x_1) \geq 0.$$

Direct calculations yield

$$\frac{\partial \phi^\rho}{\partial x_k}(x) = \frac{\partial u}{\partial x_k}(x_0) + \rho \frac{\partial \phi}{\partial x_k}(x - x_0), \quad \frac{\partial^2 \phi^\rho}{\partial x_k \partial x_l}(x) = \frac{\partial^2 \phi}{\partial x_k \partial x_l}(x - x_0).$$

Hence (4.10) implies that $u^\rho$ is a viscosity solution to (4.8). Since $X^\rho \to X(x_0)$, $a^{\rho}_{ij} \to a_{ij}(x_0)$, and $(\frac{\partial a_{ij}}{\partial x_k})^\rho \to \frac{\partial a_{ij}}{\partial x_k}(x_0)$ locally uniformly on $\mathbb{R}^n$, we observe that the family of elliptic operator $B^\rho : \mathbb{R}^n \times \mathbb{R}^n \times S^m \to \mathbb{R}$ defined by

$$B^\rho(x, p, M) := B((X_i u(x_0) + \rho \sum_{j=1}^n a^\rho_{ij}(x)p_j)_{1 \leq i \leq n}, \{ \sum_{k,l=1}^n a^\rho_{ik}(x)a^\rho_{jl}(x)M_{kl} + a^\rho_{ik}(x)(\frac{\partial a_{jl}}{\partial x_k})^\rho(x)(\frac{\partial u}{\partial x_l}(x_0) + \rho p_l) \}_{1 \leq i, j \leq m})$$

converges to $B_1(M)$ locally uniformly as $\rho \to 0$. Therefore the compactness theorem for viscosity subsolutions (see, e.g. Caffarelli-Cabré [CC], proposition 2.9, page 14) implies that $u^* \ast$ is a viscosity subsolution of (4.6). This proves Claim 4.2.

Since $u^* \ast v^* \ast$ is semiconvex and achieves its maximum at $x = 0$, we can apply Jensen’s maximum principle for semiconvex functions (see [11] and also [CIL], Lemma A.3, page 60) to conclude that there exists $x^*_s \in \mathbb{R}^n$ such that $D^2u^* \ast(x^*_s), D^2v^* \ast(x^*_s)$ both exist and $D^2(u^* \ast - v^* \ast)(x^*_s) \leq 0$. Let $M_1, M_2 : S^m \to \mathbb{R}$ be given by

$$M^i_{ij} = \sum_{k,l=1}^n \{ a_{ik}(x_0) a_{jl}(x_0) \frac{\partial^2 u^*}{\partial x_k \partial x_l}(x^*_s) + a_{ik}(x_0) \frac{\partial a_{jl}}{\partial x_k}(x_0) \frac{\partial u}{\partial x_l}(x_0) \}, 1 \leq i, j \leq m,$$

$$M^i_{ij} = \sum_{k,l=1}^n \{ a_{ik}(x_0) a_{jl}(x_0) \frac{\partial^2 v^*}{\partial x_k \partial x_l}(x^*_s) + a_{ik}(x_0) \frac{\partial a_{jl}}{\partial x_k}(x_0) \frac{\partial v}{\partial x_l}(x_0) \}, 1 \leq i, j \leq m.$$ 

Since $Du(x_0) = Dv(x_0)$, we have $Xu(x_0) = Xv(x_0)$ and, for any $p \in \mathbb{R}^m$,

$$\sum_{1 \leq i, j \leq m} (M^i_{ij} - M^i_{ij}^2)p_ip_j = \sum_{k,l=1}^n \{ \sum_{i=1}^m p_i a_{ik}(x_0) \} \{ \sum_{j=1}^m p_j a_{ij}(x_0) \} \frac{\partial^2 (u^* - v^* \ast)(x^*_s)}{\partial x_k \partial x_l} \leq 0.$$ 

Hence $M_1 \leq M_2$. Therefore the horizontal ellipticity of $B$ implies

$$B(Xu(x_0), M_1) - B(Xv(x_0), M_2) \geq 0.$$
On the other hand, we have
\begin{equation}
B_1(D^2u^*(x_0)) - B_2(D^2v^*(x_0)) = B(Xu(x_0), M_1) - B(Xv(x_0), M_2) \leq -\mu < 0.
\end{equation}
This contradicts (4.11). Hence the proof of Proposition 4.1 is complete. \hfill \Box

§5. **Auxiliary equations with horizontal gradient constraints**

Due to the degeneracy of (1.13), we establish a comparison principle for its viscosity solutions via the $L^p$-approximation scheme by [12] to construct two auxiliary equations with horizontal gradient constraints, to which supersolutions can be deformed into *strict* supersolutions under small perturbations. This section is valid for any Carnot-Carathéodory metric space generated by a family of Hörmander’s vector fields \( \{X_i\}_{i=1}^n \subset C^\infty(\Omega, R^n) \).

**Lemma 5.1.** Suppose that \( f \in C^2(R^m, R_+) \) is homogeneous of degree \( \alpha \geq 1 \). Let \( v \in C(\Omega) \) be a viscosity supersolution to
\begin{equation}
\min\{f(Xw) - \epsilon, - \sum_{i,j=1}^m f_{p_i}(Xw)f_{p_j}(Xw)X_iX_jw\} = 0, \quad \text{in } \Omega,
\end{equation}
where \( \epsilon > 0 \). Then, for any \( \delta > 0 \), there exist \( \mu = \mu(\alpha, \epsilon, \delta) > 0 \) and \( v_\delta \in C(\bar{\Omega}) \), with \( \|v_\delta - v\|_{L^\infty(\Omega)} \leq \delta \), such that \( v_\delta \) is a viscosity supersolution to
\begin{equation}
\min\{f(Xw) - \epsilon, - \sum_{i,j=1}^m f_{p_i}(Xw)f_{p_j}(Xw)X_iX_jw\} - \mu = 0, \quad \text{in } \Omega.
\end{equation}

**Proof.** It is similar to that of [12] (see also [3] and [B1]). We sketch it here. We look for \( v_\delta = g_\delta(v) \), where \( g_\delta \in C^\infty(R) \) is monotonically increasing such that \( g_\delta^{-1} \in C^\infty(R) \). To find \( g_\delta \), let \( x_0 \in \Omega \), and \( \phi \in C^2(\Omega) \) touches \( v_\delta \) from below at \( x_0 \). Let \( \phi_\delta = g_\delta^{-1}(\phi) \). Then \( \phi_\delta \) touches \( v \) from below at \( x_0 \) and
\[ \min\{f(X\phi_\delta) - \epsilon, - \sum_{i,j=1}^m f_{p_i}(X\phi_\delta)f_{p_j}(X\phi_\delta)X_iX_j\phi_\delta\} \mid_{x=x_0} \geq 0. \]

Since
\[ X_i\phi = g_\delta'(\phi_\delta)X_i\phi_\delta, \quad X_j\phi = g_\delta'(\phi_\delta)X_j\phi_\delta + g_\delta''(\phi_\delta)X_i\phi_\delta X_j\phi_\delta, \]
we have, by the \( \alpha \)-homogeneity of \( f \),
\begin{equation}
f(X\phi(x_0)) = f(g_\delta'(\phi_\delta(x_0))X\phi_\delta(x_0)) = (g_\delta'(\phi_\delta(x_0)))^\alpha f(X\phi_\delta(x_0)) \geq (g_\delta'(\phi_\delta(x_0)))^\alpha \epsilon
\end{equation}
and
\begin{equation}
\begin{aligned}
&- \sum_{i,j=1}^m f_{p_i}(X\phi)f_{p_j}(X\phi)X_iX_j\phi \mid_{x=x_0} \\
&= \{g_\delta'(\phi_\delta)^{3\alpha} \left( - \sum_{i,j=1}^m f_{p_i}(X\phi_\delta)f_{p_j}(X\phi_\delta)X_iX_j\phi_\delta \right) \\
&- g_\delta'(\phi_\delta)^{2\alpha}g_\delta''(\phi_\delta)(\sum_{i=1}^m f_{p_i}(X\phi_\delta)X_i\phi_\delta)^2 \mid_{x=x_0} \\
&\geq - g_\delta'(\phi_\delta(x_0))^{2\alpha}g_\delta''(\phi_\delta(x_0))\alpha^2 \epsilon^2,
\end{aligned}
\end{equation}
provided that $g''(\phi_0) < 0$. Here we have used (5.3) and the identity $\sum_{i=1}^{m} f_p(p) p_i = \alpha f(p)$ in the last step. Let $C_0 = 4\|v\|_{L^\infty(\Omega)} < \infty$ and define

$$g_\delta(t) = (1 + \delta)t - \frac{\delta}{4C_0}t^2, \ |t| \leq 2C_0,$$

and extend it suitably to a monotonically increasing function on $R$. Since $g'(t) \geq 1 + \frac{\delta}{2}$ and $g''(t) = -\frac{\delta}{2C_0}$ for $|t| \leq C_0$, we have

$$f(X\phi)(x_0) \geq (1 + \frac{\delta}{2})\epsilon$$

and

$$- \sum_{i,j=1}^{m} f_{p_i}(X\phi)f_{p_j}(X\phi)X_iX_j\phi|_{x=x_0} \geq \frac{\delta\alpha^2\epsilon^2}{2C_0}.$$

Therefore, if we choose $\mu = \min\{\frac{\delta\epsilon}{2}, \frac{\delta\alpha^2\epsilon^2}{2C_0}\} > 0$, then

$$\min\{f(X\phi) - \epsilon, \ - \sum_{i,j=1}^{m} f_{p_i}(X\phi)f_{p_j}(X\phi)X_iX_j\phi\}|_{x=x_0} \geq \mu.$$

The proof of Lemma 5.1 is complete. \(\square\)

Since the argument is similar, we state without proof the analogous lemma on viscosity subsolutions.

**Lemma 5.2.** Suppose that $f \in C^2(R^m, R_+)$ is homogeneous of degree $\alpha \geq 1$. Let $u \in C(\Omega)$ be a viscosity subsolution to

$$\max\{\epsilon - f(Xw), \ - \sum_{i,j=1}^{m} f_{p_i}(Xw)f_{p_j}(Xw)X_iX_jw\} = 0, \ \text{in} \ \Omega,$$

where $\epsilon > 0$. Then, for any $\delta > 0$, there are an $\mu = \mu(\alpha, \epsilon, \delta) > 0$ and $u_\delta \in C(\Omega)$, with $\|u_\delta - u\|_{L^\infty(\Omega)} \leq \delta$, such that $u_\delta$ is a viscosity subsolution to the equation

$$\max\{\epsilon - f(Xw), \ - \sum_{i,j=1}^{m} f_{p_i}(Xw)f_{p_j}(Xw)X_iX_jw\} = -\mu, \ \text{in} \ \Omega.$$

We end this section with existences of viscosity solutions to (1.13), (5.3), and (5.5). For this, we need both convexity of $f$ and $f(p) > 0$ for $p \neq 0$. More precisely,

**Lemma 5.3.** Suppose that $f \in C^2(R^m, R_+)$ is strictly convex, homogeneous of degree $\alpha \geq 1$, and $f(p) > 0$ for $p \neq 0$. Then, for any $g \in W^{1,\infty}_{cc}(\Omega)$, we have

1. There exists a viscosity solution $u \in W^{1,\infty}_{cc}(\Omega)$ to (1.13) such that $u|_{\partial\Omega} = g$.
2. There exists a viscosity solution $u_\epsilon \in W^{1,\infty}_{cc}(\Omega)$ of (5.3) such that $u_\epsilon|_{\partial\Omega} = g$.
3. There exists a viscosity solution $v_\epsilon \in W^{1,\infty}_{cc}(\Omega)$ of (5.5) such that $v_\epsilon|_{\partial\Omega} = g$.
4. There exists a continuous, nondecreasing function $\beta: R_+ \rightarrow R_+$, with $\beta(0) = 0$, such that

$$\|u_\epsilon - v_\epsilon\|_{L^\infty(\Omega)} \leq \beta(\epsilon).$$

**Proof.** The proof is inspired by the $L^k$-approximation scheme, which has been used by Bhatthacharya-Dibenedetto-Manfredi [BDM], [J2], [J], [D1] in the euclidean setting. For completeness, we outline it here. Since (1) follows from (2) with $\epsilon = 0$
and (3) can be done exactly in the way as (2), we only sketch (2) and (4) as follows. For $1 < k < \infty$, let $u_k \in W^{1,k}_X(\Omega)$ be the unique minimizer to the functional

$$F_k(v) = \int_{\Omega} (f(Xv)^k - \varepsilon^{k-1}v), \forall v \in W^{1,k}_X(\Omega), \text{ with } v|_{\partial\Omega} = g.$$  

The existence of $u_k$ can be obtained by the direct method, due to both the convexity of $f$ and $\alpha$-homogeneity of $f$, i.e. $f(p) = |p|^\alpha \max(f(p)) \geq |p|^\alpha \min_{|z|=1} f(z) \geq C|p|^\alpha$. It is easy to verify that $u_k$ satisfies the (subelliptic) $k$-Laplacian: the equation

$$(5.8) \quad -\sum_{i=1}^m X^*_i(kf(Xu_k)^{k-1}f_p(Xu)) = \varepsilon^{k-1}, \text{ in } \Omega$$

in the sense of distributions, where $X^*_i w = \sum_{j=1}^m \frac{\partial}{\partial x_j} (a_{ij}(x)w)$ is the adjoint operator of $X_i$. Let $Q$ denote the homogeneous dimension of $\mathbb{R}^n$, with respect to the vector fields $\{X_i\}_{i=1}^m$. Then it follows from the Sobolev inequality (see Hajłasz-Koskela [HK]) that $\{u_k\}_{k \geq Q+1}$ is bounded and equicontinuous. Therefore we may assume, after taking possible subsequences, that there exists a $u_\varepsilon \in W^{1,\infty}_{X\varepsilon}(\Omega)$ such that

$$u_k \to u_\varepsilon \text{ in } C^0(\overline{\Omega}) \cap Q+1 \leq k < \infty W^{1,k}_{X\varepsilon}(\Omega).$$

It is easy to see that $u_\varepsilon|_{\partial\Omega} = g$. To show that $u_\varepsilon$ is a viscosity solution to (5.3), we need

**Claim 5.4.** For $k \geq Q+1$, $u_k \in C(\overline{\Omega})$ is a viscosity solution to (5.8).

For simplicity, we only indicate that $u_k$ is a viscosity subsolution. For, otherwise, there are $x_0 \in \Omega$ and $\phi \in C^2(\Omega)$ such that

$$0 = u_k(x_0) - \phi(x_0) \geq u_k(x) - \phi(x), \forall x \in \Omega,$$

but

$$(5.9) \quad -\sum_{i=1}^m X^*_i(kf(X\phi)^{k-1}f_p(X\phi))(x_0) - \varepsilon^{k-1} = C_0 > 0.$$  

Then there exists $\delta_0 > 0$ such that

$$(5.10) \quad -\sum_{i=1}^m X^*_i(kf(X\phi)^{k-1}f_p(X\phi))(x) - \varepsilon^{k-1} \geq \frac{C_0}{2}, \forall x \in B_{\delta_0}(x_0).$$

For any small $\delta > 0$, there is a neighborhood $V_\delta(\subset B_{\delta_0}(x_0))$ of $x_0$ such that $\phi_\delta \equiv \phi - \delta$ satisfies

$$\phi_\delta(x) < u_k(x), \forall x \in V_\delta; \quad \phi_\delta(x) = u_k(x), \forall x \in \partial V_\delta.$$  

Note that $\phi_\delta$ also satisfies (5.10). Multiplying (5.8) by $u_k - \phi_\delta$ and integrating over $V_\delta$, we have

$$(5.11) \quad \sum_{i=1}^m \int_{V_\delta} k f(Xu_k)^{k-1}f_p(Xu_k)X_i(u_k - \phi_\delta) = \varepsilon^{k-1} \int_{V_\delta} (u_k - \phi_\delta).$$

On the other hand, multiplying (5.10) by $(u_k - \phi_\delta)(\geq 0)$ and integrating over $V_\delta$, we have

$$(5.12) \quad \sum_{i=1}^m \int_{V_\delta} k f(X\phi_\delta)^{k-1}f_p(X\phi_\delta)X_i(u_k - \phi_\delta) > \varepsilon^{k-1} \int_{V_\delta} (u_k - \phi_\delta).$$
Subtracting (5.11) from (5.12), we obtain
\[ 0 > k \int V_k \left( \sum_{i=1}^{m} \left( f(Xu_k)^{k-1}f_{p_i}(Xu_k) - f(X\phi_\delta)^{k-1}f_{p_i}(X\phi_\delta) \right) X_i(u_k - \phi_\delta) \right); \]
this contradicts with the convexity of \( f \). This finishes the proof of Claim 5.4.

Now we show that \( u_\varepsilon \) is a viscosity subsolution to (5.3). Let \( x \in \Omega \) and \( \phi \in C^2(\Omega) \) be such that
\[ 0 = u_\varepsilon(x) - \phi(x) \geq u_\varepsilon(y) - \phi(y), \forall y \in \Omega. \]
We need to show that
\[ \min \{ f(X\phi) - \varepsilon, - \sum_{i,j=1}^{m} f_{p_i}(X\phi)f_{p_j}(X\phi)X_iX_j\phi \} |_{x} \leq 0. \]
Since this is true if \( f(X\phi(x)) \leq \varepsilon \), we may assume that \( f(X\phi(x)) \geq (1 + 2\delta)\varepsilon \) for some \( \delta > 0 \). We know that there exist \( x_k \in \Omega \) such that \( (u_k - \phi) \) achieves its maximum at \( x_k \) and \( x_k \to x \). We may also assume that, for \( k \) sufficiently large,
\[ f(X\phi(x_k)) \geq (1 + \delta)\varepsilon. \]
It follows from Claim 5.4 that
\[ - \sum_{i=1}^{m} X_i^\ast(kf(X\phi)^{k-1}f_{p_i}(X\phi))(x_k) \leq \varepsilon^{k-1}. \]
After expansion and dividing both sides by \( k(k - 1)f(X\phi)^{k-2}(x_k) \), this gives
\[ - \sum_{i,j=1}^{m} f_{p_i}(X\phi)f_{p_j}(X\phi)X_iX_j\phi(x_k) \leq \left[ \frac{\varepsilon}{k(k - 1)} \right] \left\{ \frac{\varepsilon}{f(X\phi(x_k))} \right\}^{k-2} \left( f(X\phi(x_k)) \right) \sum_{i=1}^{m} X_i^\ast(f_{p_i}(X\phi))(x_k). \]
This, after letting \( k \) tend to \( \infty \), gives
\[ - \sum_{i,j=1}^{m} f_{p_i}(X\phi)f_{p_j}(X\phi)X_iX_j\phi(x) \leq 0. \]
One can show similarly that \( u_\varepsilon \) is also a viscosity supersolution to (5.3). This finishes the proof of (2).

Since \( v_\varepsilon \) is a limit, as \( k \to \infty \), of the minimizers \( v_k \) to
\[ G_k(v) = \int_\Omega f(Xv)^k + \varepsilon^{k-1}v, \forall v \in W^{1,k}_X(\Omega), \text{ with } u|_{\partial\Omega} = g, \]
v_\varepsilon satisfies
\[ - \sum_{i=1}^{m} X_i^\ast(kf(Xv_\varepsilon)^{k-1}f_{p_i}(Xv_\varepsilon)) = -\varepsilon^{k-1}, \text{ in } \Omega. \]
Multiplying (5.11) and (5.13) by \((u_k - v_k)\), integrating over \( \Omega \), and subtracting each other, we get
\[ \int_{\Omega} k(f(Xu_k)^{k-1}f_{p_i}(Xu_k) - f(Xv_k)^{k-1}f_{p_i}(Xv_k))X_i(u_k - v_k) \leq 4\varepsilon^{k-1}\|u_k - v_k\|_{L^1(\Omega)}. \]
Now we need

**Claim 5.5.** If \( f \in C^2(R^m) \) is strictly convex, then for any \( p, q \in R^m \)
\[
(f^{k-1}(p)f_p(p) - f^{k-1}(q)f_p(q)) \cdot (p - q) \geq C|p - q|^{\alpha(k-1) + 2}.
\]

To see (5.15), we observe that
\[
(f^{k-1}(p)f_p(p) - f^{k-1}(q)f_p(q)) \cdot (p - q)
= k^{-1} \int_0^1 \frac{d}{dt}(f^k)_p(tp + (1-t)q) dt \cdot (p - q)
\]
\[
\geq \sum_{i,j=1}^m \int_0^1 f^{k-1}(tp + (1-t)q)f_{p_i,p_j}(tp + (1-t)q) dt(p_i - q_i)(p_j - q_j)
\]
\[
\geq Ck^{-1} \int_0^1 |tp + (1-t)q|^{\alpha(k-1)} dt|p - q|^2,
\]
where we have used the strict convexity of \( f \):
\[
\sum_{i,j=1}^m f_{p_i,p_j}(v)p_i p_j \geq C_0|p - q|^2, \forall p, q, v \in R^m,
\]
the \( \alpha \)-homogeneity of \( f \) and the fact that
\[
f(v) = |v|^{\alpha} f\left(\frac{v}{|v|}\right) \geq \min_{|z|=1} f(z)|v|^\alpha \geq C|v|^\alpha, \forall v \in R^m,
\]
for some \( C > 0 \) depending only on \( f \). Since
\[
\int_0^1 |tp + (1-t)q|^{\alpha(k-1)} dt \geq C|p - q|^{\alpha(k-1)},
\]
(5.16) implies (5.15). Putting (5.15) into (5.14), we obtain
\[
k^nC^{k-1} \int_{\Omega} |Xu_k - Xv_k|^{\alpha(k-1) + 2} \leq Ck^{-1}.
\]
This, combined with the Hölder inequality, implies
\[
\int_{\Omega} |Xu_k - Xv_k| \leq k^{-\frac{1}{\alpha(k-1)+2}}(C\epsilon)^{\frac{1}{\alpha(k-1)+2}}|\Omega|^{\frac{\alpha(k-1)+2}{\alpha(k-1)+2}}.
\]
Letting \( k \) tend to \( \infty \), we have
\[
\text{(5.17)} \quad ||Xu_\epsilon - Xv_\epsilon||_{L^1(\Omega)} \leq C\epsilon^{\frac{1}{n}}.
\]
For \( u_\epsilon, v_\epsilon \in W^{1,\infty}_X(\Omega), (5.17) \), the usual interpolation inequality, and the Sobolev inequality yield that the function \( \beta \) must exist as asserted in (4). \( \Box \)

§6. Proof of Theorem C

This section is devoted to the proof of Theorem C. In this section, the \( \{X_i\}_{i=1}^m \) are horizontal vector fields of a Carnot group \( G \). Since we can identify \( G \) with \( R^n \), \( n = \dim(G) \), via the exponential map, the results in §4 and §5 are all applicable to \( G \). The idea to prove Theorem C is based on the sup/inf convolution and the comparison principle for both equations (5.1) and (5.5).
**Lemma 6.1.** Under the same assumptions as Theorem C, for any $\epsilon > 0$, if $v \in C(\overline{\Omega})$ is a viscosity subsolution to (5.1) and $w \in C(\overline{\Omega})$ is a viscosity supersolution to the (5.1), then

\[
(6.1) \quad \sup_{x \in \Omega} (v - w)(x) = \sup_{x \in \partial \Omega} (v - w)(x).
\]

**Proof.** Suppose that (6.1) were false. Then there is an $\delta_0 > 0$ such that

\[
\sup_{x \in \Omega} (v - w)(x) > \sup_{x \in \partial \Omega} (v - w)(x) + \delta_0.
\]

For any $\delta \in (0, \frac{\delta_0}{2})$, it follows from Lemma 5.1 that there are $w_\delta \in C(\overline{\Omega})$, with $\|w_\delta - w\|_{L^\infty(\Omega)} \leq \delta$, and $\mu = \mu(\delta, \epsilon, \alpha) > 0$, such that $w_\delta$ is a viscosity supersolution to (5.2). Moreover, we have

\[
(6.2) \quad \sup_{x \in \Omega} (v - w_\delta)(x) \geq \sup_{x \in \partial \Omega} (v - w_\delta)(x) + \frac{\delta_0}{4}.
\]

Now we apply Proposition 3.3 to conclude that for any $\delta \in (0, \frac{\delta_0}{2})$ there are a semiconvex $v^\delta \in W^{1,\infty}_X(\Omega)$ and a semiconcave $\tilde{w}_\delta \in W^{1,\infty}_X(\Omega)$ such that

\[
(6.3) \quad \lim_{\delta \to 0} \max \{ \|v^\delta - v\|_{L^\infty(\Omega_{C,\delta})}, \|\tilde{w}_\delta - w_\delta\|_{L^\infty(\Omega_{C,\delta})} \} = 0,
\]

where $\Omega_{C,\delta}$ is defined in §3. Moreover, $v^\delta$ is a viscosity subsolution to (5.1) and $\tilde{w}_\delta$ is a viscosity supersolution to (5.2) on $\Omega_{C,\delta}$. Therefore, we can apply Proposition 4.1 to conclude that

\[
(6.4) \quad \sup_{\Omega_{C,\delta}} (v^\delta - \tilde{w}_\delta) = \sup_{\partial \Omega_{C,\delta}} (v^\delta - \tilde{w}_\delta).
\]

Letting $\delta$ tend to zero, this yields

\[
\lim_{\delta \to 0} \sup_{\Omega_{C,\delta}} (v - w) = \lim_{\delta \to 0} \sup_{\Omega_{C,\delta}} [(v - v^\delta) + (v^\delta - \tilde{w}_\delta) + (\tilde{w}_\delta - w_\delta) + (w_\delta - w)]
\]

\[
= \lim_{\delta \to 0} \sup_{\Omega_{C,\delta}} (v^\delta - \tilde{w}_\delta)
\]

\[
= \lim_{\delta \to 0} \sup_{\partial \Omega_{C,\delta}} (v^\delta - \tilde{w}_\delta)
\]

\[
= \sup_{\partial \Omega} (v - w).
\]

This yields the desired contradiction. The proof is complete. \qed

Similarly, we have the comparison principle for (5.5).

**Lemma 6.2.** Under the same assumptions as Theorem C, for any $\epsilon > 0$, let $v \in C(\Omega)$ be a viscosity subsolution to (5.5) and $w \in C(\Omega)$ be a viscosity supersolution to (5.5). Then

\[
\sup_{x \in \Omega} (v - w)(x) = \sup_{x \in \partial \Omega} (v - w)(x).
\]

We are ready to prove the maximum principle for (1.13).

**Lemma 6.3.** Under the same assumptions as Theorem C, for a given $\phi \in W^{1,\infty}_X(\Omega)$, assume that $v \in C(\Omega)$ is a viscosity subsolution to (1.13) and $w \in C(\Omega)$ is a viscosity supersolution to (1.13) such that $v|_{\partial \Omega} = w|_{\partial \Omega} = \phi$. Then

\[
(6.5) \quad v(x) \leq w(x), \forall x \in \Omega.
\]
Proof. Let $v^+$ be a viscosity solution to (5.1), and let $w^-$ be a viscosity solution to (5.5), with $v^+|_{\partial \Omega} = w^-|_{\partial \Omega} = \phi$, obtained by Lemma 5.3. Since subsolutions of (1.13) are also subsolutions to (5.1), and supersolutions to (1.13) are also supersolutions to (5.5), we can apply Lemmas 6.1 and 6.2 to conclude that

$$
\sup_{\Omega} (v - v^+) = \sup_{\partial \Omega} (v - v^+) = 0, \quad \sup_{\Omega} (w^- - w) = \sup_{\partial \Omega} (w^- - w) = 0.
$$

Hence we have

$$
\sup_{\Omega} (v - w) \leq \sup_{\Omega} (v^+ - w^-) \leq \beta(\epsilon),
$$

where $\beta$ is given by Lemma 5.3. Since $\epsilon$ is arbitrary, this implies

$$
\sup_{\Omega} (v - w) \leq 0.
$$

This finishes the proof of Lemma 6.3. \qed

It is clear that Theorem C follows from Lemma 6.3 and hence the proof is complete. \qed

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References


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