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# A NONCOMMUTATIVE VERSION OF THE JOHN-NIRENBERG THEOREM

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Abstract. We prove a noncommutative version of the John-Nirenberg theorem for nontracial filtrations of von Neumann algebras. As an application, we obtain an analogue of the classical large deviation inequality for elements of the associated $BMO$ space.

## 1. Introduction

The John-Nirenberg theorem is an important tool in analysis and probability. It provides a characterization of $BMO$, the space of functions of *bounded mean oscillation*. In its first version it was proved by John and Nirenberg [13] in 1961. Through its connection with the theory of Hardy spaces, the John-Nirenberg result has many applications in harmonic analysis and the theory of singular integrals (see, e.g., Stein [37, 38]). We refer to the books of Bennett and Sharpley [5], Garcia-Cuerva and Rubio de Francia [9], Garnett [10] and Koosis [22] for the interval (function space) version, respectively to the work of Bass [1], Garsia [11] and Petersen [29] for the martingale version of this theorem.

In this paper we analyze analogues of the John-Nirenberg results in a noncommutative setting. We first recall the classical results. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $(\mathcal{F}_n)_{n\geq 0}$ an increasing sequence of sub-$\sigma$-algebras of $\mathcal{F}$. For $1 \leq p < \infty$ consider the norms

$$\|x\|_{BMO_p} := \sup_n \|\mathbb{E}((|x - x_{n-1}|^p)|\mathcal{F}_n)\|_\infty^{\frac{1}{p}}, \tag{1}$$

where $x_n = \mathbb{E}(x|\mathcal{F}_n)$, for all nonnegative integers $n$. The usual $BMO$ norm corresponds to $p = 2$ above, i.e., $\|x\|_{BMO} = \|x\|_{BMO_2}$. The John-Nirenberg theorem yields universal constants $C_1, C_2 > 0$ with the following property. If $\|x\|_{BMO_1} < C_2$, then

$$\sup_n \|\mathbb{E}(e^{C_1|x-x_{n-1}|}|\mathcal{F}_n)\|_\infty < 1. \tag{2}$$

Using the power series expansion for the exponential function, it follows that (2) is equivalent to

$$\|\mathbb{E}(|x - x_{n-1}|^p|\mathcal{F}_n)\|_\infty^{\frac{1}{p}} \leq Cp\, \|x\|_{BMO}, \tag{3}$$

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for all $n \geq 0$ and all $1 \leq p < \infty$, where $C > 0$ is a universal constant. Furthermore, a standard duality argument yields the equality

$$\|\mathbb{E}(|x - x_{n-1}|^p|\mathcal{F}_n)\|_\infty^{\frac{1}{p}} = \sup_{a \in L_p(\mathcal{F}_n),\, \|a\|_p \leq 1} \|(x - x_{n-1})a\|_p \,. \tag{4}$$

It can be easily seen that condition (3) implies that, for $1 \leq p < \infty$,

$$\|x\|_p \leq Cp\, \|x\|_{BMO} \,. \tag{5}$$

Moreover, the John-Nirenbeg theorem follows from the validity of (5) for arbitrary probability measures and filtrations. Indeed, let $1 \leq p < \infty$, and fix an integer $n \geq 0$. Consider a positive element $a \in L_p(\mathcal{F}_n)$ with $\|a\|_p = 1$, and denote by $\mu_a$ the probability measure defined by $d\mu_a = a^{\frac{1}{p}} d\mathbb{P}$. Then

$$\|(x - x_{n-1})a\|_p = \|x - x_{n-1}\|_{L_p(\mu_a)} \,. \tag{6}$$

Denote by $BMO((\mathcal{F}_k)_{k \geq n}, \mu_a)$ the $BMO$ space associated to the triplet $(\Omega, \mathcal{F}, \mu_a)$ and the filtration $(\mathcal{F}_k)_{k \geq n}$ of $\mathcal{F}$. Applying (5) for the probability measure $\mu_a$ and the new filtration $(\mathcal{F}_k)_{k \geq n}$ of $\mathcal{F}$, it follows that

$$\|x - x_{n-1}\|_{L_p(\mu_a)} \leq Cp\, \|x - x_{n-1}\|_{BMO((\mathcal{F}_k)_{k \geq n}, \mu_a)} \,. \tag{7}$$

Note that $\|x - x_{n-1}\|_{BMO((\mathcal{F}_k)_{k \geq n}, \mu_a)} \leq \|x\|_{BMO((\mathcal{F}_k)_{k \geq 0}, \mathbb{P})}$. An application of (6), together with formula (4) yields now the inequality (3).

In the form (5), one can formulate a noncommutative version of the John-Nirenberg theorem. First this requires an appropriate definition of $BMO$ spaces. In their seminal paper [32], Pisier and Xu proved the noncommutative analogues of the Burkholder-Gundy square function inequalities. Fermionic versions of the square function inequalities had previously been considered by Carlen and Krée [4]. In [32], Pisier and Xu also introduced the $BMO$ space for noncommutative martingales and proved the analogue of the classical Fefferman-Stein duality $BMO = (H_1)^*$. Their work triggered a rapid development in the $L_p$-theory of noncommutative martingales; see, e.g., Junge [14], Junge and Xu [18, 19, 20] and Randrianantoanina [34, 35].

In the following we assume that $\mathcal{N}$ is a von Neumann algebra, $\phi$ a normal faithful state on $\mathcal{N}$ and $(\mathcal{N}_n)_{n \geq 0}$ an increasing sequence of von Neumann subalgebras whose union generates $\mathcal{N}$ in the w*-topology. Moreover, we assume that for all positive integers $n$ there exist normal conditional expectations $\mathcal{E}_n : \mathcal{N} \to \mathcal{N}_n$ such that

$$\phi(\mathcal{E}_n(x)y) = \phi(xy) \tag{8}$$

for all $x \in \mathcal{N}$ and $y \in \mathcal{N}_n$. We recall the following definitions (see [18], [32]):

$$\|x\|_{BMO^c} \;=\; \sup_m \sup_{n \leq m} \|\mathcal{E}_n((x_m - x_{n-1})^*(x_m - x_{n-1}))\|_\infty^{\frac{1}{2}} \;,$$

where $x_n = \mathcal{E}_n(x)$, for all $n \geq 0$. Respectively,

$$\|x\|_{BMO} \;=\; \max\{\|x\|_{BMO^c}, \|x^*\|_{BMO^c}\} \,.$$

The conditioned $L_\infty$-spaces, $L_\infty^c(\mathcal{N}, \mathcal{E}_n)$, associated to the conditional expectations $\mathcal{E}_n$ were introduced in [14] (see also [36]) by defining

$$\|x\|_{L_\infty^c(\mathcal{N}, \mathcal{E}_n)} \;=\; \|\mathcal{E}_n(x^*x)\|_\infty^{\frac{1}{2}} \;.$$

We shall point out that in general, for $2 < p < \infty$, the expression $\|\mathcal{E}_n((x^*x)^{\frac{p}{2}})\|_\infty^{\frac{1}{p}}$ does not necessarily provide a norm in the noncommutative setting. Therefore, we

will use a different approach to generalize the $BMO_p$-norms. Namely, motivated by (4), we define for $2 \le p \le \infty$,

$$\|x\|_{BMO_p^c} = \sup_m \sup_{n \le m} \sup_{a \in L_p(N_n),\, \|a\|_p \le 1} \|(x_m - x_{n-1})a\|_p \,. \tag{9}$$

Respectively, we define

$$\|x\|_{BMO_p} = \max\{\|x\|_{BMO_p^c}, \|x^*\|_{BMO_p^c}\} \,. \tag{10}$$

These norms can, in fact, be obtained by interpolation between conditional $L_p$-spaces. More details are given in Section 4. (See [16] for a more general discussion of such norms.)

Our martingale version of the John-Nirenberg theorem reads as follows.

**Theorem 1.1.** *There exists a universal constant $c > 0$ such that for all $2 < p < \infty$,*

$$\|x\|_{BMO} \le \|x\|_{BMO_p} \le cp\,\|x\|_{BMO} \,. \tag{11}$$

The order $cp$ of the constant in the inequality (11) above is the same as in the commutative setting. In a preliminary version of this paper we were only able to prove that (11) holds with a constant of the order $cp^2$. The right order $cp$ was obtained using very recent results of Randrianantoanina [35] on the optimal order of growth for the constants in the noncommutative Burkholder-Gundy square function inequalities. For the proof of Theorem 1.1, note that formula (9), which is the key point in the change of density argument described in the classical setting, leads to the passage from traces to various states on the von Neumann algebra. Therefore we consider martingales with respect to states and their modular theory.

As an application of a noncommutative version of Chebychev inequality proved by Defant and Junge [8], we obtain an analogue of the classical large deviation inequality for elements of $BMO$. Namely, if $\|x\|_{BMO} < 1$, then

$$\mathbb{P}\{|x - x_0| > t\} < C_2 e^{-tC_1} \tag{12}$$

for all $t > 0$, where $C_1, C_2$ are universal constants (see [10], [11]). More precisely, we prove

**Theorem 1.2.** *There exist universal constants $c_1, c_2 > 0$ such that if $\|x\|_{BMO} < 1$, then for all $t > 0$, there exists a projection $f \in \mathcal{N}$ such that $\|(x - x_0)f\| \le t$ and*

$$\phi(1 - f) < c_2 e^{-tc_1} \,. \tag{13}$$

In the commutative case $f = 1_A$ for some measurable subset $A$ of $\Omega$. The condition $\|(x - x_0)f\| \le t$ implies that $A \subseteq \{\omega \in \Omega : |x(\omega) - x_0(\omega)| \le t\}$. Then (13) yields (12).

Our paper is organized as follows. In section 2, we explain the notation and Kosaki's interpolation results, a fundamental tool in our argument. In section 3, we discuss various inclusions of $BMO$ into $L_p$. Note that even the inclusion $BMO \subset L_2$ is not immediate in the nontracial setting. We first establish that for $2 \le p < \infty$, $(BMO, L_p)$ forms an interpolation couple and prove that $[BMO, L_p]_{\frac{p}{q}} = L_q$, $2 \le p < q < \infty$ (extending the results in [26] to the nontracial setting). By general theory of interpolation, the continuous inclusion $L_q \subset L_p$ yields a continuous inclusion $BMO \subset L_p$. The inequality (11) and the modifications required to deduce the interval version of the John-Nirenberg theorem are discussed in section 4, as well as the proof of Theorem 1.2. We are indebted to

Tao Mei and Narcisse Randrianantoanina for providing us with the preprints [25], respectively [35].

## 2. Preliminaries

We use standard notation in operator algebras. We refer to [21] and [41] for background on von Neumann algebras. In the following, we will consider a $\sigma$-finite von Neumann algebra $\mathcal{N}$ acting on a Hilbert space $H$, and a distinguished normal faithful state $\phi$ on $\mathcal{N}$. We denote by $\sigma_t = \sigma_t^\phi$ the one parameter modular automorphism group on $\mathcal{N}$ associated with $\phi$. Haagerup's abstract $L_p$-spaces are defined using the crossed-product $\mathcal{R} = \mathcal{N} \rtimes_{\sigma_t} \mathbb{R}$. We recall that $\mathcal{R} \subseteq \mathcal{B}(L_2(\mathbb{R}, H))$ is the von Neumann algebra generated by the operators $\pi(x)$, $x \in \mathcal{N}$ and, respectively, $\lambda(s)$, $s \in \mathbb{R}$, where

$$\pi(x)(\xi)(t) = \sigma_{-t}(x)\xi(t) \quad \text{and} \quad \lambda(t)(\xi)(s) = \xi(t-s)\,,$$

for all $\xi \in \mathcal{B}(L_2(\mathbb{R}, H))$ and all $t \in \mathbb{R}$. In the following, we may and will identify $\mathcal{N}$ with $\pi(\mathcal{N})$, since $\pi$ is a normal faithful representation of $\mathcal{N}$ on $L_2(\mathbb{R}, H))$. Furthermore, note that $\pi$ is invariant under the dual action

$$\theta_s(x) \;=\; W(s)xW(s)^*\,, \quad s \in \mathbb{R}\,, x \in \mathcal{R}\,,$$

and, moreover,

$$\pi(\mathcal{N}) = \{x \in \mathcal{R} : \theta_s(x) = x\,, \text{ for all } s \in \mathbb{R}\}\,.$$

Here the unitary operators $W(s)$, $s \in \mathbb{R}$, are defined by the phase shift

$$W(s)(\xi)(t) \;=\; e^{-ist}\xi(t)\,, \quad t \in \mathbb{R}\,.$$

As shown in [28], the crossed product $\mathcal{N} \rtimes_{\sigma_t} \mathbb{R}$ is semifinite and admits a unique normal semifinite trace $\tau$ such that for all $s \in \mathbb{R}$,

$$\tau(\theta_s(x)) \;=\; e^{-s}\tau(x)\,.$$

For $1 \le p \le \infty$, $L_p(\mathcal{N})$ is defined as the space of all $\tau$-measurable operators $x$ affiliated with $\mathcal{R} = \mathcal{N} \rtimes_{\sigma_t} \mathbb{R}$ such that for all $s \in \mathbb{R}$,

$$\theta_s(x) \;=\; e^{-s/p}x\,.$$

It follows from the definition that $L_\infty(\mathcal{N})$ coincides with $\mathcal{N}$. Furthermore, there is a canonical isomorphism between $L_1(\mathcal{N})$ and the predual $\mathcal{N}_*$ of $\mathcal{N}$. This requires some explanation. Following [28], every normal semifinite faithful weight (n.s.f., for short) $\psi \in (\mathcal{N}_*)_+$ is given by a density $h_\psi \in L_1(\mathcal{N})_+$ satisfying

$$\tau(h_\psi x) = \int_{\mathbb{R}} \psi(\theta_s(x))\; ds\,,$$

for all $x \in \mathcal{R}_+$. Using the polar decomposition of an arbitrary element $\psi \in \mathcal{N}_*$, this correspondence between $(\mathcal{N}_*)_+$ and $L_1(\mathcal{N})_+$ extends to a bijection between $\mathcal{N}_*$ and $L_1(\mathcal{N})$. Namely, if $\psi \in \mathcal{N}_*$, then $\psi = u|\psi|$, where $u \in \mathcal{N}$ and $|\psi|$ is the modulus of $\psi$. By construction, the corresponding $h_\psi \in L_1(\mathcal{N})$ admits the polar decomposition

$$h_\psi = u|h_\psi| = uh_{|\psi|}\,.$$

We may define a norm on $L_1(\mathcal{N})$ by

$$\|h_\psi\|_1 = |\psi|(1) = \|\psi\|_{\mathcal{N}_*}\,, \quad \psi \in \mathcal{N}_*\,.$$

In this way we obtain the isometry between $L_1(\mathcal{N})$ and $\mathcal{N}_*$. Furthermore, define a linear functional $\mathrm{tr} : L_1(\mathcal{N}) \to \mathbb{C}$, called *trace*, by

$$\mathrm{tr}(h_\psi) \;=\; \psi(1)\,.$$

It is important to note that if $\frac{1}{p} + \frac{1}{q} = 1$, then for all $x \in L_p(\mathcal{N})$ and $y \in L_q(\mathcal{N})$, we have the tracial property

$$tr(xy) \;=\; tr(yx)\,.$$

Let us now return to the special state $\phi$ fixed at the beginning. Then, as explained above, there exists a density $D_\phi \in L_1(\mathcal{N})$ such that for all $x \in \mathcal{N}$,

$$\phi(x) \;=\; \mathrm{tr}(D_\phi x).$$

In the following we drop the subscript and reserve the letter $D$ exclusively for this density. Given $1 \le p < \infty$ and $x \in L_p(\mathcal{N})$, define

$$\|x\|_p = (\mathrm{tr}(|x|^p))^{1/p}\,, \quad \text{respectively} \quad \|x\|_\infty = \|x\|_{\mathcal{N}}\,.$$

If $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$, then for all $x \in L_p(\mathcal{N})$ and $y \in L_q(\mathcal{N})$, Hölder's inequality holds, i.e.,

$$\|xy\|_r \le \|x\|_p \|y\|_q\,.$$

As a consequence, given $1 \le p < \infty$, the mapping $(x, y) \in L_p(\mathcal{N}) \times L_{p'}(\mathcal{N}) \mapsto \mathrm{tr}(xy)$, where $\frac{1}{p} + \frac{1}{p'} = 1$, defines a duality bracket between $L_p(\mathcal{N})$ and $L_{p'}(\mathcal{N})$, with respect to which we have the isometry

$$(L_p(\mathcal{N}))^* = L_{p'}(\mathcal{N})\,.$$

In the sequel we will make repeated use of the following well-known fact.

**Fact 2.1.** Let $1 \le p, q, r \le \infty$ such that $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$ and $p < \infty$. Let $x \in L_q(\mathcal{N})$, and $a \in L_r(\mathcal{N})$, $b \in L_r(\mathcal{N})$ be positive elements. Then the map

$$f(z) \;=\; a^{1-z} x b^z \in L_p(\mathcal{N})$$

is continuous on $S = \{z \in \mathbb{C} \;:\; 0 \le \mathrm{Re}(z) \le 1\}$ and analytic in the interior $S_0$ of $S$.

Consider now the 1-parameter automorphism group

$$\alpha_t(x) \;=\; D^{it} x D^{-it}\,, \quad t \in \mathbb{R}\,.$$

Note that $\alpha_t$ is strongly continuous and leaves $\mathcal{N}$ invariant since

$$\theta_s(D^{it} x D^{-it}) \;=\; e^{its} D^{it} x e^{-its} D^{its} \;=\; D^{it} x D^{-it}\,.$$

From Fact 2.1 we deduce that for every $x \in \mathcal{N}$, the map $f(z) = D^{1-z} x D^z$ is analytic, taking values in $L_1(\mathcal{N})$. In particular, it follows that the map $f_{x,y}(z) = \mathrm{tr}(D^{1-z} x D^z y)$ is analytic and satisfies

$$f_{x,y}(it) \;=\; \phi(x\alpha_t(y)) \quad \text{and} \quad f_{x,y}(1+it) \;=\; \phi(\alpha_t(y)x)\,,$$

for all $t \in \mathbb{R}$. Since

$$\phi(\alpha_t(x)) = \mathrm{tr}(D D^{it} x D^{-it}) = \phi(x)\,,$$

we deduce that $\sigma_t = \alpha_t$, for all $t \in \mathbb{R}$ (see, e.g., [39, Section 2.12]). An element $x \in \mathcal{N}$ is called analytic if the map $t \in \mathbb{R} \mapsto \sigma_t(x) \in \mathcal{N}$ extends to an analytic

function $z \in \mathbb{C} \mapsto \sigma_z(x) \in \mathcal{N}$. The family $\mathcal{N}_a$ of analytic elements in $\mathcal{N}$ is a weak$^*$-dense $*$-subalgebra of $\mathcal{N}$ (see [28]). Given $1 \le p < \infty$ and $0 \le \theta \le 1$, it can be shown (see, e.g., [14]) that the space $\mathcal{N}_a D^{1/p}$ is dense in $L_p(\mathcal{N})$ and

$$D^{\frac{1-\theta}{p}} \mathcal{N}_a D^{\frac{\theta}{p}} = \mathcal{N}_a D^{\frac{1}{p}} .$$

We will denote by $L_p(\mathcal{N})_a$ the family of elements with the property that the map $t \mapsto \sigma_t(x)$, $t \in \mathbb{R}$, extends to an analytic function on $\mathbb{C}$ with values in $L_p(\mathcal{N})$. Since $\mathcal{N}_a D^{\frac{1}{p}} \subset L_p(\mathcal{N})_a$ is dense in $L_p(\mathcal{N})$, as mentioned above, it follows that $L_p(\mathcal{N})_a$ is dense in $L_p(\mathcal{N})$.

Since interpolation is our main tool in this paper, we briefly recall in the following some basic notions concerning the complex method of interpolation due to Calderón. Our main reference for interpolation theory is Bergh and Löfström [3]. A pair of Banach spaces $(X_0, X_1)$ is called a *compatible couple* if they embed continuously in some topological vector space $X$. This allows us to consider the spaces $X_0 \cap X_1$ and $X_0 + X_1$ (by identifying them inside $X$). They are Banach spaces when equipped, respectively, with the following norms:

$$\|x\|_{X_0 \cap X_1} = \max\{\|x\|_{X_0}, \|x\|_{X_1}\} , \tag{14}$$

$$\|x\|_{X_0 + X_1} = \inf\{\|x_0\|_{X_0} + \|x_1\|_{X_1} : x = x_0 + x_1 , x_0 \in X_0 , x_1 \in X_1\} . \tag{15}$$

Following the notation from [3], let $\mathcal{F}$ be the family of all continuous and bounded functions $f : S \to X_0 + X_1$ satisfying the following properties:

(1) $f$ is analytic in $S_0$,

(2) $f(it) \in X_0$ and $f(1+it) \in X_1$ for all $t \in \mathbb{R}$,

(3) $f(it) \to 0$ and $f(1+it) \to 0$ as $t \to \infty$.

By the Phragmen-Lindelöf theorem, $\mathcal{F}$ is a Banach space under the norm

$$\|f\|_{\mathcal{F}} = \max\left\{ \sup_{t \in \mathbb{R}} \|f(it)\|_{X_0}, \sup_{t \in \mathbb{R}} \|f(1+it)\|_{X_1} \right\} .$$

For $0 \le \theta \le 1$, set $[X_0, X_1]_\theta = \{ x \in X_0 + X_1 : x = f(\theta) , \text{ for some } f \in \mathcal{F} \}$. This is called the *complex interpolation space* (of exponent $\theta$) between $X_0$ and $X_1$, and it is a Banach space under the norm

$$\|x\|_\theta = \inf\{\|f\|_{\mathcal{F}} : x = f(\theta),\ f \in \mathcal{F}\} .$$

The complex method is an exact interpolation functor; i.e., if $T : X_0 + X_1 \to Y_0 + Y_1$ is a linear operator which is bounded both from $X_0$ to $Y_0$ (with norm $M_0$) and from $X_1$ to $Y_1$ (with norm $M_1$), then $T$ is bounded from $[X_0, X_1]_\theta$ to $[Y_0, Y_1]_\theta$, with norm $\le M_0^{1-\theta} M_1^\theta$.

## 3. Interpolation results

In this paper we will make crucial use of Kosaki's interpolation results (see [23]), which will enable us to extend the results in [26] to the nontracial setting. For $1 \le p < q \le \infty$ and $0 \le \eta \le 1$ consider the mapping $I^\eta_{q,p} : L_q(\mathcal{N}) \to L_p(\mathcal{N})$ defined by

$$I^\eta_{q,p}(x) = D^{(1-\eta)(\frac{1}{p} - \frac{1}{q})} x D^{\eta(\frac{1}{p} - \frac{1}{q})} .$$

Note that $I^\eta_{q,p}$ is injective. In [23], Kosaki introduces the spaces

$$L^\eta_p(\mathcal{N}, \phi) = [I^\eta_{\infty,1}(\mathcal{N}), L_1(\mathcal{N})]_{\frac{1}{p}} , \tag{16}$$

for $1 \le p \le \infty$. We should point out that Kosaki's original notation is slightly different, namely $L_p^\eta(\mathcal{N},\phi)$ is denoted in [23] by $i_p^\eta(L_p)$, where $i_p^\eta$ denotes the mapping $I_{p,1}^\eta$. Also, in our terminology the space $L_p^\eta(\mathcal{N},\phi)$ is considered as a subspace of $L_1(\mathcal{N})$, whereas Kosaki formally works in $\mathcal{N}_*$. Using the canonical isometric isomorphism between $L_1(\mathcal{N})$ and $\mathcal{N}_*$ explained in the preliminaries, Kosaki's results may be reformulated in this language as

$$L_p^\eta(\mathcal{N},\phi) \;=\; D^{\frac{1-\eta}{p'}} L_p(\mathcal{N}) D^{\frac{\eta}{p'}} \;, \tag{17}$$

where $\frac{1}{p}+\frac{1}{p'}=1$. This is not only an equality on the level of sets, but it means that if $x \in L_p(\mathcal{N})$, then $D^{\frac{1-\eta}{p'}} x D^{\frac{\eta}{p'}} \in [I_{\infty,1}^\eta(\mathcal{N}), L_1(\mathcal{N})]_{\frac{1}{p}}$ and, furthermore,

$$\|x\|_p \;=\; \left\| D^{\frac{1-\eta}{p'}} x D^{\frac{\eta}{p'}} \right\|_{[I_{\infty,1}^\eta(\mathcal{N}), L_1(\mathcal{N})]_{\frac{1}{p}}} .$$

Moreover, every element in the interpolation space $[I_{\infty,1}^\eta(\mathcal{N}), L_1(\mathcal{N})]_{\frac{1}{p}}$ comes from an element $x \in L_p(\mathcal{N})$. In our context, we will also consider the interpolation couple $(A_0, A_1)$, where $A_0 = I_{q,p}^\eta(L_q(\mathcal{N}))$ and $A_1 = L_p(\mathcal{N})$, with $1 \le p < q \le \infty$. We will use the notation

$$\left\| I_{q,p}^\eta(x) \right\|_{A_0} \;=\; \|x\|_q \;.$$

This is to emphasize that the ambient topological vector space for the interpolation couple $(I_{q,p}^\eta(L_q(\mathcal{N})), L_p(\mathcal{N}))$ is $L_p(\mathcal{N})$, and the inclusion map of $L_q(\mathcal{N})$ into $L_p(\mathcal{N})$ is given by $I_{q,p}^\eta$. Using (16), (17) and the reiteration theorem for complex interpolation ([3, Theorem 4.6.1]), we deduce for $1 \le p < s < q \le \infty$ and $0 < \theta < 1$ with $\frac{1}{s} = \frac{1-\theta}{q} + \frac{\theta}{p}$ that

$$I_{s,p}^\eta(L_s(\mathcal{N})) \quad = \quad [I_{q,p}^\eta(L_q(\mathcal{N})), L_p(\mathcal{N})]_\theta \,. \tag{18}$$

We will now use these interpolation spaces in the context of BMO-type norms. Indeed, it is helpful to consider the following vector-valued $\ell_\infty$ space defined for sequences $(x_n)_{n\ge 0}$ in $L_p(\mathcal{N})$ as follows:

$$\|(x_n)\|_{L_p^c(\mathcal{N};\ell_\infty)} \;=\; \left\| \sup_n (x_n^* x_n) \right\|_{\frac{p}{2}}^{\frac{1}{2}} ,$$

where the right-hand side is to be understood in the sense of the following suggestive notation introduced in [14]. Namely, if $1 \le q, q' \le \infty$ with $\frac{1}{q}+\frac{1}{q'}=1$, then, for a finite sequence of positive elements in $L_p(\mathcal{N})$, set

$$\|\sup_n x_n\|_q \quad = \quad \sup\left\{ \sum_{n\ge 0} \mathrm{tr}(x_n y_n) : \; y_n \ge 0, \left\| \sum_{n\ge 0} y_n \right\|_{q'} \le 1 \right\} . \tag{19}$$

In the commutative case this coincides with the usual definition of the norm in the $l_\infty$-valued $L_q$-space. Here the actual supremum does not necessarily make sense as in the commutative case, and thus it is just a <u>notation</u>. Note that $L_p^c(\mathcal{N},\ell_\infty)$ is only well defined for $p \ge 2$.

*Remark* 3.1. Let $1 \le p < q, s < \infty$ be such that $\frac{1}{p} = \frac{1}{q} + \frac{1}{s}$. Then, for all $0 \le a, b \in L_s(\mathcal{N})$,

1) $\sup_m \| \sup_{n\le m} x_n\|_p = \| \sup_n x_n\|_p$,

2) $\|\sup_n a^{\frac{1}{2}} x_n a^{\frac{1}{2}}\|_p \leq \|a\|_s \|\sup_n x_n\|_q$,

3) $\|\sup_n x_n^{\frac{1}{2}} b x_n^{\frac{1}{2}}\|_p \leq \|b\|_\infty \|\sup_n x_n\|_p$ .

*Proof.* Item 1) is an immediate consequence of the definition (19). Item 2) follows from Hölder's inequality, namely,

$$\begin{aligned}
\|\sup_n a^{\frac{1}{2}} x_n a^{\frac{1}{2}}\|_p &= \sup\left\{\sum_{n\geq 0} \mathrm{tr}(a^{\frac{1}{2}} x_n a^{\frac{1}{2}} y_n) : y_n \geq 0, \left\|\sum_{n\geq 0} y_n\right\|_{p'} \leq 1\right\} \\
&= \sup\left\{\sum_{n\geq 0} \mathrm{tr}(x_n a^{\frac{1}{2}} y_n a^{\frac{1}{2}}) : y_n \geq 0, \left\|\sum_{n\geq 0} y_n\right\|_{p'} \leq 1\right\} \\
&\leq \|a\|_s \|\sup_n x_n\|_q \,.
\end{aligned}$$

Item 3) follows from definition (19),

$$\begin{aligned}
\|\sup_n x_n^{\frac{1}{2}} b x_n^{\frac{1}{2}}\|_p &= \sup\left\{\sum_{n\geq 0} \mathrm{tr}(x_n^{\frac{1}{2}} b x_n^{\frac{1}{2}} y_n) : y_n \geq 0, \left\|\sum_{n\geq 0} y_n\right\|_{p'} \leq 1\right\} \\
&= \sup\left\{\sum_{n\geq 0} \mathrm{tr}(b x_n^{\frac{1}{2}} y_n x_n^{\frac{1}{2}}) : y_n \geq 0, \left\|\sum_{n\geq 0} y_n\right\|_{p'} \leq 1\right\} \\
&\leq \|b\|_\infty \|\sup_n x_n\|_p \,,
\end{aligned}$$

wherein we have used the fact that

$$\mathrm{tr}(b x_n^{\frac{1}{2}} y_n x_n^{\frac{1}{2}}) \leq \|b\|_\infty \mathrm{tr}(x_n^{\frac{1}{2}} y_n x_n^{\frac{1}{2}}) = \|b\|_\infty \mathrm{tr}(x_n y_n) \,.$$

□

The following lemma, which can be proved in the same way as in the tracial case (see [26]), will enable us to use interpolation for these norms.

**Lemma 3.2.** *For $2 \leq p \leq \infty$, let $\frac{1}{q} + \frac{2}{p} = 1$. Then*

$$\|(x_n)\|_{L_p^c(\mathcal{N};\ell_\infty)} = \sup\left\{\left(\sum_{n\geq 0} \|x_n v_n\|_2^2\right)^{\frac{1}{2}} : \left\|\sum_{n\geq 0} v_n v_n^*\right\|_q \leq 1\right\} .$$

*Consequently, $L_p^c(\mathcal{N};\ell_\infty)$ is a normed space.*

In the following, we will restrict our attention to the following set-up. Let $\mathcal{N}$ be a $\sigma$-finite von Neumann algebra and $\phi$ a distinguished normal, faithful state on $\mathcal{N}$. Let $\mathcal{M} = \mathcal{N} \otimes \mathcal{B}(l_2)$ and let Tr denote the usual trace on $\mathcal{B}(l_2)$. Then $\phi \otimes \mathrm{Tr}$ is a n.s.f. weight on $\mathcal{M}$. For $1 \leq p \leq \infty$, we consider the Haagerup $L_p$-spaces $L_p(\mathcal{M})$ associated with this n.s.f. weight.

Let $V$ be the space of all infinite matrices $[x_{i,j}]_{1\leq i,j\leq\infty}$ with entries in $L_p(\mathcal{N})$, equipped with the topology of pointwise convergence in $L_p(\mathcal{N})$. More precisely, a sequence of matrices $[x_{i,j}^{(m)}]$ converges to a matrix $[x_{i,j}]$ if and only if

$$L_p(\mathcal{N}) - \lim_{m\to\infty} x_{i,j}^{(m)} = x_{i,j} \,,$$

for all $1 \leq i, j \leq \infty$. Then $V$ is a topological vector space. For $1 \leq p < q \leq \infty$ and $0 \leq \eta \leq 1$ define a mapping

$$\hat{I}^{\eta}_{q,p} = I^{\eta}_{q,p} \otimes \mathrm{Id}_{\mathcal{B}(l_2)}\,.$$

Note that $(\hat{I}^{\eta}_{q,p}(L_q(\mathcal{M})), L_p(\mathcal{M}))$ forms an interpolation couple, since both embed continuously into $V$.

**Corollary 3.3.** *Let $1 \leq p < s < q \leq \infty$ and $0 < \theta < 1$ such that $\frac{1}{s} = \frac{1-\theta}{p} + \frac{\theta}{q}$. Then, completely isometrically,*

$$[\hat{I}^{\eta}_{q,p}(L_q(\mathcal{M})), L_p(\mathcal{M})]_{\theta} \quad = \quad \hat{I}^{\eta}_{s,p}(L_s(\mathcal{M}))\,. \tag{20}$$

*Proof.* By [15], the following complete isometries hold:

$$L_p(\mathcal{M}) = S_p[L_p(\mathcal{N})] \quad \text{respectively,} \quad L_q(\mathcal{M}) = S_q[L_q(\mathcal{N})].$$

Define $A_0 = I^{\eta}_{q,p}(L_q(\mathcal{N}))$ and $A_1 = L_p(\mathcal{N})$. By the reiteration result (18) it follows that $A_\theta = I_{s,p}(L_s(\mathcal{N}))$ completely isometrically. Furthermore, by [31, Corollary 1.4], we obtain a complete isometry

$$[\, S_q[A_0], S_p[A_1]\,]_{\theta} \quad = \quad S_s[A_\theta]. \tag{21}$$

Note that for $2 \leq v < \infty$, we have

$$\|\, [x_{i,j} D^{\frac{1}{p}-\frac{1}{q}}]\, \|_{S_v[I_{v,p}(L_v(\mathcal{N}))]} \quad = \quad \|\, [x_{i,j}]\, \|_{S_v[L_v(\mathcal{N})]}.$$

Together with (21), this yields the assertion. □

For $1 \leq p < \infty$, recall the spaces $L_p(\mathcal{N}; l_2^c)$ (respectively, their row version $L_p(\mathcal{N}; l_2^r)$) defined in [18] (see also [32] for the tracial case) as the completion of the family of finite sequences $a = (a_k)_{k \geq 0}$ in $L_p(\mathcal{N})$ under the norm

$$\|a\|_{L_p(\mathcal{N}; l_2^c)} = \left\| (\sum_{k \geq 0} |a_k|^2)^{\frac{1}{2}} \right\|_p \quad \text{respectively,} \quad \|a\|_{L_p(\mathcal{N}; l_2^r)} = \left\| (\sum_{k \geq 0} |a_k^*|^2)^{\frac{1}{2}} \right\|_p .$$

In particular, for every positive integer $n$, we consider the finite-dimensional versions

$$\|a\|_{L_p(\mathcal{N}; (l_2^n)^c)} = \left\| (\sum_{k=0}^{n} |a_k|^2)^{\frac{1}{2}} \right\|_p \quad \text{respectively,} \quad \|a\|_{L_p(\mathcal{N}; (l_2^n)^r)} = \left\| (\sum_{k=0}^{n} |a_k^*|^2)^{\frac{1}{2}} \right\|_p .$$

From the above discussion, it is clear how to view the spaces $(\hat{I}^{\eta}_{q,p}(L_q(\mathcal{N}; \ell_2^c))$, $L_p(\mathcal{N}; \ell_2^c))$, where $2 \leq p < q \leq \infty$ as a compatible couple, and we obtain the following:

**Corollary 3.4.** *Let $1 \leq p < s < q \leq \infty$ and $0 < \theta < 1$ such that $\frac{1}{s} = \frac{1-\theta}{p} + \frac{\theta}{q}$. Then, completely isometrically,*

$$[\hat{I}^{\eta}_{q,p}(L_q(\mathcal{N}; \ell_2^c)), L_p(\mathcal{N}; \ell_2^c)]_{\theta} \quad = \quad \hat{I}^{\eta}_{s,p}(L_s(\mathcal{N}; \ell_2^c)), \tag{22}$$

$$[\hat{I}^{\eta}_{q,p}(L_q(\mathcal{N}; \ell_2^r)), L_p(\mathcal{N}; \ell_2^r)]_{\theta} \quad = \quad \hat{I}^{\eta}_{s,p}(L_s(\mathcal{N}; \ell_2^r)). \tag{23}$$

*Proof.* Note that $L_p(\mathcal{N}; l_2^c)$, respectively $L_p(\mathcal{N}; l_2^r)$, are 1-complemented subspaces of $L_p(\mathcal{M})$ (see [18] for details). Therefore the conclusion follows from Corollary 3.3. □

*Remark* 3.5. Corollary 3.3 holds true, more generally, in the setting of a von Neumann algebra $\mathcal{M}$ equipped with an n.f. strictly semifinite weight $\psi$. Recall that an n.s.f. weight $\psi$ is called *strictly semifinite*, if there exists an increasing sequence of projections $(p_n)_{n\geq 0}$ such that $\sigma_t^{\psi}(p_n) \ = \ p_n$ and $\psi(p_n) \ < \ \infty$, for all $n \geq 0$.

We now establish additional properties of the spaces $L_p^c(\mathcal{M}; l_\infty)$. Let $W$ be the space of infinite sequences of matrices $[x_{i,j}^{(n)}]_{1\leq i,j,n\leq\infty}$ in $L_p(\mathcal{M})$. If $2 \leq p < q \leq \infty$, then both $L_p^c(\mathcal{M}; l_\infty)$ and $L_q^c(\mathcal{M}; l_\infty)$ embed continuously into the topological vector space $W$. In fact, the map $\widehat{I}_{q,p} : L_q^c(\mathcal{M}; l_\infty) \to L_p^c(\mathcal{M}; l_\infty)$ defined by

$$\widehat{I}_{q,p}([x_{i,j}^{(n)}]) = [x_{i,j}^{(n)} D^{\frac{1}{p}-\frac{1}{q}}]$$

is a continuous inclusion from $L_q^c(\mathcal{M}; l_\infty)$ into $L_p^c(\mathcal{M}; l_\infty)$, and therefore we can interpolate between these spaces.

The following result is probably well known in interpolation theory. We include a proof for the convenience of the reader.

**Lemma 3.6.** *Let $(A_0, A_1)$ be an interpolation couple with intersection $\Delta$. Let $T_n : \Delta \to \Delta$ be a sequence of linear operators such that*

1) *each $T_n$ extends to a bounded linear operator from $A_0$ to $A_0$ such that $C_1 = \sup_n \|T_n\| < \infty$, and*
2) *each $T_n$ extends to a bounded linear operator from $A_1$ to $A_1$ such that $A_1 - \lim_{n\to\infty} T_n(x) \ = \ x$, for all $x \in \Delta$.*

*Then, for all $0 < \theta < 1$ and all $x \in A_\theta = [A_0, A_1]_\theta$, we have*

$$\lim_{n\to\infty} T_n(x) \ = \ x\,. \tag{24}$$

*Proof.* By density [3, Lemma 4.3.2] it suffices to prove (24) for elements $x \in A_\theta$ of the form

$$x \ = \ \sum_{k=0}^{m} g_k(\theta) y_k\,,$$

where $m$ is a nonnegative integer, $g_k$ are analytic functions which vanish at infinity and $y_k \in \Delta$. Let $\varepsilon > 0$ and choose $n_0 \geq 0$ such that for all $n \geq n_0$, we have

$$((1 + C_1) \sum_{k=0}^{m} \sup_t |g_k(it)| \, \|y_k\|_{A_0})^{1-\theta} (\sum_{k=0}^{m} \sup_t |g_k(1 + it)| \, \|T_n(y_k) - y_k\|_{A_1})^{\theta} < \varepsilon\,.$$

Since the complex interpolation method is an exact functor of exponent $\theta$, we deduce that

$$\begin{aligned}
&\|x - T_n(x)\|_\theta \\
&\quad \leq \sup_{t\in\mathbb{R}} \left\| \sum_{k=1}^{m} g_k(it)(y_k - T_n(y_k)) \right\|_{A_0}^{1-\theta} \sup_{t\in\mathbb{R}} \left\| \sum_{k=1}^{m} g_k(1 + it)(y_k - T_n(y_k)) \right\|_{A_1}^{\theta} \\
&\quad \leq ((1 + C_1) \sum_{k=0}^{m} \sup_t |g_k(it)| \, \|y_k\|_{A_0})^{1-\theta} (\sum_{k=0}^{m} \sup_t |g_k(1 + it)| \, \|y_k - T_n(y_k)\|_{A_1})^{\theta} \\
&\quad < \varepsilon\,.
\end{aligned}$$

This proves the assertion. □

*Remark* 3.7. Let $2 \leq p < q \leq \infty$ and let $e \in \mathcal{M}$ be a projection such that $(\phi\otimes\mathrm{Tr})(e) < \infty$. Then $e\mathcal{M}e$ is a finite von Neumann algebra, and a similar argument as in the tracial case (see [26]) shows that the following contractive inclusion holds:

$$\widehat{I}_{q,p}(L^c_q(e\mathcal{M}e; l_\infty)) \quad \subseteq \quad L^c_p(e\mathcal{M}e; l_\infty)\,. \tag{25}$$

**Proposition 3.8.** *If $2 \leq p < s < q \leq \infty$ and $0 < \theta < 1$ such that $\frac{1}{s} = \frac{1-\theta}{q} + \frac{\theta}{p}$, then, isometrically,*

$$[\widehat{I}_{q,p}(L^c_q(\mathcal{M}; \ell_\infty)), L^c_p(\mathcal{M}; \ell_\infty)]_\theta \quad = \quad \widehat{I}_{s,p}(L^c_s(\mathcal{M}; \ell_\infty)). \tag{26}$$

*Proof.* We first show that we have a contractive inclusion

$$[\widehat{I}_{q,p}(L^c_q(\mathcal{M}; \ell_\infty)), L^c_p(\mathcal{M}; \ell_\infty)]_\theta \quad \subseteq \quad \widehat{I}_{s,p}(L^c_s(\mathcal{M}; \ell_\infty)). \tag{27}$$

Note that for each positive integer $n$, $L^c_p(\mathcal{M}; l^n_\infty)$ is isomorphic (as a vector space) to the $n$-fold tensor product $L_p(\mathcal{M}) \otimes \ldots \otimes L_p(\mathcal{M})$ ($n$ times). Therefore

$$(\widehat{I}^{(n)}_{q,p}(L^c_q(\mathcal{M}; l^n_\infty)), L^c_p(\mathcal{M}; l^n_\infty))$$

is a compatible couple, where

$$\widehat{I}^{(n)}_{q,p} = \hat{I}_{q,p} \otimes \mathrm{Id}(l^n_\infty).$$

By 1) in Remark 3.1, it suffices to show that for all $n \geq 1$ we have, contractively,

$$[\widehat{I}^{(n)}_{q,p}(L^c_q(\mathcal{M}; \ell^n_\infty)), L^c_p(\mathcal{M}; \ell^n_\infty)]_\theta \subseteq \widehat{I}^{(n)}_{s,p}(L^c_s(\mathcal{M}; \ell^n_\infty)). \tag{28}$$

In the following we fix a positive integer $n$ and consider only sequences of length $n$. We claim that it suffices to prove that for all $m \geq 1$ the following contractive inclusion holds:

$$[\widehat{I}^{(n)}_{q,p}(L^c_q(\mathcal{N} \otimes M_m; l^n_\infty)), L^c_p(\mathcal{N} \otimes M_m; l^n_\infty)]_\theta \quad \subseteq \quad \widehat{I}^{(n)}_{q,p}(L^c_s(\mathcal{N} \otimes M_m; l^n_\infty))\,, \tag{29}$$

where $M_m$ is the algebra of $m\times m$ complex matrices. Note that $\mathcal{N}\otimes M_m = e_m\mathcal{M}e_m$, where $e_m = 1 \otimes p_m$ and $p_m : \mathcal{B}(l_2) \to M_m$ is the canonical orthogonal projection. Set

$$A_0 = \widehat{I}^{(n)}_{q,p}(L^c_q(\mathcal{N} \otimes M_m; l^n_\infty)) \quad \text{and} \quad A_1 = L^c_p(\mathcal{N} \otimes M_m; l^n_\infty)\,.$$

By Remark 3.7, their intersection and sum are, respectively,

$$\Delta_0 = A_0 \cap A_1 = \widehat{I}^{(n)}_{q,p}(L^c_q(\mathcal{N} \otimes M_m; l^n_\infty)), \quad \Sigma_0 = A_0 + A_1 = L^c_p(\mathcal{N} \otimes M_m; \ell^n_\infty)\,.$$

Consider the map $S_m : L^c_q(\mathcal{M}; l^n_\infty) \to L^c_q(e_m\mathcal{M}e_m; l^n_\infty)$ defined by

$$S_m((x_k)^n_{k=1}) = (e_m x_k e_m)^n_{k=1}.$$

By 2) in Remark 3.1, it follows that

$$\sup_m \|S_m\|_{A_0\to A_0} \leq 1\,.$$

Furthermore, we have $A_1 - \lim\limits_{m\to\infty} S_m(x) = x$, for all $x \in \Delta_0$ (see [14]). An application of Lemma 3.6 shows that, for all $x \in [A_0, A_1]_\theta$,

$$\lim_{m\to\infty} S_m(x) = x\,.$$

Therefore the claim is justified. We now prove (29). Let $2 \leq v < \infty$ such that $\frac{1}{v} = \frac{1}{p} - \frac{1}{q}$. Furthermore, define $f(v)$ by the relation $\frac{1}{f(v)} + \frac{1}{v} = \frac{1}{2}$ and note that $\frac{1}{f(q)} - \frac{1}{f(p)} = \frac{1}{v}$. Let

$$B_0 = \widehat{I}^{(n)}_{f(p),f(q)}(L_{f(p)}(\mathcal{N} \otimes M_m; (l^n_2)^r))\,, \quad B_1 = \widehat{I}^{(n)}_{f(q),f(q)}(L_{f(q)}(\mathcal{N} \otimes M_m; (l^n_2)^r))\,.$$

Then, their intersection and sum are, respectively,

$$\Delta_1 = \widehat{I}^{(n)}_{f(p),f(q)}(L_{f(p)}(\mathcal{N}\otimes M_m;(l_2^n)^r))\,, \quad \Sigma_1 \;=\; \widehat{I}^{(n)}_{f(q),f(q)}(L_{f(q)}(\mathcal{N}\otimes M_m;(l_2^n)^r))\,.$$

Furthermore, given finite sequences of $m\times m$ matrices $([x^{(k)}_{i,j}D^{\frac{1}{v}}]_{1\le i,j\le m})^n_{k=1}$ in $\Delta_0$ and, respectively, $([D^{\frac{1}{v}}y^{(k)}_{i,j}]_{1\le i,j\le m})^n_{k=1}$ in $\Delta_1$, define

$$T\left(\left([x^{(k)}_{i,j}D^{\frac{1}{v}}]\right)_k\times\left([D^{\frac{1}{v}}y^{(k)}_{i,j}]\right)_k\right) \;=\; \left([x^{(k)}_{i,j}D^{\frac{1}{v}}y^{(k)}_{i,j}]\right)_k\in\ell_2^n(L_2(\mathcal{N}\otimes M_m))\,.$$

This map is well defined, since for each $1\le k\le n$, $[x^{(k)}_{i,j}]_{1\le i,j\le m}\in L_q(\mathcal{N}\otimes M_m)$ and $[y^{(k)}_{i,j}]_{1\le i,j\le m}\in L_{f(p)}(\mathcal{N}\otimes M_m)$ , and, by construction, $\frac{1}{q}+\frac{1}{v}+\frac{1}{f(q)}=\frac{1}{2}$ . Moreover, we deduce from Lemma 3.2 that

$$\|T:\Delta_0\times\Delta_1\to\ell_2^n(L_2(\mathcal{N}\otimes M_m))\| \;\le\; 1\,.$$

Using multilinear complex interpolation (see [3, Theorem 4.4.1]), the map $T$ extends to a contraction

$$T:[A_0,A_1]_\theta\times[B_0,B_1]_\theta\to\ell_2^n(L_2(\mathcal{N}\otimes M_m))\,.$$

By Corollary 3.4 applied to the algebra $\mathcal{N}\otimes M_m$ , we deduce that, isometrically,

$$[B_0,B_1]_\theta=\widehat{I}^{(n)}_{f(s),f(q)}(L_{f(s)}(\mathcal{N}\otimes M_m;(l_2^n)^r))\,.$$

Therefore the extended map

$$T:[A_0,A_1]_\theta\times\widehat{I}^{(n)}_{f(s),f(q)}(L_{f(s)}(\mathcal{N}\otimes M_m;(l_2^n)^r))\to\ell_2^n(L_2(\mathcal{N}\otimes M_m))$$

is a contraction. A further application of Lemma 3.2 yields the estimate

$$\left\|T:[\widehat{I}^{(n)}_{q,p}(L^c_q(\mathcal{N}\otimes M_m;\ell_\infty^n)),L^c_p(\mathcal{N}\otimes M_m;\ell_\infty^n)]_\theta\to\widehat{I}^{(n)}_{s,p}(L^c_s(\mathcal{N}\otimes M_m;\ell_\infty^n))\right\|\le 1.$$

Thus (29) is justified. The proof of the reverse inclusion in (26) reduces, just as in the tracial case, to the interpolation result (20), via the factorization property of the spaces $L^c_p(\mathcal{M};l_\infty)$, proved in [14]. □

Let us now turn our attention to martingales invariant under the automorphism group of a given state. Indeed, we will assume that $(\mathcal{N}_n)_{n\ge0}$ is an increasing sequence of von Neumann subalgebras of $\mathcal{N}$ such that $\sigma_t(\mathcal{N}_n)\subset\mathcal{N}_n$ for all $t\in\mathbb{R}$ and all $n\ge0$ . Moreover, let $(p_n)_{n\ge0}$ be the increasing sequence of projections given by the units in $\mathcal{N}_n$ . We will further assume that $\sigma_t(p_n)=p_n$ for all $t\in\mathbb{R}$ and all $n\ge0$ . For fixed $n\in\mathbb{N}$ , according to [42] , there exists a conditional expectation $\tilde{\mathcal{E}}_n:p_n\mathcal{N}p_n\to\mathcal{N}_n$ such that

$$\tilde{\mathcal{E}}_n(\sigma_t(x)) \;=\; \sigma_t(\tilde{\mathcal{E}}_n(x)) \tag{30}$$

for all $x\in p_n\mathcal{N}p_n$ and all $t\in\mathbb{R}$ . Let us define $\mathcal{E}_n(x) \;=\; \tilde{\mathcal{E}}_n(p_nxp_n)$ and note that $\mathcal{E}_n$ still satisfies the invariance property (30). Then the martingale differences are defined by

$$d_n(x) \;=\; \mathcal{E}_n(x)-\mathcal{E}_{n-1}(x)\,,$$

where $\mathcal{E}_{-1}(x)=0$. Moreover, for $1\le p<\infty$ , the conditional expectation and the martingale difference operators extend in a natural way to $L_p(\mathcal{N})$ (see [18]), such that for all $0<\theta<1$ the relation

$$\mathcal{E}_n(D^{\frac{1-\theta}{p}}xD^{\frac{\theta}{p}}) \;=\; D^{\frac{1-\theta}{p}}\mathcal{E}_n(x)D^{\frac{\theta}{p}} \tag{31}$$

holds for all $x \in \mathcal{N}$. Furthermore, if $\frac{1}{p} = \frac{1}{q} + \frac{1}{s} + \frac{1}{t}$, then for all $a \in L_s(\mathcal{N}_n)$, $b \in L_t(\mathcal{N}_n)$ and $x \in L_q(\mathcal{N})$, we have

$$\mathcal{E}_n(axb) \;=\; a\mathcal{E}_n(x)b\,. \tag{32}$$

Recall also that if $1 \le p \le \infty$ and $x_\infty \in L_p(\mathcal{N})$, then the sequence $(x_n)_{n\ge 0}$ defined by $x_n = \mathcal{E}_n(x_\infty)$ is a bounded $L_p(\mathcal{N})$-martingale which converges to $x_\infty$ in $L_p(\mathcal{N})$ (respectively, in the w*-topology if $p = \infty$). Conversely, for $1 < p < \infty$, we deduce from the uniform convexity of $L_p(\mathcal{N})$ that every bounded $L_p(\mathcal{N})$-martingale converges to some element $x_\infty \in L_p(\mathcal{N})$ and hence is of the above form. Therefore, we will often identify a martingale with its limit, whenever this exists.

For convenience, we will also assume that the predual $\mathcal{N}_*$ is separable. Our results will hold without this assumption, and we refer to the Appendix in [12] for techniques deducing the general case from the $\sigma$-finite one. However, assuming that $\mathcal{N}_*$ is separable, we will find a separable $\sigma$-weakly dense subalgebra $\mathcal{A}_0$ of $\mathcal{N}_a$. Following [14], we can then consider the countably generated Hilbert $C^*$-module $F_n$ generated by $\mathcal{N}_n$ and $\mathcal{A}_0$, and find a right module isomorphism $u_n : F_n \to C_\infty(\mathcal{N}_n)$ such that for all $x, y \in F_n$, we have

$$\mathcal{E}_n(y^*x) \;=\; u_n(y)^*u_n(x)\,. \tag{33}$$

Moreover, for $1 \le p < \infty$ the map $u_n^p : \mathcal{N}_{n,a}D^{\frac{1}{p}} \to C_\infty(\mathcal{N}_{n,a})D^{\frac{1}{p}}$ defined by

$$u_n^p(xD^{\frac{1}{p}}) \;=\; u_n(x)D^{\frac{1}{p}}\,, \quad x \in \mathcal{N}_{n,a},$$

is well defined and isometric if we equip $\mathcal{N}_{n,a}D^{\frac{1}{p}}$ with the norm

$$\left\|xD^{\frac{1}{p}}\right\|_{L_p(\mathcal{N},\mathcal{E}_n)} \;=\; \left\|D^{\frac{1}{p}}\mathcal{E}_n(x^*x)D^{\frac{1}{p}}\right\|_{\frac{p}{2}}^{\frac{1}{2}}\,.$$

Indeed, in [14] this was considered only in the faithful case. However, by working with the algebra $\mathcal{N}_n + \mathbb{C}(1-p_n)$ and the corresponding $\tilde{u}_n^p$, and then defining $u_n^p(xD^{\frac{1}{p}}) = \tilde{u}_n^p(xD^{\frac{1}{p}})p_n$, we can extend the construction to the nonfaithful case (see Section 8 in [18] for details). For $2 \le p \le \infty$, we will consider the space $L_p^cMO(\mathcal{N})$ of martingale difference sequences $x \cong (d_k)_{k\ge 0}$ satisfying

$$\|x\|_{L_p^cMO(\mathcal{N})} \;=\; \sup_m \left\| \sup_{n\le m} \mathcal{E}_n\left(\sum_{k=n}^m d_k^*d_k\right)\right\|_{\frac{p}{2}}^{\frac{1}{2}} < \infty\,. \tag{34}$$

Note that if $p = \infty$, then $L_\infty^cMO(\mathcal{N}) = BMO_c(\mathcal{N})$.

Recall also the following norms:

$$\|x\|_{H_p^c(\mathcal{N})} = \|dx\|_{L_p(\mathcal{N};l_2^c)}\,. \tag{35}$$

The next results concerning the spaces $L_p^cMO(\mathcal{N})$ will be very useful in the sequel. The following noncommutative version of Stein's inequality is essentially contained in [18].

**Lemma 3.9.** *Let $1 < p < \infty$. Define a map $Q$ on all finite sequences $x = (x_n)_{n\ge 0}$ in $L_p(\mathcal{N})$ by $Q(x) = (\mathcal{E}_n(x_n))_{n\ge 0}$. Then*

$$\|Q(x)\|_{L_p(\mathcal{N};l_2^c)} \le \gamma_p\|x\|_{L_p(\mathcal{N};l_2^c)}, \quad \textit{respectively} \quad \|Q(x)\|_{L_p(\mathcal{N};l_2^r)} \le \gamma_p\|x\|_{L_p(\mathcal{N};l_2^r)}\,.$$

*Thus $Q$ extends to a bounded projection on $L_p(\mathcal{N};l_2^c)$ and $L_p(\mathcal{N};l_2^r)$, respectively.*

Furthermore, as an application of Stein's inequality, Doob's inequality and the duality between $L^c_p MO(\mathcal{N})$ and $H^c_{p'}(\mathcal{N})$, where $\frac{1}{p}+\frac{1}{p'}=1$, the following was proved in [14] (see also Section 8 in [18]).

**Theorem 3.10.** *Let $1<p<\infty$, then, with equivalent norms,*

$$H^c_p(\mathcal{N}) = L^c_p MO(\mathcal{N})\,.$$

Let $2\le p<q\le\infty$. As already observed in [18], the map $I_{q,p}: L^c_q MO(\mathcal{N})\to L^c_p MO(\mathcal{N})$ is a contraction. However, for $0<\eta<1$, the inclusion map $I^\eta_{q,p}: L^c_q MO(\mathcal{N})\to L^c_p MO(\mathcal{N})$ is not necessarily continuous anymore, but is well defined on finite sequences. Since we would like to consider interpolation between these spaces, let $\mathcal{S}=\prod_{n\in\mathbb{N}}\prod_{n\le m} L_p(\mathcal{N}_m)$, endowed with the product topology. For $p<v<q$ with $\frac{1}{p}=\frac{1}{q}+\frac{1}{v}$, we may consider the embedding $\tilde{I}^\eta_{q,p}: L^c_q MO(\mathcal{N})\to\mathcal{S}$ defined by

$$\tilde{I}^\eta_{q,p}((x_m)_m) \;=\; (D^{\frac{1-\eta}{v}}(x_m-x_{n-1})D^{\frac{\eta}{v}})_{m,n}, \tag{36}$$

where $(x_m)_m$ is a finite martingale. Note that by Theorem 3.10, it follows that finite martingales are dense in $L^c_q MO(\mathcal{N})$ for $q<\infty$, since this is true for the space $H^c_q(\mathcal{N})$.

The following proposition is a nontracial modification of the result in [26].

**Proposition 3.11.** *Let $2\le p<s<q\le\infty$ and $0\le\theta\le 1$ such that $\frac{1}{s}=\frac{1-\theta}{q}+\frac{\theta}{p}$. Then, isometrically*

$$[\tilde{I}_{q,p}(L^c_q MO(\mathcal{N})), L^c_p MO(\mathcal{N})]_\theta \;=\; \tilde{I}_{s,p}(L^c_s MO(\mathcal{N}))\,. \tag{37}$$

*Proof.* For $2\le v\le\infty$, we define a map $\Phi_v: L^c_v MO(\mathcal{N})\to L^c_v(\mathcal{M};\ell_\infty)$ by

$$\Phi_v(x) = (u^v_n(x-\mathcal{E}_{n-1}(x)))_{n\ge 0}\,.$$

We claim that $\Phi_v$ is an isometry. Indeed, an application of Lemma 3.2, together with (33), shows that

$$\begin{aligned}
\|\Phi_v(x)\|_{L^c_v(\mathcal{M};\ell_\infty)} &= \left\|\sup_n u^v_n(x-\mathcal{E}_{n-1}(x))^* u^v_n(x-\mathcal{E}_{n-1}(x))\right\|^{\frac12}_{\frac{v}{2}}\\
&= \left\|\sup_n \mathcal{E}_n((x-\mathcal{E}_{n-1}(x))^*(x-\mathcal{E}_{n-1}(x)))\right\|^{\frac12}_{\frac{v}{2}}\\
&= \|x\|_{L^c_v MO(\mathcal{N})}\,.
\end{aligned}$$

Moreover, we observe that by definition of the maps $u^v_n$ we have

$$\Phi_p(xD^{\frac1p-\frac1s}) \;=\; \Phi_s(x)D^{\frac1p-\frac1s}\,.$$

This implies that

$$\Phi_p\tilde{I}_{s,p} \;=\; \widehat{I}_{s,p}\Phi_s\,.$$

Hence, the family $\Phi_v$ is compatible with the interpolation couples

$$(\tilde{I}_{q,p}(L^c_q MO(\mathcal{N})), L^c_p MO(\mathcal{N})) \quad\text{and}\quad (\widehat{I}_{q,p}(L^c_q(\mathcal{M},\ell_\infty)), L^c_p(\mathcal{M},\ell_\infty))\,.$$

We deduce that

$$\Phi_p: [\tilde{I}_{q,p}(L^c_q MO(\mathcal{N})), L^c_p MO(\mathcal{N})]_\theta \to [\widehat{I}_{q,p}(L^c_q(\mathcal{M};\ell_\infty)), L^c_p(\mathcal{M};\ell_\infty)]_\theta$$

is a contraction. An application of Proposition 3.8 yields the contraction

$$\widehat{I}_{s,p}\Phi_p: [\tilde{I}_{q,p}(L^c_q MO(\mathcal{N})), L^c_p MO(\mathcal{N})]_\theta \to \widehat{I}_{s,p}(L^c_s(\mathcal{M};\ell_\infty))\,.$$

By considering elements in the intersection $\Delta = (L_q^c MO(\mathcal{N}))D^{\frac{1}{p}-\frac{1}{q}} \subset L_p^c MO(\mathcal{N})$, we deduce that the image of $\widehat{I}_{s,p}\Phi_p$ is contained in $\widehat{I}_{s,p}\Phi_s(L_s^c MO(\mathcal{N}))$. Since $\Phi_s$ is isometric, we obtain a contractive inclusion

$$j_s : [\tilde{I}_{q,p}(L_q^c MO(\mathcal{N})), L_p^c MO(\mathcal{N})]_\theta \to \tilde{I}_{s,p}(L_s^c MO(\mathcal{N}))\,,$$

densely defined by the formula

$$j_s(xD^{\frac{1}{p}-\frac{1}{q}}) \;=\; xD^{\frac{1}{p}-\frac{1}{s}}\;.$$

A similar argument as in the proof of Proposition 3.4 in [26] yields the converse inclusion. The assertion is proved. □

As an application, we prove the result announced in [18].

**Corollary 3.12.** *Let $4 < p < \infty$; then for every $x \in L_p(\mathcal{N})$, we have*

$$\|x\|_{L_p^c MO(\mathcal{N})} \;\le\; c_4^{4/p}\;\|x\|_p\;.$$

*Here $c_4$ is an absolute constant.*

*Proof.* Fix $0 < \theta < 1$ such that $\frac{1}{p} = \frac{1-\theta}{\infty} + \frac{\theta}{4}$. Let $x \in L_p(\mathcal{N})$ with $\|x\|_p \le 1$. Consider the polar decomposition $x = u|x|$ with $u \in \mathcal{N}$ and $|x| = (x^*x)^{\frac{1}{2}} \in L_p(\mathcal{N})$. It follows that $\||x|\|_p = \|x\|_p \le 1$. Given $\delta > 0$ and $C \ge 1$ define a function

$$f(z) \;=\; C^{-z}\exp(\delta(\theta - z)^2)\;u|x|^{\frac{zp}{4}}D^{\frac{(1-z)}{4}}\;,$$

for all $z$ in the unit strip. According to Fact 2.1 in the preliminaries, it follows that $f$ is analytic and $f(z) \in L_4(\mathcal{N})$. Furthermore, for all $t \in \mathbb{R}$, note that $|x|^{pit}D^{-it} \in \mathcal{N}$. Hence

$$\left\|f(it)D^{-\frac{1}{4}}\right\|_\infty \;\le\; \exp(\delta\theta^2)\,, \tag{38}$$

and, respectively,

$$\|f(1+it)\|_4 \;\le\; C^{-1}\;\exp(\delta\theta^2)\;. \tag{39}$$

According to Theorem 3.10 (see [18] for details), there exist absolute constants $\beta_4\,,\gamma_4 > 0$ such that for all $y \in L_4(\mathcal{N})$,

$$\|y\|_{L_4^c MO(\mathcal{N})} \;\le\; 2\gamma_4\;\|y\|_{H_4^c(\mathcal{N})} \;\le\; 2\gamma_4\beta_4\,\|y\|_4\;.$$

Trivially, we have for $y \in \mathcal{N}$ that

$$\|y\|_{BMO_c(\mathcal{N})} \;\le\; 2\,\|y\|_\infty\;.$$

Choose $C = 2\gamma_4\beta_4 := c_4$. It follows that $f(\theta) \in [\tilde{I}_{\infty,4}(BMO_c(\mathcal{N})), L_4^c MO(\mathcal{N})]_\theta$, with

$$\|f(\theta)\|_\theta \le (\exp(\delta\theta^2))^{1-\theta}(\exp(\delta\theta^2))^\theta = \exp(\delta)\,.$$

According to Proposition 3.11, we deduce that

$$\left\|\tilde{I}_{p,4}(f(\theta))\right\|_{L_p^c MO(\mathcal{N})} \;\le\; \exp(\delta)\;.$$

Since $\delta > 0$ was arbitrarily chosen and $f(\theta) = {c_4}^{-\frac{4}{p}}xD^{\frac{1}{4}-\frac{1}{p}}$, the assertion follows. □

**Lemma 3.13.** *Let $2 \le p < q \le \infty$ and $0 < \theta < 1$. Then $\bigcup_{m\ge 0}\tilde{I}_{q,p}((\mathcal{N}_{m,a})D^{\frac{1}{q}})$ is norm dense in the interpolation space $[\tilde{I}^{\eta}_{q,p}(L_q^c MO(\mathcal{N})), L_p^c MO(\mathcal{N})]_\theta$.*

*Proof.* First we note that the sequence $(\mathcal{E}_m)_{m\geq 0}$ given by the conditional expectations satisfies the assumptions of Lemma 3.6. Thus the space of finite martingales is norm dense in the interpolation space, since this is true for the space $L_p^c MO(\mathcal{N})$, as observed before. Given an integer $m \geq 0$, we note that the inclusions

$$L_s(\mathcal{N}_m; l_2^c) \subset L_s^c MO(\mathcal{N}_m) \subset L_s^c MO(\mathcal{N})$$

are continuous, since

$$\|d_k(x)\|_{L_s(\mathcal{N}_m)} \ \leq \ 2\min\{\|x\|_{L_s^c MO(\mathcal{N}_m)}, \|x\|_{L_s(\mathcal{N}_m)}\}$$

for all $0 \leq k \leq m$. The intersection space is

$$\tilde{I}_{q,p}^{\eta}(L_q^c MO(\mathcal{N}_m)) \cap L_p^c MO(\mathcal{N}_m) \ = \ \tilde{I}_{q,p}^{\eta}(L_q^c MO(\mathcal{N}_m)) \, .$$

Since $\mathcal{N}_m D^{\frac{1}{q}}$ is norm dense in $L_q(\mathcal{N}_m)$, we deduce that $\tilde{I}_{q,p}^{\eta}(\mathcal{N}_m D^{\frac{1}{q}})$ is also dense in the interpolation space. In order to obtain analytic elements we recall the approximation operators $R_k : L_p(\mathcal{N}) \to L_p(\mathcal{N})$ defined by

$$R_k(x) \ = \ k^{\frac{1}{2}} \pi^{-\frac{1}{2}} \int_{\mathbb{R}} \exp(-kt^2) D^{it} x D^{-it} \, dt \, .$$

Since $\sigma_t(x) = D^{it} x D^{-it}$ is strongly continuous on $\mathcal{N}$, we deduce by approximation that $\sigma_t$ is norm continuous on $L_p(\mathcal{N})$ for all $2 \leq p < \infty$ (see [14], [17]). It then follows that

$$L_p - \lim_{k\to\infty} R_k(x) \ = \ x$$

for every $x \in L_p(\mathcal{N})$. Therefore, the same is true for the space $L_p^c MO(\mathcal{N}_m)$. Clearly, the map $\sigma_t : L_p^c MO(\mathcal{N}) \to L_p^c MO(\mathcal{N})$ is a contraction. It follows that $R_k$ is a contraction on $L_p^c MO(\mathcal{N})$, for all $2 \leq p \leq \infty$. In view of Lemma 3.6, we deduce that elements of the form $\tilde{I}_{q,p}^{\eta}(R_k(x) D^{\frac{1}{q}})$ are norm dense in the interpolation space $[\tilde{I}_{q,p}^{\eta}(L_q^c MO(\mathcal{N}_m)), L_p^c MO(\mathcal{N}_m)]_\theta$, for all $x \in \mathcal{N}_m$. Using the fact that the image $R_k(\mathcal{N}_m)$ consists of analytic elements and

$$R_k(x D^{\frac{1}{q}}) \ = \ R_k(x) D^{\frac{1}{q}} \ ,$$

we deduce the assertion. □

In order to establish the inclusion $I_{\infty,p}^1(BMO(\mathcal{N})) = I_{\infty,p}^1(BMO_c \cap BMO_r) \subset L_p(\mathcal{N})$, we have to consider and analyze the space $I_{\infty,p}^1(BMO_r(\mathcal{N}))$. In contrast with the inclusion $I_{\infty,p}^1(BMO_c(\mathcal{N})) \subset L_p^c MO(\mathcal{N})$, which follows easily by the definition, for the row version we are multiplying by $D^{\frac{1}{p}}$ on the wrong side. Alternatively, we may consider the space $I_{\infty,p}^0(BMO_c(\mathcal{N}))$. In this case we can use Kosaki's change of density argument (see [23]) in order to investigate the interpolation space $[I_{\infty,p}^0(BMO_c(\mathcal{N})), L_p^c MO(\mathcal{N})]_\theta$, for $0 < \theta < 1$.

**Proposition 3.14.** *Let $2 \leq p < q \leq \infty$, $p \leq v < \infty$ be such that $\frac{1}{p} = \frac{1}{q} + \frac{1}{v}$ and $0 < \theta < 1$. Then for all $0 \leq \eta < 1$, the spaces $[\tilde{I}_{q,p}^{\eta}(L_q^c MO(\mathcal{N})), L_p^c MO(\mathcal{N})]_\theta$ and $[\tilde{I}_{q,p}^{1}(L_q^c MO(\mathcal{N})), L_p^c MO(\mathcal{N})]_\theta$ are isometrically isomorphic. The isomorphism*

$$T_\theta : [\tilde{I}_{q,p}^{1}(L_q^c MO(\mathcal{N})), L_p^c MO(\mathcal{N})]_\theta \to [\tilde{I}_{q,p}^{\eta}(L_q^c MO(\mathcal{N})), L_p^c MO(\mathcal{N})]_\theta$$

*is defined by*

$$T_\theta((x_m D^{\frac{1}{v}})_m) \;=\; \Big(D^{\frac{(1-\eta)(1-\theta)}{v}} x_m D^{\frac{1}{v}-\frac{(1-\eta)(1-\theta)}{v}}\Big)_m \;,$$

*for all adapted finite sequences* $(x_m)_m \in L^c_q MO(\mathcal{N})$ .

*Proof.* We will first construct a contraction

$$T_\theta : [\tilde{I}^1_{q,p}(L^c_q MO(\mathcal{N})), L^c_p MO(\mathcal{N})]_\theta \to [\tilde{I}^\eta_{q,p}(L^c_q MO(\mathcal{N})), L^c_p MO(\mathcal{N})]_\theta \;.$$

According to Lemma 3.13, it suffices to define $T_\theta$ on the dense set $\bigcup_{m\geq 0} \tilde{I}^1_{q,p}(\mathcal{N}_{m,a} D^{\frac{1}{q}})$ and show that the restriction is contractive. Furthermore, by the density result [3, Lemma 4.3.2], it is enough to consider elements of the form

$$x \;=\; f(\theta) D^{\frac{1}{p}} \,,$$

where

$$f(z) \;=\; \sum_{k=1}^{l} g_k(z) y_k$$

with $y_k \in \mathcal{N}_{m,a}$, for some $m$, and $g_k$ analytic functions which vanish at infinity such that

$$\max\left\{\sup_{t\in\mathbb{R}} \left\| f(it) D^{\frac{1}{q}} \right\|_{L^c_q MO(\mathcal{N})}, \sup_{t\in\mathbb{R}} \left\| f(1+it) D^{\frac{1}{p}} \right\|_{L^c_p MO(\mathcal{N})}\right\} \;<\; 1.$$

For all $z$ in the unit strip $S$ we define

$$g(z) = D^{\frac{(1-\eta)(1-z)}{v}} f(z) D^{\frac{1}{q}+\frac{\eta}{v}+\frac{(1-\eta)z}{v}} = \sum_{k=1}^{l} g_k(z) (D^{\frac{(1-\eta)(1-z)}{v}} y_k D^{\frac{(1-\eta)z}{v}} D^{\frac{1}{q}+\frac{\eta}{v}}) \;.$$

According to Fact 2.1, this function is analytic, taking values in $L_p(\mathcal{N}_m) \subset L^c_p(\mathcal{N}_m; l_\infty)$. Then, we observe that $\sigma_t$ is an isometry on $L^c_p MO(\mathcal{N})$, and therefore

$$\begin{aligned} \|g(1+it)\|_{L^c_p MO(\mathcal{N})} &= \left\| D^{\frac{-it(1-\eta)}{v}} f(1+it) D^{\frac{1}{q}+\frac{\eta}{v}+\frac{1-\eta}{v}} D^{\frac{(1-\eta)it}{v}} \right\|_{L^c_p MO(\mathcal{N})} \\ &= \left\| f(1+it) D^{\frac{1}{p}} \right\|_{L^c_p MO(\mathcal{N})} < 1 \;. \end{aligned}$$

On the other hand, by (36),

$$\begin{aligned} \left\| D^{-\frac{1-\eta}{v}} g(it) D^{-\frac{\eta}{v}} \right\|_{L^c_q MO(\mathcal{N})} &= \left\| D^{\frac{-it(1-\eta)}{v}} f(it) D^{\frac{(1-\eta)it}{v}} D^{\frac{1}{q}} \right\|_{L^c_q MO(\mathcal{N})} \\ &= \left\| D^{\frac{-it(1-\eta)}{v}} f(it) D^{\frac{1}{q}} D^{\frac{(1-\eta)it}{v}} \right\|_{L^c_q MO(\mathcal{N})} \\ &= \left\| f(it) D^{\frac{1}{q}} \right\|_{L^c_q MO(\mathcal{N})} < 1 \;. \end{aligned}$$

Therefore, for $x = f(\theta) D^{\frac{1}{p}}$ it follows that

$$\begin{aligned} T_\theta(x) = g(\theta) \;&=\; \sum_{k=1}^{l} g_k(\theta) D^{\frac{(1-\eta)(1-\theta)}{v}} y_k D^{\frac{(1-\eta)\theta}{v}} D^{\frac{1}{q}+\frac{\eta}{v}} \\ &= D^{\frac{(1-\eta)(1-\theta)}{v}} f(\theta) D^{\frac{1}{q}} D^{\frac{(1-\eta)\theta}{v}+\frac{\eta}{v}} \;=\; D^{\frac{(1-\eta)(1-\theta)}{v}} f(\theta) D^{\frac{1}{p}-\frac{(1-\eta)(1-\theta)}{v}} \end{aligned} \tag{40}$$

is an element in the interpolation space $[\tilde{I}^{\eta}_{q,p}(L^c_q MO(\mathcal{N})), L^c_p MO(\mathcal{N})]_\theta$. Hence $T_\theta$ defined by (40) extends to a contraction. Conversely, a similar argument shows that the map $\tilde{T}_\theta$ defined by

$$\tilde{T}_\theta(D^{\frac{1-\eta}{v}} f(\theta) D^{\frac{1}{q}} D^{\frac{\eta}{v}}) = D^{\frac{(1-\eta)\theta}{v}} f(\theta) D^{\frac{1}{q}+\frac{\eta}{v}+\frac{(1-\eta)(1-\theta)}{v}}$$

is a contraction. Since $f$ is analytic, there exists $y \in \mathcal{N}_{m,a}$ such that

$$D^{\frac{(1-\eta)\theta}{v}} f(\theta) \ = \ y D^{\frac{(1-\eta)\theta}{v}} \ .$$

Furthermore, we deduce that

$$\begin{aligned} T_\theta \tilde{T}_\theta(D^{\frac{1-\eta}{v}} f(\theta) D^{\frac{1}{q}} D^{\frac{\eta}{v}}) &= T_\theta(y D^{\frac{1}{p}}) \\ &= D^{\frac{(1-\eta)(1-\theta)}{v}} y D^{\frac{1}{p}-\frac{(1-\eta)(1-\theta)}{v}} \\ &= D^{\frac{(1-\eta)(1-\theta)}{v}} y D^{\frac{(1-\eta)\theta}{v}} D^{\frac{1}{q}+\frac{\eta}{v}} \\ &= D^{\frac{1-\eta}{v}} f(\theta) D^{\frac{1}{q}+\frac{\eta}{v}} \ . \end{aligned}$$

Hence $T_\theta \tilde{T}_\theta = \mathrm{id}$. Similarly, we may show that $\tilde{T}_\theta T_\theta = \mathrm{id}$, and the assertion follows. □

**Corollary 3.15.** *Let $2 < p < s < q \leq \infty$, $0 < \theta < 1$ such that $\frac{1}{s} = \frac{1-\theta}{q} + \frac{\theta}{p}$, and $0 \leq \eta \leq 1$. Then, isometrically,*

$$[\tilde{I}^{\eta}_{q,p}(L^c_q MO(\mathcal{N})), L^c_p MO(\mathcal{N})]_\theta \ = \ \tilde{I}^{\eta}_{s,p}(L^c_s MO(\mathcal{N})) \, .$$

*Proof.* Let us note first that $\frac{1}{p} - \frac{1}{s} = \frac{1-\theta}{v}$, where $\frac{1}{v} = \frac{1}{p} - \frac{1}{q}$. Then, we observe that

$$T_\theta(x D^{\frac{1}{p}-\frac{1}{s}}) \ = \ D^{(1-\eta)\frac{1-\theta}{v}} x D^{\frac{1}{p}-\frac{1}{s}-(1-\eta)\frac{1-\theta}{v}} \ = \ D^{(1-\eta)(\frac{1}{p}-\frac{1}{s})} x D^{\eta(\frac{1}{p}-\frac{1}{s})} \ .$$

The assertion now follows from Proposition 3.11 and Proposition 3.14. □

Recall now the row version of the martingale norm (34)

$$\|x\|_{L^r_p MO(\mathcal{N})} \ = \ \sup_m \left\| \sup_{n \leq m} \mathcal{E}_n \left( \sum_{k=n}^m d_k(x) d_k(x)^* \right) \right\|^{\frac{1}{2}}_{\frac{p}{2}} \ .$$

Interchanging the roles of columns and rows we immediately get the following result.

**Proposition 3.16.** *Let $2 < p < t < q \leq \infty$, $0 < \theta < 1$ such that $\frac{1}{s} = \frac{1-\theta}{q} + \frac{\theta}{p}$, and $0 \leq \eta \leq 1$. Then, isometrically,*

$$[\tilde{I}^{\eta}_{q,p}(L^r_q MO(\mathcal{N})), L^r_p MO(\mathcal{N})]_\theta = \tilde{I}^{\eta}_{s,p}(L^r_s MO(\mathcal{N})) \, .$$

For $2 < p \leq \infty$, the following martingale norms were defined in [18]:

$$\|x\|_{L_p MO(\mathcal{N})} \ = \ \max\{\|x\|_{L^c_p MO(\mathcal{N})}, \|x\|_{L^r_p MO(\mathcal{N})}\} \, ,$$

respectively,

$$\|x\|_{H_p(\mathcal{N})} = \max\{\|x\|_{H^c_p(\mathcal{N})}, \|x^*\|_{H^c_p(\mathcal{N})}\} \, .$$

We now state the first version of our main result.

**Theorem 3.17.** *Let $2 < p < s < q \leq \infty$, $0 < \theta < 1$ such that $\frac{1}{s} = \frac{1-\theta}{q} + \frac{\theta}{p}$, and $0 \leq \eta \leq 1$. Then, with equivalent norms,*

$$[I^{\eta}_{q,p}(L_q MO(\mathcal{N})), L_p(\mathcal{N})]_\theta = I^{\eta}_{s,p}(L_s(\mathcal{N})) \, .$$

*Proof.* Recall that, by Corollary 3.12, we have

$$\|x\|_{L_p^c MO(\mathcal{N})} \leq \lambda_p \|x\|_p \,,$$

where $\lambda_p = O(1)$ as $p \to \infty$. Combining this inequality with Corollary 3.15, Proposition 3.16 and Corollary 5.6 of [35], it follows that

$$\begin{aligned}
[I^\eta_{q,p}(L_q MO(\mathcal{N})), L_p(\mathcal{N})]_\theta &\subseteq [\tilde{I}^\eta_{q,p}(L_q^c MO), L_p^c MO]_\theta \cap [\tilde{I}^\eta_{q,p}(L_q^r MO), L_p^r MO]_\theta \\
&\subseteq \tilde{I}^\eta_{s,p}(L_s^c MO) \cap \tilde{I}^\eta_{s,p}(L_s^r MO) \\
&= \tilde{I}^\eta_{s,p}(L_s^c MO \cap L_s^r MO) \\
&\subseteq \tilde{I}^\eta_{s,p}(L_s(\mathcal{N}))\,,
\end{aligned}$$

where the norm of the last inclusion above is $\lambda'_s \leq Cs$ as $s \to \infty$; here $C$ is an absolute constant. The reverse inclusion follows by applying Kosaki's interpolation result (18), together with Corollary 3.15 and Proposition 3.16, namely

$$\begin{aligned}
I^\eta_{s,p}(L_s(\mathcal{N})) &\subseteq [I^\eta_{q,p}(L_q(\mathcal{N})), L_p(\mathcal{N}))]_\theta \\
&\subseteq [I^\eta_{q,p}(L_q^c MO \cap L_q^r MO), L_p(\mathcal{N})]_\theta \\
&= [I^\eta_{q,p}(L_q MO), L_p(\mathcal{N})]_\theta \,.
\end{aligned}$$

□

As an application, we obtain our main result of this section. The following continuous inclusion is proved using the fact that we have an interpolation scale of spaces.

**Theorem 3.18.** *Let $1 \leq p < \infty$ and $0 \leq \eta \leq 1$. Then the inclusion map*

$$I^\eta_{\infty,p}(BMO) \subset L_p(\mathcal{N})$$

*is bounded, with norm $c(p) \leq cp$, where $c$ is an absolute constant.*

*Proof.* Assume that $p \geq 8$. Let $4 < s < p/2 < \infty$ and define $0 < \theta < 1$ such that $\frac{1}{p} = \frac{1-\theta}{\infty} + \frac{\theta}{s}$. Since the complex interpolation method is an exact interpolation functor of exponent $\theta$, we deduce from (the proof of) Theorem 3.17 that for a finite martingale difference sequence $x = (d_k)_{k=0}^n \in BMO$ we have

$$\begin{aligned}
\|D^{\frac{1-\eta}{p}} x D^{\frac{\eta}{p}}\|_p &\leq c(p) \|D^{\frac{1-\eta}{s}} x D^{\frac{\eta}{s}}\|_{[I^\eta_{\infty,s}(BMO), L_s(\mathcal{N})]_\theta} \\
&\leq c(p) \|x\|^{1-\theta}_{BMO} \|D^{\frac{1-\eta}{s}} x D^{\frac{\eta}{s}}\|^\theta_s \leq c(p) \|x\|^{1-\theta}_{BMO} \|D^{\frac{1-\eta}{p}} x D^{\frac{\eta}{p}}\|^\theta_p \,,
\end{aligned}$$

where $c(p) \leq Cp$, with $C$ an absolute constant. Thus, we get

$$\|D^{\frac{1-\eta}{p}} x D^{\frac{\eta}{p}}\|_p \leq (c(p))^{\frac{1}{1-\theta}} \|x\|_{BMO} \,. \tag{41}$$

Under the hypotheses on $p$ and $s$, a simple computation shows that

$$(c(p))^{\frac{1}{1-\theta}} \leq cp \,,$$

where $c$ is an absolute constant. Therefore (41) implies that

$$\|D^{\frac{1-\eta}{p}} x D^{\frac{\eta}{p}}\|_p \leq cp \; \|x\|_{BMO} \,.$$

For an arbitrary element $x = (d_k)_{k \geq 0}$ in $BMO$, we consider the sequence of projections $P_n(x)$, where

$$P_n(x) = (d_k)_{k=0}^n \,,$$

whose images $I^\eta_{\infty,p}(P_n(x))$ are uniformly bounded and thus converge in $L_p(\mathcal{N})$ to $I^\eta_{\infty,p}(x)$. This yields the assertion for $p \geq 8$. Since $L_8(\mathcal{N})$ embeds into $L_p(\mathcal{N})$ for all $1 \leq p < 8$, the proof is complete. □

Note that the symmetric embedding

$$\Psi_p(x) \;=\; D^{\frac{1}{2p}} x D^{\frac{1}{2p}}$$

is positivity preserving, and thus it can be considered as the natural embedding of $\mathcal{N}$ into $L_p(\mathcal{N})$. As a special case of Theorem 3.18 we obtain the following.

**Corollary 3.19.** *If $1 \le p < \infty$, then the inclusion map*

$$\Psi_p(BMO) \subset L_p(\mathcal{N})$$

*is bounded, with norm $c(p) \le cp$, where $c$ is an absolute constant.*

*Remark* 3.20. The following example, which answers a question raised by Tao Mei, shows that the above inclusion does not hold if we only restrict to column, respectively row versions of the $BMO$ norm. Let $n$ be a positive integer and consider the von Neumann algebra

$$\mathcal{N} = L_\infty([0,1])\bar{\otimes} M_n\,,$$

where $M_n$ is the algebra of $n \times n$ complex matrices. For $k \ge 1$ let $\Sigma_k$ be the $\sigma$-algebra generated by dyadic intervals in $[0,1]$ of length $2^{-k}$. Denote by $\mathcal{N}_k$ the subalgebra $L_\infty([0,1],\Sigma_k)\bar{\otimes} M_n$ of $\mathcal{N}$ and let $\mathcal{E}_k = \mathbb{E}_k \otimes \mathrm{Id}_{M_n}$ be the conditional expectation onto $\mathcal{N}_k$. Let $\varepsilon_k \in \{-1,1\}$ and define

$$x = \sum_{k=1}^{n} \varepsilon_k \otimes e_{1k}\,.$$

Then $x$ is a martingale relative to the filtration $(\mathcal{N}_k)$. The martingale differences are given by $d_k(x) = \varepsilon_k \otimes e_{1k}$. A simple computation shows that $\|x\|_{L_p(\mathcal{N})} = n^{\frac{1}{2}-\frac{1}{p}}$, while

$$\|x\|_{BMO_c} = \sup_m \left\| \sum_{k=m}^{n} \mathcal{E}_m(d_k^*(x) d_k(x)) \right\|_\infty^{\frac{1}{2}} = \sup_k \|\varepsilon_k^2\|_\infty^{\frac{1}{2}} = 1\,.$$

Assume that $p > 2$. Then for any $C > 0$, there exists $n \ge 1$ such that $n^{\frac{1}{2}-\frac{1}{p}} > C$. This implies that $BMO_c(\mathcal{N})$ is not contained in $L_p(\mathcal{N})$.

## 4. Main results

In this section, we will show how to derive the John-Nirenberg type result from Theorem 3.18. Let us start with an immediate application.

**Corollary 4.1.** *Let $1 \le p < \infty$, a positive integer $n$ and an element $a \in L_p(\mathcal{N}_n)$. Then, there exists an absolute constant $c > 0$ such that for all $x \in \mathcal{N}$,*

$$\|(x - x_{n-1})a\|_p \;\le\; cp\,\|x\|_{BMO}\,\|a\|_p\,. \tag{42}$$

*Proof.* Assume that $p \ge 2$. It suffices to show (42) for positive elements $a \in L_p(\mathcal{N}_n)$ with $\|a\|_p = 1$. Furthermore, by approximation with elements of the form $(a^p + \varepsilon D)^{\frac{1}{p}}$, we may assume that $a$ has full support. Define a new state $\phi_a$ on $\mathcal{N}$ by

$$\phi_a(x) \;=\; \mathrm{tr}(a^p x)\,, \quad x \in \mathcal{N}\,.$$

Denote by $L_p(\mathcal{N},\phi_a)$ the noncommutative $L_p$ space associated to the state $\phi_a$. Note that

$$\|x - x_{n-1}\|_{L_p(\mathcal{N},\phi_a)} = \|(x - x_{n-1})a\|_p\,. \tag{43}$$

For all nonnegative integers $k$ define $\widetilde{\mathcal{N}}_k = \mathcal{N}_{n+k}$. Then $\widetilde{\mathcal{N}}_0 = \mathcal{N}_n$ and $(\widetilde{\mathcal{N}}_k)_{k\geq 0}$ is a filtration of $\mathcal{N}$. Moreover, if $m \geq n$ and $x \in \mathcal{N}$, then $xa^p \in L_1(\mathcal{N}_m)$. Hence,

$$\phi_a(yx) \;=\; \mathrm{tr}(a^p yx) \;=\; \mathrm{tr}(yxa^p) \;=\; \mathrm{tr}(\mathcal{E}_m(y)xa^p) \;=\; \phi_a(\mathcal{E}_m(y)x)\,, \tag{44}$$

for all $y \in \mathcal{N}_m$. Denote by $BMO((\widetilde{\mathcal{N}}_k)_{k\geq n}\,,\phi_a)$ the $BMO$ space associated to the filtration $(\widetilde{\mathcal{N}}_k)_{k\geq 0}$ of $\mathcal{N}$ and the state $\phi_a$. From Theorem 3.18 it follows that

$$\|x - x_{n-1}\|_{L_p(\mathcal{N},\phi_a)} \;\leq\; cp\,\|x - x_{n-1}\|_{BMO((\widetilde{\mathcal{N}}_k)_{k\geq n}\,,\phi_a)}\,. \tag{45}$$

The modular automorphism group of $\phi_a$ is given by

$$\sigma_t^{\phi_a}(x) \;=\; a^{ipt}xa^{-ipt}\,, \quad t \in \mathbb{R}\,.$$

From (44) we deduce that for all $k \geq 0$, the conditional expectations $\tilde{\mathcal{E}}_k = \mathcal{E}(\cdot|\widetilde{\mathcal{N}}_k)$ associated with the state $\phi_a$ are given by the original conditional expectations, namely, $\tilde{\mathcal{E}}_k \;=\; \mathcal{E}_{k+n}$. Therefore,

$$\tilde{\mathcal{E}}_k(x - x_{n-1}) \;=\; \mathcal{E}_{k+n}(x - x_{n-1}) \;=\; x_{k+n} - x_{n-1}\,.$$

Hence, denoting $x - x_{n-1}$ by $y$, we obtain

$$\tilde{\mathcal{E}}_k((y - \tilde{\mathcal{E}}_{k-1}(y))^*(y - \tilde{\mathcal{E}}_{k-1}(y))) = \mathcal{E}_{n+k}((x - x_{n+k-1})^*(x - x_{n+k-1}))\,.$$

By the definition of the $BMO$ norm, it follows that

$$\|x - x_{n-1}\|_{BMO((\widetilde{\mathcal{N}}_k)_{k\geq n}\,,\phi_a)} \;\leq\; \|x\|_{BMO}\,.$$

Together with (45) and (43), this yields the assertion in the case $p \geq 2$. Furthermore, if $1 \leq p < 2$ and $0 \leq a \in L_p(\mathcal{N})$ with $\|a\|_p = 1$, consider $a_1 = a^{\frac{1}{2}}$. Then $0 \leq a_1 \in L_{2p}(\mathcal{N})$ and $\|a_1\|_{2p} = 1$. By Hölder's inequality it follows that

$$\|(x - x_{n-1})a\|_p \leq \|(x - x_{n-1})a_1\|_{2p}\,. \tag{46}$$

Note that $2p \geq 2$. Applying (42) and (46) to $a_1$, we obtain the conclusion. □

*Remark* 4.2. Note that for any integer $n \geq 0$, the inclusion $\mathcal{N} \subset L^c_\infty(\mathcal{N},\mathcal{E}_n)$ is injective. Thus, given $2 \leq p < \infty$, we can consider the space $[\mathcal{N}, L^c_\infty(\mathcal{N},\mathcal{E}_n)]^{\frac{2}{p}}$ obtained by the upper method of complex interpolation of exponent $\frac{2}{p}$. These are particular examples of conditional $L_p$-spaces considered in [16]. Following ideas of Pisier [30], it is proved in [16] that

$$\|x\|_{[\mathcal{N},L^c_\infty(\mathcal{N},\mathcal{E}_n)]^{\frac{2}{p}}} = \sup_{\|a\|_{L_p(\mathcal{N}_n)}\leq 1} \|xa\|_{L_p(\mathcal{N})}\,. \tag{47}$$

Therefore, as already mentioned in the Introduction, the $BMO^c_p$ norms defined by (9) are in fact interpolation norms, namely

$$\|x\|_{BMO^c_p} = \|x - x_{n-1}\|_{[\mathcal{N},L^c_\infty(\mathcal{N},\mathcal{E}_n)]^{\frac{2}{p}}}\,. \tag{48}$$

*Proof of Theorem* 1.1. Let $2 < p < \infty$. Trivially, we have

$$[\mathcal{N}, L^c_\infty(\mathcal{N},\mathcal{E}_n)]^{\frac{2}{p}} \subset L^c_\infty(\mathcal{N},\mathcal{E}_n)\,.$$

Therefore, we deduce that

$$\|\mathcal{E}_n((x - x_{n-1})^*(x - x_{n-1}))\|_\infty^{\frac{1}{2}} \;\leq\; \|x - x_{n-1}\|_{[\mathcal{N},L^c_\infty(\mathcal{N},\mathcal{E}_n)]^{\frac{2}{p}}}\,.$$

Combining this with (48), it follows immediately that

$$\|x\|_{BMO} \;\leq\; \|x\|_{BMO_p}\,.$$

For the converse, we deduce from (9) and Corollary 4.1 that

$$\|x\|_{BMO_p^c} \;=\; \sup_{\|a\|_{L_p(\mathcal{N}_n)}\le 1} \|(x-x_{n-1})a\|_p \;\le\; cp\,\|x\|_{BMO}\ .$$

Applying the same argument to $x^*$, we deduce that

$$\|x\|_{BMO_p} \;\le\; cp\,\|x\|_{BMO}\ ,$$

which concludes the proof. □

*Remark* 4.3. As an application of Theorem 1.1 we now prove that the noncommutative analogue of the classical result

$$BMO \subset L_{\exp} \tag{49}$$

holds in the setting of a semifinite von Neumann algebra $\mathcal{N}$ equipped with a normal, faithful trace $\tau$. In this context the space $L_{\exp}(\mathcal{N})$ can be defined following the general scheme of *symmetric spaces* associated to $(\mathcal{N},\tau)$ and a rearrangement invariant Banach function space developed in [6] and [7]. Therefore, motivated by the classical definition of the Zygmund space $L_{\exp}$, we define

$$\|x\|_{L_{\exp}(\mathcal{N})} = \inf\{\lambda>0 : \tau(e^{\frac{|x|}{\lambda}-1})\le 1\}\,. \tag{50}$$

Suppose that $\|x\|_{BMO}\le 1$ and let $\lambda>0$. Then, using the power series expansion for the exponential, it follows from Theorem 1.1 that

$$\begin{aligned}\tau(e^{\frac{|x|}{\lambda}-1}) = \frac{1}{e}\sum_{k=0}^{\infty}\frac{\tau(|x|^k)}{\lambda^k k!} \;&=\; \frac{1}{e}\left(1+\sum_{k=1}^{\infty}\frac{\|x\|_k^k}{\lambda^k k!}\right)\\ &\le\; \frac{1}{e}\left(1+\sum_{k=1}^{\infty}\frac{c^k k^k}{\lambda^k k!}\right)\\ &\le\; \frac{1}{e}\left(1+\sum_{k=1}^{\infty}\frac{c^k e^k}{\lambda^k}\right) = \frac{1}{e}\sum_{k=0}^{\infty}\left(\frac{ce}{\lambda}\right)^k .\end{aligned}$$

For the last inequality we have used the fact, which can be verified by induction, that $k^k\le k!e^k$ for all integers $k\ge 1$. A simple computation shows that if $\frac{1}{\lambda}\le \frac{1}{ce}\left(1-\frac{1}{e}\right)$, then $\frac{1}{e}\sum_{k=0}^{\infty}\left(\frac{ce}{\lambda}\right)^k\le 1$. By (50), it follows that

$$\|x\|_{L_{\exp}(\mathcal{N},\tau)} \le \frac{1}{ce}\left(1-\frac{1}{e}\right)\ .$$

Hence $x\in L_{\exp}(\mathcal{N})$ and (49) is proved.

**Open problem.** Is there a universal constant $C$ such that if $x\in BMO$, then

$$\sup_n \|\mathcal{E}_n(|x-x_{n-1}|^p)\|_\infty^{\frac{1}{p}} \le Cp\|x\|_{BMO}\,,$$

for all $1\le p<\infty$?

The noncommutative versions of function space $BMO$ have been developed by Mei [25]. Namely, for a function $x:\mathbb{R}\to\mathcal{N}$, define the following norm:

$$\|x\|_{BMO_c} \;=\; \sup_{\emptyset\ne I}\left\|\int_I (x(t)-x_I)^*(x(t)-x_I)\frac{dt}{|I|}\right\|_{\mathcal{N}}\ ,$$

where $x_I = \int_I x(t)\frac{dt}{|I|}$, for every nonempty interval $I$. Furthermore, define

$$\|x\|_{BMO_r} \;=\; \|x^*\|_{BMO_c}$$

and

$$\|x\|_{BMO} \;=\; \max\{\|x\|_{BMO_c}, \|x\|_{BMO_r}\}\,.$$

As in the martingale setting, we use interpolation to define the $BMO_p$-norms in this context. Namely, for $2 < p < \infty$ and a fixed nonempty interval $I$, let $\mu_I = \frac{dt}{|I|}$ and define the space

$$\mathcal{N}(L_p^c(\mu_I)) \;=\; [\mathcal{N}\bar{\otimes}L_\infty(\mu_I), \mathcal{N}\bar{\otimes}L_2^c(\mu_I)]^{\frac{2}{p}}$$

by (the upper method of) complex interpolation. Recall that

$$\mathcal{N}\bar{\otimes}L_\infty(\mu_I) = L_\infty(I, \mu_I; \mathcal{N})$$

is the usual space of measurable, essentially bounded $\mathcal{N}$-valued functions, while the space $\mathcal{N}\bar{\otimes}L_2^c(\mu_I)$ is defined following the operator space tradition of viewing $L_2(\mu_I)$ implemented as a column in $B(L_2(\mu_I))$. Equivalently,

$$\|x\|_{\mathcal{N}\bar{\otimes}L_2^c(\mu_I)} \;=\; \left\| \int_I (x(t))^* x(t) d\mu_I(t) \right\|_{\mathcal{N}}^{\frac{1}{2}}$$

for all $\mathcal{N}$-valued functions which are measurable with respect to the strong operator topology. Note that $\mathcal{E}_I(x) \;=\; \int_I x(t)d\mu_I(t)$ is a conditional expectation onto $\mathcal{N}_I = \mathcal{N}\bar{\otimes}L_\infty(\mu_I)$. Therefore, the spaces $\mathcal{N}(L_p^c(\mu_I))$ are special cases of conditional $L_p$-spaces considered in [16]. In particular the following norm estimate holds:

$$\|1_I x\|_{\mathcal{N}(L_p^c(\mu_I))} \;=\; \sup_{\|a\|_{L_p(\mathcal{N}_I)} \le 1} \left( \int_I \|x(t)a\|^p_{L_p(\mathcal{N})}\, d\mu_I(t) \right)^{\frac{1}{p}} . \tag{51}$$

As in [25], we will use the following result to transfer martingale results to intervals.

**Lemma 4.4** (T. Mei)**.** *Let $I \subset \mathbb{R}$ be an interval. Then there exists a dyadic interval $J = [\frac{k-1}{2^n}, \frac{k}{2^n}]$ or $J' = [\frac{1}{3\cdot 2^{n'}} + \frac{k'-1}{2^{n'}}, \frac{1}{3\cdot 2^{n'}} + \frac{k'}{2^{n'}}]$ such that*

$$I \subset J \quad \textit{and} \quad |J| \;\le\; 6|I|$$

*or*

$$I \subset J' \quad \textit{and} \quad |J'| \;\le\; 6|I|\,.$$

**Lemma 4.5.** *For a nonempty interval $I \subset \mathbb{R}$ and an element $x \in \mathcal{N}\bar{\otimes}L_\infty(\mathbb{R})$ let*

$$\|x\|_{p,I} \;=\; \sup_{\|a\|_{L_p(\mathcal{N})} \le 1} \|(1_I x - x_I)a\|_{L_p(I,\mu_I;L_p(\mathcal{N}))}\,.$$

*Then, for all intervals $I \subset J \subset \mathbb{R}$,*

$$\|x\|_{p,I} \;\le\; 2\left(\frac{|J|}{|I|}\right)^{\frac{1}{p}} \|x\|_{p,J}\,.$$

*Proof.* This is of course a standard argument. Let $a \in L_p(\mathcal{N})$. Then we have

$$\|(x_I - x_J)a\|_p = \left\| \int_I [(x(s) - x_J)a]\, d\mu_I(s) \right\|_p \le \left( \int_I \|(x(t) - x_J)a\|_p^p \, d\mu_I(t) \right)^{\frac{1}{p}} .$$

Therefore, we obtain

$$\begin{aligned} \|(1_I x - x_I)a\|_{L_p(I,\mu_I;L_p(\mathcal{N}))} &\le \|(1_I x - x_J)a\|_{L_p(I,\mu_I;L_p(\mathcal{N}))} + \|(x_I - x_J)a\|_p \\ &\le 2 \left( \int_I \|(x(t) - x_J)a\|_p^p \, d\mu_I(t) \right)^{\frac{1}{p}} \\ &\le 2 \left( \frac{|J|}{|I|} \right)^{\frac{1}{p}} \left( \int_J \|(x(t) - x_J)a\|_p^p \, d\mu_J(t) \right)^{\frac{1}{p}} . \end{aligned}$$

Taking supremum over all elements $a$ in the unit ball of $L_p(\mathcal{N})$ implies the assertion. □

We are now ready to formulate a noncommutative interval version of the John-Nirenberg theorem.

**Theorem 4.6.** *For all $2 < p < \infty$, there exists an absolute constant $c > 0$ such that*

$$\begin{aligned} \|x\|_{BMO} &\le \sup_{\emptyset \neq I \ \text{interval}} \max\{\|x - x_I\|_{\mathcal{N}(L_p^c(\mu_I))}, \|x^* - x_I^*\|_{\mathcal{N}(L_p^c(\mu_I))}\} \\ &\le cp\, \|x\|_{BMO} . \end{aligned}$$

*Proof.* Denote by $\widetilde{\mathcal{N}}$ the von Neumann algebra tensor product $\mathcal{N}\bar{\otimes}L_\infty[0,1]$. For each integer $n \ge 0$, let $\Sigma_n$ be the $\sigma$-algebra generated by dyadic intervals in $[0,1]$ of length $2^{-n}$. Denote by $\widetilde{\mathcal{N}}_n$ the subalgebra $\mathcal{N}\bar{\otimes}L_\infty([0,1],\Sigma_n)$ of $\widetilde{\mathcal{N}}$ and let $\tilde{\mathcal{E}}_n$ be the conditional expectation onto $\widetilde{\mathcal{N}}_n$. Consider the $BMO$ space associated to the filtration $(\widetilde{\mathcal{N}}_n)_{n\ge 0}$ of $\widetilde{\mathcal{N}}$. For $x \cong (x_n)_{n\ge 0}$, where $x_n = \tilde{\mathcal{E}}_n(x)$, we deduce that

$$\sup_n \max\{\|(x - x_n)\|_{L_p^c(\tilde{\mathcal{N}},\tilde{\mathcal{E}}_n)}, \|(x - x_n)^*\|_{L_p^c(\tilde{\mathcal{N}},\tilde{\mathcal{E}}_n)}\} \le \|x\|_{BMO} .$$

An application of (51) shows that

$$\begin{aligned} \|x - x_n\|^p_{L_p^c(\tilde{\mathcal{N}},\tilde{\mathcal{E}}_n)} &= \sup_{\sum_k \|a_k\|_p^p |I_k| \le 1} \sum_k |I_k| \int_{I_k} \|(x(t) - x_{I_k})a_k\|_p^p \, d\mu_{I_k}(t) \\ &= \sup_k \|1_{I_k} x - x_{I_k}\|^p_{\mathcal{N}(L_p^c(\mu_{I_k}))} . \end{aligned}$$

Here the supremum is taken over all intervals $I_k \subset [0,1]$ of length $2^{-n}$. Therefore, we may reformulate the martingale result in Theorem 1.1 as

$$\begin{aligned} &\sup_{I \ \text{dyadic} \subset [0,1]} \max\{\|1_I x - x_I\|_{\mathcal{N}(L_p^c(\mu_I))}, \|(1_I x - x_I)^*\|_{\mathcal{N}(L_p^c(\mu_I))}\} \\ &\le cp \sup_{I \ \text{dyadic} \subset [0,1]} \max\{\|1_I x - x_I\|_{\mathcal{N}(L_2^c(\mu_I))}, \|(1_I x - x_I)^*\|_{\mathcal{N}(L_2^c(\mu_I))}\} . \end{aligned}$$

Using the dilation and translation invariance properties of the Lebesgue measure, it follows for all integers $k \geq 0$ that

$$\sup_{I \text{ dyadic } \subset [2^{-k},2^k]} \max\{\|1_I x - x_I\|_{\mathcal{N}(L_p^c(\mu_I))}, \|(1_I x - x_I)^*\|_{\mathcal{N}(L_p^c(\mu_I))}\}$$
$$\leq cp \sup_{I \text{ dyadic } \subset [2^{-k},2^k]} \max\{\|1_I x - x_I\|_{\mathcal{N}(L_2^c(\mu_I))}, \|(1_I x - x_I)^*\|_{\mathcal{N}(L_2^c(\mu_I))}\}.$$

Taking the supremum over $k$, we obtain the desired inequality for arbitrary dyadic martingales. Applying Mei's result 4.4, together with Lemma 4.5, the assertion follows. □

We now discuss an analogue of the classical large deviation inequality (12) in this setting. Our main tool is the following corollary to a noncommutative version of Chebychev inequality proved by Defant and Junge (see [8]).

**Corollary 4.7** (Defant-Junge)**.** *Let $2 \leq p < \infty$ and $x \in L_p(\mathcal{N})$. Then, for every $\varepsilon > 0$, there exists a projection $f \in \mathcal{N}$ with $\phi(1-f) \leq \varepsilon$, such that whenever $yD^{\frac{1}{p}} = wx$, where $w, y \in \mathcal{N}$, then*

$$\sup_n \|f\mathcal{E}_n(y^*y)f\|^{\frac{1}{2}} \leq (1+\sqrt{10})\varepsilon^{-\frac{1}{p}}\|w\|_\infty\|x\|_p. \tag{52}$$

*Moreover,*

$$\sup_n \|\mathcal{E}_n(y)f\| \leq (1+\sqrt{10})\varepsilon^{-\frac{1}{p}}\|w\|_\infty\|x\|_p. \tag{53}$$

*Proof of Theorem* 1.2. Recall that the state $\phi$ satisfies condition (8). Let $c_2 = e$. Furthermore, consider $t \geq \frac{1}{c_1}$ and let $\varepsilon := e^{-tc_1}$, where $c_1$ is a constant to be made precise later. Define $p = 4\ln(\varepsilon^{-1}) = 4tc_2$. By construction it follows that $4 \leq p < \infty$. Now let $x \in BMO$ with $\|x\|_{BMO} \leq 1$. Denote by $y$ the element $(x-x_0)D^{\frac{1}{p}}$. Note that $y \in L_p(\mathcal{N})$. By the noncommutative Doob's inequality (see [14]) and Theorem 1.1, we get

$$\|\sup_n \mathcal{E}_n(y^*y)\|_{\frac{p}{2}} \leq d_{\frac{p}{2}}\|y\|_p^2 \leq d_{\frac{p}{2}}\, c^2p^2\, \|x-x_0\|_{BMO}^2, \tag{54}$$

where $c$ is the universal constant in Theorem 1.1 and $d_{\frac{p}{2}}$ is the constant in the noncommutative Doob's inequality. Since $\frac{p}{2} \geq 2$, it follows that $d_{\frac{p}{2}} = 2$ (see [14]). Furthermore, note that

$$\|x-x_0\|_{BMO} \leq \|x\|_{BMO}. \tag{55}$$

Indeed, we have the estimates

$$\begin{aligned}\mathcal{E}_0((x-x_0)^*(x-x_0)) &= \mathcal{E}_0(x^*x - x^*x_0 - \mathcal{E}_0(x^*)x + \mathcal{E}_0(x^*)\mathcal{E}_0(x)) \\ &= \mathcal{E}_0(x^*x) - \mathcal{E}_0(x^*)\mathcal{E}_0(x).\end{aligned}$$

By Kadison's inequality it follows that

$$\|\mathcal{E}_0((x-x_0)^*(x-x_0))\| \leq \|\mathcal{E}_0(x^*x)\|. \tag{56}$$

Moreover, for $n \geq 1$ we have

$$(x-\mathcal{E}_0(x)) - \mathcal{E}_{n-1}(x-\mathcal{E}_0(x)) = (x-\mathcal{E}_0(x)) - (\mathcal{E}_{n-1}(x)-\mathcal{E}_0(x)) = x - \mathcal{E}_{n-1}(x).$$

Combining this with (56) we deduce that

$$\|x-x_0\|_{BMO_c} \leq \|x\|_{BMO_c},$$

which yields the inequality (55). A further application of Kadison's inequality and (54), together with (55), implies that

$$\|\sup_m \mathcal{E}_m(y^*)\mathcal{E}_m(y)\|_{\frac{p}{2}} \le 2c^2p^2\,.$$

Then by [14], Remark 3.7, there exists $0 \le a \in L_p(\mathcal{N})$ and a sequence of positive contractions $w_m \in \mathcal{N}$ such that

$$\mathcal{E}_m(y^*)\mathcal{E}_m(y) = aw_ma\,, \qquad \|a\|_p^2 \sup_m \|w_m\|_\infty = \|\sup_m \mathcal{E}_m(y^*)\mathcal{E}_m(y)\|_{\frac{p}{2}} \le \sqrt{2}cp\,.$$

Using the polar decomposition of $\mathcal{E}_m(y)$, we find partial isometries $u_m$ such that, denoting $u_m w_m^{\frac{1}{2}}$ by $z_m$, we obtain

$$\mathcal{E}_m(y) = z_m a\,, \qquad \|a\|_p \sup_m \|z_m\|_\infty \le \sqrt{K}p\,. \tag{57}$$

Note that $\mathcal{E}_m(x) - x_0 \in \mathcal{N}$, for all nonnegative integers $m$. Indeed, for all $n \ge 0$,

$$1 \ge \|x\|_{BMO_c}^2 = \sup_n \left\| \sum_{k\ge n} \mathcal{E}_n(d_k^* d_k) \right\|_\infty \ge \|\mathcal{E}_n(d_n^* d_n)\|_\infty = \|d_n^* d_n\|_\infty\,.$$

This implies that $d_n^* d_n \in \mathcal{N}$ and hence $d_n \in \mathcal{N}$. Therefore $\mathcal{E}_m(x) - x_0 = \sum_{k=1}^m d_k \in \mathcal{N}$. Since

$$\mathcal{E}_m(y) = (\mathcal{E}_m(x) - x_0)D^{\frac{1}{p}}\,,$$

we are now in the condition to apply Corollary 4.7 and deduce the existence of a projection $f \in \mathcal{N}$ with $\phi(1-f) \le \varepsilon$ such that for all $m \ge 0$,

$$\|(\mathcal{E}_m(x) - x_0)f\| \le (1+\sqrt{10})\varepsilon^{-\frac{1}{p}}\|z_m\|_\infty \|a\|_p\,.$$

Using the estimates (57) we deduce that

$$\|(\mathcal{E}_m(x) - x_0)f\| \le Kp\,\varepsilon^{-\frac{1}{p}}\,,$$

where $K = (1+\sqrt{10})\sqrt{2}c$. Therefore $\|(\mathcal{E}_m(x) - x_0)h\| \le Kp\,\varepsilon^{-\frac{1}{p}}$, for all $h \in fH$. Since the sequence $(\mathcal{E}_m)_{m\ge 0}$ converges to $\mathrm{Id}_{\mathcal{N}}$ in the strong operator topology, it follows that

$$\|(x - x_0)f\| \le Kp\,\varepsilon^{-\frac{1}{p}}\,. \tag{58}$$

We now make precise the constant $c_1$. Namely, let

$$c_1 := \frac{1}{4Ke^{1/4}} = \frac{1}{4\sqrt{2}(1+\sqrt{10})ce^{1/4}}\,.$$

A simple computation yields the equality $Kp\varepsilon^{-\frac{1}{p}} = t$. By (58) we deduce the inequality $\|(x-x_0)f\| \le t$. Moreover, $\phi(1-f) \le \varepsilon < c_2e^{-tc_1}$, which yields the assertion. For $0 < t < \frac{1}{c_1}$, let $f = 0$. The condition $\|(x-x_0)f\| \le t$ is then automatically satisfied. Furthermore, note that

$$\phi(1-f) = \phi(1) = 1 < \frac{e}{e^{tc_1}} = c_2e^{-tc_1}\,.$$

This completes the proof. □

*Remark* 4.8. After completing this paper, we were informed that Tao Mei has obtained a simple proof of the inequalities (11), but only in the tracial setting, based on the interpolation result proved in [26].

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