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ON THE ROLE OF QUADRATIC OSCILLATIONS IN NONLINEAR SCHRÖDINGER EQUATIONS II. THE L^2-CRITICAL CASE

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Abstract. We consider a nonlinear semi-classical Schrödinger equation for which quadratic oscillations lead to focusing at one point, described by a nonlinear scattering operator. The relevance of the nonlinearity was discussed by R. Carles, C. Fermanian–Kammerer and I. Gallagher for L^2-supercritical power-like nonlinearities and more general initial data. The present results concern the L^2-critical case, in space dimensions 1 and 2; we describe the set of non-linearizable data, which is larger, due to the scaling. As an application, we make precise a result by F. Merle and L. Vega concerning finite time blow up for the critical Schrödinger equation. The proof relies on linear and nonlinear profile decompositions.

1. Introduction

Consider the initial value problem

$$i\varepsilon\partial_t u^\varepsilon + \frac{1}{2}\varepsilon^2\Delta u^\varepsilon = \varepsilon^{n\sigma}|u^\varepsilon|^{2\sigma}u^\varepsilon, \quad u^\varepsilon_{|t=0} = u^\varepsilon_0\,, \tag{1.1}$$

where $x \in \mathbb{R}^n$ and $\varepsilon \in]0,1]$. Our aim is to understand the relevance of the nonlinearity in the limit $\varepsilon \to 0$, according to the properties of the initial data u^ε_0. In [7], the case $\sigma > 2/n$, with $\sigma < 2/(n-2)$ if $n \geq 3$ and u^ε_0, $\varepsilon\nabla_x u^\varepsilon_0$ bounded in $L^2(\mathbb{R}^n)$ uniformly for $\varepsilon \in]0,1]$, was studied. Note that under these assumptions, global existence in $H^1(\mathbb{R}^n)$ for fixed $\varepsilon > 0$ is well known (see e.g. [8]). It was proven that the nonlinearity has a leading order influence in the limit $\varepsilon \to 0$ if and only if the initial data include a quadratic oscillation of the form

$$f(x - x^\varepsilon)e^{-i\frac{|x-x^\varepsilon|^2}{2\varepsilon t^\varepsilon}}\,,$$

for some $x^\varepsilon \in \mathbb{R}^n$ and $t^\varepsilon > 0$, with $\limsup_{\varepsilon\to 0} t^\varepsilon/\varepsilon \in]0,+\infty[$ (see [7, Theorem 1.2] for a precise statement). Two things have to be said about this property. First, it shows that the presence of quadratic oscillations is necessary for the nonlinearity to have a leading order influence; it was established in [6] that it is sufficient. Recall that when the initial data contains a highly oscillatory quadratic phase, rays of geometric optics (also known as classical trajectories) are lines that meet at one

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point (essentially, $(t,x) = (\lim t^\varepsilon, \lim x^\varepsilon)$ when these limits exist): this is focusing, in the semi-classical régime (see [6] for details). Second, only one scale is involved in such initial profiles, that is, ε. In the present paper, we study the L^2-critical case, $\sigma = 2/n$. We prove that quadratic oscillations are not necessary to have a leading order nonlinear behavior, if we assume that the initial data satisfy the same assumptions as in [7]; scales other than ε have to be taken into account, because $\sigma = 2/n$ corresponds to the critical scaling at the L^2 level. To see this, consider a solution $\mathtt{U}$ to the nonlinear Schrödinger equation

$$i\partial_t \mathtt{U} + \frac{1}{2}\Delta \mathtt{U} = \lambda |\mathtt{U}|^{4/n}\mathtt{U}\,, \tag{1.2}$$

with $\lambda = 1$ and $\mathtt{U}_{|t=0} = \phi$. If $\phi \in \Sigma$, where

$$\Sigma := \left\{\phi \in H^1(\mathbb{R}^n)\,;\ |x|\phi \in L^2(\mathbb{R}^n)\right\}\,, \tag{1.3}$$

then the solution $\mathtt{U}$ of (1.2) is defined globally in time, with $\mathtt{U} \in C(\mathbb{R}_t;\Sigma)$ (see e.g. [8]). Let $(t_0,x_0) \in \mathbb{R}\times\mathbb{R}^n$. It is straightforward to see that

$$u^\varepsilon(t,x) = \frac{1}{\varepsilon^{n/4}}\mathtt{U}\left(t-t_0, \frac{x-x_0}{\sqrt{\varepsilon}}\right) \tag{1.4}$$

solves (1.1) with $\sigma = 2/n$, and that $u^\varepsilon(0,\cdot)$ and $\varepsilon\nabla_x u^\varepsilon(0,\cdot)$ are bounded in $L^2(\mathbb{R}^n)$, uniformly in $\varepsilon \in]0,1]$. This particular solution is such that the nonlinearity in (1.1) is relevant at leading order, at any (finite) time, near $x = x_0$. This is in contrast with the L^2-supercritical case $\sigma > 2/n$, where only profiles of the form

$$u^\varepsilon(t,x) = \frac{1}{\varepsilon^{n/2}}\mathtt{U}\left(\frac{t-t^\varepsilon}{\varepsilon}, \frac{x-x^\varepsilon}{\varepsilon}\right) \tag{1.5}$$

were relevant. The solutions (1.4) are deduced from the solutions (1.5) by scaling. If $\mathtt{U}$ solves (1.2), then so does $\widetilde{\mathtt{U}}$, given by

$$\widetilde{\mathtt{U}}(t,x) = \lambda^{n/2}\mathtt{U}\left(\lambda^2 t, \lambda x\right)\,,$$

for any real λ: the case $\sigma = 2/n$ is L^2-critical. Applying this transform to solutions (1.5) with $\lambda = \sqrt{\varepsilon}$ yields solutions (1.4), with $t_0 = t^\varepsilon/\varepsilon$ and $x_0 = x^\varepsilon/\sqrt{\varepsilon}$.

Before going further into details, we fix some notations and introduce a definition. We consider initial value problems

$$i\varepsilon\partial_t u^\varepsilon + \frac{1}{2}\varepsilon^2\Delta u^\varepsilon = \lambda\varepsilon^2|u^\varepsilon|^{4/n}u^\varepsilon, \quad u^\varepsilon_{|t=0} = u_0^\varepsilon\,, \tag{1.6}$$

with $\lambda \in \{-1,+1\}$, that is, we consider the L^2-critical case of (1.1), with possibly focusing nonlinearities ($\lambda = -1$). As in [7], we define the free evolution v^ε of u_0^ε,

$$i\varepsilon\partial_t v^\varepsilon + \frac{1}{2}\varepsilon^2\Delta v^\varepsilon = 0, \quad v^\varepsilon_{|t=0} = u_0^\varepsilon\,. \tag{1.7}$$

We resume some notations used in [7].

Notation. i) For a family $(a^\varepsilon)_{0<\varepsilon\le 1}$ of functions in $H^1(\mathbb{R}^n)$, define

$$\|a^\varepsilon\|_{H^1_\varepsilon} := \|a^\varepsilon\|_{L^2} + \|\varepsilon\nabla a^\varepsilon\|_{L^2}\,.$$

We will say that a^ε is bounded (resp. goes to zero) in H^1_ε if

$$\limsup_{\varepsilon\to 0} \|a^\varepsilon\|_{H^1_\varepsilon} < \infty \text{ (resp. } = 0).$$

ii) If $(\alpha^\varepsilon)_{0<\varepsilon\leq 1}$ and $(\beta^\varepsilon)_{0<\varepsilon\leq 1}$ are two families of positive numbers, we write

$$\alpha^\varepsilon \lesssim \beta^\varepsilon \tag{1.8}$$

if there exists C independent of $\varepsilon \in]0,1]$ such that for any $\varepsilon \in]0,1]$,

$$\alpha^\varepsilon \leq C\beta^\varepsilon.$$

From now on, u^ε (resp. v^ε) stands for the solution to (1.6) (resp. (1.7)), with $\lambda = -1$ or $+1$ indifferently, unless precisely specified.

Definition 1.1 (Linearizability)**.** Let $u_0^\varepsilon \in L^2(\mathbb{R}^n)$ be bounded in $L^2(\mathbb{R}^n)$, and let I^ε be an interval of $\mathbb{R}$, possibly depending on ε.

i) The solution u^ε is linearizable on I^ε in L^2 if

$$\limsup_{\varepsilon\to 0} \sup_{t\in I^\varepsilon} \|u^\varepsilon(t) - v^\varepsilon(t)\|_{L^2(\mathbb{R}^n)} = 0\,.$$

ii) If in addition $u_0^\varepsilon \in H^1(\mathbb{R}^n)$ and u_0^ε is bounded in H^1_ε, we say that u^ε is linearizable on I^ε in H^1_ε if

$$\limsup_{\varepsilon\to 0} \sup_{t\in I^\varepsilon} \Big(\|u^\varepsilon(t) - v^\varepsilon(t)\|_{L^2(\mathbb{R}^n)} + \|\varepsilon\nabla_x u^\varepsilon(t) - \varepsilon\nabla_x v^\varepsilon(t)\|_{L^2(\mathbb{R}^n)} \Big) = 0\,.$$

We prove the following result. Note that we have to restrict to the case of space dimensions 1 and 2 (see Remark 3.5 below).

Theorem 1.2. *Assume $n = 1$ or 2. Let u_0^ε be bounded in $L^2(\mathbb{R}^n)$, and let $I^\varepsilon \ni 0$ be a time interval.*

- *u^ε is linearizable on I^ε in L^2 if and only if*

$$\limsup_{\varepsilon\to 0} \varepsilon \|v^\varepsilon\|^{2+4/n}_{L^{2+4/n}(I^\varepsilon\times\mathbb{R}^n)} = 0\,. \tag{1.9}$$

- *Assume in addition that $u_0^\varepsilon \in H^1$ and u_0^ε is bounded in H^1_ε. Then u^ε is linearizable on I^ε in H^1_ε if and only if* (1.9) *holds.*

Note that a similar result was proven in [7], in the L^2-supercritical case, with a different linearizability condition,

$$\limsup_{\varepsilon\to 0} \varepsilon^2 \sup_{t\in I^\varepsilon} \|v^\varepsilon(t)\|^{2+4/n}_{L^{2+4/n}(\mathbb{R}^n)} = 0\,. \tag{1.10}$$

The fact that this condition is necessary for u^ε to be linearizable in H^1_ε is easy to see, from the classical conservations of mass and energy, which we write in the case $\sigma = 2/n$ (in the general case, the powers $2 + 4/n$ are replaced by $2\sigma + 2$):

$$\begin{aligned}
&\text{Mass: } \frac{d}{dt}\|u^\varepsilon(t)\|_{L^2} = \frac{d}{dt}\|v^\varepsilon(t)\|_{L^2} = 0,\\
&\text{Linear energy: } \frac{d}{dt}\|\varepsilon\nabla_x v^\varepsilon(t)\|_{L^2} = 0,\\
&\text{Nonlinear energy: } \frac{d}{dt}\left(\frac{1}{2}\|\varepsilon\nabla_x u^\varepsilon(t)\|^2_{L^2} + \frac{\lambda\varepsilon^2}{2+4/n}\|u^\varepsilon(t)\|^{2+4/n}_{L^{2+4/n}}\right) = 0\,.
\end{aligned} \tag{1.11}$$

The proof that condition (1.10) implies linearizability in H^1_ε (which is a stronger property than linearizability in L^2) involves Strichartz estimates, and seems to rely in an unnatural way on the assumption $\sigma > 2/n$. Example (1.4) shows that this assumption is relevant: the solution v^ε associated to u^ε in (1.4) is given by

$$v^\varepsilon(t,x) = \frac{1}{\varepsilon^{n/4}}\mathtt{V}\left(t-t_0, \frac{x-x_0}{\sqrt{\varepsilon}}\right), \text{ where } \mathtt{V} = e^{i\frac{t+t_0}{2}\Delta}\mathtt{U}(-t_0)\,.$$

For any $T>0$ independent of ε, it satisfies (1.10) with $I^\varepsilon = [0,T]$, but u^ε is not linearizable on $[0,T]$ in L^2; note that v^ε does not satisfy (1.9), which is reassuring.

The proof that (1.9) is necessary for linearizability in L^2 relies on profile decomposition for L^2 solutions of (1.2). It was established in [22] for the case $n=2$. We prove it in the one-dimensional case in Section 3.

Definition 1.3. If $(h_j^\varepsilon, t_j^\varepsilon, x_j^\varepsilon, \xi_j^\varepsilon)_{j\in\mathbb{N}}$ is a family of sequences in $\mathbb{R}_+\setminus\{0\}\times\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n$, then we say that $(h_j^\varepsilon, t_j^\varepsilon, x_j^\varepsilon, \xi_j^\varepsilon)_{j\in\mathbb{N}}$ is an orthogonal family if

$$\limsup_{\varepsilon\to 0}\left(\frac{h_j^\varepsilon}{h_k^\varepsilon}+\frac{h_k^\varepsilon}{h_j^\varepsilon}+\frac{|t_j^\varepsilon-t_k^\varepsilon|}{(h_j^\varepsilon)^2}+\left|\frac{x_j^\varepsilon-x_k^\varepsilon}{h_j^\varepsilon}+\frac{t_j^\varepsilon\xi_j^\varepsilon-t_k^\varepsilon\xi_k^\varepsilon}{h_j^\varepsilon}\right|\right)=\infty\,,\quad \forall j\neq k.$$

Theorem 1.4 (Linear profiles)**.** *Let $n=1$ or 2, and let $\mathtt{U}_0^\varepsilon$ be a bounded family in $L^2(\mathbb{R}^n)$.*

i) *Up to extracting a subsequence, there exist an orthogonal family*

$$(h_j^\varepsilon, t_j^\varepsilon, x_j^\varepsilon, \xi_j^\varepsilon)_{j\in\mathbb{N}}$$

in $\mathbb{R}_+\setminus\{0\}\times\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n$, and a family $(\phi_j)_{j\in\mathbb{N}}$ bounded in $L^2(\mathbb{R}^n)$, such that for every $\ell\geq 1$,

$$e^{i\frac{t}{2}\Delta}\mathtt{U}_0^\varepsilon = \sum_{j=1}^{\ell}\mathtt{H}_j^\varepsilon(\phi_j)(t,x)+r_\ell^\varepsilon(t,x)\,,$$

$$\textit{where}\quad \mathtt{H}_j^\varepsilon(\phi_j)(t,x) = e^{i\frac{t}{2}\Delta}\left(e^{ix\cdot\xi_j^\varepsilon}e^{-i\frac{t_j^\varepsilon}{2}\Delta}\frac{1}{(h_j^\varepsilon)^{n/2}}\phi_j\left(\frac{x-x_j^\varepsilon}{h_j^\varepsilon}\right)\right),$$

$$\textit{and}\quad \limsup_{\varepsilon\to 0}\|r_\ell^\varepsilon\|_{L^{2+4/n}(\mathbb{R}\times\mathbb{R}^n)}\underset{\ell\to+\infty}{\longrightarrow}0\,.$$

Furthermore, for every $\ell\geq 1$, we have

$$\|\mathtt{U}_0^\varepsilon\|^2_{L^2(\mathbb{R})} = \sum_{j=1}^{\ell}\|\phi_j\|^2_{L^2(\mathbb{R})}+\|r_\ell^\varepsilon\|^2_{L^2(\mathbb{R})}+o(1)\quad \textit{as } \varepsilon\to 0\,. \tag{1.12}$$

ii) *If in addition the family $(\mathtt{U}_0^\varepsilon)_{0<\varepsilon\leq 1}$ is bounded in $H^1(\mathbb{R}^n)$, or more generally if*

$$\limsup_{\varepsilon\to 0}\int_{|\xi|>R}\left|\widehat{\mathtt{U}}_0^\varepsilon(\xi)\right|^2 d\xi\to 0\quad \textit{as } R\to+\infty\,, \tag{1.13}$$

then for every $j\geq 1$, $h_j^\varepsilon\geq 1$, and $(\xi_j^\varepsilon)_\varepsilon$ is bounded, $|\xi_j^\varepsilon|\leq C_j$.

To state the nonlinear analog to that result, we introduce the following definition:

Definition 1.5. Let $\Gamma^\varepsilon = (h^\varepsilon, t^\varepsilon, x^\varepsilon, \xi^\varepsilon)$ be a sequence in $\mathbb{R}_+\setminus\{0\}\times\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n$ such that $t^\varepsilon/(h^\varepsilon)^2$ has a limit in $[-\infty,+\infty]$ as ε goes to zero. For $\phi\in L^2(\mathbb{R}^n)$, we

define the nonlinear profile $\mathtt{U}$ associated to $(\phi, \Gamma^\varepsilon)$ as the unique maximal solution of the nonlinear equation (1.2) satisfying

$$\left\| \mathtt{U}\left(\frac{-t^\varepsilon}{(h^\varepsilon)^2}\right) - e^{-i\frac{t^\varepsilon}{2(h^\varepsilon)^2}\Delta}\phi \right\|_{L^2(\mathbb{R}^n)} \underset{\varepsilon\to 0}{\longrightarrow} 0\,.$$

Essentially, ϕ is a Cauchy data for $\mathtt{U}$ if $t^\varepsilon/(h^\varepsilon)^2$ has a finite limit, and an asymptotic state (scattering data) otherwise.

Theorem 1.6 (Nonlinear profiles)**.** *Let $n = 1$ or 2, let $\mathtt{U}_0^\varepsilon$ be a bounded family in $L^2(\mathbb{R}^n)$ and let $\mathtt{U}^\varepsilon$ be the solution to* (1.2) *with initial datum $\mathtt{U}_0^\varepsilon$. Let $(\phi_j, \Gamma_j^\varepsilon)_{j\in\mathbb{N}^*}$ be the family of linear profiles given by Theorem* 1.4*, and let $(\mathtt{U}_j)_{j\in\mathbb{N}^*}$ be the family given by Definition* 1.5 *(up to the extraction of a subsequence).*

Let $I^\varepsilon \subset \mathbb{R}$ be a family of open intervals containing the origin. The following statements are equivalent:

(i) *For every $j \geq 1$, we have*

$$\limsup_{\varepsilon\to 0} \|\mathtt{U}_j\|_{L^{2+4/n}(I_j^\varepsilon\times\mathbb{R}^n)} < +\infty\,, \quad \text{where } I_j^\varepsilon := (h_j^\varepsilon)^{-2}\left(I^\varepsilon - t_j^\varepsilon\right)\,.$$

(ii) $\limsup_{\varepsilon\to 0} \|\mathtt{U}^\varepsilon\|_{L^{2+4/n}(I^\varepsilon\times\mathbb{R}^n)} < +\infty$.

Moreover, if (i) *or* (ii) *holds, then $\mathtt{U}^\varepsilon = \sum_{j=1}^\ell \mathtt{U}_j^\varepsilon + r_\ell^\varepsilon + \rho_\ell^\varepsilon$, where r_ℓ^ε is given by Theorem* 1.4*, and*

$$\limsup_{\varepsilon\to 0}\left(\|\rho_\ell^\varepsilon\|_{L^{2+4/n}(I^\varepsilon\times\mathbb{R}^n)} + \|\rho_\ell^\varepsilon\|_{L^\infty(I^\varepsilon;L^2(\mathbb{R}^n))}\right) \underset{\ell\to+\infty}{\longrightarrow} 0\,, \tag{1.14}$$

$$\mathtt{U}_j^\varepsilon(t,x) = e^{ix\cdot\xi_j^\varepsilon - i\frac{t}{2}(\xi_j^\varepsilon)^2}\frac{1}{(h_j^\varepsilon)^{n/2}}\mathtt{U}_j\left(\frac{t-t_j^\varepsilon}{(h_j^\varepsilon)^2}, \frac{x-x_j^\varepsilon - t\xi_j^\varepsilon}{h_j^\varepsilon}\right)\,. \tag{1.15}$$

We give two applications to these results, besides the proof of Theorem 1.2. The first one is the equivalent of [7, Theorem 1.2], which characterizes the obstructions to linearizability. The second one concerns the properties of blowing up solutions, in the same spirit as [22].

The equivalent to [7, Theorem 1.2] is the following.

Corollary 1.7. *Assume $n = 1$ or 2, and let u_0^ε be bounded in $L^2(\mathbb{R}^n)$. Let $T > 0$ and assume that* (1.9) *is not satisfied with $I^\varepsilon = [0, T]$. Then up to the extraction of a subsequence, there exist an orthogonal family $(h_j^\varepsilon, t_j^\varepsilon, x_j^\varepsilon, \xi_j^\varepsilon)_{j\in\mathbb{N}}$ and a family $(\phi_j)_{j\in\mathbb{N}}$, bounded in $L^2(\mathbb{R}^n)$, such that*

$$\begin{aligned} u_0^\varepsilon(x) &= \sum_{j=1}^\ell \widetilde{H}_j^\varepsilon(\phi_j)(x) + w_\ell^\varepsilon(x)\,, \\ \text{where } \widetilde{H}_j^\varepsilon(\phi_j)(x) &= e^{ix\cdot\xi_j^\varepsilon/\sqrt{\varepsilon}} e^{-i\varepsilon\frac{t_j^\varepsilon}{2}\Delta}\left(\frac{1}{(h_j^\varepsilon\sqrt{\varepsilon})^{n/2}}\phi_j\left(\frac{x-x_j^\varepsilon}{h_j^\varepsilon\sqrt{\varepsilon}}\right)\right)\,, \\ \text{and } \limsup_{\varepsilon\to 0}\varepsilon\|e^{i\varepsilon\frac{t}{2}\Delta}w_\ell^\varepsilon\|^{2+4/n}_{L^{2+4/n}(\mathbb{R}\times\mathbb{R}^n)} &\underset{\ell\to+\infty}{\longrightarrow} 0\,. \end{aligned} \tag{1.16}$$

We have $\liminf t_j^\varepsilon/(h_j^\varepsilon)^2 \neq -\infty$, $\liminf (T - t_j^\varepsilon)/(h_j^\varepsilon)^2 \neq -\infty$ (as $\varepsilon \to 0$), and $h_j^\varepsilon \leq 1$ for every $j \in \mathbb{N}$.

If $t_j^\varepsilon/(h_j^\varepsilon)^2 \to +\infty$ as $\varepsilon \to 0$, then we also have

$$\widetilde{H}_j^\varepsilon(\phi_j)(x) = e^{i\frac{x\cdot\xi_j^\varepsilon}{\sqrt{\varepsilon}}+in\frac{\pi}{4}} e^{-i\frac{|x-x_j^\varepsilon|^2}{2\varepsilon t_j^\varepsilon}} \left(\frac{h_j^\varepsilon}{t_j^\varepsilon\sqrt{\varepsilon}}\right)^{n/2} \widehat{\phi}_j\left(-\frac{h_j^\varepsilon}{t_j^\varepsilon\sqrt{\varepsilon}}(x-x_j^\varepsilon)\right) + o(1) \text{ in } L^2(\mathbb{R}^n) \text{ as } \varepsilon \to 0\,, \tag{1.17}$$

where $\widehat{\phi}$ stands for the Fourier transform of ϕ: $\widehat{\phi}(\xi) = (2\pi)^{-n/2}\int e^{-ix\cdot\xi}\phi(x)dx$. If in addition u_0^ε is bounded in H^1_ε, then we have $h_j^\varepsilon \geq \sqrt{\varepsilon}$.

Remark. Even if u_0^ε is bounded in H^1_ε, we cannot say more than $\phi_j \in L^2(\mathbb{R}^n)$, while in [7], the H^1_ε assumption implied $\phi_j \in H^1(\mathbb{R}^n)$. This is due to the fact that several scales of concentrations must be taken into account in the present case, while in [7], only the scale ε was relevant. In that case, the profile decomposition in the homogeneous space $\dot{H}^1(\mathbb{R}^n)$ performed in [18] could be used to deduce properties in the *inhomogeneous* Sobolev space H^1. In our case, we cannot compare the L^2 and $\dot{H}^1$ profile decompositions.

Remark. Compare this result with [7, Theorem 1.2].

• Scales. As we already mentioned, not only must the scale ε be considered in the obstructions to the linearizability in H^1_ε, but every scale between ε and $\sqrt{\varepsilon}$. Examples (1.4) and (1.5) can thus be considered as two borderline cases.

• Quadratic oscillations. The asymptotic expansion (1.17) highlights quadratic oscillations in the initial data, which are exactly ε-oscillatory, unless $t_j^\varepsilon/(h_j^\varepsilon)^2$ is bounded. That case corresponds to initial focusing for u^ε (see for instance (1.4)). In [7], this phenomenon was excluded by the assumption

$$\varepsilon^2\|u_0^\varepsilon\|^{2\sigma+2}_{L^{2\sigma+2}} \underset{\varepsilon\to 0}{\longrightarrow} 0\,,$$

because the only relevant concentrating scale was ε. In the present case, every profile such that $\sqrt{\varepsilon} \ll h_j^\varepsilon \leq 1$ satisfies the above property, and concentrates with the scale $h_j^\varepsilon\sqrt{\varepsilon} \neq \varepsilon$ at time $t = t_j^\varepsilon$. It also concentrates with the same scale at time $t = 0$ if $t_j^\varepsilon/(h_j^\varepsilon)^2$ is bounded. So it is a matter of choice to consider whether or not quadratic oscillations are necessary to have a leading order nonlinear behavior, according to the way one treats initial focusing.

• Properties of t_j^ε. The localization of the cores in time is not as precise as in [7], where we had $\limsup t_j^\varepsilon \in [0,T]$. We actually have the same condition from the properties $\liminf t_j^\varepsilon/(h_j^\varepsilon)^2 \neq -\infty$ and $\liminf (T-t_j^\varepsilon)/(h_j^\varepsilon)^2 \neq -\infty$, provided that the scale h_j^ε goes to zero as $\varepsilon \to 0$. When h_j^ε is constant, we cannot say much about t_j^ε; see (1.4).

The second application of Theorem 1.4 concerns finite time blow up, which may occur for H^1-solutions of (1.2) when $\lambda = -1$ (not when $\lambda = 1$, from the conservation of energy). For solutions $\mathtt{U}$ in L^2 and not necessarily in H^1, the conservation of mass shows that the only obstruction to global existence in L^2 is the unboundness of $\|\mathtt{U}\|_{L^{2+4/n}([0,T]\times\mathbb{R}^n)}$ (see e.g. [9]).

Corollary 1.8. *Assume $n = 1$ or 2. Let $\mathtt{U}$ be an L^2-solution to (1.2), and assume that $\mathtt{U}$ blows up at time $T > 0$ (not before),*

$$\int_0^T\int_{\mathbb{R}^n} |\mathtt{U}(t,x)|^{2+\frac{4}{n}}dxdt = +\infty\,.$$

Let $(t_k)_{k\in\mathbb{N}}$ be an increasing sequence going to T as $k\to+\infty$. Then up to a subsequence, there exist $x_j^k, y_j^k\in\mathbb{R}^n$, $\rho_j^k, h_j^k>0$, $t_j^k\geq 0$ and a family $(\mathtt{U}_j,\widetilde{\mathtt{U}}_j)_{j\in\mathbb{N}}$ bounded in L^2 such that

$$\begin{aligned}\mathtt{U}(t_k,x)&=\sum_{j=1}^{\ell}e^{ix\cdot y_j^k}\frac{1}{(\rho_j^k)^{n/2}}\mathtt{U}_j\left(\frac{x-x_j^k}{\rho_j^k}\right)\\&+\sum_{j=1}^{\ell}e^{ix\cdot y_j^k}e^{-i\frac{|x-x_j^k|^2}{2(T-t_k)t_j^k}}\frac{1}{(\widetilde{\rho}_j^k)^{n/2}}\widetilde{\mathtt{U}}_j\left(\frac{x-x_j^k}{\widetilde{\rho}_j^k}\right)+\mathtt{W}_\ell^k(x)\,,\end{aligned}\tag{1.18}$$

with

$$\limsup_{k\to+\infty}\left\|e^{i\frac{t}{2}\Delta}\mathtt{W}_\ell^k\right\|_{L^{2+4/n}(\mathbb{R}\times\mathbb{R}^n)}\underset{\ell\to+\infty}{\longrightarrow}0\quad,\quad\widetilde{\rho}_j^k=\frac{t_j^k\sqrt{T-t_k}}{h_j^k}\,,$$

and the additional properties, for every $j\in\mathbb{N}$,

$$\lim_{k\to+\infty}\frac{T-t_k}{(\rho_j^k)^2}\geq 1\,,\tag{1.19}$$

$$\text{the sequence } (t_j^k)_{k\in\mathbb{N}} \text{ is bounded, and } \lim_{k\to+\infty}\frac{t_j^k}{(h_j^k)^2}=+\infty\,.\tag{1.20}$$

Moreover the terms in the sum (1.18) *are pairly orthogonal in the limit $k\to+\infty$, each term being orthogonal to $\mathtt{W}_\ell^k$.*

Remark. For the profiles associated to $\mathtt{U}_j$, (1.19) shows that the blow up rate is bounded from below by $(T-t)^{-1/2}$ in the L^2 case. In the H^1 case, this property is well known (see [10] or [8]). For the profiles $\widetilde{\mathtt{U}}_j$, it is less clear. Assume that $\widetilde{\mathtt{U}}_j$ is smooth; then the $\dot{H}^1$ norm of the profiles associated to $\widetilde{\mathtt{U}}_j$ is of order

$$\left\|\frac{e^{ix\cdot\xi_j^k-i\frac{|x-x_j^k|^2}{2(T-t_k)t_j^k}}}{(\widetilde{\rho}_j^k)^{n/2}}\widetilde{\mathtt{U}}_j\left(\frac{x-x_j^k}{\widetilde{\rho}_j^k}\right)\right\|_{\dot{H}^1}\sim|\xi_j^k|+\frac{1}{h_j^k\sqrt{T-t_k}}+\frac{(h_j^k)^2}{t_j^k}\frac{1}{h_j^k\sqrt{T-t_k}}\,.$$

The second term is due to quadratic oscillations, and dominates the last term, obtained by differentiating $\widetilde{\mathtt{U}}_j$, from (1.20). Since from (1.20) $h_j^k\to 0$ as $k\to+\infty$, this suggests that the blow up rate for the profiles associated to $\widetilde{\mathtt{U}}_j$ is also bounded from below by $(T-t)^{-1/2}$ (and is large compared to this minimal rate).

Remark. Some blowing up solutions are known explicitly [32]. They are of the form

$$\mathtt{U}(t,x)=e^{-i\frac{|x|^2}{2(T-t)}+\frac{i}{T-t}}\frac{1}{(T-t)^{n/2}}Q\left(\frac{x}{T-t}\right)\,,$$

where Q denotes the unique spherically symmetric solution of (see [28], [19])

$$-\frac{1}{2}\Delta Q+Q=-\lambda|Q|^{4/n}Q\,,\quad Q>0\text{ in }\mathbb{R}^n\,.$$

It is proven in [20] that up to the invariants of (1.2), these are the only H^1 blowing up solutions with minimal mass $\|\mathtt{U}\|_{L^2}=\|Q\|_{L^2}$. This yields

$$\mathtt{U}(T-\varepsilon,x)=e^{-i\frac{|x|^2}{2\varepsilon}+\frac{i}{\varepsilon}}\frac{1}{\varepsilon^{n/2}}Q\left(\frac{x}{\varepsilon}\right),$$

which is equivalent, up to the extraction of a subsequence, to

$$e^{-i\frac{|x|^2}{2\varepsilon}+i\theta}\frac{1}{\varepsilon^{n/2}}Q\left(\frac{x}{\varepsilon}\right)$$

for some $\theta\in\mathbb{R}$. This term may look like a profile $\widetilde{\mathtt{U}}_j$, because it contains a quadratic phase, with $t_j^\varepsilon=1$. However, the quadratic oscillation is not relevant in the profile decomposition, because the profile is already concentrated to:

$$e^{-i\frac{|x|^2}{2\varepsilon}+i\theta}\frac{1}{\varepsilon^{n/2}}Q\left(\frac{x}{\varepsilon}\right)=e^{i\theta}\frac{1}{\varepsilon^{n/2}}Q\left(\frac{x}{\varepsilon}\right)+o(1)\quad\text{in } L^2\,,$$

and small terms in L^2 are linearizable from Strichartz estimates (see (2.3) below). Note that it is only in the régime that we consider whether quadratic oscillations become negligible (as time is sufficiently close to the critical time): they play a decisive role in igniting the blow up phenomenon, at least in this case (the explicit formula for these solutions seems to rely on very rigid properties; see [5, 26, 21]). The quadratic oscillations gather some mass of u near one point, and start the blow up phenomenon: these oscillations, which appear after a pseudo-conformal transform (see e.g. [25, 14, 33, 20]), turn a nondispersive solution (a solitary wave) into a self-focusing solution. A similar explicit formula is available in the semi-classical limit for (1.1); see [6, p. 485]. In the WKB asymptotics, the phase dictates the *geometry* of the propagation, and the solution solves an ordinary differential equation along these rays at leading order (see [6]). When WKB methods cease to be valid (close to the blow up time), the solution seems to be so concentrated that giving a geometric interpretation is a delicate issue.

2. Preliminary estimates

First, note that the dependence upon ε in (1.6) can be "removed" by the change of unknown function

$$u^\varepsilon(t,x)=\frac{1}{\varepsilon^{n/4}}\mathtt{U}^\varepsilon\left(t,\frac{x}{\sqrt{\varepsilon}}\right)\,. \tag{2.1}$$

One checks that u^ε solves (1.6) on I^ε if and only if $\mathtt{U}^\varepsilon$ solves (1.2) on I^ε, and

$$\|u^\varepsilon(t)\|_{L^2}=\|\mathtt{U}^\varepsilon(t)\|_{L^2},\qquad \varepsilon\|u^\varepsilon(t)\|_{L^{2+4/n}}^{2+4/n}=\|\mathtt{U}^\varepsilon(t)\|_{L^{2+4/n}}^{2+4/n}\,. \tag{2.2}$$

In this section, we recall the classical Strichartz estimates, then we establish a refined Strichartz inequality in the space dimension one case.

2.1. Classical Strichartz estimates.

The original Strichartz estimate [29, 15], which holds in any space dimension, states the following: there exists a constant C such that for any $\phi\in L^2(\mathbb{R}^n)$,

$$\left\|e^{i\frac{t}{2}\Delta}\phi\right\|_{L^{2+4/n}(\mathbb{R}\times\mathbb{R}^n)}\leq C\|\phi\|_{L^2(\mathbb{R}^n)}\,. \tag{2.3}$$

In the case of inhomogeneous Schrödinger equations, we have a similar estimate, which was first proved in [34]. Denote

$$\boxed{\gamma=2+\frac{4}{n}\,,}$$

and γ' its Hölder-conjugate exponent. There exists a constant C such that for any time interval $I \ni 0$ and any $\psi \in L^{\gamma'}(I \times \mathbb{R}^n)$,

$$\begin{aligned} \left\| \int_0^t e^{i\frac{t-s}{2}\Delta}\psi(s)ds \right\|_{L^\gamma(I\times\mathbb{R}^n)} &\leq C\|\psi\|_{L^{\gamma'}(I\times\mathbb{R}^n)}\,, \\ \left\| \int_0^t e^{i\frac{t-s}{2}\Delta}\psi(s)ds \right\|_{L^\infty(I;L^2(\mathbb{R}^n))} &\leq C\|\psi\|_{L^{\gamma'}(I\times\mathbb{R}^n)}\,. \end{aligned} \tag{2.4}$$

Other estimates are available, but here we shall use only the three recalled above.

2.2. **A refined Strichartz estimate.** Following [3], a refined Strichartz inequality was proved in [24] for space dimension two:

$$\begin{aligned} &\left\| e^{i\frac{t}{2}\Delta}u_0 \right\|_{L^4(\mathbb{R}_t\times\mathbb{R}^2_x)} \lesssim \|\widehat{u_0}\|_{\chi_p}\,, \quad \text{for } p > \frac{12}{7}, \quad \text{where} \\ &\|f\|_{\chi_p} = \left(\sum_{j\in\mathbb{Z}} \sum_{\tau\in C_j} 2^{4j} \left(\frac{1}{2^{2j}} \int_\tau |f|^p \right)^{4/p} \right)^{1/4}. \end{aligned} \tag{2.5}$$

Here τ denotes a square with side length 2^j, and C_j denotes a corresponding grid of the plane. This estimate was used in [4, 22]. We prove its (simpler) analog in space dimension one.

Proposition 2.1. *Let $p > 1$. There exists C_p such that for any $f \in L^2(\mathbb{R})$,*

$$\left\| e^{i\frac{t}{2}\partial_x^2} f \right\|_{L^6(\mathbb{R}_t\times\mathbb{R}_x)} \leq C_p \left(\sup_{\substack{\tau>0 \\ \xi_0\in\mathbb{R}}} \tau^{\frac{1}{2}-\frac{1}{p}} \left\| \widehat{f} \right\|_{L^p([\xi_0-\tau,\xi_0+\tau])} \right)^{1/3} \|f\|^{2/3}_{L^2(\mathbb{R})}\,.$$

Proof. The proof follows very closely the argument used in [16] in the context of KdV equation.

By using the explicit formula for the fundamental solution $e^{i\frac{t}{2}\partial_x^2}$, we get

$$\left| e^{i\frac{t}{2}\partial_x^2} f \right|^2 = \iint_{\mathbb{R}^2} e^{it(\eta^2-\xi^2)+ix(\xi-\eta)} \widehat{f}(\xi)\widehat{f}(\eta) d\xi d\eta\,.$$

Introduce the change of variables $u = \eta^2 - \xi^2$ and $v = \eta - \xi$:

$$\left| e^{i\frac{t}{2}\partial_x^2} f \right|^2 = \iint_{\mathbb{R}^2} e^{itu-ixv} \widehat{f}(\xi)\widehat{f}(\eta) \frac{dudv}{|\xi-\eta|^{1/2}}\,.$$

We use the usual trick $\|e^{i\frac{t}{2}\partial_x^2} f\|^6_{L^6(\mathbb{R}^2)} = \||e^{i\frac{t}{2}\partial_x^2} f|^2\|^3_{L^3(\mathbb{R}^2)}$. From Hausdorff–Young's inequality and the inverse change of variable, we infer

$$\left\| e^{i\frac{t}{2}\partial_x^2} f \right\|^2_{L^6(\mathbb{R}^2)} \lesssim \left(\iint_{\mathbb{R}^2} \left| \widehat{f}(\xi) \right|^{3/2} \left| \widehat{f}(\eta) \right|^{3/2} \frac{d\xi d\eta}{|\xi-\eta|^{1/2}} \right)^{2/3}.$$

The end of the proof is analogous to that of [16, Theorem 3]. Cauchy–Schwarz inequality yields

$$\left\| e^{i\frac{t}{2}\partial_x^2} f \right\|^3_{L^6(\mathbb{R}^2)} \lesssim \left\| \widehat{f} \right\|_{L^2(\mathbb{R})} \left(\int_{\mathbb{R}} \left| I_{1/2} \left(\left| \widehat{f}(\xi) \right|^{3/2} \right) \right|^2 \left| \widehat{f}(\xi) \right| d\xi \right)^{1/2},$$

where $I_{1/2}$ stands for the fractional integration $\frac{1}{|x|^{1/2}}*$. By Fefferman–Phong's weighted inequality [11], we get

$$\left(\int_{\mathbb{R}}\left|I_{1/2}\left(\left|\widehat{f}(\xi)\right|^{3/2}\right)\right|^2\left|\widehat{f}(\xi)\right|d\xi\right)^{1/2}\leq C_p\sup_{\substack{\tau>0\\ \xi_0\in\mathbb{R}}}\tau^{\frac{1}{2}-\frac{1}{p}}\left\|\widehat{f}\right\|_{L^p([\xi_0-\tau,\xi_0+\tau])}\|\widehat{f}\|_{L^2(\mathbb{R})},$$

which completes the proof of the proposition. □

3. Proof of Theorem 1.4: Linear profile decomposition

In this section, we prove Theorem 1.4 in the case $n=1$. The case $n=2$ was established in [22]. For the benefit of the reader, we give a complete proof in the one-dimensional case. We follow essentially the same lines as in [1, 22, 18, 12]. The idea relies on an exhaustion algorithm inspired from [23], and first used for such contexts as the present one in [13].

The beginning of the proof does not rely on the assumption $n=1$. We thus write it with a general $n\geq 1$, and we point out the steps which are bound to the case $n=1$ (see Remark 3.5). We resume some notation used in the Introduction.

Notation. For a sequence $\Gamma_j^\varepsilon=(h_j^\varepsilon,t_j^\varepsilon,x_j^\varepsilon,\xi_j^\varepsilon)$ in $\mathbb{R}_+\setminus\{0\}\times\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n$, we denote

$$\mathtt{H}_j^\varepsilon(\phi_j)(t,x)=e^{i\frac{t}{2}\Delta}\left(e^{ix\cdot\xi_j^\varepsilon}e^{-i\frac{t_j^\varepsilon}{2}\Delta}\frac{1}{(h_j^\varepsilon)^{n/2}}\phi_j\left(\frac{x-x_j^\varepsilon}{h_j^\varepsilon}\right)\right). \tag{3.1}$$

For a sequence $\widetilde{\Gamma}_j^\varepsilon=(h_j^\varepsilon,x_j^\varepsilon,\xi_j^\varepsilon)$ in $\mathbb{R}_+\setminus\{0\}\times\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n$, we denote

$$\widetilde{\mathtt{H}}_j^\varepsilon(\phi_j)(x)=\mathtt{H}_j^\varepsilon(\phi_j)(0,x)=e^{ix\cdot\xi_j^\varepsilon}e^{-i\frac{t_j^\varepsilon}{2}\Delta}\frac{1}{(h_j^\varepsilon)^{n/2}}\phi_j\left(\frac{x-x_j^\varepsilon}{h_j^\varepsilon}\right). \tag{3.2}$$

The following identity is straightforward:

$$\begin{aligned}\mathtt{H}_j^\varepsilon(\phi_j)(t,x)&=e^{ix\cdot\xi_j^\varepsilon-i\frac{t}{2}(\xi_j^\varepsilon)^2}\frac{1}{(h_j^\varepsilon)^{n/2}}\mathtt{V}_j\left(\frac{t-t_j^\varepsilon}{(h_j^\varepsilon)^2},\frac{x-x_j^\varepsilon-t\xi_j^\varepsilon}{h_j^\varepsilon}\right),\\ &\text{where } \mathtt{V}_j(t)=e^{i\frac{t}{2}\Delta}\phi_j\,.\end{aligned} \tag{3.3}$$

Remark 3.1. If two sequences $\Gamma_j^\varepsilon=(h_j^\varepsilon,t_j^\varepsilon,x_j^\varepsilon,\xi_j^\varepsilon)$ and $\Gamma_k^\varepsilon=(h_k^\varepsilon,t_k^\varepsilon,x_k^\varepsilon,\xi_k^\varepsilon)$ are not orthogonal, then, up to a subsequence, $(\widetilde{\mathtt{H}}_j^\varepsilon)^{-1}\widetilde{\mathtt{H}}_k^\varepsilon\to\widetilde{\mathtt{H}}$ strongly as $\varepsilon\to 0$, where $\widetilde{\mathtt{H}}$ is isometric on $L^2(\mathbb{R}^n)$.

Let $\mathtt{U_0}=(\mathtt{U}_0^\varepsilon)_{0<\varepsilon\leq 1}$ be a bounded sequence in $L^2(\mathbb{R}^n)$. We denote by $\mathcal{V}(\mathtt{U_0})$ the set of weak limits of subsequences of the form $(\widetilde{\mathtt{H}}^\varepsilon)^{-1}\mathtt{U}_0^\varepsilon$ for some $\Gamma^\varepsilon=(h^\varepsilon,t^\varepsilon,x^\varepsilon,\xi^\varepsilon)$ in $\mathbb{R}_+\setminus\{0\}\times\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n$:

$$\mathcal{V}(\mathtt{U_0})=\left\{w-\lim_{k\to+\infty}(\widetilde{\mathtt{H}}^{\varepsilon_k})^{-1}\mathtt{U}_0^{\varepsilon_k}\ ;\ \varepsilon_k\underset{k\to+\infty}{\longrightarrow}0,\ \Gamma^{\varepsilon_k}\in\mathbb{R}_+\setminus\{0\}\times\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n\right\}.$$

We denote

$$\eta(\mathtt{U_0})=\sup\left\{\|\phi\|_{L^2(\mathbb{R}^n)}\quad;\quad\phi\in\mathcal{V}(\mathtt{U_0})\right\}.$$

We obviously have[1]

$$\eta(\mathtt{U_0})\leq\limsup_{\varepsilon\to 0}\|\mathtt{U}_0^\varepsilon\|_{L^2(\mathbb{R}^n)}\ .$$

[1] We have the lim sup – not the lim inf – because we consider all possible subsequences ε_k.

We prove that there exist a sequence $(\phi_j)_{j\geq 1}$ and a family of pairwise orthogonal sequences $\Gamma_j^\varepsilon = (h_j^\varepsilon, t_j^\varepsilon, x_j^\varepsilon, \xi_j^\varepsilon)$ such that, up to extracting a subsequence,

$$\mathtt{U}_0^\varepsilon = \sum_{j=1}^{\ell} \widetilde{\mathtt{H}}_j^\varepsilon(\phi_j) + \mathtt{V}_\ell^\varepsilon\,, \quad \text{with } \eta(\mathtt{V}_\ell) \underset{\ell\to+\infty}{\longrightarrow} 0\,, \tag{3.4}$$

and with the almost orthogonality identity,

$$\|\mathtt{U}_0^\varepsilon\|^2_{L^2(\mathbb{R}^n)} = \sum_{j=1}^{\ell} \|\phi_j\|^2_{L^2(\mathbb{R}^n)} + \|\mathtt{V}_\ell^\varepsilon\|^2_{L^2(\mathbb{R}^n)} + o(1) \quad \text{as } \varepsilon \to 0\,. \tag{3.5}$$

Indeed, if $\eta(\mathtt{U}_{\mathbf{0}}) = 0$, then we can take $\phi_j \equiv 0$ for all j. Otherwise, we choose $\phi_1 \in \mathcal{V}(\mathtt{U}_{\mathbf{0}})$ such that

$$\|\phi_1\|_{L^2(\mathbb{R}^n)} \geq \frac{1}{2}\eta(\mathtt{U}_{\mathbf{0}}) > 0\,.$$

By definition, there exists some sequence $\Gamma_1^\varepsilon = (h_1^\varepsilon, t_1^\varepsilon, x_1^\varepsilon, \xi_1^\varepsilon)$ such that, up to extracting a subsequence, we have

$$\left(\widetilde{\mathtt{H}}_1^\varepsilon\right)^{-1} \mathtt{U}_0^\varepsilon \rightharpoonup \phi_1\,.$$

We set $\mathtt{V}_1^\varepsilon = \mathtt{U}_0^\varepsilon - \widetilde{\mathtt{H}}_1^\varepsilon(\phi_1)$, and we get

$$\|\mathtt{U}_0^\varepsilon\|^2_{L^2(\mathbb{R}^n)} = \|\phi_1\|^2_{L^2(\mathbb{R}^n)} + \|\mathtt{V}_1^\varepsilon\|^2_{L^2(\mathbb{R}^n)} + o(1) \quad \text{as } \varepsilon \to 0\,.$$

Now, we replace $\mathtt{U}_0^\varepsilon$ with $\mathtt{V}_1^\varepsilon$, and repeat the same process. If $\eta(\mathtt{V}_{\mathbf{1}}) > 0$, we get ϕ_2, Γ_2^ε and $\mathtt{V}_{\mathbf{2}}$. Moreover, Γ_1^ε and Γ_2^ε are orthogonal. Otherwise, up to extracting a subsequence, we use Remark 3.1: $(\widetilde{\mathtt{H}}_2^\varepsilon)^{-1}\widetilde{\mathtt{H}}_1^\varepsilon \to \widetilde{\mathtt{H}}$ strongly as $\varepsilon \to 0$, where $\widetilde{\mathtt{H}}$ is isometric on $L^2(\mathbb{R}^n)$. Since

$$(\widetilde{\mathtt{H}}_2^\varepsilon)^{-1}\mathtt{V}_1^\varepsilon = \left((\widetilde{\mathtt{H}}_2^\varepsilon)^{-1}\widetilde{\mathtt{H}}_1^\varepsilon\right)(\widetilde{\mathtt{H}}_1^\varepsilon)^{-1}\mathtt{V}_1^\varepsilon$$

and $(\widetilde{\mathtt{H}}_1^\varepsilon)^{-1}\mathtt{V}_1^\varepsilon$ converges weakly to zero, this implies $\phi_2 \equiv 0$, hence $\eta(\mathtt{V}_{\mathbf{1}}) = 0$, which yields a contradiction.

Iterating this argument, a diagonal process yields a family of pairwise orthogonal sequences Γ_j^ε, and $(\phi_j)_{j\geq 1}$ satisfying (3.5). Since $(\mathtt{U}_0^\varepsilon)_{0<\varepsilon\leq 1}$ is bounded in $L^2(\mathbb{R}^n)$, (3.5) yields

$$\sum_{j=1}^{\ell} \|\phi_j\|^2_{L^2(\mathbb{R}^n)} \leq \limsup_{\varepsilon\to 0} \|\mathtt{U}_0^\varepsilon\|^2_{L^2(\mathbb{R}^n)}\,.$$

Since the bound is independent of $\ell \geq 1$, the series $\sum \|\phi_j\|^2_{L^2(\mathbb{R}^n)}$ is convergent, and

$$\|\phi_j\|_{L^2(\mathbb{R}^n)} \to 0 \quad \text{as } j \to +\infty\,.$$

Furthermore, we have by construction

$$\eta(\mathtt{V}_\ell) \leq \|\phi_{\ell-1}\|_{L^2(\mathbb{R}^n)}\,,$$

which yields (3.4).

When the initial data satisfy (1.13), we alter the above algorithm. We *impose* the lower bound on the scales and the boundedness of the cores in the Fourier side:

$$\widetilde{\mathcal{V}}(\mathtt{U}_{\mathbf{0}}) = \Big\{ w - \lim_{k\to+\infty} (\widetilde{\mathtt{H}}^{\varepsilon_k})^{-1}\mathtt{U}_0^{\varepsilon_k}\ ;\ \varepsilon_k \underset{k\to+\infty}{\longrightarrow} 0,\ \Gamma^{\varepsilon_k} \in [1,+\infty[\times\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n\,, \\ \text{with } |\xi^\varepsilon| \lesssim 1 \Big\}.$$

Note that the assumption $h^\varepsilon \geq 1$ is nothing but boundedness away from zero. Up to an ε-independent dilation of the profiles, we may always assume that a scale bounded away from zero is bounded from below by 1.

Repeating the same algorithm as above, the property (1.13) remains at each step. Note that the stronger assumption $\mathtt{U}_0^\varepsilon \in H^1$ is not stable: we may not have $\phi_1 \in H^1$. The same lines yield (3.4) and (3.5), with η replaced by $\widetilde{\eta}$ with the natural definition for $\widetilde{\eta}$. In particular, $\mathtt{V}_\ell^\varepsilon$ satisfies (1.13) for any $\ell \geq 1$.

Theorem 1.4 stems from the following proposition:

Proposition 3.2. *We assume $n = 1$. Let $(\mathtt{U}_0^\varepsilon)_{0<\varepsilon\leq 1}$ be a family of $L^2(\mathbb{R})$ such that*

$$\|\mathtt{U}_0^\varepsilon\|_{L^2(\mathbb{R})} \leq M \quad \textit{and} \quad \left\|e^{i\frac{t}{2}\partial_x^2}\mathtt{U}_0^\varepsilon\right\|_{L^6(\mathbb{R}_t\times\mathbb{R}_x)} \geq m > 0\,.$$

There exists $\Gamma^\varepsilon = (h^\varepsilon, t^\varepsilon, x^\varepsilon, \xi^\varepsilon)$ such that, up to a subsequence,

$$\left(\widetilde{\mathtt{H}}^\varepsilon\right)^{-1}(\mathtt{U}_0^\varepsilon) \rightharpoonup \phi\,, \quad \textit{where} \quad \left\|e^{i\frac{t}{2}\partial_x^2}\phi\right\|_{L^6(\mathbb{R}_t\times\mathbb{R}_x)} \geq \beta(m) > 0\,.$$

Moreover, if $(\mathtt{U}_0^\varepsilon)_{0<\varepsilon\leq 1}$ satisfies (1.13), then one can choose $h^\varepsilon \geq 1$ and $|\xi^\varepsilon| \lesssim 1$.

Remark. The dependence of β upon M is not mentioned in the above statement. Simply recall that from Strichartz inequality (2.3), $m \lesssim M$.

This proposition, together with (3.4), yields Theorem 1.4. Indeed, if $\mathtt{V}_\ell^\varepsilon = r_\ell^\varepsilon(0,x)$ was such that

$$\limsup_{\varepsilon\to 0}\left\|e^{i\frac{t}{2}\partial_x^2}\mathtt{V}_\ell^\varepsilon\right\|_{L^6(\mathbb{R}^2)} = \limsup_{\varepsilon\to 0}\|r_\ell^\varepsilon\|_{L^6(\mathbb{R}^2)} \not\to 0 \quad \text{as } \ell \to +\infty\,,$$

then there would exist $\ell_k \to +\infty$ as $k \to +\infty$, and $m > 0$, such that for any $k \in \mathbb{N}$,

$$\limsup_{\varepsilon\to 0}\left\|e^{i\frac{t}{2}\partial_x^2}\mathtt{V}_{\ell_k}^\varepsilon\right\|_{L^6(\mathbb{R}^2)} \geq m\,.$$

From (3.5), we have

$$\limsup_{\varepsilon\to 0}\left\|\mathtt{V}_{\ell_k}^\varepsilon\right\|_{L^2(\mathbb{R})} \leq \limsup_{\varepsilon\to 0}\|\mathtt{U}_0^\varepsilon\|_{L^2(\mathbb{R})} =: M\,.$$

From Proposition 3.2, there exists $\varphi_\ell \in \eta(\mathtt{V}_\ell)$ with

$$\left\|e^{i\frac{t}{2}\Delta}\varphi_{\ell_k}\right\|_{L^6(\mathbb{R}_t\times\mathbb{R}_x)} \geq \beta(m) > 0\,.$$

This implies $\eta(\mathtt{V}_{\ell_\mathbf{k}}) \geq \beta(m) > 0$ for any $k \in \mathbb{N}$, which contradicts (3.4).

Proof. The proof of Proposition 3.2 relies on several intermediary results. First, we extract scales h_j^ε and cores on the Fourier side ξ_j^ε and obtain a remainder arbitrarily small thanks to the refined Strichartz estimate.

Lemma 3.3. *Let $(\mathtt{U}_0^\varepsilon)_\varepsilon$ be a bounded sequence in $L^2(\mathbb{R})$. Then for every $\delta > 0$, there exist $N = N(\delta)$, a family $(h_j^\varepsilon, \theta_j^\varepsilon)_{1\leq j\leq N} \in \mathbb{R}_+ \setminus \{0\} \times \mathbb{R}$, and a family $(\mathbf{g}_j)_{1\leq j\leq N}$ of bounded sequences in $L^2(\mathbb{R})$ such that, up to a subsequence,*

(i) *If $j \neq k$, $\dfrac{h_j^\varepsilon}{h_k^\varepsilon} + \dfrac{h_k^\varepsilon}{h_j^\varepsilon} + \left|\theta_k^\varepsilon - \dfrac{h_k^\varepsilon}{h_j^\varepsilon}\theta_j^\varepsilon\right| \underset{\varepsilon\to 0}{\longrightarrow} +\infty$.*

(ii) *For every $1 \leq j \leq N$, there exists F_j bounded, compactly supported, such that*

$$\sqrt{h_j^\varepsilon}\left|\widehat{g}_j^\varepsilon\left(h_j^\varepsilon\xi + \theta_j^\varepsilon\right)\right| \leq F_j(\xi)\,. \tag{3.6}$$

(iii) *For every $\ell \geq 1$ and $x \in \mathbb{R}$,*

$$\mathtt{U}_0^\varepsilon = \sum_{j=1}^N g_j^\varepsilon + q^\varepsilon\,, \quad \textit{with} \quad \left\| e^{i\frac{t}{2}\partial_x^2} q^\varepsilon \right\|_{L^6(\mathbb{R}^2)} \leq \delta\,. \tag{3.7}$$

Moreover, we have the almost orthogonality identity:

$$\|\mathtt{U}_0^\varepsilon\|_{L^2}^2 = \sum_{j=1}^N \left\|g_j^\varepsilon\right\|_{L^2}^2 + \|q^\varepsilon\|_{L^2}^2 + o(1) \quad \textit{as } \varepsilon \to 0\,. \tag{3.8}$$

Proof of Lemma 3.3. For $\gamma^\varepsilon = (h^\varepsilon, \theta^\varepsilon) \in \mathbb{R}_+ \setminus \{0\} \times \mathbb{R}$, we define

$$\mathtt{G}^\varepsilon(f)(\xi) = \sqrt{h^\varepsilon} f^\varepsilon\left(h^\varepsilon \xi + \theta^\varepsilon\right)\,.$$

If $\left\| e^{i\frac{t}{2}\partial_x^2} \mathtt{U}_0^\varepsilon \right\|_{L^6(\mathbb{R}^2)} \leq \delta$, then nothing is to be proved. Otherwise, up to extracting a subsequence, $\left\| e^{i\frac{t}{2}\partial_x^2} \mathtt{U}_0^\varepsilon \right\|_{L^6(\mathbb{R}^2)} > \delta$. Apply Proposition 2.1 with $p = 4/3$; there exists a family of intervals $I^\varepsilon = [\theta^\varepsilon - h^\varepsilon, \theta^\varepsilon + h^\varepsilon]$ such that

$$\int_{I^\varepsilon} \left|\widehat{\mathtt{U}}_0^\varepsilon\right|^{4/3} \geq C\delta^4 (h^\varepsilon)^{1/3}\,,$$

where the constant C is uniform since $(\mathtt{U}_0^\varepsilon)_\varepsilon$ is bounded in L^2. For any $A > 0$, we have

$$\int_{I^\varepsilon \cap \{|\widehat{\mathtt{U}}_0^\varepsilon| > A\}} \left|\widehat{\mathtt{U}}_0^\varepsilon\right|^{4/3} \leq A^{-2/3} \left\|\widehat{\mathtt{U}}_0^\varepsilon\right\|_{L^2}^2\,.$$

Taking $A = C'/\left(\sqrt{h^\varepsilon}\delta^6\right)$ yields

$$\int_{I^\varepsilon \cap \{|\widehat{\mathtt{U}}_0^\varepsilon| \leq A\}} \left|\widehat{\mathtt{U}}_0^\varepsilon\right|^{4/3} \gtrsim \delta^4 (h^\varepsilon)^{1/3}\,.$$

From Hölder's inequality, we infer

$$\int_{I^\varepsilon \cap \{|\widehat{\mathtt{U}}_0^\varepsilon| \leq A\}} \left|\widehat{\mathtt{U}}_0^\varepsilon\right|^2 \geq C''\delta^6\,,$$

for some uniform constant C''. Define $\mathtt{v}_1^\varepsilon$ and γ_1^ε by

$$\widehat{\mathtt{v}}_1^\varepsilon = \widehat{\mathtt{U}}_0^\varepsilon \mathbb{1}_{I^\varepsilon \cap \{|\widehat{\mathtt{U}}_0^\varepsilon| \leq A\}}, \quad \gamma_1^\varepsilon = (h^\varepsilon, \theta^\varepsilon)\,.$$

We have

$$|\mathtt{G}_1^\varepsilon(\widehat{\mathtt{v}}_1^\varepsilon)(\xi)| \leq C(\delta)\mathbb{1}_{[-1,1]}(\xi)\,,$$

which is (3.6) with g_j^ε replaced by $\mathtt{v}_1^\varepsilon$. Furthermore,

$$\|\mathtt{U}_0^\varepsilon\|_{L^2}^2 = \|\mathtt{U}_0^\varepsilon - \mathtt{v}_1^\varepsilon\|_{L^2}^2 + \|\mathtt{v}_1^\varepsilon\|_{L^2}^2\,,$$

since the supports are disjoint from the Fourier side.

We repeat the same argument with $\mathtt{U}_0^\varepsilon - \mathtt{v}_1^\varepsilon$ in place of $\mathtt{U}_0^\varepsilon$. At each step, the L^2 norm decreases of at least $(C'')^{1/2}\delta^3$, with the same constant C'' as for the first step. After $N(\delta)$ steps, we obtain $(\mathtt{v}_j^\varepsilon)_{1 \leq j \leq N(\delta)}$ and $(\gamma_j^\varepsilon)_{1 \leq j \leq N(\delta)}$ satisfying (3.6), such that

$$\mathtt{U}_0^\varepsilon = \sum_{j=1}^{N(\delta)} \mathtt{v}_j^\varepsilon + q^\varepsilon\,, \quad \text{with } \left\| e^{i\frac{t}{2}\partial_x^2} q^\varepsilon \right\|_{L^6(\mathbb{R}^2)} \leq \delta\,, \tag{3.9}$$

and $\|\mathtt{U}_0^\varepsilon\|_{L^2}^2 = \sum_{j=1}^{N(\delta)} \left\|\mathtt{v}_j^\varepsilon\right\|_{L^2}^2 + \|q^\varepsilon\|_{L^2}^2 + o(1)$ as $\varepsilon \to 0$. However, the properties of the first point of the lemma need not be satisfied. To obtain these properties, we reorganize the decomposition. We say that γ_j^ε and γ_k^ε are orthogonal if

$$\frac{h_j^\varepsilon}{h_k^\varepsilon} + \frac{h_k^\varepsilon}{h_j^\varepsilon} + \left|\theta_k^\varepsilon - \frac{h_k^\varepsilon}{h_j^\varepsilon}\theta_j^\varepsilon\right| \to +\infty \quad \text{as } \varepsilon \to 0\,.$$

Define

$$g_1^\varepsilon = \sum_{j=1}^{N(\delta)} \mathtt{v}_j^\varepsilon - \sum_{\gamma_j^\varepsilon \perp \gamma_1^\varepsilon} \mathtt{v}_j^\varepsilon\,.$$

If there exists $2 \le j_0 \le N(\delta)$ such that $\gamma_{j_0}^\varepsilon$ is orthogonal to γ_1^ε, then we define

$$g_2^\varepsilon = \sum_{j=1}^{N(\delta)} \mathtt{v}_j^\varepsilon - \sum_{\substack{\gamma_j^\varepsilon \perp \gamma_1^\varepsilon \\ \gamma_j^\varepsilon \perp \gamma_{j_0}^\varepsilon}} \mathtt{v}_j^\varepsilon\,.$$

Repeating this argument a finite number of times, we rearrange the above sum. The almost orthogonality relation (3.8) holds, since the supports of the functions we consider are disjoint from the Fourier side. Finally, we must make sure that up to an extraction, the first point of the lemma is satisfied, and that (3.6) holds.

The $\mathtt{v}_j^\varepsilon$'s kept in the definition of g_1^ε are such that the γ_j^ε are not orthogonal one to another. It is sufficient to show that up to an extraction, $\mathtt{G}_1^\varepsilon(\mathtt{v}_j^\varepsilon)$ is bounded by a compactly supported bounded function, for such j's. By construction, $\mathtt{G}_j^\varepsilon(\mathtt{v}_j^\varepsilon)$ is bounded by a compactly supported bounded function; we have

$$\mathtt{G}_1^\varepsilon(\mathtt{G}_j^\varepsilon)^{-1} f(\xi) = \sqrt{\frac{h_1^\varepsilon}{h_j^\varepsilon}} f\left(\frac{h_1^\varepsilon}{h_j^\varepsilon}\xi + \theta_1^\varepsilon - \frac{h_1^\varepsilon}{h_j^\varepsilon}\theta_j^\varepsilon\right).$$

Since $\gamma_j^\varepsilon \not\perp \gamma_1^\varepsilon$, up to an extraction, $h_1^\varepsilon/h_j^\varepsilon \to \lambda_{1j} \in \mathbb{R}_+ \setminus \{0\}$, and $\theta_1^\varepsilon - \frac{h_1^\varepsilon}{h_j^\varepsilon}\theta_j^\varepsilon$ is bounded as $\varepsilon \to 0$, which yields the desired estimate for $\mathtt{G}_1^\varepsilon(\mathtt{v}_j^\varepsilon)$. Reasoning the same way for the other terms proves (i) and (ii), and completes the proof of the lemma. □

Next, we study sequences whose scale h^ε is fixed, equal to 1, and extract cores in space-time.

Proposition 3.4. *Let* $\mathbf{P} = (P^\varepsilon)_{0<\varepsilon\le 1}$ *be a sequence such that*

$$\left|\widehat{P^\varepsilon}(\xi)\right| \le F(\xi)\,, \tag{3.10}$$

where $F \in L^\infty(\mathbb{R})$ *is compactly supported. Then there exist a subsequence of* P^ε *(still denoted* P^ε*), a family* $(\mathbf{x}_\alpha, \mathbf{s}_\alpha)_{\alpha\ge 1}$ *of sequences in* $\mathbb{R}\times\mathbb{R}$*, and a sequence* $(\phi_\alpha)_{\alpha\ge 1}$ *of* L^2 *functions, such that:*

(i) *If* $\alpha \ne \beta$, $|x_\alpha^\varepsilon - x_\beta^\varepsilon| + |s_\alpha^\varepsilon - s_\beta^\varepsilon| \to +\infty$ *as* $\varepsilon \to 0$.

(ii) *For every $A \geq 1$ and every $x \in \mathbb{R}$, we have*

$$P^\varepsilon(x) = \sum_{\alpha=1}^{A} e^{-is_\alpha^\varepsilon \partial_x^2} \phi_\alpha(x - x_\alpha^\varepsilon) + P_A^\varepsilon(x)\,, \quad \textit{with}$$

$$\limsup_{\varepsilon\to 0} \left\| e^{i\frac{t}{2}\partial_x^2} P_A^\varepsilon \right\|_{L^6(\mathbb{R}^2)} \underset{A\to\infty}{\longrightarrow} 0\,, \quad \textit{and} \tag{3.11}$$

$$\|P^\varepsilon\|_{L^2}^2 = \sum_{\alpha=1}^{A} \|\phi_\alpha\|_{L^2}^2 + \|P_A^\varepsilon\|_{L^2}^2 + o(1) \quad \textit{as } \varepsilon \to 0\,. \tag{3.12}$$

Proof of Proposition 3.4. Let $\mathcal{W}(\mathbf{P})$ be the set of weak limits of subsequences of $\mathbf{P}$ after translation in the phase space:

$$\mathcal{W}(\mathbf{P}) = \left\{ w - \lim_{k\to+\infty} e^{is^{\varepsilon_k}\partial_x^2} P^\varepsilon(\cdot + x^{\varepsilon_k})\ ;\ \varepsilon_k \underset{k\to+\infty}{\longrightarrow} 0,\ (x^\varepsilon, s^\varepsilon) \in \mathbb{R}\times\mathbb{R} \right\}.$$

We denote

$$\mu(\mathbf{P}) = \sup\left\{ \|\phi\|_{L^2}\ ;\ \phi \in \mathcal{W}(\mathbf{P}) \right\}.$$

As in the beginning of this section, we have

$$\mu(\mathbf{P}) \leq \limsup_{\varepsilon\to 0} \|P^\varepsilon\|_{L^2}\ ,$$

and, up to extracting a subsequence, we can write

$$P^\varepsilon(x) = \sum_{\alpha=1}^{A} e^{-is_\alpha^\varepsilon \partial_x^2} \phi_\alpha(x - x_\alpha^\varepsilon) + P_A^\varepsilon(x)\,, \quad \mu(\mathbf{P}_A) \underset{A\to+\infty}{\longrightarrow} 0\,,$$

with the almost orthogonality identity (3.12). To complete the proof of Proposition 3.4, we have to prove (3.11).

Notice that the orthogonality argument yields a result more precise than (3.12): for every $\alpha \geq 1$ and every $\psi \in \mathcal{F}(C_0^\infty(\mathbb{R}))$,

$$\left\| \widehat{\psi}\widehat{P}^\varepsilon \right\|_{L^2}^2 = \sum_{\alpha=1}^{A} \left\| \widehat{\psi}\widehat{\phi}_\alpha \right\|_{L^2}^2 + \left\| \widehat{\psi}\widehat{P}_A^\varepsilon \right\|_{L^2}^2 + o(1) \quad \text{as } \varepsilon \to 0\,.$$

This fact, together with the assumption (3.10), proves that for every $A \geq 1$, $\widehat{P}_A^\varepsilon$ is supported in $\operatorname{supp} F$, and

$$\limsup_{\varepsilon\to 0} \left\| \widehat{P}_A^\varepsilon \right\|_{L^\infty} \leq \left\| \widehat{F} \right\|_{L^\infty}\ . \tag{3.13}$$

Introduce a cut-off $\chi(t,x) = \chi_1(t)\chi_2(x)$, with $\chi_j \in \mathcal{S}(\mathbb{R})$, such that

$$|\widehat{\chi}_1| + |\widehat{\chi}_2| \leq 2, \qquad \widehat{\chi}_2 \equiv 1 \ \text{ on } \operatorname{supp} F, \qquad \widehat{\chi}_1\left(\frac{-\xi^2}{2}\right) \equiv 1 \ \text{ on } \operatorname{supp} \widehat{\chi}_2\,.$$

Let $*$ denote the convolution in (t,x), and $\psi_A^\varepsilon(t,x) = e^{i\frac{t}{2}\partial_x^2} P_A^\varepsilon$. The function $\chi * \psi_A^\varepsilon$ solves the linear Schrödinger equation, so

$$\mathcal{F}\left(\chi * \psi_A^\varepsilon\big|_{t=0}\right)(\xi) = \widehat{\chi}_1\left(\frac{-\xi^2}{2}\right)\widehat{\chi}_2(\xi)\widehat{P_A^\varepsilon}(\xi) = \widehat{P_A^\varepsilon}(\xi)\,,$$

from the assumptions on χ_1 and χ_2. Therefore, $\chi * \psi_A^\varepsilon = \psi_A^\varepsilon$. We use a restriction result in space dimension 1 (see e.g. [30]): for every $4 < q < 6$ and every $\widehat{G} \in$

$L^\infty(B(0,R))$,

$$\left\| \int_{B(0,R)} e^{i\frac{t}{2}|\xi|^2+ix\cdot\xi}\widehat{G}(\xi)d\xi \right\|_{L^q(\mathbb{R}^2)} \leq C(q,R)\left\|\widehat{G}\right\|_{L^\infty}\,. \tag{3.14}$$

Fix $4<q<6$. Using (3.13) and (3.14), we have

$$\begin{aligned}\limsup_{\varepsilon\to 0}\|\chi*\psi_A^\varepsilon\|_{L^6(\mathbb{R}^2)} &\leq \limsup_{\varepsilon\to 0}\|\chi*\psi_A^\varepsilon\|_{L^q(\mathbb{R}^2)}^{q/6}\limsup_{\varepsilon\to 0}\|\chi*\psi_A^\varepsilon\|_{L^\infty(\mathbb{R}^2)}^{1-q/6}\\ &\leq \|F\|_{L^\infty(\mathbb{R})}^{q/6}\limsup_{\varepsilon\to 0}\|\chi*\psi_A^\varepsilon\|_{L^\infty(\mathbb{R}^2)}^{1-q/6}\,.\end{aligned}$$

On the other hand, the definition of $\mathcal{W}(\mathbf{P}_A)$ implies

$$\limsup_{\varepsilon\to 0}\|\chi*\psi_A^\varepsilon\|_{L^\infty(\mathbb{R}^2)} \leq \sup\left\{\left|\iint \chi(-t,-x)e^{i\frac{t}{2}\partial_x^2}\phi\,dxdt\right|\ ;\ \phi\in\mathcal{W}(\mathbf{P}_A)\right\}.$$

Using Hölder's inequality, then the Strichartz estimate, we obtain

$$\begin{aligned}\limsup_{\varepsilon\to 0}\|\chi*\psi_A^\varepsilon\|_{L^\infty(\mathbb{R}^2)} &\leq \|\chi\|_{L^{6/5}(\mathbb{R}^2)}\sup\left\{\left\|e^{i\frac{t}{2}\partial_x^2}\phi\right\|_{L^6(\mathbb{R}^2)}\ ;\ \phi\in\mathcal{W}(\mathbf{P}_A)\right\}\\ &\lesssim \|\chi\|_{L^{6/5}(\mathbb{R}^2)}\,\mu(\mathbf{P}_A)\,.\end{aligned}$$

Therefore,

$$\limsup_{\varepsilon\to 0}\left\|e^{i\frac{t}{2}\partial_x^2}P_A^\varepsilon\right\|_{L^6(\mathbb{R}^2)} \lesssim \mu(\mathbf{P}_A)^{1-\frac{q}{6}}\to 0 \quad \text{as } A\to+\infty\,,$$

which completes the proof of Proposition 3.4. □

We can now finish the proof of Proposition 3.2. Back to the decomposition (3.7), we set, for $1\leq j\leq N$,

$$P_j^\varepsilon(x) = e^{-ix\theta_j^\varepsilon/h_j^\varepsilon}\sqrt{h_j^\varepsilon}g_j^\varepsilon\left(h_j^\varepsilon x\right).$$

Since g_j^ε satisfies (3.6), the sequence $(P_j^\varepsilon)_{0<\varepsilon\leq 1}$ satisfies the assumptions of Proposition 3.4. Thus, for every $1\leq j\leq N$, there exists a family $(\phi_{j,\alpha})_{\alpha\geq 1}$ of L^2 functions, and a family $(y_{j,\alpha}^\varepsilon, s_{j,\alpha}^\varepsilon)\in\mathbb{R}\times\mathbb{R}$, such that

$$P_j^\varepsilon(x) = \sum_{\alpha=1}^A e^{-is_{j,\alpha}^\varepsilon\partial_x^2}\phi_{j,\alpha}\left(x-y_{j,\alpha}^\varepsilon\right)+P_{j,A}^\varepsilon(x)\,, \tag{3.15}$$

together with (3.11) and (3.12). For each $1\leq j\leq N$, choose A_j such that for $A\geq A_j$,

$$\limsup_{\varepsilon\to 0}\left\|e^{i\frac{t}{2}\partial_x^2}P_{j,A}^\varepsilon\right\|_{L^6(\mathbb{R}^2)} \leq \frac{\delta}{N}\,.$$

In terms of g_j^ε, (3.15) reads

$$g_j^\varepsilon = \sum_{\alpha=1}^A \widetilde{\mathtt{H}}_{j,\alpha}^\varepsilon\left(\phi_{j,\alpha}\right)+\mathtt{w}_{j,A}^\varepsilon\,,\quad \text{where}$$

$$\Gamma_{j,\alpha}^\varepsilon = \left(h_j^\varepsilon, 2s_{j,\alpha}^\varepsilon, h_j^\varepsilon y_{j,\alpha}^\varepsilon, \frac{\theta_j^\varepsilon}{h_j^\varepsilon}\right),\quad \mathtt{w}_{j,A}^\varepsilon(x) = \frac{e^{-ix\xi_j^\varepsilon}}{\sqrt{h_j^\varepsilon}}P_{j,A}^\varepsilon\left(\frac{x}{h_j^\varepsilon}\right).$$

Using (3.7), it follows that

$$\mathtt{U}_0^\varepsilon = \sum_{j=1}^N \left(\sum_{\alpha=1}^{A_j} \widetilde{\mathtt{H}}_{j,\alpha}^\varepsilon \left(\phi_{j,\alpha}\right) + \mathtt{w}_{j,A}^\varepsilon \right) + q^\varepsilon \,.$$

Relabeling the pairs (j,α), we get

$$\mathtt{U}_0^\varepsilon = \sum_{j=1}^K \widetilde{\mathtt{H}}_j^\varepsilon \left(\phi_j\right) + \mathtt{W}^\varepsilon \,,$$

where $K = \sum_{j=1}^N A_j$ and $\mathtt{W}^\varepsilon = \sum_{j=1}^N \mathtt{w}_{j,A_j}^\varepsilon + q^\varepsilon$. The remainder satisfies

$$\limsup_{\varepsilon\to 0} \left\| e^{i\frac{t}{2}\partial_x^2} \mathtt{W}^\varepsilon \right\|_{L^6(\mathbb{R}^2)} \leq 2\delta \,.$$

It is clear that the Γ_j^ε's are pairwise orthogonal. Combining (3.8) and (3.12), we obtain

$$\|\mathtt{U}_0^\varepsilon\|_{L^2}^2 = \sum_{j=1}^N \left(\sum_{\alpha=1}^{A_j} \|\phi_{j,\alpha}\|_{L^2}^2 + \left\|\mathtt{w}_{j,A}^\varepsilon\right\|_{L^2}^2 \right) + \|q^\varepsilon\|_{L^2}^2 + o(1) \quad \text{as } \varepsilon \to 0 \,.$$

Thus,

$$\sum_{j=1}^K \|\phi_j\|_{L^2}^2 \leq \limsup_{\varepsilon\to 0} \|\mathtt{U}_0^\varepsilon\|_{L^2}^2 \leq M^2 \,. \tag{3.16}$$

Since $\left\| e^{i\frac{t}{2}\partial_x^2} \mathtt{U}_0^\varepsilon \right\|_{L^6(\mathbb{R}^2)} \geq m > 0$, choose δ small enough so that

$$\frac{1}{2} \left\| e^{i\frac{t}{2}\partial_x^2} \mathtt{U}_0^\varepsilon \right\|_{L^6(\mathbb{R}^2)}^6 \leq \left\| \sum_{j=1}^K \mathtt{H}_j^\varepsilon \left(\phi_j\right) \right\|_{L^6(\mathbb{R}^2)}^6 \leq \left\| e^{i\frac{t}{2}\partial_x^2} \mathtt{U}_0^\varepsilon \right\|_{L^6(\mathbb{R}^2)}^6 \,.$$

A classical argument of orthogonality (see e.g. [13]) yields, as $\varepsilon \to 0$,

$$\left\| \sum_{j=1}^K \mathtt{H}_j^\varepsilon \left(\phi_j\right) \right\|_{L^6(\mathbb{R}^2)}^6 = \sum_{j=1}^K \left\| \mathtt{H}_j^\varepsilon \left(\phi_j\right) \right\|_{L^6(\mathbb{R}^2)}^6 + o(1) = \sum_{j=1}^K \left\| e^{i\frac{t}{2}\partial_x^2} \phi_j \right\|_{L^6(\mathbb{R}^2)}^6 + o(1) \,.$$

Let j_0 be such that

$$\left\| e^{i\frac{t}{2}\partial_x^2} \phi_{j_0} \right\|_{L^6(\mathbb{R}^2)} = \max_{1\leq j\leq K} \left\| e^{i\frac{t}{2}\partial_x^2} \phi_j \right\|_{L^6(\mathbb{R}^2)} \,.$$

Using the Strichartz estimate, we infer

$$\begin{aligned} \frac{m^6}{2} &\leq \sum_{j=1}^K \left\| e^{i\frac{t}{2}\partial_x^2} \phi_j \right\|_{L^6(\mathbb{R}^2)}^6 \leq \left\| e^{i\frac{t}{2}\partial_x^2} \phi_{j_0} \right\|_{L^6(\mathbb{R}^2)}^4 \sum_{j=1}^K \left\| e^{i\frac{t}{2}\partial_x^2} \phi_j \right\|_{L^6(\mathbb{R}^2)}^2 \\ &\lesssim \left\| e^{i\frac{t}{2}\partial_x^2} \phi_{j_0} \right\|_{L^6(\mathbb{R}^2)}^4 \sum_{j=1}^K \|\phi_j\|_{L^2(\mathbb{R}^2)}^2 \lesssim M^2 \left\| e^{i\frac{t}{2}\partial_x^2} \phi_{j_0} \right\|_{L^6(\mathbb{R}^2)}^4 \,, \end{aligned}$$

where the last estimate follows from (3.16). Thus,

$$\left\| e^{i\frac{t}{2}\partial_x^2} \phi_{j_0} \right\|_{L^6(\mathbb{R}^2)} \geq \beta \approx \frac{m^{3/2}}{M} \,.$$

The pairwise orthogonality of the Γ_j^ε's yields

$$\left(\widetilde{\mathtt{H}}_{j_0}^\varepsilon\right)^{-1}\mathtt{U}_0^\varepsilon \rightharpoonup \phi = \phi_{j_0} + \mathtt{W}\,,$$

where $\mathtt{W}$ is the weak limit of $(\widetilde{\mathtt{H}}_{j_0}^\varepsilon)^{-1}\mathtt{W}^\varepsilon$. Since

$$\left\|e^{i\frac{t}{2}\partial_x^2}\mathtt{W}\right\|_{L^6(\mathbb{R}^2)} \leq \limsup_{\varepsilon\to 0}\left\|e^{i\frac{t}{2}\partial_x^2}\mathtt{W}^\varepsilon\right\|_{L^6(\mathbb{R}^2)} \leq 2\delta\,,$$

we get

$$\left\|e^{i\frac{t}{2}\partial_x^2}\phi\right\|_{L^6(\mathbb{R}^2)} \geq \frac{\beta}{2}\,,$$

provided that $\delta > 0$ is sufficiently small. This completes the proof of Proposition 3.2 in the general case.

When $(\mathtt{U}_0^\varepsilon)_\varepsilon$ satisfies (1.13), there exists $R = R(\delta)$ such that for every ε,

$$\left\|\widehat{\mathtt{U}}_0^\varepsilon \mathbb{1}_{|\xi|\leq R}\right\|_{L^2(\mathbb{R})} \geq \left\|\widehat{\mathtt{U}}_0^\varepsilon\right\|_{L^2(\mathbb{R})} - \frac{\delta}{2}\,.$$

In the proof of Lemma 3.3 (this is the step where the scales h^ε and cores in the Fourier side appear), we can therefore consider $\widehat{\mathtt{U}}_0^\varepsilon \mathbb{1}_{|\xi|\leq R}$ in place of $\widehat{\mathtt{U}}_0^\varepsilon$. This implies that for any j, $-\theta_j^\varepsilon/h_j^\varepsilon$ (the center of the balls we extract) and $1/h_j^\varepsilon$ (the radius of the balls we extract) are uniformly bounded. This means exactly that the sequence $(\xi_j^\varepsilon)_\varepsilon$ is bounded for every j, and that h_j^ε is bounded away from zero. As mentioned already, up to an ε-independent dilation of the profiles ϕ_j, we deduce $h_j^\varepsilon \geq 1$. □

Remark 3.5. Why do we suppose $n = 1$ or 2 only? Essentially to have a refined Strichartz estimate, as in [24] in the case of space dimension two, and in Proposition 2.1 for the one-dimensional case. Note that the proof uses the fact that $2 + \frac{4}{n}$ is an even integer, to decompose the $L^{2+\frac{4}{n}}$ norm as a product. The restriction estimate (3.14) holds in higher dimensions. It is proved in [2] that if the space dimension is $n \geq 3$, then such an estimate holds for some $q < 2 + \frac{4}{n}$, which is what we use in the above computations (it holds more generally for $q > 2 + \frac{4}{n-1}$; see [31]).

4. Proof of Theorem 1.6: Nonlinear profile decomposition

Roughly speaking, Theorem 1.6 is essentially a consequence of Theorem 1.4 and of Strichartz inequalities, and is based on a perturbative analysis. This result has no exact counterpart in [22]. Note that one of the key ingredients is Theorem 1.4, and this is the only reason why we have to restrict the space dimension. Since the approach is very similar to [1, 18, 12], we shall only sketch the proof (see [17] for more details).

We prove the equivalence (i)⇔(ii); since the profiles $\mathtt{U}_j$ are given by Theorem 1.4 and Definition 1.5, and r_ℓ^ε is given by Theorem 1.4, only (1.14) has to be proved. It follows from the perturbative argument of the proof (i)⇔(ii).

(i)⇒(ii). Recall that I_j^ε is defined by $I_j^\varepsilon := (h_j^\varepsilon)^{-2}\left(I^\varepsilon - t_j^\varepsilon\right)$, and that $\mathtt{U}_j^\varepsilon$ is given by (1.15). We shall also denote $\mathtt{V}_j^\varepsilon$ for the functions defined as in (1.15), with $\mathtt{U}_j$ replaced by $e^{i\frac{t}{2}\Delta}\phi_j$, given by Theorem 1.4, that is, $\mathtt{V}_j^\varepsilon = \mathtt{H}_j^\varepsilon(\phi_j)$ (see (3.3)).

The function ρ_ℓ^ε is *defined* by $\rho_\ell^\varepsilon = \mathtt{U}^\varepsilon - \sum_{j=1}^\ell \mathtt{U}_j^\varepsilon - r_\ell^\varepsilon$. Denote $F(z) = \lambda |z|^{4/n} z$. The (expected) remainder ρ_ℓ^ε solves

$$i\partial_t \rho_\ell^\varepsilon + \frac{1}{2}\Delta \rho_\ell^\varepsilon = f_\ell^\varepsilon, \qquad \rho_\ell^\varepsilon\big|_{t=0} = \sum_{j=1}^\ell \left(\mathtt{V}_j^\varepsilon - \mathtt{U}_j^\varepsilon\right)\big|_{t=0},$$

where

$$f_\ell^\varepsilon = F\left(\rho_\ell^\varepsilon + \sum_{j=1}^\ell \mathtt{U}_j^\varepsilon + r_\ell^\varepsilon\right) - \sum_{j=1}^\ell F\left(\mathtt{U}_j^\varepsilon\right).$$

We use the orthogonality of the Γ_j^ε's and the assumption (i) to prove that (1.14) holds, that is, $\limsup_{\varepsilon\to 0}\left(\|\rho_\ell^\varepsilon\|_{L^\gamma(I^\varepsilon\times\mathbb{R}^n)} + \|\rho_\ell^\varepsilon\|_{L^\infty(I^\varepsilon;L^2(\mathbb{R}^n))}\right) \underset{\ell\to+\infty}{\longrightarrow} 0$.

Once proved, this property implies (ii), since for some ℓ_0 sufficiently large,

$$\limsup_{\varepsilon\to 0}\|\mathtt{U}^\varepsilon\|_{L^\gamma(I^\varepsilon\times\mathbb{R}^n)} \le \sum_{j=1}^{\ell_0}\limsup_{\varepsilon\to 0}\|\mathtt{U}_j\|_{L^\gamma(I_j^\varepsilon\times\mathbb{R}^n)} + 1 < +\infty,$$

by assumption (i). For $J^\varepsilon = [a^\varepsilon, b^\varepsilon] \subset I^\varepsilon$, Strichartz inequalities yield

$$\|\rho_\ell^\varepsilon\|_{L^\gamma(J^\varepsilon\times\mathbb{R}^n)} + \|\rho_\ell^\varepsilon\|_{L^\infty(J^\varepsilon;L^2(\mathbb{R}^n))} \lesssim \|\rho_\ell^\varepsilon(a^\varepsilon)\|_{L^2(\mathbb{R}^n)} + \|f_\ell^\varepsilon\|_{L^{\gamma'}(J^\varepsilon\times\mathbb{R}^n)}.$$

From triangle and Hölder's inequalities,

$$\|f_\ell^\varepsilon\|_{L^{\gamma'}(J^\varepsilon\times\mathbb{R}^n)} \lesssim \|\rho_\ell^\varepsilon\|_{L^\gamma}^\gamma + \left\|\sum_{j=1}^\ell \mathtt{U}_j^\varepsilon + r_\ell^\varepsilon\right\|_{L^\gamma}^{\gamma-1}\|\rho_\ell^\varepsilon\|_{L^\gamma} \tag{4.1}$$

$$+\left\|\sum_{j=1}^\ell F(\mathtt{U}_j^\varepsilon) - F\left(\sum_{j=1}^\ell \mathtt{U}_j^\varepsilon\right)\right\|_{L^{\gamma'}} + \left\|F\left(\sum_{j=1}^\ell \mathtt{U}_j^\varepsilon + r_\ell^\varepsilon\right) - F\left(\sum_{j=1}^\ell \mathtt{U}_j^\varepsilon\right)\right\|_{L^{\gamma'}}. \tag{4.2}$$

The terms in (4.2) are small by assumption (i), Hölder's inequality and orthogonality (see for instance [13], and [7] when γ is not an integer). The first term in (4.1) is treated by a bootstrap argument. We have to take care of the second term in (4.1). The next lemma is proved in [17]. It allows us to absorb this linear term, thanks to a suitable partition of the interval I^ε.

Lemma 4.1. *For every $\delta > 0$, there exists an ε–dependent* finite *partition of I^ε,*

$$I^\varepsilon = \bigcup_{k=1}^{p(\delta)} J_k^\varepsilon,$$

such that for every $1 \le k \le p(\delta)$ and every $\ell \ge 1$,

$$\limsup_{\varepsilon\to 0}\left\|\sum_{j=1}^\ell \mathtt{U}_j^\varepsilon\right\|_{L^\gamma(J_k^\varepsilon\times\mathbb{R}^n)} \le \delta.$$

Sketch of the proof. By orthogonality, for every $\ell \ge 1$,

$$\limsup_{\varepsilon\to 0}\left\|\sum_{j=1}^\ell \mathtt{U}_j^\varepsilon\right\|_{L^\gamma(I^\varepsilon\times\mathbb{R}^n)}^\gamma = \sum_{j=1}^\ell \limsup_{\varepsilon\to 0}\|\mathtt{U}_j^\varepsilon\|_{L^\gamma(I^\varepsilon\times\mathbb{R}^n)}^\gamma.$$

On the other hand, the almost L^2–orthogonality (1.12) and the conservation of mass for (1.2) imply that for some $\ell(\delta)$,

$$\|\mathtt{U}_j\|_{L^2(\mathbb{R}^n)} \le \delta, \quad \forall j \ge \ell(\delta).$$

Using global existence results for small L^2 data (see e.g. [8]), $\mathtt{U}_j$ is then defined globally in time, and from the Strichartz estimate,

$$\|\mathtt{U}_j\|_{L^\gamma(\mathbb{R}\times\mathbb{R}^n)} \lesssim \|\mathtt{U}_j\|_{L^2(\mathbb{R}^n)} = \|\phi_j\|_{L^2(\mathbb{R}^n)}\,.$$

Since $\gamma > 2$ for any $n \geq 1$, we infer

$$\sum_{j\geq \ell(\delta)} \|\mathtt{U}_j\|^\gamma_{L^\gamma(\mathbb{R}\times\mathbb{R}^n)} < +\infty\,.$$

Using this and orthogonality, we infer

$$\limsup_{\varepsilon\to 0} \left\| \sum_{j=1}^{\ell} \mathtt{U}_j^\varepsilon \right\|^\gamma_{L^\gamma(I^\varepsilon\times\mathbb{R}^n)} \leq \sum_{j=1}^{\ell(\delta)} \limsup_{\varepsilon\to 0} \left\|\mathtt{U}_j^\varepsilon\right\|^\gamma_{L^\gamma(I^\varepsilon\times\mathbb{R}^n)} + \frac{\delta}{2}\,.$$

Thus, it suffices to construct a family of partial decompositions as in the statement of the lemma, for every $1 \leq j \leq \ell(\delta)$ and such that

$$\limsup_{\varepsilon\to 0} \left\|\mathtt{U}_j^\varepsilon\right\|^\gamma_{L^\gamma(J_k^\varepsilon\times\mathbb{R}^n)} \leq \frac{\delta}{2\ell(\delta)}\,, \quad \forall 1 \leq k \leq p(\delta)\,.$$

The final decomposition is obtained by intersecting all the partial ones. Consider the case $j = 1$, and denote by I_1 the maximal interval of existence of $\mathtt{U}_1$. One checks that there exists a closed interval J_1 such that

$$\limsup_{\varepsilon\to 0} I_1^\varepsilon = J_1\,, \quad \|\mathtt{U}_1\|_{L^\gamma(J_1\times\mathbb{R}^n)} < +\infty\,.$$

We decompose J_1 as $J_1 = \bigcup_{k=1}^{p_1(\delta)} J_{1k}$ so that

$$\|\mathtt{U}_1\|_{L^\gamma(J_{1k}\times\mathbb{R}^n)} < \frac{\delta}{2\ell(\delta)}, \quad \forall 1 \leq k \leq p_1(\delta)\,.$$

At this first step, the intervals J_1^ε are then obtained by scaling

$$J_k^\varepsilon = I^\varepsilon \cap \left((h_1^\varepsilon)^2 J_{1k} + t_1^\varepsilon\right)\,.$$

Repeating this argument on each J_k^ε a finite number of times yields the lemma. □

Choosing $\delta > 0$ sufficiently small, Lemma 4.1 allows us to prove that

$$\limsup_{\varepsilon\to 0} \left(\|\rho_\ell^\varepsilon\|_{L^\gamma(I^\varepsilon\times\mathbb{R}^n)} + \|\rho_\ell^\varepsilon\|_{L^\infty(I^\varepsilon;L^2(\mathbb{R}^n))} \right) \underset{\ell\to+\infty}{\longrightarrow} 0\,,$$

thanks to an absorption argument for the linear term (4.1), orthogonality in the source term (4.2), and a bootstrap argument.

<u>(ii)⇒(i)</u>. By assumption, there exists $M > 0$ such that

$$\limsup_{\varepsilon\to 0} \|\mathtt{U}^\varepsilon\|_{L^\gamma(I^\varepsilon\times\mathbb{R}^n)} \leq \frac{M}{2}\,.$$

Assume that (i) does not hold. Reorganizing the family of profiles, we may assume that for some $\ell_0 \geq 1$, $\mathtt{U}_j$ is not global, that is, $\|\mathtt{U}_j\|_{L^\gamma(\mathbb{R}\times\mathbb{R}^n)} = \infty$ – if $1 \leq j \leq \ell_0$, and $\mathtt{U}_j$ is global for $j > \ell_0$. Indeed, if all the profiles are defined globally in time, the problem is trivial. Thus, we only have to consider a finite family of profiles, thanks to the small data global existence results mentioned above.

Let I_j denote the maximal interval of existence of $\mathtt{U}_j$, for $1 \le j \le \ell_0$. The failure of (i) means that there exists some intervals $I_j(M)$ such that

$$\frac{-t_j^\varepsilon}{(h_j^\varepsilon)^2} \in I_j(M) \subset I_j \cap I_j^\varepsilon \quad \text{for } \varepsilon \ll 1 \ ; \ M \le \|\mathtt{U}_j\|_{L^\gamma(I_j(M)\times\mathbb{R}^n)} < \infty \,, \quad 1 \le j \le \ell_0 \,.$$

Denote $I_j^\varepsilon(M) = (h_j^\varepsilon)^2 I_j(M) + t_j^\varepsilon$. Then $0 \in I_j^\varepsilon(M) \subset I^\varepsilon$ for $\varepsilon \ll 1$ and

$$M \le \limsup_{\varepsilon\to 0} \left\|\mathtt{U}_j^\varepsilon\right\|_{L^\gamma(I_j^\varepsilon(M)\times\mathbb{R}^n)} < \infty \,, \quad 1 \le j \le \ell_0 \,. \tag{4.3}$$

By permutation, extraction of a subsequence and considering the backward and inward problems separately, we may take

$$I_1^\varepsilon(M) \subset I_2^\varepsilon(M) \subset \ldots \subset I_{\ell_0}^\varepsilon(M) \,.$$

We infer

$$\left\|\mathtt{U}_j^\varepsilon\right\|_{L^\gamma(I_1^\varepsilon(M)\times\mathbb{R}^n)} < \infty \,, \quad 1 \le j \le \ell_0 \,.$$

We have

$$\limsup_{\varepsilon\to 0} \|\mathtt{U}^\varepsilon\|_{L^\gamma(I_1^\varepsilon(M)\times\mathbb{R}^n)} \le \limsup_{\varepsilon\to 0} \|\mathtt{U}^\varepsilon\|_{L^\gamma(I^\varepsilon\times\mathbb{R}^n)} \le \frac{M}{2} \,. \tag{4.4}$$

Since $\mathtt{U}_j$ is global for $j > \ell_0$, (i) is satisfied with I^ε replaced by $I_1^\varepsilon(M)$, and the first part of the proof yields (1.14). By orthogonality,

$$\begin{aligned} \limsup_{\varepsilon\to 0} \|\mathtt{U}^\varepsilon\|^\gamma_{L^\gamma(I_1^\varepsilon(M)\times\mathbb{R}^n)} &= \limsup_{\ell\to\infty} \left(\limsup_{\varepsilon\to 0} \Big\| \sum_{j=1}^{\ell} \mathtt{U}_j^\varepsilon \Big\|^\gamma_{L^\gamma(I_1^\varepsilon(M)\times\mathbb{R}^n)} \right) \\ &= \sum_{j=1}^{\infty} \limsup_{\varepsilon\to 0} \left\|\mathtt{U}_j^\varepsilon\right\|^\gamma_{L^\gamma(I_1^\varepsilon(M)\times\mathbb{R}^n)} \,. \end{aligned} \tag{4.5}$$

In particular, (4.4) and (4.5) yield

$$\limsup_{\varepsilon\to 0} \|\mathtt{U}_1^\varepsilon\|_{L^\gamma(I_1^\varepsilon(M)\times\mathbb{R}^n)} \le \frac{M}{2} \,,$$

which contradicts (4.3). Thus (i) holds, and we saw in the first part of the proof that it implies (1.14).

5. Proof of Theorem 1.2: Linearizability

Using the scaling (2.1), we restate Theorem 1.2. Define

$$\mathtt{U}_0^\varepsilon := \mathtt{U}^\varepsilon_{|t=0} \ \text{ and } V^\varepsilon := e^{i\frac{t}{2}\Delta}\mathtt{U}_0^\varepsilon \,.$$

Then u^ε and $\mathtt{U}^\varepsilon$ are simultaneously linearizable on I^ε in L^2. Moreover, u^ε is linearizable on I^ε in H^1_ε if and only if $\mathtt{U}^\varepsilon$ is linearizable on I^ε in $H^1_{\sqrt{\varepsilon}}$. We now have to prove:

Theorem 5.1. *Assume $n = 1$ or 2. Let $\mathtt{U}_0^\varepsilon$ be bounded in $L^2(\mathbb{R}^n)$, and let $I^\varepsilon \ni 0$ be a time interval.*

- *$\mathtt{U}^\varepsilon$ is linearizable on I^ε in L^2 if and only if*

$$\limsup_{\varepsilon\to 0} \|\mathtt{V}^\varepsilon\|^{2+4/n}_{L^{2+4/n}(I^\varepsilon\times\mathbb{R}^n)} = 0 \,. \tag{5.1}$$

- *Assume in addition that $\mathtt{U}_0^\varepsilon \in H^1$ and $\mathtt{U}_0^\varepsilon$ is bounded in $H^1_{\sqrt{\varepsilon}}$. Then $\mathtt{U}^\varepsilon$ is linearizable on I^ε in $H^1_{\sqrt{\varepsilon}}$ if and only if (5.1) holds.*

Proof. We first prove that condition (5.1) is sufficient for linearizability, thanks to the classical Strichartz estimates. In particular, no restriction on the space dimension is necessary at this stage. Denote $\mathtt{W}^\varepsilon = \mathtt{U}^\varepsilon - \mathtt{V}^\varepsilon$. It solves

$$i\partial_t \mathtt{W}^\varepsilon + \frac{1}{2}\Delta \mathtt{W}^\varepsilon = \lambda |\mathtt{U}^\varepsilon|^{4/n}\mathtt{U}^\varepsilon\,, \quad \mathtt{W}^\varepsilon_{|t=0} = 0\,. \tag{5.2}$$

Since $\gamma = 2 + 4/n$, we have

$$\frac{1}{\gamma'} = \frac{1}{\gamma} + \frac{4/n}{\gamma}\,.$$

Applying Strichartz estimate (2.4) to (5.2), along with Hölder's inequality, we have, for $t \in I^\varepsilon$,

$$\begin{aligned}\|\mathtt{W}^\varepsilon\|_{L^\gamma([0,t]\times\mathbb{R}^n)} &\lesssim \||\mathtt{U}^\varepsilon|^{4/n}\mathtt{U}^\varepsilon\|_{L^{\gamma'}([0,t]\times\mathbb{R}^n)} \lesssim \|\mathtt{U}^\varepsilon\|^{1+\frac{4}{n}}_{L^\gamma([0,t]\times\mathbb{R}^n)}\\ &\lesssim \|\mathtt{V}^\varepsilon\|^{1+\frac{4}{n}}_{L^\gamma(I^\varepsilon\times\mathbb{R}^n)} + \|\mathtt{W}^\varepsilon\|^{1+\frac{4}{n}}_{L^\gamma([0,t]\times\mathbb{R}^n)}\,.\end{aligned}$$

Using assumption (5.1), we apply a bootstrap argument: for ε sufficiently small,

$$\|\mathtt{W}^\varepsilon\|_{L^\gamma(I^\varepsilon\times\mathbb{R}^n)} \lesssim \|\mathtt{V}^\varepsilon\|^{1+\frac{4}{n}}_{L^\gamma(I^\varepsilon\times\mathbb{R}^n)}\,.$$

We infer that for ε sufficiently small,

$$\|\mathtt{U}^\varepsilon\|_{L^\gamma(I^\varepsilon\times\mathbb{R}^n)} \lesssim \|\mathtt{V}^\varepsilon\|_{L^\gamma(I^\varepsilon\times\mathbb{R}^n)} + \|\mathtt{V}^\varepsilon\|^{1+\frac{4}{n}}_{L^\gamma(I^\varepsilon\times\mathbb{R}^n)}\,, \tag{5.3}$$

and (5.1) holds with $\mathtt{V}^\varepsilon$ replaced by $\mathtt{U}^\varepsilon$. Applying the second part of Strichartz estimate (2.4) yields

$$\|\mathtt{W}^\varepsilon\|_{L^\infty(I^\varepsilon;L^2(\mathbb{R}^n))} \lesssim \||\mathtt{U}^\varepsilon|^{4/n}\mathtt{U}^\varepsilon\|_{L^{\gamma'}(I^\varepsilon\times\mathbb{R}^n)} \lesssim \|\mathtt{U}^\varepsilon\|^{1+\frac{4}{n}}_{L^\gamma(I^\varepsilon\times\mathbb{R}^n)} \underset{\varepsilon\to 0}{\longrightarrow} 0\,,$$

which is linearizability on I^ε in L^2.

Now assume that $\mathtt{U}^\varepsilon_0 \in H^1(\mathbb{R}^n)$ is bounded in $H^1_{\sqrt{\varepsilon}}$. Differentiating (5.2) with respect to the space variable, we have

$$\begin{aligned}\|\sqrt{\varepsilon}\nabla_x \mathtt{W}^\varepsilon\|_{L^\gamma(I^\varepsilon\times\mathbb{R}^n)} &\lesssim \||\mathtt{U}^\varepsilon|^{4/n}\sqrt{\varepsilon}\nabla_x\mathtt{U}^\varepsilon\|_{L^{\gamma'}(I^\varepsilon\times\mathbb{R}^n)}\\ &\lesssim \|\mathtt{U}^\varepsilon\|^{4/n}_{L^\gamma(I^\varepsilon\times\mathbb{R}^n)}\|\sqrt{\varepsilon}\nabla_x\mathtt{U}^\varepsilon\|_{L^\gamma(I^\varepsilon\times\mathbb{R}^n)}\\ &\lesssim \|\mathtt{U}^\varepsilon\|^{4/n}_{L^\gamma(I^\varepsilon\times\mathbb{R}^n)}\left(\|\sqrt{\varepsilon}\nabla_x\mathtt{V}^\varepsilon\|_{L^\gamma(I^\varepsilon\times\mathbb{R}^n)} + \|\sqrt{\varepsilon}\nabla_x\mathtt{W}^\varepsilon\|_{L^\gamma(I^\varepsilon\times\mathbb{R}^n)}\right)\,.\end{aligned}$$

From (5.3) and (5.1), the term in $\sqrt{\varepsilon}\nabla_x\mathtt{W}^\varepsilon$ on the right-hand side can be absorbed by the left-hand side for ε sufficiently small. The uniform boundedness of $\sqrt{\varepsilon}\nabla_x\mathtt{V}^\varepsilon$ in $L^\gamma(\mathbb{R}\times\mathbb{R}^n)$, which stems from the boundedness of its data in L^2 and Strichartz estimate (2.3), shows that

$$\|\sqrt{\varepsilon}\nabla_x\mathtt{W}^\varepsilon\|_{L^\gamma(I^\varepsilon\times\mathbb{R}^n)} \underset{\varepsilon\to 0}{\longrightarrow} 0\,.$$

Applying inhomogeneous Strichartz estimate (2.4) yields

$$\|\sqrt{\varepsilon}\nabla_x\mathtt{W}^\varepsilon\|_{L^\infty(I^\varepsilon;L^2(\mathbb{R}^n))} \lesssim \|\mathtt{U}^\varepsilon\|^{4/n}_{L^\gamma(I^\varepsilon\times\mathbb{R}^n)}\|\sqrt{\varepsilon}\nabla_x\mathtt{U}^\varepsilon\|_{L^\gamma(I^\varepsilon\times\mathbb{R}^n)} \underset{\varepsilon\to 0}{\longrightarrow} 0\,,$$

which proves that $\mathtt{U}^\varepsilon$ is linearizable on I^ε in $H^1_{\sqrt{\varepsilon}}$.

We complete the proof of Theorem 5.1 by showing that condition (5.1) is necessary for linearizability in L^2 (hence for linearizability in $H^1_{\sqrt{\varepsilon}}$). The proof relies on the profile decompositions stated in Theorems 1.4 and 1.6. We consider two cases.

First case. The family $(\mathtt{U}^\varepsilon)_{0<\varepsilon\leq 1}$ is uniformly bounded in $L^\gamma(I^\varepsilon\times\mathbb{R}^n)$. In this case, we can use Theorems 1.4 and 1.6 to deduce the following lemma.

Lemma 5.2. *Assume $n=1$ or 2. Let $\mathtt{U}_0^\varepsilon$ be bounded in $L^2(\mathbb{R}^n)$, let $I^\varepsilon=[0,T^\varepsilon[$ be a (possibly unbounded) time interval, and assume that $\mathtt{U}^\varepsilon$ is bounded in $L^\gamma(I^\varepsilon\times\mathbb{R}^n)$. Then up to the extraction of a subsequence, there exist an orthogonal family $(h_j^\varepsilon,t_j^\varepsilon,x_j^\varepsilon,\xi_j^\varepsilon)_{j\in\mathbb{N}}$ in $\mathbb{R}_+\setminus\{0\}\times\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n$ and a family $(\phi_j)_{j\in\mathbb{N}}$ bounded in $L^2(\mathbb{R}^n)$, such that if $\mathtt{V}_j=e^{i\frac{t}{2}\Delta}\phi_j$ and $\mathtt{U}_j$ is given by Definition 1.5, we have*

$$\limsup_{\varepsilon\to 0}\|\mathtt{U}^\varepsilon-\mathtt{V}^\varepsilon\|^\gamma_{L^\gamma(I^\varepsilon\times\mathbb{R}^n)}=\sum_{j=1}^\infty\limsup_{\varepsilon\to 0}\|\mathtt{U}_j-\mathtt{V}_j\|^\gamma_{L^\gamma(I_j^\varepsilon\times\mathbb{R}^n)}\,, \tag{5.4}$$

where $I_j^\varepsilon=(h_j^\varepsilon)^{-2}(I^\varepsilon-t_j^\varepsilon)$. In addition, for every fixed $\varepsilon>0$, none of the terms in the series is zero.

Proof of Lemma 5.2. From Theorems 1.4 and 1.6, there exist an orthogonal family $(h_j^\varepsilon,t_j^\varepsilon,x_j^\varepsilon,\xi_j^\varepsilon)_{j\in\mathbb{N}}$ in $\mathbb{R}_+\setminus\{0\}\times\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n$ and a family $(\phi_j)_{j\in\mathbb{N}}$ bounded in $L^2(\mathbb{R}^n)$, such that if $\mathtt{V}_j=e^{i\frac{t}{2}\Delta}\phi_j$ and $\mathtt{U}_j$ is given by Definition 1.5 (up to the extraction of a subsequence), we have, for any $\ell\in\mathbb{N}$,

$$\mathtt{U}^\varepsilon(t,x)-\mathtt{V}^\varepsilon(t,x)=\sum_{j=1}^\ell\mathtt{H}_j^\varepsilon\left(\mathtt{U}_j\big|_{t=0}-\mathtt{V}_j\big|_{t=0}\right)(t,x)+\rho_\ell^\varepsilon(t,x)\,, \tag{5.5}$$

with $\limsup_{\varepsilon\to 0}\|\rho_\ell^\varepsilon\|_{L^\gamma(I^\varepsilon\times\mathbb{R}^n)}\to 0$ as $\ell\to+\infty$. The scales h_j^ε, cores $(t_j^\varepsilon,x_j^\varepsilon,\xi_j^\varepsilon)$ and initial profiles ϕ_j are the same for $\mathtt{U}^\varepsilon$ and $\mathtt{V}^\varepsilon$, since they are given by the profile decomposition for the initial data $\mathtt{U}^\varepsilon_{|t=0}=\mathtt{V}^\varepsilon_{|t=0}=\mathtt{U}_0^\varepsilon$. Since the family $(h_j^\varepsilon,t_j^\varepsilon,x_j^\varepsilon,\xi_j^\varepsilon)_{j\in\mathbb{N}}$ is orthogonal, we have, for any ℓ,

$$\begin{aligned}\limsup_{\varepsilon\to 0}\|\mathtt{U}^\varepsilon-\mathtt{V}^\varepsilon\|^\gamma_{L^\gamma(I^\varepsilon\times\mathbb{R}^n)}=&\sum_{j=1}^\ell\limsup_{\varepsilon\to 0}\|\mathtt{U}_j-\mathtt{V}_j\|^\gamma_{L^\gamma(I_j^\varepsilon\times\mathbb{R}^n)}\\&+\limsup_{\varepsilon\to 0}\|\rho_\ell^\varepsilon\|^\gamma_{L^\gamma(I^\varepsilon\times\mathbb{R}^n)}.\end{aligned}$$

Letting $\ell\to+\infty$ yields (5.4). Now assume that for a fixed $\varepsilon>0$, one of the terms in the series (5.4) is zero. This means that two solutions of the nonlinear Schrödinger equation (1.2) and of the Schrödinger equation respectively coincide on the nontrivial time interval $I_{j_0}^\varepsilon$. Uniqueness for these two equations shows that necessarily $\mathtt{U}_{j_0}=\mathtt{V}_{j_0}\equiv 0$, in which case the family $(\mathtt{U}_j,\mathtt{V}_j)_j$ can be relabeled to avoid null terms. □

Definition 5.3. Let $\delta^\varepsilon>0$ and $a^\varepsilon\in\mathbb{R}$. We say that the interval $]a^\varepsilon,a^\varepsilon+\delta^\varepsilon[$ is *asymptotically trivial* in either of the following cases:

- $a^\varepsilon\to+\infty$ as $\varepsilon\to 0$, or
- $a^\varepsilon+\delta^\varepsilon\to-\infty$ as $\varepsilon\to 0$, or
- $\delta^\varepsilon\to 0$ as $\varepsilon\to 0$.

Lemma 5.4. *Under the assumptions of Lemma* 5.2*, if $\|\mathtt{U}^\varepsilon-\mathtt{V}^\varepsilon\|_{L^\gamma(I^\varepsilon\times\mathbb{R}^n)}\to 0$ as $\varepsilon\to 0$, then $\|\mathtt{V}^\varepsilon\|_{L^\gamma(I^\varepsilon\times\mathbb{R}^n)}\to 0$ as $\varepsilon\to 0$.*

Proof of Lemma 5.4. From Lemma 5.2, if $\|\mathtt{U}^\varepsilon - \mathtt{V}^\varepsilon\|_{L^\gamma(I^\varepsilon\times\mathbb{R}^n)} \to 0$, then every interval I_j^ε is asymptotically trivial. The profile decomposition for $\mathtt{V}^\varepsilon$ yields

$$\mathtt{V}^\varepsilon(t,x) = \sum_{j=1}^{\ell} \mathtt{H}_j^\varepsilon\left(\mathtt{V}_j\right)(t,x) + r_\ell^\varepsilon(t,x)\,,$$

with $\limsup_{\varepsilon\to 0}\|r_\ell^\varepsilon\|_{L^\gamma(I^\varepsilon\times\mathbb{R}^n)} \to 0$ as $\ell\to+\infty$. Fix $\ell\in\mathbb{N}$. We infer from the orthogonality of $(h_j^\varepsilon, t_j^\varepsilon, x_j^\varepsilon, \xi_j^\varepsilon)_{j\in\mathbb{N}}$ that

$$\limsup_{\varepsilon\to 0}\|\mathtt{V}^\varepsilon\|^\gamma_{L^\gamma(I^\varepsilon\times\mathbb{R}^n)} = \sum_{j=1}^{\ell}\limsup_{\varepsilon\to 0}\|\mathtt{V}_j\|^\gamma_{L^\gamma(I_j^\varepsilon\times\mathbb{R}^n)} + \limsup_{\varepsilon\to 0}\|r_\ell^\varepsilon\|^\gamma_{L^\gamma(I^\varepsilon\times\mathbb{R}^n)}\,.$$

Since all the intervals I_j^ε are asymptotically trivial, every term in the sum is zero, and we have

$$\limsup_{\varepsilon\to 0}\|\mathtt{V}^\varepsilon\|_{L^\gamma(I^\varepsilon\times\mathbb{R}^n)} = \limsup_{\varepsilon\to 0}\|r_\ell^\varepsilon\|_{L^\gamma(I^\varepsilon\times\mathbb{R}^n)}\,.$$

Since the left-hand side is independent of ℓ, we conclude that both terms are zero, which completes the proof of Lemma 5.4. □

We can now complete the proof of Theorem 5.1 in the case where the family $(\mathtt{U}^\varepsilon)_{0<\varepsilon\le 1}$ is uniformly bounded in $L^\gamma(I^\varepsilon\times\mathbb{R}^n)$. Assume that $\mathtt{U}^\varepsilon$ is linearizable on I^ε in L^2. From Lemma 5.4, it is enough to prove that

$$\|\mathtt{U}^\varepsilon - \mathtt{V}^\varepsilon\|_{L^\gamma(I^\varepsilon\times\mathbb{R}^n)} \underset{\varepsilon\to 0}{\longrightarrow} 0\,.$$

If it were not so, then from Lemma 5.2, there would exist j_0 such that the interval $I_{j_0}^\varepsilon$ is not asymptotically trivial. Up to the extraction of a subsequence, we can assume that there exist $a<b$ independent of ε and ε_0 such that for $0<\varepsilon\le\varepsilon_0$,

$$]a,b[\subset I_{j_0}^\varepsilon = \left[-\frac{t_{j_0}^\varepsilon}{(h_{j_0}^\varepsilon)^2}, \frac{T^\varepsilon - t_{j_0}^\varepsilon}{(h_{j_0}^\varepsilon)^2}\right[\,.$$

Let $\ell > j_0$. Apply the operator $(\mathtt{H}_{j_0}^\varepsilon)^{-1}$ to (5.5), and take the weak limit in $\mathcal{D}'(]a,b[\times\mathbb{R}^n)$. By orthogonality,

$$w\text{–}\lim(\mathtt{H}_{j_0}^\varepsilon)^{-1}(\mathtt{U}^\varepsilon - \mathtt{V}^\varepsilon) = (\mathtt{U}_{j_0} - \mathtt{V}_{j_0})\mathbf{1}_{]a,b[}(t) + w\text{–}\lim(\mathtt{H}_{j_0}^\varepsilon)^{-1}\rho_\ell^\varepsilon\,. \tag{5.6}$$

Denote $\mathtt{w}_\ell := w\text{–}\lim(H_{j_0}^\varepsilon)^{-1}\rho_\ell^\varepsilon$. We have

$$\|\mathtt{w}_\ell\|_{L^\gamma(]a,b[\times\mathbb{R}^n)} \le \liminf_{\varepsilon\to 0}\|(\mathtt{H}_{j_0}^\varepsilon)^{-1}\rho_\ell^\varepsilon\|_{L^\gamma(]a,b[\times\mathbb{R}^n)} \le \liminf_{\varepsilon\to 0}\|\rho_\ell^\varepsilon\|_{L^\gamma(I^\varepsilon\times\mathbb{R}^n)} \underset{\ell\to+\infty}{\longrightarrow} 0\,.$$

In (5.6), $\mathtt{w}_\ell$ is the only term possibly depending on ℓ, therefore it is zero, and $w\text{–}\lim(\mathtt{H}_{j_0}^\varepsilon)^{-1}(\mathtt{U}^\varepsilon - \mathtt{V}^\varepsilon) \neq 0$. Since $\mathtt{H}_{j_0}^\varepsilon$ is unitary on $L^2(\mathbb{R}^n)$, we have, for $0<\varepsilon\le\varepsilon_0$,

$$\left\|(\mathtt{H}_{j_0}^\varepsilon)^{-1}(\mathtt{U}^\varepsilon - \mathtt{V}^\varepsilon)\right\|_{L^\infty(]a,b[;L^2(\mathbb{R}^n))} \le \|\mathtt{U}^\varepsilon - \mathtt{V}^\varepsilon\|_{L^\infty(I^\varepsilon;L^2(\mathbb{R}^n))}\,.$$

The right-hand side goes to zero as $\varepsilon\to 0$ since $\mathtt{U}^\varepsilon$ is assumed to be linearizable on I^ε in L^2. Therefore the left hand side goes to zero. This is impossible, since the weak limit is not zero. This contradiction shows that we can apply Lemma 5.4, and complete the proof of Theorem 5.1 in the case where the family $(\mathtt{U}^\varepsilon)_{0<\varepsilon\le 1}$ is uniformly bounded in $L^\gamma(I^\varepsilon\times\mathbb{R}^n)$.

Second case. There exists a subsequence of $(\mathtt{U}^\varepsilon)_{0<\varepsilon\le 1}$, still denoted $\mathtt{U}^\varepsilon$, such that

$$\|\mathtt{U}^\varepsilon\|_{L^\gamma(I^\varepsilon\times\mathbb{R}^n)} \underset{\varepsilon\to 0}{\longrightarrow} +\infty\,.$$

Then there exists $\tau^\varepsilon \in I^\varepsilon$ such that for every $\varepsilon \in]0,1]$,

$$\|\mathtt{U}^\varepsilon\|_{L^\gamma([0,\tau^\varepsilon[\times\mathbb{R}^n)} = 1\,. \tag{5.7}$$

We can mimic the proof of the first case on the time interval $[0,\tau^\varepsilon[$. Lemma 5.4 shows that $\|\mathtt{U}^\varepsilon\|_{L^\gamma([0,\tau^\varepsilon[\times\mathbb{R}^n)} \to 0$ as $\varepsilon \to 0$, which contradicts (5.7). Therefore the second case never occurs, and the proof of Theorem 5.1 is complete. □

Remark. The above proof of linearizability relies on the profile decompositions (linear and nonlinear). Note that in [7], the proof of linearizability used only the conservations of mass and energy, and Strichartz inequalities. Only after the linearizability criterion had been proved was a (linear) profile decomposition used.

6. Obstructions to linearizability

6.1. **Profile decomposition.** In this subsection, we show how to deduce Corollary 1.7 from Theorem 1.4.

Resuming the scaling (2.1), (1.16) is exactly the result given by the first part of Theorem 1.4 on the time interval $[0,T]$ when considering the trace $t=0$. We use the first part of Theorem 1.4 because Theorem 1.2 reduces our problem to the study of a solution to the *linear* Schrödinger equation. Note that even if we considered a defocusing nonlinearity ($\lambda=+1$), with u_0^ε bounded in H^1_ε, we could not claim that $\mathtt{U}^\varepsilon$ is uniformly bounded in $L^\gamma([0,T]\times\mathbb{R}^n)$. This is because we do not know that H^1 solutions to (1.2) with $\lambda=+1$ decay like solutions to the free equations as time goes to infinity (this is known in Σ); this issue is related to the asymptotic completeness of wave operators in H^1.

Working with the functions $\mathtt{U}^\varepsilon$ and $\mathtt{V}^\varepsilon$, (1.16) writes

$$\begin{gathered}\mathtt{U}_0^\varepsilon(x) = \sum_{j=1}^{\ell} \widetilde{\mathtt{H}}_j^\varepsilon(\phi_j)(x) + \mathtt{W}_\ell^\varepsilon(x)\,,\\ \text{with } \limsup_{\varepsilon\to 0}\left\|e^{i\frac{t}{2}\Delta}\mathtt{W}_\ell^\varepsilon\right\|^{2+4/n}_{L^{2+4/n}(\mathbb{R}\times\mathbb{R}^n)} \underset{\ell\to+\infty}{\longrightarrow} 0\,.\end{gathered} \tag{6.1}$$

From (3.3),

$$\left\|e^{i\frac{t}{2}\Delta}\widetilde{\mathtt{H}}_j^\varepsilon(\phi_j)\right\|_{L^\gamma([0,T]\times\mathbb{R}^n)} = \|\mathtt{V}_j\|_{L^\gamma(I_j^\varepsilon\times\mathbb{R}^n)}\,,\ \text{with } I_j^\varepsilon = \left[\frac{-t_j^\varepsilon}{(h_j^\varepsilon)^2}, \frac{T-t_j^\varepsilon}{(h_j^\varepsilon)^2}\right]\,.$$

If I_j^ε is asymptotically trivial, then $\widetilde{\mathtt{H}}_j^\varepsilon(\phi_j)$ can be incorporated into the remainder term $\mathtt{W}_\ell^\varepsilon$, a case which can be excluded, up to relabeling our family of sequences. This means that we can assume

$$\frac{-t_j^\varepsilon}{(h_j^\varepsilon)^2} \not\to +\infty\,,\quad \frac{T-t_j^\varepsilon}{(h_j^\varepsilon)^2} \not\to -\infty\,,\quad \text{and}\quad \frac{T}{(h_j^\varepsilon)^2} \not\to 0\,.$$

The first two points imply the properties on t_j^ε stated in Corollary 1.7. We infer from the last point that h_j^ε is bounded, by 1 up to the extraction of a subsequence and an ε-independent dilation of the profiles ϕ_j.

Now suppose that $u_0^\varepsilon \in H^1$ and is bounded in H^1_ε. Then for every j, $\xi_j^\varepsilon = \mathcal{O}(\varepsilon^{-1/2})$ as $\varepsilon \to 0$. To see this, introduce the scaling

$$\psi^\varepsilon(t,x) = \varepsilon^{n/2} u^\varepsilon(\varepsilon t, \varepsilon x)\,. \tag{6.2}$$

The function ψ^ε solves (1.2), and the family $(\psi^\varepsilon(0,x))_{0<\varepsilon\leq 1}$ is bounded in H^1. The plane oscillations in the decomposition for ψ^ε are, from (6.2), $e^{ix\cdot\xi_j^\varepsilon\sqrt{\varepsilon}}$. From the second point of Theorem 1.4, we infer that $\xi_j^\varepsilon\sqrt{\varepsilon}=\mathcal{O}(1)$. We also deduce the lower bound $h_j^\varepsilon\geq\sqrt{\varepsilon}$.

Finally, (1.17) is obtained from (1.16) *via* the classical formula (see e.g. [27])

$$e^{i\frac{t}{2}\Delta}\phi=e^{in\frac{\pi}{4}}e^{i\frac{x^2}{2t}}\frac{1}{|t|^{n/2}}\widehat{\phi}\left(\frac{x}{t}\right)+o(1)\quad\text{in } L^2(\mathbb{R}^n),\ \text{as } t\to-\infty\,. \tag{6.3}$$

6.2. **Nonlinear superposition.** We now assume $\lambda=+1$. The decomposition (1.16) is necessary for the nonlinear term in (1.6) to have a leading order influence on finite term intervals. The aim of this section is to provide an argument suggesting that it is sufficient. As mentioned before, the gap between belief and proof is related to the asymptotic completeness of wave operators in H^1.

Suppose the initial data u_0^ε has the form (1.16) for a fixed ℓ and a linearizable remainder: there exists $T>0$ such that

$$\begin{aligned}
&u_0^\varepsilon(x)=\sum_{j=1}^{\ell}\widetilde{H}_j^\varepsilon(\phi_j)(x)+w^\varepsilon(x)\,,\\
&\text{where } \widetilde{H}_j^\varepsilon(\phi_j)(x)=e^{ix\cdot\xi_j^\varepsilon/\sqrt{\varepsilon}}e^{-i\varepsilon\frac{t_j^\varepsilon}{2}\Delta}\left(\frac{1}{(h_j^\varepsilon\sqrt{\varepsilon})^{n/2}}\phi_j\left(\frac{x-x_j^\varepsilon}{h_j^\varepsilon\sqrt{\varepsilon}}\right)\right)\\
&\text{and } \limsup_{\varepsilon\to0}\varepsilon\|e^{i\varepsilon\frac{t}{2}\Delta}w^\varepsilon\|^{2+4/n}_{L^{2+4/n}([0,T]\times\mathbb{R}^n)}=0\,.
\end{aligned}\tag{6.4}$$

If we assume that $\phi_j\in\Sigma$ for every $j\in\{1,\dots,\ell\}$, then we can take advantage of the global well-posedness and the existence of a complete scattering theory for (1.2) in Σ when $\lambda=+1$. Moreover, we may assume that $t_j^\varepsilon/(h_j^\varepsilon)^2$ converges as $\varepsilon\to0$ for every j. Let v_j^ε be the solution of the initial value problem

$$i\varepsilon\partial_t v_j^\varepsilon+\frac{1}{2}\varepsilon^2\Delta v_j^\varepsilon=\varepsilon^2|v_j^\varepsilon|^{4/n}v_j^\varepsilon,\qquad v_{j|t=0}^\varepsilon=\widetilde{H}_j^\varepsilon(\phi_j)\,. \tag{6.5}$$

For every j, the following asymptotics holds in $L^\infty(\mathbb{R};L^2)$ as ε goes to zero:

$$v_j^\varepsilon(t,x)=e^{i\frac{x\cdot\xi_j^\varepsilon}{\sqrt{\varepsilon}}-i\frac{t}{2}(\xi_j^\varepsilon)^2}\frac{1}{(h_j^\varepsilon\sqrt{\varepsilon})^{n/2}}\mathtt{V}_j\left(\frac{t-t_j^\varepsilon}{(h_j^\varepsilon)^2},\frac{x-x_j^\varepsilon-t\xi_j^\varepsilon}{h_j^\varepsilon\sqrt{\varepsilon}}\right)+o(1),$$

where $\mathtt{V}_j$ is given by

$$i\partial_t\mathtt{V}_j+\frac{1}{2}\Delta\mathtt{V}_j=|\mathtt{V}_j|^{4/n}\mathtt{V}_j,\qquad e^{-i\frac{t}{2}\Delta}\mathtt{V}_j(t)\big|_{t=-\lim t_j^\varepsilon/(h_j^\varepsilon)^2}=\phi_j\,.$$

Note that the above problem may be an initial value problem or a scattering problem according to the value of $\lim t_j^\varepsilon/(h_j^\varepsilon)^2$. We see that v_j^ε has a genuine nonlinear behavior on $[0,T]$ if $\lim t_j^\varepsilon/(h_j^\varepsilon)^2\neq-\infty$ and $\lim(T-t_j^\varepsilon)/(h_j^\varepsilon)^2\neq-\infty$ (compare with Corollary 1.7).

Following the lines of [1] and [7], the next result can be shown, thanks to the linearizability criterion given by Theorem 1.2. We leave out the proof here, for it bears no new idea.

Theorem 6.1. *Assume $n=1$ or 2, $\lambda=+1$, and let u_0^ε be given by* (6.4) *with $\phi_j\in\Sigma$ and an orthogonal family $(h_j^\varepsilon,t_j^\varepsilon,x_j^\varepsilon,\xi_j^\varepsilon)_{j\in\mathbb{N}}$ such that $\sqrt{\varepsilon}\leq h_j^\varepsilon\leq1$. Then*

the following asymptotics hold in $L^\infty([0,T];L^2)$ *as* ε *goes to zero,*

$$u^\varepsilon = \sum_{j=1}^{\ell} v_j^\varepsilon + e^{i\varepsilon\frac{t}{2}\Delta} w^\varepsilon + o(1)\,,$$

where each v_j^ε *solves* (6.5).

7. Blowing up solutions

Assume $n=1$ or 2. Let $\mathtt{U}$ be an L^2-solution to (1.2) which blows up[2] at time $T>0$ (not before),

$$\int_0^T\!\!\int_{\mathbb{R}^n} |\mathtt{U}(t,x)|^{2+\frac{4}{n}}\,dxdt = +\infty\,. \tag{7.1}$$

Let $(t_k)_{k\in\mathbb{N}}$ be a sequence going to T as $k\to+\infty$, with $t_k<T$ for every k. Denote $\varepsilon_k = T-t_k$, and define

$$u^\varepsilon(t,x) = \mathtt{U}(\varepsilon t + T - \varepsilon, x)\,,$$

where the notation ε stands for ε_k. Then u^ε solves (1.6). The function $\mathtt{U}$ blows up at time T if and only if u^ε is not linearizable on $[0,1]$ (in L^2), from Theorem 1.2 and its proof. The function v^ε is given by

$$v^\varepsilon(t,x) = e^{i\varepsilon\frac{t}{2}\Delta} u_0^\varepsilon(x) = e^{i\varepsilon\frac{t}{2}\Delta}\mathtt{U}(T-\varepsilon,x)\,.$$

Define

$$\mathtt{V}^\varepsilon(t,x) = v^\varepsilon\left(\frac{t-T}{\varepsilon}+1, x\right)\,.$$

Since u^ε is not linearizable on $[0,1]$, we have $\liminf_{\varepsilon\to0}\varepsilon\|v^\varepsilon\|^\gamma_{L^\gamma([0,1]\times\mathbb{R}^n)}>0$. From Corollary 1.7, up to the extraction of a subsequence,

$$u_0^\varepsilon(x) = \sum_{j=1}^{\ell}\widetilde{H}_j^\varepsilon(\phi_j)(x) + w_\ell^\varepsilon(x)\,,$$

$$\text{where } \widetilde{H}_j^\varepsilon(\phi_j)(x) = e^{ix\cdot\xi_j^\varepsilon/\sqrt{\varepsilon}} e^{-i\varepsilon\frac{t_j^\varepsilon}{2}\Delta}\left(\frac{1}{(h_j^\varepsilon\sqrt{\varepsilon})^{n/2}}\phi_j\left(\frac{x-x_j^\varepsilon}{h_j^\varepsilon\sqrt{\varepsilon}}\right)\right)$$

$$\text{and } \limsup_{\varepsilon\to0}\varepsilon\|e^{i\varepsilon\frac{t}{2}\Delta}w_\ell^\varepsilon\|^{2+4/n}_{L^{2+4/n}(\mathbb{R}\times\mathbb{R}^n)} \underset{\ell\to+\infty}{\longrightarrow} 0\,.$$

Recall that from (3.3),

$$\widetilde{H}_j^\varepsilon(\phi_j)(x) = e^{ix\cdot\xi_j^\varepsilon/\sqrt{\varepsilon}}\frac{1}{(h_j^\varepsilon\sqrt{\varepsilon})^{n/2}}\mathtt{V}_j\left(\frac{-t_j^\varepsilon}{(h_j^\varepsilon)^2}, \frac{x-x_j^\varepsilon}{h_j^\varepsilon\sqrt{\varepsilon}}\right),\ \text{where } \mathtt{V}_j(t) = e^{i\frac{t}{2}\Delta}\phi_j\,.$$

Moreover, we can assume

$$\frac{-t_j^\varepsilon}{(h_j^\varepsilon)^2}\not\to+\infty\,,\quad \frac{1-t_j^\varepsilon}{(h_j^\varepsilon)^2}\not\to-\infty\,,\quad \text{and}\quad \frac{1}{(h_j^\varepsilon)^2}\not\to 0\,,$$

[2]The general consensus is that even in the L^2 framework, this can occur only in the attractive case $\lambda<0$.

for otherwise, the corresponding profile may be incorporated into the remainder w_ℓ^ε. This implies that for every j, $(h_j^\varepsilon)_{j\in\mathbb{N}}$ and $(t_j^\varepsilon)_{j\in\mathbb{N}}$ are bounded sequences. Up to extracting a subsequence, we distinguish two cases:

$$\frac{t_j^\varepsilon}{(h_j^\varepsilon)^2} \to \lambda \in \mathbb{R} \text{ as } \varepsilon \to 0\,, \quad \text{or} \quad \frac{t_j^\varepsilon}{(h_j^\varepsilon)^2} \to +\infty\,.$$

In the first case, we set $y_j^k = \xi_j^\varepsilon/\sqrt{\varepsilon}$, $x_j^k = x_j^\varepsilon$, $\rho_j^k = h_j^\varepsilon\sqrt{\varepsilon} = h_j^\varepsilon\sqrt{T-t_k} \le \sqrt{T-t_k}$ and $\mathtt{U}_j = \mathtt{V}_j(-\lambda)$. In the second case, we infer from (6.3) that in L^2,

$$\widetilde{H}_j^\varepsilon(\phi_j)(x) \underset{\varepsilon\to 0}{\sim} e^{in\frac{\pi}{4}+ix\cdot\xi_j^\varepsilon/\sqrt{\varepsilon}} e^{-i\frac{|x-x_j^\varepsilon|^2}{2\varepsilon t_j^\varepsilon}} \left(\frac{h_j^\varepsilon}{t_j^\varepsilon\sqrt{\varepsilon}}\right)^{n/2} \widehat{\phi}_j\left(\frac{h_j^\varepsilon}{t_j^\varepsilon\sqrt{\varepsilon}}(x-x_j^\varepsilon)\right)\,.$$

We set $y_j^k = \xi_j^\varepsilon/\sqrt{\varepsilon}$, $x_j^k = x_j^\varepsilon$, and $\widetilde{\mathtt{U}}_j = e^{in\pi/4}\widehat{\phi}_j$, and the proof of Corollary 1.8 is complete, up to relabeling the family of sequences and possibly taking some $\mathtt{U}_j$ or some $\widetilde{\mathtt{U}}_j$ equal to zero.

Remark. When only one profile is present, quadratic oscillations are not relevant near the blow up time. Assume

$$\begin{aligned} u_0^\varepsilon(x) &= \widetilde{H}^\varepsilon(\phi)(x) + w^\varepsilon(x)\,,\\ \text{where } \widetilde{H}^\varepsilon(\phi)(x) &= e^{ix\cdot\xi^\varepsilon/\sqrt{\varepsilon}} e^{-i\varepsilon\frac{t^\varepsilon}{2}\Delta}\left(\frac{1}{(h^\varepsilon\sqrt{\varepsilon})^{n/2}}\phi\left(\frac{x-x^\varepsilon}{h^\varepsilon\sqrt{\varepsilon}}\right)\right)\\ \text{and } \limsup_{\varepsilon\to 0} \varepsilon\|e^{i\varepsilon\frac{t}{2}\Delta}w^\varepsilon\|^{2+4/n}_{L^{2+4/n}(\mathbb{R}\times\mathbb{R}^n)} &\underset{\ell\to+\infty}{\longrightarrow} 0\,. \end{aligned}$$

Since there is blow up at time T,

$$\liminf_{\varepsilon\to 0} \varepsilon\|v^\varepsilon\|^\gamma_{L^\gamma([0,1]\times\mathbb{R}^n)} > 0\,. \tag{7.2}$$

On the other hand, we also have

$$\liminf_{\varepsilon\to 0} \|\mathtt{V}^\varepsilon\|^\gamma_{L^\gamma([0,T-\varepsilon]\times\mathbb{R}^n)} = \liminf_{\varepsilon\to 0} \varepsilon\|v^\varepsilon\|^\gamma_{L^\gamma([1-T/\varepsilon,0]\times\mathbb{R}^n)} > 0\,. \tag{7.3}$$

If this limit was zero, then v^ε would be linearizable in L^2 on $[1-T/\varepsilon, 0]$, and

$$\liminf_{\varepsilon\to 0} \varepsilon\|u^\varepsilon\|^\gamma_{L^\gamma([1-T/\varepsilon,0]\times\mathbb{R}^n)} = 0 = \liminf_{\varepsilon\to 0} \|\mathtt{U}\|^\gamma_{L^\gamma([0,T-\varepsilon]\times\mathbb{R}^n)}\,,$$

which contradicts (7.1). Recall

$$\widetilde{H}^\varepsilon(\phi)(x) = e^{ix\cdot\xi^\varepsilon/\sqrt{\varepsilon}}\frac{1}{(h^\varepsilon\sqrt{\varepsilon})^{n/2}}\mathtt{V}\left(\frac{-t^\varepsilon}{(h_j^\varepsilon)^2}, \frac{x-x^\varepsilon}{h^\varepsilon\sqrt{\varepsilon}}\right), \text{ where } \mathtt{V}(t) = e^{i\frac{t}{2}\Delta}\phi\,.$$

From (7.2), we have $-t^\varepsilon/(h_j^\varepsilon)^2 \not\to +\infty$, and from (7.3), $-t^\varepsilon/(h_j^\varepsilon)^2 \not\to -\infty$. Therefore, up to an extraction, $-t^\varepsilon/(h_j^\varepsilon)^2 \to \lambda \in \mathbb{R}$, and we are left with a profile only, and no quadratic oscillation.

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