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THE PSEUDOHYPERBOLIC METRIC AND BERGMAN SPACES IN THE BALL

PETER DUREN AND RACHEL WEIR

Abstract. The pseudohyperbolic metric is developed for the unit ball of $\mathbb{C}^n$ and is applied to a study of uniformly discrete sequences and Bergman spaces of holomorphic functions on the ball.

The pseudohyperbolic metric plays an important role in the study of Bergman spaces over the unit disk. It is defined in terms of Möbius self-mappings of the disk. As is well known, these conformal automorphisms generalize nicely to the unit ball in $\mathbb{C}^n$. This suggests a natural extension of the pseudohyperbolic metric that retains all of the key properties it enjoys in the disk. Another basic tool is a Möbius-invariant generalization of hyperbolic area. These devices allow us to derive various results for Bergman spaces over the ball by methods very similar to those for the disk. Although many of the results are known, the sources are scattered and the proofs are often indirect and complicated. Here we offer a straightforward and unified treatment that is essentially self-contained. In addition to expository contributions, we obtain several results that appear to be new in the setting of the ball.

§1. Introduction

We begin with a brief review of relevant concepts and results in one complex variable. For $0 < p < \infty$, the *Bergman space* A^p consists of all functions f analytic in $\mathbb{D}$ with finite area integral

$$\|f\|_p^p = \int_{\mathbb{D}} |f(z)|^p \, d\nu(z) \,,$$

where ν is Lebesgue measure normalized so that $\nu(\mathbb{D}) = 1$.

The *pseudohyperbolic metric* in the unit disk $\mathbb{D}$ is defined by the formula

$$\rho(z,w) = |\varphi_w(z)| \,, \qquad \varphi_w(z) = \frac{w-z}{1-\overline{w}z} \,,$$

where $z, w \in \mathbb{D}$. It is a true metric; in fact, the triangle inequality takes the strong form

$$\rho(z,w) \le \frac{\rho(z,\alpha)+\rho(\alpha,w)}{1+\rho(z,\alpha)\rho(\alpha,w)} \,. \tag{1}$$

It is also Möbius-invariant in the sense that

$$\rho(\varphi_\alpha(z), \varphi_\alpha(w)) = \rho(z,w)$$

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for every point $\alpha \in \mathbb{D}$. For $\alpha \in \mathbb{D}$ and $0 < r < 1$, the set

$$\Delta(\alpha, r) = \{z \in \mathbb{D} : \rho(z, \alpha) < r\}$$

is known as a *pseudohyperbolic disk* with *center* α and *radius* r. It is a true Euclidean disk since Möbius transformations preserve circles, but α is not its Euclidean center and r is not its Euclidean radius unless $\alpha = 0$. The Euclidean center and radius of $\Delta(\alpha, r)$ are found to be

$$\beta = \frac{(1-r^2)\alpha}{1-r^2|\alpha|^2} \qquad \text{and} \qquad R = \frac{r(1-|\alpha|^2)}{1-r^2|\alpha|^2}\,,$$

respectively. Using the formula

$$|\varphi_\alpha'(z)|^2 = \frac{(1-|\alpha|^2)^2}{|1-\overline{\alpha}z|^4}$$

for the Jacobian of the mapping φ_α, one easily calculates the (normalized) area

$$\nu(\Delta(\alpha, r)) = \left(\frac{r(1-|\alpha|^2)}{1-r^2|\alpha|^2}\right)^2 = R^2\,.$$

A sequence $\Gamma = \{z_k\}$ of points in $\mathbb{D}$ is said to be *uniformly discrete* if $\rho(z_j, z_k) \geq \delta$ for some constant $\delta > 0$ and all pairs of distinct indices j and k. The *counting function* $N(\Gamma, \alpha, r)$ is defined as the number of points of Γ that lie in the pseudohyperbolic disk $\Delta(\alpha, r)$. It can be shown (see [3], p. 68) that

$$N(\Gamma, \alpha, r) \leq \left(\tfrac{2}{\delta} + 1\right)^2 \frac{1}{1-r^2}\,.$$

In particular, $N(\Gamma, \alpha, r) = O(1/(1-r))$ as $r \to 1$. Uniformly discrete sequences are intimately related to Bergman spaces. For instance, for any choice of $p > 0$, the inequality

$$\sum_{k=1}^{\infty} (1-|z_k|^2)^2 |f(z_k)|^p \leq C\,\|f\|_p^p$$

holds for some constant C and all $f \in A^p$, if and only if Γ is a finite union of uniformly discrete sequences. (See [3], Section 2.11.)

A finite positive Borel measure μ on $\mathbb{D}$ is called a *Carleson measure for* A^p if

$$\int_{\mathbb{D}} |f(z)|^p\, d\mu \leq C\,\|f\|_p^p\,, \qquad f \in A^p\,,$$

for some constant C depending only on p. It is known that the Carleson measures are the same for every space A^p and are characterized by the property

$$\mu(\Delta(\alpha, r)) \leq C\,\nu(\Delta(\alpha, r))\,, \qquad \alpha \in \mathbb{D}\,,$$

for some constant C depending only on r. If such an inequality holds for some radius r, then it holds for each $r \in (0, 1)$. (See [3], Section 2.10.)

The *hyperbolic area* of a measurable set $\Omega \subset \mathbb{D}$ is defined by

$$\tau(\Omega) = \int_\Omega \frac{d\nu(z)}{(1-|z|^2)^2}\,.$$

Using the formula for the Jacobian $|\varphi_\alpha'(z)|^2$, one can verify that hyperbolic area is Möbius-invariant: $\tau(\varphi_\alpha(\Omega)) = \tau(\Omega)$, for every $\alpha \in \mathbb{D}$. This shows in particular

that $\tau(\Delta(\alpha,r)) = \tau(\Delta(0,r))$, whereupon a simple integration gives the formula

$$\tau(\Delta(\alpha,r)) = \frac{r^2}{1-r^2}$$

for the hyperbolic area of a pseudohyperbolic disk. Note that the area depends only on the radius, not on the center.

Further information on all of these topics can be found in the book by Duren and Schuster [3]. We shall refer to Rudin [13] for notation and basic facts about Möbius self-mappings of the ball.

§2. The pseudohyperbolic metric for the ball

In standard notation, the *norm* $|z|$ of a point $z = (z_1, \dots, z_n) \in \mathbb{C}^n$ is defined by $|z|^2 = |z_1|^2 + \cdots + |z_n|^2$. The *inner product* is

$$\langle z, w\rangle = z_1\overline{w_1} + \cdots + z_n\overline{w_n}\,, \qquad z, w \in \mathbb{C}^n\,.$$

Thus $|\langle z,w\rangle| \le |z|\,|w|$ and $\langle z,z\rangle = |z|^2$. The *unit ball* $\mathbb{B}_n$ is the set of points $z \in \mathbb{C}^n$ with $|z| < 1$.

The linear isometries of $\mathbb{C}^n$ are called *unitary transformations*. These are the linear surjections U that preserve inner products:

$$\langle U(z), U(w)\rangle = \langle z, w\rangle\,, \qquad z, w \in \mathbb{C}^n\,.$$

Their restrictions to $\mathbb{B}_n$ can be viewed as rotations of the ball.

Möbius self-mappings of the disk have a natural generalization to $\mathbb{B}_n$ (*cf.* Rudin [13], Section 2.2). Fix a point $\alpha \in \mathbb{B}_n$ and let P_α be the orthogonal projection of $\mathbb{C}^n$ onto the subspace

$$[\alpha] = \{\lambda\alpha : \lambda \in \mathbb{C}\}$$

generated by α. Thus $P_0(z) = 0$ and

$$P_\alpha(z) = \frac{\langle z,\alpha\rangle}{\langle \alpha,\alpha\rangle}\,\alpha\,, \qquad \alpha \ne 0\,.$$

Let $Q_\alpha(z) = z - P_\alpha(z)$ be the projection onto the orthogonal complement of $[\alpha]$, and let $s_\alpha = (1-|\alpha|^2)^{1/2}$. Now define

$$\varphi_\alpha(z) = \frac{\alpha - P_\alpha(z) - s_\alpha Q_\alpha(z)}{1-\langle z,\alpha\rangle}\,, \qquad \alpha \in \mathbb{B}_n\,.$$

Note that $P_\alpha(z) = z$ when $n = 1$, so this expression agrees with our previous definition for the disk. Observe also that φ_α is a holomorphic mapping of $\overline{\mathbb{B}_n}$ into $\mathbb{C}^n$, since $\langle z,\alpha\rangle \ne 1$ for $|z| < 1/|\alpha|$. Moreover, φ_α is a biholomorphic mapping of $\mathbb{B}_n$ onto itself, also called an *automorphism* of $\mathbb{B}_n$, with the following properties (*cf.* Rudin [13], Theorem 2.2.2):

(*i*) $\varphi_\alpha(0) = \alpha\,, \quad \varphi_\alpha(\alpha) = 0\,.$

(*ii*) φ_α is an involution: $\varphi_\alpha(\varphi_\alpha(z)) = z\,.$

(*iii*) $1 - |\varphi_\alpha(z)|^2 = \dfrac{(1-|\alpha|^2)(1-|z|^2)}{|1-\langle z,\alpha\rangle|^2}\,.$

(*iv*) $1 - \langle \varphi_\alpha(z), \varphi_\alpha(w)\rangle = \dfrac{(1-\langle\alpha,\alpha\rangle)(1-\langle z,w\rangle)}{(1-\langle z,\alpha\rangle)(1-\langle\alpha,w\rangle)}\,.$

In fact, every automorphism of $\mathbb{B}_n$ has the form $U\varphi_\alpha$ for some $\alpha \in \mathbb{B}_n$ and some unitary transformation U (*cf.* Rudin [13], Theorem 2.2.5). We will refer to the automorphisms φ_α as *Möbius self-mappings* of the ball.

The *pseudohyperbolic metric for the ball* is defined by

$$\rho(z,w) = |\varphi_w(z)|\,, \qquad z,w \in \mathbb{B}_n\,.$$

We will show it is a true metric. Clearly, $\rho(z,w) \geq 0$ and $\rho(z,w) = 0$ if and only if $z = w$, since $\varphi_w(z) = 0$ only for $z = w$. The symmetry property $\rho(z,w) = \rho(w,z)$ follows from the identity (*iii*). The triangle inequality is less obvious, but it actually holds in the same strong form (1) as for $n = 1$, and this will be important for our purposes. The pseudohyperbolic metric is rotation-invariant in the sense that $\rho(U(z),U(w)) = \rho(z,w)$ for unitary transformations U. It is also useful to know that the pseudohyperbolic metric for the ball remains Möbius-invariant. We state these various properties as a theorem.

Theorem 1. *The pseudohyperbolic metric of the ball $\mathbb{B}_n$ has the properties*

(*a*) $\rho(U(z),U(w)) = \rho(z,w)$ *for all* $z,w \in \mathbb{B}_n$ *and all unitary* U.

(*b*) $\rho(\varphi_\alpha(z),\varphi_\alpha(w)) = \rho(z,w)$ *for all* $z,w,\alpha \in \mathbb{B}_n$.

(*c*) $\dfrac{|\rho(z,\alpha)-\rho(\alpha,w)|}{1-\rho(z,\alpha)\rho(\alpha,w)} \leq \rho(z,w) \leq \dfrac{\rho(z,\alpha)+\rho(\alpha,w)}{1+\rho(z,\alpha)\rho(\alpha,w)}$ *for all* $z,w,\alpha \in \mathbb{B}_n$.

Before passing to the proof, we introduce some additional terminology. The *pseudohyperbolic ball* with *center* $\alpha \in \mathbb{B}_n$ and *radius* $r \in (0,1)$ is

$$\Delta(\alpha,r) = \{z \in \mathbb{B}_n : \rho(z,\alpha) < r\}\,.$$

Note that $\rho(z,0) = |z|$ since $\varphi_0(z) = -z$, so $\Delta(0,r)$ is the true Euclidean ball $|z| < r$. It follows from the Möbius invariance asserted by Theorem 1(*b*) that

$$\Delta(\alpha,r) = \varphi_\alpha(\Delta(0,r))\,, \qquad \alpha \in \mathbb{B}_n\,,$$

since $\varphi_\alpha(0) = \alpha$. In contrast with the one-variable case, however, $\Delta(\alpha,r)$ is not a Euclidean ball when $\alpha \neq 0$, but is the ellipsoid defined by

$$\frac{|P_\alpha(z)-\beta|^2}{r^2t^2} + \frac{|Q_\alpha(z)|^2}{r^2t} < 1\,, \tag{2}$$

where

$$\beta = \frac{(1-r^2)\alpha}{1-r^2|\alpha|^2}\,, \qquad t = \frac{1-|\alpha|^2}{1-r^2|\alpha|^2}\,.$$

This can be confirmed by straightforward calculation (*cf.* Rudin [13], p. 29). The point β is seen to be the center of the ellipsoid $\Delta(\alpha,r)$, since $\beta \in [\alpha]$ and P_α is the projection onto $[\alpha]$, while Q_α is the orthogonal projection. The intersection of $\Delta(\alpha,r)$ with $[\alpha]$ is a (one-dimensional) disk of radius rt. The intersection of $\Delta(\alpha,r)$ with the orthocomplement of $[\alpha]$ is an $(n-1)$-dimensional Euclidean ball of radius $r\sqrt{t}$.

Proof of Theorem 1. (*a*). Because unitary transformations preserve inner products, the identity

$$1-\rho(U(z),U(w))^2 = 1-\rho(z,w)^2$$

follows directly from the property (*iii*).

(b). In view of the properties (iii) and (iv), we find

$$1-\rho(\varphi_\alpha(z),\varphi_\alpha(w))^2 = 1-\left|\varphi_{\varphi_\alpha(w)}(\varphi_\alpha(z))\right|^2 = \frac{\big(1-|\varphi_\alpha(w)|^2\big)\big(1-|\varphi_\alpha(z)|^2\big)}{\left|1-\langle\varphi_\alpha(z),\varphi_\alpha(w)\rangle\right|^2}$$
$$= \frac{\big(1-|w|^2\big)\big(1-|z|^2\big)}{\left|1-\langle z,w\rangle\right|^2} = 1-|\varphi_w(z)|^2 = 1-\rho(z,w)^2\,,$$

which proves (b). Combining (a) and (b), we see that ρ is invariant under every automorphism of the ball.

(c). Since ρ is Möbius-invariant, we can assume that $\alpha=0$. Thus, we need to show that

$$\frac{|\,|z|-|w|\,|}{1-|z|\,|w|} \le |\varphi_w(z)| \le \frac{|z|+|w|}{1+|z|\,|w|}\,. \tag{3}$$

By symmetry, we may assume that $|z|\le|w|$. Let $r=|z|$, and recall that φ_w maps the ball $\Delta(0,r)$ onto the ellipsoid $\Delta(w,r)$ defined by

$$\frac{|P_w(\zeta)-\beta|^2}{r^2t^2} + \frac{|Q_w(\zeta)|^2}{r^2t} < 1\,,$$

where

$$\beta = \frac{(1-r^2)w}{1-r^2|w|^2}\,, \qquad t = \frac{1-|w|^2}{1-r^2|w|^2}\,.$$

Therefore, we need to find the maximum and minimum of $|\zeta| = |\varphi_w(z)|$ for all points ζ on the boundary of the ellipsoid $\Delta(w,r)$. Since $\Delta(w,r)$ intersects the subspace $[w]$ in a disk with center β and radius rt, it is geometrically clear that the sharp bounds will be

$$|\beta|-rt \le |\zeta| \le |\beta|+rt\,,$$

which transform to the desired inequalities (3). Note that the assumption $r\le|w|$ implies $|\beta|-rt\ge 0$. A more formal proof can be based on the method of Lagrange multipliers.

Alternatively, one can apply the Cauchy–Schwarz inequality to see that

$$1-|z|\,|w| \le |1-\langle z,w\rangle| \le 1+|z|\,|w|\,,$$

whereupon (3) follows by appeal to the basic identity (iii). This relatively simple proof of the strong triangle inequality (c) is implicit in a paper of MacCluer, Stroethoff, and Zhao ([9], Lemma 3). □

Our next project is to calculate the volume of a pseudohyperbolic ball. Let ν_n denote Lebesgue volume measure in $\mathbb{R}^{2n}$, normalized so that $\nu_n(\mathbb{B}_n)=1$, and let σ_n be the corresponding measure on the surface of the unit sphere

$$\mathbb{S}_n = \{z\in\mathbb{C}^n : |z|=1\}\,,$$

normalized so that $\sigma_n(\mathbb{S}_n)=1$. Then (*cf.* Rudin [13], p. 13)

$$\int_{\mathbb{C}^n} f(z)\,d\nu_n(z) = 2n\int_0^\infty r^{2n-1}\int_{\mathbb{S}_n} f(r\zeta)\,d\sigma_n(\zeta)\,dr \tag{4}$$

for any function $f\in L^1$.

For $0<p<\infty$, the *Bergman space* $A^p(\mathbb{B}_n)$ consists of all functions f holomorphic in $\mathbb{B}_n$ for which the volume integral

$$\|f\|_p^p = \int_{\mathbb{B}_n} |f(z)^p\,d\nu_n(z)$$

is finite. The *Bergman kernel function* for $\mathbb{B}_n$ is

$$K_n(z,\zeta) = \frac{1}{(1-\langle z,\zeta\rangle)^{n+1}}\,, \qquad z,\zeta \in \mathbb{B}_n\,.$$

It has the reproducing property

$$f(z) = \int_{\mathbb{B}_n} K_n(z,\zeta) f(\zeta)\, d\nu_n(\zeta)\,, \qquad z \in \mathbb{B}_n\,,$$

for every function $f \in A^1(\mathbb{B}_n)$. (See Rudin [13], Theorem 3.1.3.) As a particular application, we find

$$\int_{\mathbb{B}_n} |K_n(z,\zeta)|^2\, d\nu_n(\zeta) = K_n(z,z)$$

or

$$\int_{\mathbb{B}_n} \frac{d\nu_n(\zeta)}{|1-\langle z,\zeta\rangle|^{2(n+1)}} = \frac{1}{(1-|z|^2)^{n+1}}\,, \qquad z \in \mathbb{B}_n\,. \tag{5}$$

Viewed as a mapping from $\mathbb{R}^{2n}$ to $\mathbb{R}^{2n}$, the Möbius transformation φ_α has a (real) Jacobian

$$(J\varphi_\alpha)(z) = \left(\frac{1-|\alpha|^2}{|1-\langle z,\alpha\rangle|^2}\right)^{n+1}\,, \qquad z \in \mathbb{B}_n\,. \tag{6}$$

(See Rudin [13], Theorem 2.2.6.) Using (6), we can now calculate the volume of a pseudohyperbolic ball $\Delta(\alpha,r)$. The formula is

$$\nu_n(\Delta(\alpha,r)) = r^{2n}\left(\frac{1-|\alpha|^2}{1-r^2|\alpha|^2}\right)^{n+1}\,. \tag{7}$$

To prove (7), use the transformation $w = \varphi_\alpha(z)$ and apply the relation (5) to write

$$\begin{aligned}
\nu_n(\Delta(\alpha,r)) &= \int_{\Delta(\alpha,r)} d\nu_n(w) = \int_{\Delta(0,r)} (J\varphi_\alpha)(z)\, d\nu_n(z) \\
&= \int_{\Delta(0,r)} \left(\frac{1-|\alpha|^2}{|1-\langle z,\alpha\rangle|^2}\right)^{n+1} d\nu_n(z) \\
&= r^{2n}(1-|\alpha|^2)^{n+1} \int_{\mathbb{B}_n} \frac{d\nu_n(\zeta)}{|1-\langle r\zeta,\alpha\rangle|^{2(n+1)}} \\
&= r^{2n}\left(\frac{1-|\alpha|^2}{1-r^2|\alpha|^2}\right)^{n+1}\,.
\end{aligned}$$

This proves (7). As a special case, $\nu_n(\Delta(0,r)) = r^{2n}$.

We now turn to the generalization of hyperbolic area. The *hyperbolic volume* of a measurable set $\Omega \subset \mathbb{B}_n$ is defined by

$$\tau_n(\Omega) = \int_\Omega \frac{d\nu_n(z)}{(1-|z|^2)^{n+1}}\,.$$

It can be shown (*cf.* Rudin [13], Theorem 2.2.6) that τ_n is Möbius-invariant. In other words,

$$\tau_n(\varphi_\alpha(\Omega)) = \tau_n(\Omega)\,, \qquad \alpha \in \mathbb{B}_n\,.$$

The proof makes use of the formula (6) for the Jacobian of φ_α. In particular,

$$\tau_n(\Delta(\alpha,r)) = \tau_n(\Delta(0,r))\,, \qquad \alpha \in \mathbb{B}_n\,.$$

This leads to the formula

$$\tau_n(\Delta(\alpha,r)) = \frac{r^{2n}}{(1-r^2)^n}\,, \qquad \alpha \in \mathbb{B}_n, \tag{8}$$

for the hyperbolic volume of a pseudohyperbolic ball.

To prove (8), one can use the relation (4) to calculate

$$\begin{aligned}\tau_n(\Delta(\alpha,r)) = \tau_n(\Delta(0,r)) &= \int_{\Delta(0,r)} \frac{d\nu_n(z)}{(1-|z|^2)^{n+1}} \\ &= 2n\int_0^r \frac{t^{2n-1}}{(1-t^2)^{n+1}} \int_{\mathbb{S}_n} d\sigma_n(\zeta)\,dt \\ &= 2n\int_0^r \frac{t^{2n-1}}{(1-t^2)^{n+1}}\,dt = \frac{r^{2n}}{(1-r^2)^n}\,.\end{aligned}$$

§3. Carleson measures for Bergman space

A positive Borel measure μ on the unit ball $\mathbb{B}_n$ is said to be a *Carleson measure for the Bergman space* $A^p(\mathbb{B}_n)$ if

$$\int_{\mathbb{B}_n} |f(z)|^p\,d\mu(z) \le C\,\|f\|_p^p \tag{9}$$

for some constant C depending only on p and for all functions $f \in A^p(\mathbb{B}_n)$. The following theorem provides a geometric description.

Theorem 2. *Let μ be a finite positive Borel measure on $\mathbb{B}_n$. Then for each fixed p $(0<p<\infty)$, the following three statements are equivalent:*

(i) μ is a Carleson measure for $A^p(\mathbb{B}_n)$.

(ii) An inequality $\mu(\Delta(\alpha,r)) \le C\,\nu_n(\Delta(\alpha,r))$ holds for some r $(0<r<1)$, for some constant C, and for all pseudohyperbolic balls $\Delta(\alpha,r)$.

(iii) An inequality $\mu(\Delta(\alpha,r)) \le C\,\nu_n(\Delta(\alpha,r))$ holds for every r $(0<r<1)$, for some constant C depending only on r, and for all pseudohyperbolic balls $\Delta(\alpha,r)$.

Corollary. *The Carleson measures for the Bergman space $A^p(\mathbb{B}_n)$ are the same for every index p in the interval $0<p<\infty$.*

Theorem 2 has a long history. Carleson measures for Bergman spaces were first described by Hastings [5], and independently by Oleĭnik and Pavlov [12] and Oleĭnik [11]. Subsequently, Cima and Wogen [2] adapted the result to the ball, using spherical caps at the boundary instead of pseudohyperbolic balls. The statement in terms of pseudohyperbolic balls, essentially due to Luecking [7], is more natural for Bergman spaces, as it facilitates both the proof of the theorem and its applications. The present formulation is due to Axler [1] for the unit disk and was adapted by Duren and Schuster [3]. The proof generalizes easily to the ball, but we have not found it in the literature, so we include a sketch for the sake of completeness.

The implication $(iii) \implies (ii)$ is trivial, and the implication $(i) \implies (iii)$ will be contained in the proof of Theorem 3 below. The main difficulty is to show that $(ii) \implies (i)$. This is accomplished with the help of two lemmas.

Lemma 1. *For each pseudohyperbolic radius r $(0<r<1)$ there exist a sequence $\{\alpha_k\}$ of points in $\mathbb{B}_n$ and an integer m such that*

$$\bigcup_{k=1}^{\infty} \Delta(\alpha_k,r) = \mathbb{B}_n,$$

and no point of $\mathbb{B}_n$ *belongs to more than* m *of the dilated balls* $\Delta(\alpha_k, R)$*, where* $R = \frac{1}{2}(1+r)$*.*

Lemma 2. *Let* $0 < p < \infty$ *and* $0 < r < 1$*, and define* $R = \frac{1}{2}(1+r)$*. Then for each* $\alpha \in \mathbb{B}_n$ *and all* $z \in \Delta(\alpha, r)$*, the inequality*

$$|f(z)|^p \le K \int_{\Delta(\alpha,R)} |f(\zeta)|^p \, d\nu_n(\zeta)$$

holds for some constant K *depending only on* r *and every function* f *holomorphic in* $\mathbb{B}_n$*.*

The lemmas can be proved by straightforward generalizations of methods used for the disk (*cf.* [3], pp. 62–64). The details will not be pursued here.

Proof of Theorem 2. Using the two lemmas, we now show that $(ii) \implies (i)$. A proof that $(i) \implies (iii)$ is given at the end of this section.

If $\Delta(\alpha_k, r)$ are the pseudohyperbolic balls of Lemma 1, then

$$\int_{\mathbb{B}_n} |f(z)|^p \, d\mu(z) \le \sum_{k=1}^{\infty} \int_{\Delta(\alpha_k,r)} |f(z)|^p \, d\mu(z) \, .$$

But Lemma 2 and the hypothesis (ii) give

$$\begin{aligned} \int_{\Delta(\alpha_k,r)} |f(z)|^p \, d\mu(z) &\le K\mu(\Delta(\alpha_k, r)) \int_{\Delta(\alpha_k,R)} |f(\zeta)|^p \, d\nu_n(\zeta) \\ &\le CK\nu_n(\Delta(\alpha_k, r)) \int_{\Delta(\alpha_k,R)} |f(\zeta)|^p \, d\nu_n(\zeta) \\ &\le CK \int_{\Delta(\alpha_k,R)} |f(\zeta)|^p \, d\nu_n(\zeta) \, . \end{aligned}$$

Therefore, we can infer from Lemma 1 that

$$\int_{\mathbb{B}_n} |f(z)|^p \, d\mu(z) \le CK \sum_{k=1}^{\infty} \int_{\Delta(\alpha_k,R)} |f(\zeta)|^p \, d\nu_n(\zeta) \le CKm \, \|f\|_p^p$$

for every function $f \in A^p(\mathbb{B}_n)$, which is the property (i). □

On the basis of Theorem 2, we can give yet another description of Carleson measures. The *Berezin transform* of a finite measure μ on $\mathbb{B}_n$ is the function

$$B\mu(\alpha) = \int_{\mathbb{B}_n} |k_\alpha(z)|^2 \, d\mu(z) \, , \qquad \alpha \in \mathbb{B}_n \, ,$$

where

$$k_\alpha(z) = \frac{K_n(z, \alpha)}{\|K_n(\,\cdot\,, \alpha)\|_2} = \frac{(1 - |\alpha|^2)^{\frac{n+1}{2}}}{(1 - \langle z, \alpha \rangle)^{n+1}}$$

is the normalized kernel function for $\mathbb{B}_n$.

Theorem 3. *A finite positive Borel measure on* $\mathbb{B}_n$ *is a Carleson measure for* $A^p(\mathbb{B}_n)$ *if and only if its Berezin transform is bounded.*

The proof will require a special estimate on the kernel function.

Lemma 3. *For each pseudohyperbolic ball $\Delta(\alpha, r)$, the kernel function satisfies the sharp inequalities*

$$\left(\frac{1-r|\alpha|}{1-|\alpha|^2}\right)^{n+1} \le |K_n(z,\alpha)| \le \left(\frac{1+r|\alpha|}{1-|\alpha|^2}\right)^{n+1}, \qquad z \in \Delta(\alpha, r)\,.$$

Proof. Let $z = \varphi_\alpha(w)$. Then $z \in \Delta(\alpha, r)$ if and only if $|w| < r$. Thus, we need to estimate $|K_n(\varphi_\alpha(w), \alpha)|$ for $|w| < r$. Now observe that

$$\begin{aligned} K_n(\varphi_\alpha(w), \alpha) &= \frac{1}{(1-\langle \varphi_\alpha(w), \alpha\rangle)^{n+1}} = \frac{1}{(1-\langle \varphi_\alpha(w), \varphi_\alpha(0)\rangle)^{n+1}} \\ &= \left(\frac{1-\langle w, \alpha\rangle}{1-|\alpha|^2}\right)^{n+1}. \end{aligned}$$

But the sharp inequalities

$$1 - r|\alpha| \le |1 - \langle w, \alpha\rangle| \le 1 + r|\alpha|$$

hold for $|w| \le r$, so the desired result follows. □

Proof of Theorem 3. If μ is a Carleson measure, then

$$B\mu(\alpha) = \int_{\mathbb{B}_n} |k_\alpha(z)|^2 \, d\mu(z) \le C \, \|k_\alpha\|^2 = C$$

for all $\alpha \in \mathbb{B}_n$, so $B\mu$ is bounded. Conversely, if the Berezin transform $B\mu$ is bounded, we have

$$\int_{\Delta(\alpha,r)} \frac{(1-|\alpha|^2)^{n+1}}{|1-\langle z, \alpha\rangle|^{2(n+1)}} \, d\mu(z) \le B\mu(\alpha) \le C$$

for every $\alpha \in \mathbb{B}_n$ and $0 < r < 1$. But by Lemma 3,

$$\frac{1}{|1-\langle z, \alpha\rangle|^{n+1}} \ge \left(\frac{1-r|\alpha|}{1-|\alpha|^2}\right)^{n+1} \ge \left(\frac{1-r}{1-|\alpha|^2}\right)^{n+1}$$

for $z \in \Delta(\alpha, r)$, so it follows that

$$\frac{(1-r)^{2(n+1)}}{(1-|\alpha|^2)^{n+1}} \, \mu(\Delta(\alpha, r)) \le C\,, \qquad \alpha \in \mathbb{B}_n\,.$$

On the other hand, formula (7) gives

$$\nu_n(\Delta(\alpha, r)) = r^{2n} \left(\frac{1-|\alpha|^2}{1-r^2|\alpha|^2}\right)^{n+1} \ge r^{2n}(1-|\alpha|^2)^{n+1}\,,$$

so we conclude that

$$\mu(\Delta(\alpha, r)) \le C \, \frac{(1-|\alpha|^2)^{n+1}}{(1-r)^{2(n+1)}} \le \frac{C}{r^{2n}(1-r)^{2(n+1)}} \, \nu_n(\Delta(\alpha, r))\,,$$

which implies that μ is a Carleson measure, in view of Theorem 2. □

Implicit in the preceding argument is a proof that $(i) \implies (iii)$ in Theorem 2. Indeed, if μ is a Carleson measure for $A^p(\mathbb{B}_n)$, then by inserting the function $f_\alpha = k_\alpha^{2/p}$ into (9) we conclude that the Berezin transform of μ is bounded, and (iii) follows as above.

§4. Uniformly discrete sequences in the ball

Henceforth we will use the notation $\{z_k\}$ for a sequence of points $z_k \in \mathbb{B}_n$. The conflict with our previous notation $z = (z_1, \dots, z_n)$ should cause no confusion.

As in the one-dimensional case, a sequence $\Gamma = \{z_k\}$ in $\mathbb{B}_n$ is said to be *uniformly discrete* if $\rho(z_j, z_k) \geq \delta > 0$ for all $j \neq k$. The number $\delta(\Gamma) = \inf_{j \neq k} \rho(z_j, z_k)$ is called the *separation constant* of Γ.

Lemma 4. *If $\{z_k\}$ is a uniformly discrete sequence in $\mathbb{B}_n$ with separation constant δ, then*

$$\sum_{k=1}^{\infty} (1 - |z_k|^2)^{n+1} |f(z_k)|^p \leq \left(\tfrac{2}{\delta}\right)^{2n} \int_{\mathbb{B}_n} |f(z)|^p \, d\nu_n(z)$$

for every function $f \in A^p(\mathbb{B}_n)$.

Proof. By the triangle inequality, the pseudohyperbolic balls $\Delta(z_k, \delta/2)$ are pairwise disjoint. Thus, since each function $|f \circ \varphi_{z_k}|^p J\varphi_{z_k}$ is plurisubharmonic, where $J\varphi_\alpha$ is the Jacobian given by (6), we have

$$\begin{aligned}
\int_{\mathbb{B}_n} |f(z)|^p \, d\nu_n(z) &\geq \sum_{k=1}^{\infty} \int_{\Delta(z_k, \delta/2)} |f(z)|^p \, d\nu_n(z) \\
&= \sum_{k=1}^{\infty} \int_{\Delta(0, \delta/2)} |f(\varphi_{z_k}(\zeta))|^p \left(\frac{1 - |z_k|^2}{|1 - \langle \zeta, z_k \rangle|^2} \right)^{n+1} d\nu_n(z) \\
&\geq \left(\tfrac{\delta}{2}\right)^{2n} \sum_{k=1}^{\infty} (1 - |z_k|^2)^{n+1} |f(\varphi_{z_k}(0))|^p \\
&= \left(\tfrac{\delta}{2}\right)^{2n} \sum_{k=1}^{\infty} (1 - |z_k|^2)^{n+1} |f(z_k)|^p \,.
\end{aligned}$$

□

Corollary. *If $\{z_k\}$ is a uniformly discrete sequence in $\mathbb{B}_n$ with separation constant δ, then*

$$\sum_{k=1}^{\infty} (1 - |z_k|^2)^{n+1} \leq \left(\tfrac{2}{\delta}\right)^{2n} .$$

Proof. Take $f(z) \equiv 1$. □

The proof of Lemma 4 is well known for the disk and was previously adapted to the ball by Jevtić, Massaneda, and Thomas [6].

For $\alpha \in \mathbb{B}_n$ and $0 < r < 1$, the *counting function* $N(\Gamma, \alpha, r)$ is the number of points in Γ that lie in the pseudohyperbolic ball $\Delta(\alpha, r)$. The estimate for the disk can be generalized as follows.

Lemma 5. *If $\Gamma = \{z_k\}$ is a uniformly discrete sequence in $\mathbb{B}_n$ with separation constant δ, then its counting function satisfies*

$$N(\Gamma, \alpha, r) < \left(\tfrac{2}{\delta} + 1\right)^{2n} \frac{1}{(1 - r^2)^n} \,.$$

In particular,

$$N(\Gamma, \alpha, r) = O\left(\frac{1}{(1 - r)^n} \right) , \qquad r \to 1 \,.$$

Proof. In view of the Möbius invariance of the pseudohyperbolic metric, the sequence $\varphi_\alpha(\Gamma) = \{\varphi_\alpha(z_k)\}$ is again uniformly discrete with separation constant δ, so there is no loss of generality in taking $\alpha = 0$. As in the proof of Lemma 4, the balls $\Delta(z_k, \frac{\delta}{2})$ are pairwise disjoint. We claim that

$$\Delta(\zeta, \tfrac{\delta}{2}) \subset \Delta(0, R)\,, \qquad R = \frac{r + \frac{\delta}{2}}{1 + \frac{r\delta}{2}}\,,$$

whenever $\zeta \in \Delta(0, r)$. Indeed, if $\zeta \in \Delta(0, r)$ and $z \in \Delta(\zeta, \frac{\delta}{2})$, then by the strong form of the triangle inequality

$$|z| = \rho(0, z) \le \frac{\rho(0, \zeta) + \rho(\zeta, z)}{1 + \rho(0, \zeta)\rho(\zeta, z)} < \frac{r + \frac{\delta}{2}}{1 + \frac{r\delta}{2}} = R\,,$$

since $\rho(0, \zeta) < r$ and $\rho(\zeta, z) < \frac{\delta}{2}$. Here we have used the elementary fact that

$$\frac{x + y}{1 + xy} < \frac{a + b}{1 + ab} \qquad \text{for } \ 0 \le x < a\,,\ 0 \le y < b\,.$$

This shows that $z \in \Delta(0, R)$, which proves the claim. We now conclude that

$$\sum_{z_k \in \Delta(0,r)} \tau_n(\Delta(z_k, \tfrac{\delta}{2})) \le \tau_n(\Delta(0, R))\,,$$

since the balls $\Delta(z_k, \frac{\delta}{2})$ are disjoint. Invoking formula (8) for hyperbolic volume of a pseudohyperbolic ball, we infer that

$$N(\Gamma, 0, r)\, \frac{\left(\frac{\delta}{2}\right)^{2n}}{\left(1 - \frac{\delta^2}{4}\right)^n} \le \frac{R^{2n}}{(1 - R^2)^n}\,,$$

which reduces to the inequality

$$N(\Gamma, 0, r) \le \left(\tfrac{2r}{\delta} + 1\right)^{2n} \frac{1}{(1 - r^2)^n} < \left(\tfrac{2}{\delta} + 1\right)^{2n} \frac{1}{(1 - r^2)^n}\,.$$

□

The next result is a partial converse. The proof is exactly the same as for the disk (*cf.* [3], pp. 69–70) and is omitted here.

Lemma 6. *Let $\Gamma = \{z_k\}$ be a sequence of points in the unit ball $\mathbb{B}_n$ such that for some fixed radius $r > 0$, each pseudohyperbolic ball $\Delta(\alpha, r)$ contains at most N points z_k. Then Γ is the disjoint union of at most N uniformly discrete sequences.*

We are now in a position to establish a theorem that describes finite unions of uniformly discrete sequences in several ways. Jevtić, Massaneda, and Thomas [6] have pointed out that the equivalence of statements (i), (iv), (v), and (vi) in the following theorem is implicit in an earlier paper of Massaneda [10], whose methods are different from ours.

Theorem 4. *For a sequence $\Gamma = \{z_k\}$ of distinct points in $\mathbb{B}_n$, the following six statements are equivalent:*

(i) *Γ is a finite union of uniformly discrete sequences.*
(ii) $\sup_{\alpha \in \mathbb{B}_n} N(\Gamma, \alpha, r) < \infty$ *for some $r \in (0, 1)$.*
(iii) $\sup_{\alpha \in \mathbb{B}_n} N(\Gamma, \alpha, r) < \infty$ *for each $r \in (0, 1)$.*

(iv) *For some $p \in (0,\infty)$, there exists a constant C such that*

$$\sum_{k=1}^{\infty}(1-|z_k|^2)^{n+1}|f(z_k)|^p \le C\,\|f\|_p^p\,, \qquad f \in A^p(\mathbb{B}_n)\,.$$

(v) *For each $p \in (0,\infty)$, there exists a constant C such that*

$$\sum_{k=1}^{\infty}(1-|z_k|^2)^{n+1}|f(z_k)|^p \le C\,\|f\|_p^p\,, \qquad f \in A^p(\mathbb{B}_n)\,.$$

(vi)

$$\sup_{\alpha\in\mathbb{B}_n}\sum_{k=1}^{\infty}(1-|\varphi_\alpha(z_k)|^2)^{n+1} < \infty\,.$$

Proof. $(i) \iff (ii) \iff (iii)$. Lemma 5 shows that $(i) \implies (iii)$. Trivially, $(iii) \implies (ii)$. Lemma 6 shows that $(ii) \implies (i)$.

$(i) \implies (iv) \iff (v)$. Lemma 4 shows that $(i) \implies (v)$. But (v) says that the discrete measure $\mu = \sum(1-|z_k|^2)^{n+1}\delta_{z_k}$, with mass $(1-|z_k|^2)^{n+1}$ at the point z_k for each k, is a Carleson measure for $A^p(\mathbb{B}_n)$. The Carleson measures for $A^p(\mathbb{B}_n)$ are independent of p, by the corollary to Theorem 2, so $(iv) \iff (v)$.

$(v) \implies (vi)$. Choose the function

$$f_\alpha(z) = k_\alpha(z)^{2/p} = \left\{\frac{1-|\alpha|^2}{(1-\langle z,\alpha\rangle)^2}\right\}^{\frac{n+1}{p}}\,,$$

for which $\|f_\alpha\|_p = 1$. Inserting this function into the inequality of (v) and appealing to a basic identity for Möbius mappings φ_α (*cf.* Section 1, property (iii)), we conclude that

$$\begin{aligned}\sum_{k=1}^{\infty}(1-|\varphi_\alpha(z_k)|^2)^{n+1} &= \sum_{k=1}^{\infty}\frac{(1-|\alpha|^2)^{n+1}(1-|z_k|^2)^{n+1}}{|1-\langle z_k,\alpha\rangle|^{2(n+1)}}\\ &= \sum_{k=1}^{\infty}(1-|z_k|^2)^{n+1}|f_\alpha(z_k)|^p \le C\,\|f_\alpha\|_p^p = C\end{aligned}$$

for every $\alpha \in \mathbb{B}_n$, which implies (vi).

$(vi) \implies (iii)$. Fix any $r \in (0,1)$ and let $\alpha \in \mathbb{B}_n$. Since $|\varphi_\alpha(z_k)| < r$ for $n(\Gamma,\alpha,r)$ points z_k, we infer from (vi) that

$$\begin{aligned}n(\Gamma,\alpha,r)(1-r^2)^{n+1} &\le \sum_{z_k\in\Delta(\alpha,r)}(1-|\varphi_\alpha(z_k)|^2)^{n+1}\\ &\le \sum_{k=1}^{\infty}(1-|\varphi_\alpha(z_k)|^2)^{n+1} \le C\end{aligned}$$

for all $\alpha \in \mathbb{B}_n$, which implies (iii). □

We remark that condition (vi) says that the Berezin transform of the discrete measure $\mu = \sum(1-|z_k|^2)^{n+1}\delta_{z_k}$ is bounded. By Theorem 3, it is equivalent to say that μ is a Carleson measure for $A^p(\mathbb{B}_n)$.

Finally, we note that the corollary of Lemma 4 can be improved. The following theorem generalizes a result obtained by Duren, Schuster, and Vukotić [4] for the disk.

Theorem 5. *If a sequence $\{z_k\}$ in $\mathbb{B}_n$ is uniformly discrete and $z_k \neq 0$ for all k, then*

$$\sum_{k=1}^{\infty}(1-|z_k|)^n\left(\log\frac{1}{1-|z_k|}\right)^{-(1+\varepsilon)} < \infty$$

for each $\varepsilon > 0$.

Proof. For convenience, denote the counting function $N(\Gamma, 0, r)$ by $N(r)$, and recall that $N(r) = O(1/(1-r)^n)$ by Lemma 5. Let

$$\lambda(r) = (1-r)^n\left(\log\frac{1}{1-r}\right)^{-(1+\varepsilon)}, \qquad 0 < r < 1\,,$$

and observe that $N(r)\lambda(r) \to 0$ as $r \to 0$ and as $r \to 1$. Therefore, an integration by parts gives

$$\sum_{k=1}^{\infty}(1-|z_k|)^n\left(\log\frac{1}{1-|z_k|}\right)^{-(1+\varepsilon)} = \int_0^1 \lambda(r)\,dN(r) = -\int_0^1 N(r)\lambda'(r)\,dr\,.$$

But

$$\lambda'(r) = -(1-r)^{n-1}\left(\log\frac{1}{1-r}\right)^{-(1+\varepsilon)}\left\{n + (1+\varepsilon)\left(\log\frac{1}{1-r}\right)^{-1}\right\},$$

so the estimate of Lemma 5 implies that

$$-\int_0^1 N(r)\lambda'(r)\,dr < \infty\,.$$

□

Theorem 5 is best possible in the sense that the series may diverge for $\varepsilon = 0$. This was proved for the disk ($n = 1$) by an example constructed in [4]. MacCluer [8] has now carried out a similar construction for the ball ($n > 1$).

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Department of Mathematics, University of Michigan, Ann Arbor, Michigan 48109–1109

E-mail address: duren@umich.edu

Department of Mathematics, University of Virginia, Charlottesville, Virginia 22904

E-mail address: rw8t@virginia.edu

Current address: Department of Mathematics, Allegheny College, 520 N. Main Street, Meadville, Pennsylvania 16335