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CLASSIFICATION OF HOMOMORPHISMS AND DYNAMICAL SYSTEMS

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Dedicated to George Elliott on his 60th birthday

ABSTRACT. Let A be a unital simple C^*-algebra, with tracial rank zero and let X be a compact metric space. Suppose that $h_1, h_2 : C(X) \to A$ are two unital monomorphisms. We show that h_1 and h_2 are approximately unitarily equivalent if and only if

$$[h_1] = [h_2] \text{ in } KL(C(X), A) \quad \text{and} \quad \tau \circ h_1(f) = \tau \circ h_2(f)$$

for every $f \in C(X)$ and every trace τ of A. Inspired by a theorem of Tomiyama, we introduce a notion of approximate conjugacy for minimal dynamical systems. Let X be a compact metric space and let $\alpha, \beta : X \to X$ be two minimal homeomorphisms. Using the above-mentioned result, we show that two dynamical systems are approximately conjugate in that sense if and only if a K-theoretical condition is satisfied. In the case that X is the Cantor set, this notion coincides with the strong orbit equivalence of Giordano, Putnam and Skau, and the K-theoretical condition is equivalent to saying that the associate crossed product C^*-algebras are isomorphic.

Another application of the above-mentioned result is given for C^*-dynamical systems related to a problem of Kishimoto. Let A be a unital simple AH-algebra with no dimension growth and with real rank zero, and let $\alpha \in Aut(A)$. We prove that if α^r fixes a large subgroup of $K_0(A)$ and has the tracial Rokhlin property, then $A \rtimes_\alpha \mathbb{Z}$ is again a unital simple AH-algebra with no dimension growth and with real rank zero.

1. INTRODUCTION

Let X be a compact metric space and let $\alpha, \beta : X \to X$ be homeomorphisms. Suppose that α and β are minimal, i.e., neither α nor β has non-trivial invariant closed subsets. Recall that α and β are conjugate if there exists a homeomorphism $\sigma : X \to X$ such that $\alpha = \sigma \circ \beta \circ \sigma^{-1}$. They are flip conjugate if either α and β are conjugate or α and β^{-1} are conjugate. It was proved by Jun Tomiyama ([56]) that α and β are flip conjugate if and only if there exists an isomorphism from $C(X) \rtimes_\alpha \mathbb{Z}$ onto $C(X) \rtimes_\beta \mathbb{Z}$ such that it maps $C(X)$ onto $C(X)$, where $C(X) \rtimes_\alpha \mathbb{Z}$ and $C(X) \rtimes_\beta \mathbb{Z}$ are the associated crossed product C^*-algebras or the transformation group C^*-algebras. It is speculated that C^*-algebra theory may help to understand the minimal dynamical systems (X, α) and (X, β). This is further demonstrated in Giordano, Putnam and Skau's work ([16]) on minimal Cantor systems. They show, among other things, that two minimal Cantor systems (X, α) and (X, β) are strong

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orbit equivalent if and only if the associated crossed products are isomorphic. It is worth noting that under the assumption that X is an infinite set and α and β are minimal, the associated crossed product C^*-algebras are amenable simple C^*-algebras. Given the recent development in the classification of amenable simple C^*-algebras, it seems possible to have a K-theoretical description of a useful equivalence relation for minimal dynamical systems as demonstrated in Giordano, Putnam and Skau's work. The author has proposed to study a version of approximate conjugacy for minimal dynamical systems (see [43]). The original purpose of this research is to give a K-theoretical description of approximate conjugacy in minimal dynamical systems.

Diverging from dynamical systems, consider homomorphisms from $C(X)$, the C^*-algebra of continuous functions on a compact metric space, to a unital simple C^*-algebra A. It is fundamentally important in topology to study homomorphisms from $C(X)$ to $C(Y)$. In C^*-algebra theory, it is also fundamentally important to understand homomorphisms from $C(X)$ into a unital C^*-algebra. The earliest study of this kind is the classical Brown-Douglass-Fillmore theory (see [3] and [4]). The BDF-theory classified monomorphisms from $C(X)$ into the Calkin algebra $B(l^2)/\mathcal{K}(l^2)$. The original motivation was to classify essentially normal operators. There is no doubt that BDF-theory plays a crucial role in the development of C^*-algebra theory, in particular, in the aspect of C^*-algebra theory related to K-theory and KK-theory. One may note that the Calkin algebra is a very special (non-separable) C^*-algebra. But it is a simple C^*-algebra with real rank zero. It becomes clear that the study of monomorphisms from $C(X)$ into a separable simple C^*-algebra with real rank zero is also very important (see [8], [31], [32], [33], [34], [18], [19] and [20]). In this paper, we prove that, under the assumption that A is a unital simple C^*-algebra with tracial rank zero, then monomorphisms from $C(X)$ into A can be classified up to approximate unitary equivalence by their K-theoretical information. We believe that this result is potentially very useful. We will demonstrate this by presenting two applications.

Returning to minimal dynamical systems, it is known that many crossed product C^*-algebras associated with minimal dynamical systems are simple C^*-algebras with tracial rank zero. For example it is known (see for example Theorem 1.15 of [16]) that $C(X) \rtimes_\alpha \mathbb{Z}$ is an $A\mathbb{T}$-algebra of real rank zero if X is the Cantor set. It follows (see [13] and [35]) that they have tracial rank zero. More recently it is shown by Q. Lin and N.C. Phillips ([48]) that crossed products resulted from minimal diffeomorphisms on a manifold are inductive limits of subhomogeneous C^*-algebras. Consequently, by [41], they have tracial topological rank zero. Thus by the classification theorem of [39] these simple C^*-algebras are classified by their K-theory. The classification of monomorphisms from $C(X)$ to those simple C^*-algebras leads to the notion of approximate (flip) conjugacy in minimal dynamical systems. Briefly speaking, two minimal dynamical systems (X,α) and (X,β) are approximately K-conjugate if there exist two sequence of homeomorphisms $\sigma_n, \gamma_n : X \to X$ such that

$$\lim_{n\to\infty} f\circ\sigma_n\circ\beta\circ\sigma_n^{-1} = f\circ\alpha \quad \text{and} \quad \lim_{n\to\infty} f\circ\gamma_n\circ\alpha\circ\gamma_n^{-1} = f\circ\beta$$

for all $f \in C(X)$ and both σ_n and γ_n satisfy a K-theoretical constrain. A preliminary result ([45]) shows that when X is a Cantor set, α and β are approximately

K-conjugate if and only if $C(X) \rtimes_\alpha \mathbb{Z} \cong C(X) \rtimes_\beta \mathbb{Z}$. In this paper, we define a C^*-version of approximate flip conjugacy and use the classification of monomorphisms from $C(X)$ into a unital simple C^*-algebra of tracial rank zero to give a K-theoretical condition for two minimal dynamical systems being approximate flip conjugate in that sense.

We present another related application of the above-mentioned classification of monomorphisms from $C(X)$. We study C^*-dynamical systems (A, α), where A is a unital simple C^*-algebra with tracial rank zero and $\alpha \in Aut(A)$ which satisfies a certain Rokhlin property. The Rokhlin property in ergodic theory was first adopted into operator algebras in the context of von Neumann algebras by A. Connes ([7]). It was adopted by Herman and Ocneanu ([29]), then by M. Rørdam ([53]), A. Kishimoto and more recently by M. Isumi ([22]) in a much more general context of C^*-algebras. Kishimoto has studied the problem when a crossed product of a simple A$\mathbb{T}$-algebra A of real rank zero by an automorphism $\alpha \in Aut(A)$ is again an A$\mathbb{T}$-algebra of real rank zero. A more general question is when $A \rtimes_\alpha \mathbb{Z}$ is a unital simple AH-algebra with real rank zero if A is a unital simple AH-algebra. Given the classification theorem for simple separable amenable C^*-algebra with tracial rank zero, a similar question is under what condition $A \rtimes_\alpha \mathbb{Z}$ has tracial rank zero.

In order to make reasonable sense, one has to assume that α is sufficiently outer. As proposed by A. Kishimoto ([26]), the right description of "sufficiently outer" is that α has a Rokhlin property. One version of Rokhlin property was introduced in [50] called "tracial Rokhlin property" which is closely related to, but slightly weaker than, the so-called approximate Rokhlin property used in [24] (see Definition 3.12, below). It was shown in [50] that the tracial Rokhlin property occurs quite often. Tracial cyclic Rokhlin property was introduced in [47] which is a strong Rokhlin property (see Definition 3.13 below). For example, if α has the tracial cyclic Rokhlin property, then α_{*0} fixes a large subset of $K_0(A)$. It is shown in [44] and [47] that if A has tracial rank zero and α has the tracial cyclic Rokhlin property, then $A \rtimes_\alpha \mathbb{Z}$ has tracial Rokhlin property. Thus one may apply classification theorem in [39] to these simple crossed products. This leads to the question of when an automorphism α has tracial cyclic Rokhlin property. We will apply the results of classification of monomorphisms from $C(X)$ into a unital simple C^*-algebra with tracial rank zero to this problem. Among other things, we will show that, if $\alpha_{*0}^r|_G = \mathrm{id}_G$ for some subgroup $G \subset K_0(A)$ for which $\rho_A(G) = \rho_A(K_0(A))$ and α has tracial Rokhlin property, then α has tracial cyclic Rokhlin property. This result implies that, under the same condition, $A \rtimes_\alpha \mathbb{Z}$ is a unital simple AH-algebra with no dimension growth and with real rank zero if A is. This solves the generalized version of the Kishimoto problem.

The paper is organized as follows. In Section 2, we list some conventions that will be used in this paper. In Section 3, we present the main results. We first give (in Subsection 2.1) the classification of monomorphisms from $C(X)$ into a unital simple C^*-algebra with tracial topological rank zero. We then give two applications of the theorem, one for minimal dynamical systems (Subsection 2.2) and the other for the C^*-dynamical systems and the Rokhlin property (Subsection 2.3). In Section 4, we give the proof of the classification theorem mentioned above and also give proofs of several approximate version of it. In Section 5, we present the proof of the theorems presented in Subsection 3.2. Finally, in Section 6, we give the proof of the main results in Subsection 2.3.

2. Notation

We will use the following conventions:

(1) Let A be a C^*-algebra. $A_{s.a.}$ is the set of all selfadjoint elements of A, and A_+ is the positive cone of A.

(2) Let A be a C^*-algebra and let $a \in A$. We write $\mathrm{Her}(a)$ for the hereditary C^*-subalgebra generated by a, i.e., $\mathrm{Her}(a) = \overline{aAa}$.

(3) Let A be a C^*-algebra and let $p, q \in A$ be two projections. We write $p \sim q$ and say p is equivalent to q if there exists a partial isometry $v \in A$ such that $v^*v = p$ and $vv^* = q$. If $a \in A_+$, we write $[p] \le [a]$ if $p \sim q$ for some projection $q \in \mathrm{Her}(a)$.

(4) Let A be a C^*-algebra. We denote by $Aut(A)$ the automorphism group of A. If A is unital and $u \in A$ is a unitary, we denote by $\mathrm{ad}\, u$ the inner automorphism defined by $\mathrm{ad}\, u(a) = u^*au$ for all $a \in A$.

(5) $T(A)$ is the tracial state space of A. Denote by $\rho_A : K_0(A) \to Aff(T(A))$ the homomorphism induced by $\rho_A([p])(\tau) = \tau(p)$ for $\tau \in T(A)$.

Furthermore, we also use $\rho_A : A_{s.a} \to Aff(T(A))$ for the homomorphism defined by $\rho_A(a)(\tau) = \tau(a)$ for $a \in A_{s.a}$.

(6) Let A and B be two C^*-algebras and let $\phi, \psi : A \to B$ be two maps. Let $\varepsilon > 0$ and $\mathcal{F} \subset A$ be a finite subset. We write

$$\phi \approx_\varepsilon \psi \text{ on } \mathcal{F}$$

if

$$\|\phi(a) - \psi(a)\| < \varepsilon \text{ for all } a \in \mathcal{F}.$$

If B is unital and there is a unitary $u \in B$ such that

$$\|\mathrm{ad}\, u \circ \phi(a) - \psi(a)\| < \varepsilon \text{ for all } a \in A,$$

then we write

$$\phi \overset{u}{\sim}_\varepsilon \psi \text{ on } \mathcal{F}.$$

(7) Let $x \in A$, $\varepsilon > 0$ and $\mathcal{F} \subset A$. We write $x \in_\varepsilon \mathcal{F}$ if $\mathrm{dist}(x, \mathcal{F}) < \varepsilon$ or there is $y \in \mathcal{F}$ such that $\|x - y\| < \varepsilon$.

Let $\mathcal{F}$ and $\mathcal{G}$ be subsets of a C^*-algebra A; we write $\mathcal{F} \subset_\varepsilon \mathcal{G}$, if for every $x \in \mathcal{F}$, $x \in_\varepsilon \mathcal{G}$.

(8) Let A be a separable amenable C^*-algebra. We say that A satisfies the Universal Coefficient Theorem (UCT) if for any σ-unital C^*-algebra B one has the following short exact sequence:

$$0 \to ext_{\mathbb{Z}}(K_{*-1}(A), K_*(B)) \to KK^*(A, B) \to Hom(K_*(A), K_*(B)) \to 0.$$

Every C^*-algebra A in the so-called "bootstrap" class $\mathcal{N}$ satisfies the UCT.

Let G and F be abelian groups. Denote by $Pext(G, F)$ the group of pure group extensions in $ext_{\mathbb{Z}}(G, F)$, i.e., those extensions of F by G so that every finitely generated subgroup of F lifts. Denote $KL(A, B) = KK(A, B)/Pext(K_{*-1}(A), K_*(B))$.

(9) Let C_n be a commutative C^*-algebra with $K_0(C_n) = \mathbb{Z}/n\mathbb{Z}$ and $K_1(C_n) = 0$. Suppose that A is a C^*-algebra. Then $K_i(A, \mathbb{Z}/k\mathbb{Z}) = K_i(A \otimes C_k)$. Let $\mathbf{P}(A)$ be the set of equivalence classes of projections in $M_\infty(A)$, $M_\infty(C(S^1) \otimes A)$, $M_\infty(\widetilde{(A \otimes C_m)})$ and $M_\infty(\widetilde{(C(S^1) \otimes A \otimes C_m)})$. We have the following commutative diagram ([55]):

$$\begin{array}{ccccc} K_0(A) & \to & K_0(A, \mathbb{Z}/k\mathbb{Z}) & \to & K_1(A) \\ \uparrow_{\mathbf{k}} & & & & \downarrow_{\mathbf{k}} \\ K_0(A) & \leftarrow & K_1(A, \mathbb{Z}/k\mathbb{Z}) & \leftarrow & K_1(A) \end{array}$$

As in [10], we use the notation

$$\underline{K}(A) = \bigoplus_{i=0,1,n\in\mathbb{Z}_+} K_i(A;\mathbb{Z}/n\mathbb{Z}).$$

By $Hom_\Lambda(\underline{K}(A), \underline{K}(B))$ we mean all homomorphisms from $\underline{K}(A)$ to $\underline{K}(B)$ which respect the direct sum decomposition and the so-called Bockstein operations (see [10]). It follows from [10] that if A satisfies the Universal Coefficient Theorem, then $Hom_\Lambda(\underline{K}(A), \underline{K}(B)) \cong KL(A,B)$.

(10) Let $\{A_n\}$ be a sequence of C^*-algebras. Set $l^\infty(\{B_n\}) = \prod_{n=1}^\infty B_n$ (C^*-product of $\{B_n\}$) and $c_0(\{B_n\}) = \bigoplus_{n=1}^\infty B_n$ (C^*-direct sum). We will use $q_\infty(\{B_n\})$ for the quotient $l^\infty(\{B_n\})/c_0(\{B_n\})$.

(11) Let $A = \lim_{n\to\infty}(A_n, \phi_n)$, where $\phi_n : A_n \to A_{n+1}$ is the connecting homomorphism. We denote by $\phi_{n,\infty} : A_n \to A$ the homomorphism induced by the inductive system. A is said to be an A$\mathbb{T}$-algebra if each A_n has the form $C(\mathbb{T})\otimes F_n$, for some finite-dimensional C^*-subalgebra F_n. A is said to be an AH-algebra if each A_n has the form $P_nM_{k(n)}(C(X_n))P_n$, where X_n is a finite CW-complex and $P_n \in M_{k(n)}(C(X_n))$ is a projection. We say that A has no dimension growth if there is an integer N such that $\dim X_n \le N$.

(12) Let A and B be two C^*-algebras and let $\phi : A \to B$ be a contractive completely positive linear map. Let $\varepsilon > 0$ and let $\mathcal{F} \subset A$ be a subset. The map ϕ is said to be $\mathcal{F}$-ε-multiplicative if

$$\|\phi(ab) - \phi(a)\phi(b)\| < \varepsilon \text{ for all } a, b \in \mathcal{F}.$$

(13) Let A be a C^*-algebra, let $\{B_n\}$ be a sequence of C^*-algebras and let $\phi_n : A \to B_n$ be a sequence of contractive completely positive linear maps. We say that $\{\phi_n\}$ is a sequentially asymptotic morphism if

$$\lim_{n\to\infty} \|\phi_n(ab) - \phi_n(a)\phi_n(b)\| = 0 \text{ for all } a \in A.$$

Let $\Phi : A \to l^\infty(\{B_n\})$ be defined by $\Phi(a) = \{\phi_n\}$ and let $\phi = \pi\circ\Phi : A \to q_\infty(\{B_n\})$, where $\pi : l^\infty(\{B_n\}) \to q_\infty(\{B_n\})$ is the quotient map. Then ϕ is a homomorphism. In particular, $[\phi]$ gives an element in $Hom(\underline{K}(A), \underline{K}(q_\infty(\{B_n\})))$. It follows that, for any finite subset $\mathcal{P} \subset \mathbf{P}(A)$, for sufficiently large n, $[\phi_n]|_{\mathcal{P}}$ is well defined partial map to $\underline{K}(B_n)$.

Thus, given any finite subset $\mathcal{P} \subset \mathbf{P}(A)$, there is $\delta > 0$ and a finite subset $\mathcal{F} \subset A$, such that, if $\phi : A \to B$ is an $\mathcal{F}$-δ-multiplicative contractive completely positive linear map, then $[\phi]|_{\mathcal{P}}$ is well defined.

In what follows, given a finite subset $\mathcal{P} \subset \mathbf{P}(A)$, and ϕ is an $\mathcal{F}$-δ-multiplicative contractive completely positive linear map, when we write $[\phi]|_{\mathcal{P}}$ we mean it is well defined.

(14) Let $h : C(X) \to A$ be a homomorphism and let $O \subset X$ be an open subset. We write

$$\mathrm{Her}(h(O)) = \{h(f) : f(t) = 0 \ \ t \in X \setminus O\}.$$

(15) Let A be a C^*-algebra and let $\{a_n\}$ be a sequence of elements in A. We say that $\{a_n\}$ is a *central sequence* if

$$\lim_{n\to\infty} \|a_nx - xa_n\| = 0 \text{ for all } x \in A.$$

These conventions will be used throughout the paper without further explanation.

3. The main results

3.1. **Monomorphisms from $C(X)$.** In the 1970's, Brown, Douglass and Fillmore ([3], [4] and [5]) proved that two unital monomorphisms h_1 and h_2 from $C(X)$ into the Calkin algebra $B(l^2)/\mathcal{K}(l^2)$, where $B(l^2)$ is the C^*-algebra of bounded operators and $\mathcal{K}(l^2)$ is the C^*-algebra of compact operators on the Hilbert space l^2, are unitarily equivalent if and only if they induce the same KK-element. It should be noted that the Calkin algebra is a simple C^*-algebra with real rank zero. M. Dadarlat showed that two monomorphisms from $C(X)$ to a unital purely infinite simple C^*-algebra are approximately unitarily equivalent if and only if they give the same element in $KL(C(X), A)$. We consider two monomorphisms $h_1, h_2 : C(X) \to A$, where A is a unital simple C^*-algebra with tracial (topological) rank zero. One important previous result was obtained in [19], where we assume that A has a unique trace.

We recall the definition of the tracial (topological) rank of C*-algebras.

Definition 3.1. Let A be a unital simple C^*-algebra and $k \in \mathbb{N}$. Then A is said to have *tracial (topological) rank zero* if and only if for any finite set $\mathcal{F} \subset A$, and $\varepsilon > 0$ and any non-zero positive element $a \in A$, there exists a finite-dimensional C^*-subalgebra $B \subset A$ with $\mathrm{id}_B = p$ such that

(1) $\|[x, p]\| < \varepsilon$ for all $x \in \mathcal{F}$,
(2) $pxp \in_\varepsilon B$ for all $x \in \mathcal{F}$,
(3) $[1 - p] \leq [a]$.

We write $\mathrm{TR}(A) = 0$ if A has tracial (topological) rank zero.

Recall that a C^*-algebra A is said to have the Fundamental Comparison Property, if, for any two projections p, $q \in A$, $\tau(p) < \tau(q)$ for all $\tau \in T(A)$ implies that $p \sim q' \leq q$. If A has the Fundamental Comparison Property, then condition (3) above can be replaced by

(3′) $\tau(1 - p) < \varepsilon$ for all $\tau \in T(A)$.

It is proved in [36] that if $\mathrm{TR}(A) = 0$, then A has real rank zero, stable rank one and weakly unperforated $K_0(A)$. Every simple AH-algebra with slow dimension growth and with real rank zero has tracial rank zero. Other simple C^*-algebras that are inductive limits of type I are also proved to have tracial rank zero (see [41]). Separable simple amenable C^*-algebras with tracial rank zero which satisfy the UCT are classified by their K-theory (see [39] and [38]).

Definition 3.2. Let A be a C^*-algebra and let B be a unital C^*-algebra. Suppose that $h_1, h_2 : A \to B$ are two maps. We say h_1 and h_2 are approximately unitarily equivalent if

$$h_1 \overset{u}{\sim}_\varepsilon h_2 \ \text{ on } \ \mathcal{F}$$

for every finite subset $\mathcal{F}$ and $\varepsilon > 0$. In other words, there exists a sequence of unitaries $u_n \in B$ such that

$$\lim_{n\to\infty} \|\mathrm{ad}\, u_n \circ h_1(a) - h_2(a)\| = 0 \ \text{ for all } \ a \in A.$$

Suppose that A has tracial states and both h_1 and h_2 are homomorphisms. Let $\tau \in T(A)$. It is clear that if h_1 and h_2 are approximately unitarily equivalent, then $\tau \circ h_1 = \tau \circ h_2$. In other words, $\tau \circ h_1$ and $\tau \circ h_2$ induce the same Borel measure on X.

The first main result of this paper is the following.

Theorem 3.3. *Let X be a compact metric space and let A be a unital separable simple C^*-algebra with $TR(A) = 0$. Suppose that $h_1 : C(X) \to A$ is a unital monomorphism. For any $\varepsilon > 0$ and any finite subset $\mathcal{F} \subset C(X)$, there exist $\delta > 0$ and a finite subset $\mathcal{G} \subset C(X)$ satisfying the following: if $h_2 : C(X) \to A$ is another unital monomorphism such that*

$$[h_1] = [h_2] \text{ in } KL(C(X), A) \quad \text{and} \quad |\tau \circ h_1(f) - \tau \circ h_2(f)| < \delta$$

for all $f \in \mathcal{G}$ and $\tau \in T(A)$, then there exists a unitary $u \in A$ such that

$$h_1 \overset{u}{\sim}_{\varepsilon} h_2 \text{ on } \mathcal{F}.$$

Consequently, we have

Theorem 3.4. *Let X be a compact metric space and let A be a unital separable simple C^*-algebra with $TR(A) = 0$. Suppose that $h_1, h_2 : C(X) \to A$ are two unital monomorphisms. Then h_1 and h_2 are approximately unitarily equivalent if and only if*

$$[h_1] = [h_2] \text{ in } KL(C(X), A)$$

and $\tau \circ h_1(a) = \tau \circ h_2(a)$ for all $a \in C(X)$.

A special case when A has a unique tracial state was obtained in [19].

Corollary 3.5. *Let X be a compact metric space with torsion free $K_i(C(X))$ ($i = 0, 1$) and let A be a unital separable simple C^*-algebra with $TR(A) = 0$. Suppose that $h_1, h_2 : C(X) \to A$ are two unital monomorphisms. Then h_1 and h_2 are approximately unitarily equivalent if and only if*

$$(h_1)_{*i} = (h_2)_{*i}, \quad i = 0, 1,$$

and $\tau \circ h_1(a) = \tau \circ h_2(a)$ for all $a \in C(X)$.

We will give two interesting applications of these results. Some approximate versions of it will be give in Section 4.

3.2. **Minimal dynamical systems.** Let X be a compact metric space and let $\alpha : X \to X$ be a homeomorphism. Recall that α is said to be minimal if $\{\alpha^n(x) : n \in \mathbb{Z}\}$ is dense in X. We assume that X is an infinite set. The corresponding transformation group C^*-algebra denoted by $A_\alpha = C(X) \rtimes_\alpha \mathbb{Z}$ is a unital simple C^*-algebra. It is also amenable and separable. It is in the so-called "Bootstrap" class of C^*-algebras. Therefore it satisfies the Universal Coefficient Theorem.

Many C^*-algebras described above have tracial rank zero. For example, if X is a connected manifold and α is a diffeomorphism, then $TR(A_\alpha) = 0$ if and only if $\rho_{A_\alpha}(K_0(A_\alpha))$ is dense in $Aff(T(A_\alpha))$ (see [49] and [41]).

In what follows, we will use A_α for $C(X) \rtimes_\alpha \mathbb{Z}$ and $j_\alpha : C(X) \to A_\alpha$ for the obvious embedding.

A theorem of Tomiyama (see [56]) establishes the following important relation between C^*-algebra theory and topological dynamics.

Theorem 3.6 (J. Tomiyama)**.** *Let X be a compact metric space and let $\alpha, \beta : X \to X$ be homeomorphisms. Suppose that (X, α) and (X, β) are topologically transitive. Then α and β are flip conjugate if and only if there is an isomorphism $\phi : C(X) \rtimes_\alpha \mathbb{Z} \to C(X) \rtimes_\beta \mathbb{Z}$ such that $\phi \circ j_\alpha = j_\beta \circ \chi$ for some isomorphism $\chi : C(X) \to C(X)$.*

It should be noted that all minimal dynamical systems are transitive.

In the light of Tomiyama's theorem, we introduce the following version of approximate flip conjugacy (for the case that $TR(A_\alpha) = TR(A_\beta) = 0$).

Definition 3.7. Let (X, α) and (X, β) be two minimal dynamical systems such that $TR(A_\alpha) = TR(A_\beta) = 0$. We say that (X, α) and (X, β) are *C^*-strongly approximately flip conjugate* if there exists a sequence of isomorphisms $\phi_n : A_\alpha \to A_\beta$ and a sequence of isomorphisms $\chi_n : C(X) \to C(X)$ such that $[\phi_n] = [\phi_1]$ in $KL(A_\alpha, A_\beta)$ for all n and

$$\lim_{n\to\infty} \|\phi_n \circ j_\alpha(f) - j_\beta \circ \chi_n(f)\| = 0 \text{ for all } f \in C(X).$$

For the general case that the crossed products are not assumed to have tracial rank zero, a modified definition is given in [46].

In Theorem 3.6, let $\theta = [\phi]$ in $KK(A_\alpha, A_\beta)$. Let $\Gamma(\theta)$ be the induced element in $Hom(K_*(A_\alpha), K_*(A_\beta))$ which preserves the order and the unit. Then one has

$$[j_\alpha] \times \theta = [j_\beta \circ \chi]$$

Suppose that $TR(A_\alpha) = TR(A_\beta) = 0$. Then $\rho_{A_\alpha}(K_0(A_\alpha))$ and $\rho_{A_\beta}(K_0(A_\beta))$ are dense in $Aff(T(A_\alpha))$ and $Aff(T(A_\beta))$, respectively. Thus $\Gamma(\theta)$ induces an order and unit preserving affine isomorphism $\theta_\rho : Aff(T(A_\alpha)) \to Aff(T(A_\beta))$. Recall $\rho_{A_\alpha} : (A_\alpha)_{s.a} \to Aff(T(A_\alpha))$ by $\rho_{A_\alpha}(a)(\tau) = \tau(a)$ for $\tau \in T(A)$. Therefore, in terms of K-theory and KK-theory, one has the following: If α and β are flip conjugate, then there is an isomorphism $\chi : C(X) \to C(X)$ such that

$$[j_\alpha] \times \theta = [j_\beta \circ \chi] \text{ in } KK(C(X), A_\beta) \quad \text{and} \quad \theta_\rho \circ \rho_{A_\alpha} \circ j_\alpha = \rho_{A_\beta} \circ j_\beta \circ \chi. \tag{e 3.1}$$

The following theorem gives a K-theoretical description of C^*-strong approximate flip conjugacy.

Theorem 3.8. *Let (X, α) and (X, β) be two minimal dynamical systems such that A_α and A_β have tracial rank zero. Then α and β are C^*-strongly approximately flip conjugate if and only if the following hold: There is a sequence of isomorphisms $\chi_n : C(X) \to C(X)$ and $\theta \in KL(A_\alpha, A_\beta)$ such that $\Gamma(\theta)$ gives an isomorphism from $(K_0(A_\alpha), K_0(A_\alpha)_+, [1], K_1(A_\alpha))$ to $(K_0(A_\beta), K_0(A_\beta)_+, [1], K_1(A_\beta))$, for any finitely generated subgroup $G \subset \underline{K}(C(X))$,*

$$[j_\alpha] \times \theta|_G = [j_\beta \circ \chi_n]|_G \text{ in } KL(C(X), A_\beta) \text{ for all sufficiently large } n \tag{e 3.2}$$

and

$$\lim_{n\to\infty} \|\rho_{A_\beta} \circ j_\beta \circ \chi_n(f) - \theta_\rho \circ \rho_{A_\alpha} \circ j_\alpha(f)\| = 0 \text{ for all } f \in C(X)_{s.a}. \tag{e 3.3}$$

Corollary 3.9. *Let X be a compact metric space with torsion free K-theory. Let (X, α) and (X, β) be two minimal dynamical systems such that $TR(A_\alpha) = TR(A_\beta) = 0$. Suppose that there is a unit preserving order isomorphism*

$$\gamma : (K_0(A_\alpha), K_0(A_\alpha)_+, [1_{A_\alpha}], K_1(A_\alpha)) \to (K_0(A_\beta), K_0(A_\beta)_+, [1_{A_\beta}], K_1(A_\beta)), \tag{e 3.4}$$

$$[j_\alpha] \times \theta = [j_\beta \circ \chi] \text{ in } KL(C(X), A_\beta) \quad \text{and} \quad \gamma_\rho \circ j_\alpha = \rho_{A_\beta} \circ j_\beta \circ \chi \tag{e 3.5}$$

for some isomorphism $\chi : C(X) \to C(X)$. Then (X, α) and (X, β) are C^*-strongly approximately flip conjugate.

Remark 3.10. In Definition 3.7, we require that $[\phi_n] = [\phi_1]$ for all n. It will be only a marginal gain by removing this condition from the definition. If both $K_i(A_\alpha)$ and $K_i(A_\beta)$ are finitely generated ($i = 0, 1$), then there are only finitely many order isomorphisms which preserve the identity. Moreover, in this case, $ext_{\mathbb{Z}}(K_{i-1}(A_\alpha), K_i(A_\beta))$ has only finitely many elements. Thus, in this case, there are only finitely many elements $\theta \in KL(A_\alpha, A_\beta)$ which gives order and unit preserving isomorphisms from $K_i(A_\alpha)$ to $K_i(A_\beta)$ ($i = 0, 1$). Hence, there is a subsequence $\{n_k\}$ such that $[\phi_{n_k}] = \theta$ for some θ. Therefore one may well assume that $[\phi_n] = [\phi_1]$ for all n.

In the case when X is the Cantor set, $K_0(C(X)) = C(X, \mathbb{Z})$. It follows that, if there is $\theta : K_i(A_\alpha) \to K_i(A_\beta)$ that is an order and unit preserving isomorphism, then there exists $\chi : C(X) \to C(X)$ such that

$$\theta \circ (j_\alpha)_{*0} = (j_\beta \circ \chi)_{*0}$$

(see Theorem 2.6 of [45]). Moreover, it implies that

$$\theta_\rho \circ \rho_{A_\alpha} \circ j_\alpha = \rho_{A_\beta} \circ j_\beta \circ \chi.$$

In other words, in the case that X is the Cantor set condition (e 3.3) or (e 3.5) is automatic.

Furthermore, if X is the Cantor set, two minimal homeomorphisms α and β are C^*-strongly approximately conjugate if and only if they are approximately K-conjugate. More precisely we have the following theorem which is also related to the work of Giordano, Putnam and Skau in [16].

Theorem 3.11 (Theorem 5.4 of [45]). *Let X be the Cantor set and let α and β be minimal homeomorphisms. Then the following are equivalent:*

(i) *α and β are C^*-strongly approximately flip conjugate,*

(ii) *α and β are approximately K-conjugate,*

(iii) *A_α and A_β are isomorphic,*

(iv) $(K_0(A_\alpha), K_0(A_\alpha)_+, [1_{A_\alpha}]) \cong (K_0(A_\beta), K_0(A_\beta)_+, [1_{A_\beta}])$,

(v) *there exists a sequence $\gamma_n \in [[\alpha]]$ and $\sigma_n \in [[\beta]]$, and a homeomorphism $\chi : X \to X$ such that*

$$f \circ \alpha = \lim_{n\to\infty} f \circ \chi \circ \sigma_n \circ \beta \circ \sigma_n^{-1} \circ \chi^{-1} \text{ and}$$
$$f \circ \beta = \lim_{n\to\infty} f \circ \chi^{-1} \circ \gamma_n \circ \alpha \circ \gamma_n^{-1} \circ \chi$$

for all $f \in C(X)$.

(vi) *α and β are strong orbit equivalent.*

In Theorem 3.11, $[[\alpha]]$ is the set of topological full group with respect to α, i.e., the group of all homeomorphisms $\gamma : X \to X$ such that $\gamma(x) = \alpha^{n(x)}(x)$ for all $x \in X$, where $n \in C(X, \mathbb{Z})$. We will explain other terminologies used in Theorem 3.11 in Section 5. Further discussion of approximate conjugacy will be given in Section 5.

3.3. **C^*-dynamical systems and the Rokhlin property.** By a C^*-dynamical system we mean a pair (A, α), where A is a C^*-algebra and $\alpha \in Aut(A)$. Let A be a unital simple $A\mathbb{T}$-algebra with real rank zero and $\alpha \in Aut(A)$ be a "sufficiently outer" automorphism. A. Kishimoto studied the problem when the associated crossed product is again an $A\mathbb{T}$-algebra of real rank zero ([24], [25], [26] and [27]).

In particular, Kishimoto studied the case that A has a unique tracial state and α is approximately inner. Kishimoto also suggested that the appropriate notion for "sufficiently outer" is the Rokhlin property. A more general question is: Let A be a unital simple AH-algebra with no dimension growth and with real rank zero. Suppose that $\alpha \in Aut(A)$. When is $A \rtimes_\alpha \mathbb{Z}$ again a unital simple AH-algebra with no dimension growth and with real rank zero?

With the classification theorem for simple C^*-algebras of tracial topological rank zero, an important question related to the crossed products is the following: Let A be a unital separable simple C^*-algebra with tracial rank zero and α an (outer) automorphism. When does $A \rtimes_\alpha \mathbb{Z}$ have tracial rank zero?

The following is defined in [50, Definition 2.1].

Definition 3.12. Let A be a simple unital C^*-algebra and let $\alpha \in \mathrm{Aut}(A)$. We say α has the *tracial Rokhlin property* if for every finite set $F \subset A$, every $\varepsilon > 0$, every $n \in \mathbb{N}$, and every nonzero positive element $x \in A$, there are mutually orthogonal projections $e_0, e_1, \dots, e_n \in A$ such that:

(1) $\|\alpha(e_j) - e_{j+1}\| < \varepsilon$ for $0 \le j \le n-1$.
(2) $\|e_j a - a e_j\| < \varepsilon$ for $0 \le j \le n$ and all $a \in F$.
(3) With $e = \sum_{j=0}^n e_j$, $[1-e] \le [x]$.

Recall that a unital simple C^*-algebra A is said to have the Fundamental Comparison Property if for any two projections $p, q \in A$, $p \sim q' \le q$, if $\tau(p) < \tau(q)$ for all $\tau \in T(A)$. In Definition 3.12, if A has the Fundamental Comparison Property, then condition (3) can be replaced by

(3′) $\tau(1-e) < \varepsilon$ for all $\tau \in A$.

This is the reason why it called tracial Rokhlin property.

A much stronger version of Rokhlin property was recently introduced in [47].

Definition 3.13. Let A be a simple unital C^*-algebra and let $\alpha \in \mathrm{Aut}(A)$. We say α has the *tracial cyclic Rokhlin property* if for every finite set $F \subset A$, every $\varepsilon > 0$, every $n \in \mathbb{N}$, and every non-zero positive element $x \in A$, there are mutually orthogonal projections $e_0, e_1, \dots, e_n \in A$ such that

(1) $\|\alpha(e_j) - e_{j+1}\| < \varepsilon$ for $0 \le j \le n$, where $e_{n+1} = e_0$.
(2) $\|e_j a - a e_j\| < \varepsilon$ for $0 \le j \le n$ and all $a \in F$.
(3) With $e = \sum_{j=0}^n e_j$, $[1-e] \le [x]$.

Note that if α has the tracial cyclic Rokhlin property, then $K_0(A)$ must have a "dense" subset which is invariant under α_{*0}.

It is shown (see Theorem 2.9 of [47] and also [44]) that if α has tracial cyclic Rokhlin property, then indeed the crossed product $A \rtimes_\alpha \mathbb{Z}$ has tracial rank zero. It is proved in [50] that tracial Rokhlin property occurs more often than one may think. For example, assuming that A is a unital separable simple C^*-algebra with a unique tracial state and with tracial rank zero, they prove in [50] that $A \rtimes_\alpha \mathbb{Z}$ has tracial Rokhlin property if and only if $A \rtimes_\alpha \mathbb{Z}$ has a unique tracial state, or $A \rtimes_\alpha \mathbb{Z}$ has real rank zero. It is proved in [47] that if α^r is approximately inner for some integer $r > 0$ and α has the tracial Rokhlin property, then α has the tracial cyclic Rokhlin property. Consequently, $A \rtimes_\alpha \mathbb{Z}$ has tracial rank zero. In particular, by the classification theorem for simple unital separable amenable C^*-algebras with tracial rank zero (see [39]), if A is a unital $A\mathbb{T}$-algebra of real rank zero, in this case, $A \rtimes_\alpha \mathbb{Z}$ is again a unital $A\mathbb{T}$-algebra with real rank zero provided that $A \rtimes_\alpha \mathbb{Z}$ has torsion-free K-theory. This solves Kishimoto's problem in this setting. By applying an approximate version of Theorem 3.3 (see Corollary 4.8), we have the following theorem.

Theorem 3.14. *Let A be a unital separable simple amenable C^*-algebra with $TR(A) = 0$ which satisfies the UCT and let $\alpha \in Aut(A)$. Suppose that α satisfies the tracial Rokhlin property. If there is an integer $r > 0$ such that $\alpha^r_{*0}|_G = \mathrm{id}_G$ for some subgroup $G \subset K_0(A)$ for which $\rho_A(G) = \rho_A(K_0(A))$, then α satisfies the tracial cyclic Rokhlin property.*

From this we obtain:

Theorem 3.15. *Let A be a unital separable simple amenable C^*-algebra with $TR(A) = 0$ which satisfies the UCT and let $\alpha \in Aut(A)$. Suppose that α satisfies the tracial Rokhlin property and $\alpha^r_{*0}|_G = \mathrm{id}_G$ for some subgroup $G \subset K_0(A)$ for which $\rho_A(G) = \rho_A(K_0(A))$. Then $A \rtimes_\alpha \mathbb{Z}$ is a unital simple AH-algebra with no dimension growth and with real rank zero.*

As a consequence, combining the results in [50], we also have the following.

Theorem 3.16. *Let A be a unital separable simple amenable C^*-algebra with $TR(A) = 0$ and with a unique tracial state which satisfies the UCT, and let $\alpha \in Aut(A)$ be such that $\alpha^k_{*0}|G = \mathrm{id}_G$ for some subgroup $G \subset K_0(A)$ for which $\rho_A(G) = \rho_A(K_0(A))$. Then the following are equivalent:*

(i) *α has the tracial Rokhlin property;*
(ii) *$A \rtimes_\alpha \mathbb{Z}$ has a unique tracial state;*
(iii) *$A \rtimes_\alpha \mathbb{Z}$ has real rank zero;*
(iv) *α has the tracial cyclic Rokhlin property;*
(v) *$A \rtimes_\alpha \mathbb{Z}$ has tracial rank zero;*
(vi) *$A \rtimes_\alpha \mathbb{Z}$ is a unital simple AH-algebra with unique tracial state and real rank zero.*

Finally we would like to mention the Furstenberg transformations on irrational rotation algebras studied recently by H. Osaka and N. C. Phillips. These also include the transformation group C^*-algebras of minimal Furstenberg transformations on the torus.

Definition 3.17. Let A_θ be the usual rotation algebra generated by unitaries u and v satisfying $vu = e^{2\pi i\theta}uv$. Let $\theta, \gamma \in \mathbb{R}$, let $d \in \mathbb{Z}$, and let $f : S^1 \to \mathbb{R}$ be a continuous function. The *Furstenberg transformation* on A_θ determined by (θ, γ, d, f) is the automorphism $\alpha_{\theta,\gamma,d,f}$ of A_θ such that

$$\alpha_{\theta,\gamma,d,f}(u) = e^{2\pi i\gamma}u \quad \text{and} \quad \alpha_{\theta,\gamma,d,f}(v) = \exp(2\pi i f(u))u^d v.$$

The resulted crossed product is denoted by $A_\theta \rtimes_\alpha \mathbb{Z}$.

Combining the results in [50] and Theorem 3.16, we obtain the following.

Theorem 3.18. *Let $\theta, \gamma \in \mathbb{R}$ and suppose that $1, \theta, \gamma$ are linearly independent over $\mathbb{Q}$. Let $d \in \mathbb{Z}$ and let $\alpha_{\theta,\gamma,d,f} \in Aut(A_\theta)$ be as defined in Definition* 3.17. *Then*

(1) *$A_\theta \rtimes_\alpha \mathbb{Z}$ is a unital simple AH-algebra with no dimension growth, with real rank zero and with a unique tracial state.*

(2) *If, in addition, $d = 0$, then $A_\theta \rtimes_\alpha \mathbb{Z}$ is an $A\mathbb{T}$-algebra with real rank zero.*

4. Maps from $C(X)$

In this section we will prove Theorem 3.3 and Theorem 3.4. We do this by proving an approximate version of these (Theorem 4.6). We will also prove Theorem 4.8, which we use in Section 5.

Lemma 4.1. *Let A be a unital C^*-algebra. For any $\varepsilon > 0$ and any finite subset $\mathcal{F} \subset A$, there exists $\delta > 0$ and $\mathcal{G} \subset A$ which satisfy the following:*

If B is another unital C^*-algebra, $\tau \in T(B)$ and $\phi_n : A \to B$ is a unital contractive completely positive linear map which is $\mathcal{G}$-δ-multiplicative, then there exists a tracial state $\sigma \in T(A)$ such that

$$|\tau \circ \phi_n(a) - \sigma(a)| < \varepsilon \ \textit{for all} \ a \in \mathcal{F}.$$

Proof. Suppose that the lemma is false. Then there exist $\varepsilon_0 > 0$ and a finite subset $\mathcal{F}_0 \subset A$, there exists a $\mathcal{G}_n$-δ_n-multiplicative contractive completely positive linear map $\phi_n : A \to B_n$, where B_n are unital C^*-algebras and $\bigcup_{n=1}^{\infty} G_n$ is dense in A and $\sum_{n=1}^{\infty} \delta_n < \infty$, and a sequence of tracial states $\tau_n \in T(B_n)$ such that

$$\inf_{n \in N} \{\inf\{\sup\{|\tau_n \circ \phi_n(a) - \sigma(a)| : a \in \mathcal{F}_0\} : \sigma \in T(A)\}\} \geq \varepsilon_0.$$

Take a weak limit σ of $\{\tau_n \circ \phi_n\}$. Then σ is a tracial state of A. There is subsequence $\phi_{n(k)}$ such that

$$\sigma(a) = \lim_{k \to \infty} \tau \circ \phi_{n(k)}(a) \ \text{for all} \ a \in A.$$

This gives a contradiction. □

Definition 4.2. Let X be a compact metric space and let A be a unital C^*-algebra. Suppose that $\phi : C(X) \to A$ is a unital positive linear map and suppose that $\tau \in T(A)$. Then $\tau \circ \phi$ is a positive linear functional on $C(X)$. We use $\mu_{\tau \circ \phi}$ for the induced probability Borel regular measure.

Let X be a compact metric space and $\eta > 0$. Then there are finitely many (distinct) points $x_1, x_2, ..., x_m \in X$ such that $\{x_1, x_2, ..., x_m\}$ is an η-dense subset. There is an integer $s > 0$, such that

$$O_i \cap O_j = \emptyset, \quad i \neq j,$$

where $O_i = \{x \in X : \text{dist}(x, x_i) < \eta/2s\}$, $i = 1, 2, ..., m$. The integer s depends on the choice of $\{x_1, x_2, ..., x_m\}$.

Lemma 4.3. *Let X be a compact metric space, let $\varepsilon > 0$ and let $\mathcal{F} \subset C(X)$ be a finite subset. Let $L > 0$ be an integer and let $\eta > 0$ be such that $|f(x) - f(x')| < \varepsilon/8$ if $\text{dist}(x, x') < \eta$. Then, for any integer $s > 0$, any finite $\eta/2$-dense subset $\{x_1, x_2, ..., x_m\}$ of X for which $O_i \cap O_j = \emptyset$,* if *$i \neq j$, where*

$$O_i = \{\xi \in X : \text{dist}(\xi, x_i) < \eta/2s\}$$

and any $1/2s > \sigma > 0$, there exist a finite subset $\mathcal{G} \subset C(X)$ and $\delta > 0$ satisfying the following:

For any unital separable stably finite C^*-algebra A with real rank zero, $\tau \in T(A)$ and any $\mathcal{G}$-δ-multiplicative contractive completely positive linear map $\phi : C(X) \to A$, if $\mu_{\tau \circ \phi}(O_i) > \sigma \cdot \eta$, for all i, then there are mutually orthogonal projections

$p_1, p_2, ..., p_m$ *in* A *such that*

$$\|\phi(f) - ((1-p)\phi(f)(1-p) + \sum_{i=1}^{m} f(x_i)p_i)\| < \varepsilon \text{ for all } f \in \mathcal{F}$$
$$\text{and} \quad \|(1-p)\phi(f) - \phi(f)(1-p)\| < \varepsilon,$$

where $p = \sum_{i=1}^{m} p_i$,

$$\tau(p_k) > (4mL+4)\tau(1 - \sum_{i=1}^{m} p_i) \quad \text{and} \quad \tau(p_k) > \frac{\sigma}{2} \cdot \eta, \ k = 1, 2, ..., m.$$

(In the above statement, η does not depend on the choice of σ.)

Proof. Suppose that there exists $\varepsilon_0 > 0$ and a finite subset $\mathcal{F}_0 \subset C(X)$ so that the lemma is false. Fix $\varepsilon > 0$ so that $\varepsilon < \varepsilon_0/4$. Let $\eta > 0$ so that

$$|f(x) - f(y)| < \varepsilon/8 \text{ if } \operatorname{dist}(x, y) < \eta, \ x, y \in X,$$

for all $f \in \mathcal{F}_0$. Suppose that $\{x_1, x_2, ..., x_m\}$ is an $\eta/2$-dense subset of X such that $O_i \cap O_j = \emptyset$, if $i \neq j$, where

$$O_i = \{x \in X : \operatorname{dist}(x, x_i) < \eta/2s\}, \ i = 1, 2, ..., m,$$

for some integer $s > 0$. Let $1/2s > \sigma > 0$. Since the lemma is false (for a choice of the aboved-mentioned $x_1, ..., x_m$, s and σ), there exist a sequence of unital separable stably finite C^*-algebra B_n of real rank zero, a sequence of unital $\mathcal{G}_n$-δ_n-multiplicative contractive completely positive linear maps $\phi_n : C(X) \to B_n$, where $\bigcup_{n=1}^{\infty} G_n$ is dense in $C(X)$ and $\sum_{n=1}^{\infty} \delta_n < \infty$, and there are $\tau_n \in T(B_n)$ such that $\mu_{\tau \circ \phi_n}(O_i) > \sigma \cdot \eta$ satisfying the following:

$$\inf\{\sup\{\|\phi_n(f) - [(1-p_n)\phi_n(1-p_n) + \sum_{i=1}^{m} f(x_i)p(i,n)]\| : f \in \mathcal{F}_0\} : n\} \geq \varepsilon_0$$

where the infimum is taken among all mutually orthogonal projections $p(1, n)$, $p(2, n), ..., p(m, n)$ which satisfy

$$\tau_n(p(i,n)) > (4mL+4)\tau_n(1-p_n) \quad \text{and} \quad \tau_n(p(i,n)) > \sigma\eta/2,$$

where $p_n = \sum_{i=1}^{m} p(i, n)$.

Define $\Phi : C(X) \to l^{\infty}(\{B_n\})$ by $\Phi(f) = \{\phi_n(f)\}$ and let $\pi : l^{\infty}(\{B_n\}) \to q_{\infty}(\{B_n\})$. Then $\pi \circ \Phi : C(X) \to q_{\infty}(\{B_n\})$ is a homomorphism.

By passing to a subsequence, if necessary, we may also assume that there is $\tau \in l^{\infty}(\{B_n\})$ such that

$$\lim_{n\to\infty} \tau_n(a_n) = \tau(\{a_n\})$$

for any $\{a_n\} \in l^{\infty}(\{B_n\})$. Moreover, since, for each $\{a_n\} \in c_0(\{B_n\})$, $\lim_{n\to\infty} \tau_n(a_n) = 0$, we may view τ as a tracial state of $q_{\infty}(\{B_n\})$. It follows that

$$\mu_{\tau, \pi\circ\Phi}(O_i) > \sigma \cdot \eta,$$

$i = 1, 2, ..., m$.

Exactly as in the proof of 2.11 of [34], there are $r_i \in (0, 1)$ such that

$$\mu_{\tau, \pi\circ\Phi}(C_{r_i}) = 0, \tag{e 4.6}$$

where

$$C_{r_i} = \{\zeta \in X : \operatorname{dist}(\zeta, x_i) = (1 + r_i)(\eta/2)\}.$$

Let $\Omega_1, \Omega_2, ..., \Omega_{K'}$ be disjoint open subsets such that $\bigcup_{i=1}^{K'} \Omega_i = X \setminus \bigcup_{i=1}^m C_{r_i}$. Therefore

$$\bigcup_{i=1}^{K'} \Omega_i \cup (\bigcup_{i=1}^{m} C_{r_i}) = X \quad \text{and} \quad \operatorname{diam}(\Omega_i) < \eta/2.$$

(Note that $K' \leq m^m$.) Since $O_i \cap O_j = \emptyset$, if $i \neq j$, we may assume that $O_i \subset \Omega_i$, $i = 1, 2, ..., m$, and $\mu_{\tau \circ \pi \circ \Phi}(\Omega_i) > 0$, $i = 1, 2, ..., K$ (and $\mu_{\tau \circ \pi \circ \Phi}(\Omega_i) = 0$ if $i > K$ and $K' \geq K \geq m$). Let $B_{\Omega_i} = \mathrm{Her}(\pi \circ \Phi(\Omega_j))$, $i = 1, 2, ..., K'$. Since B_n has real rank zero, $q_\infty(\{B_n\})$ also has real rank zero. Hence there is an approximate identity $\{e_i(n)\}$ for B_{O_i} for each i. Then

$$\tau(e_i^{(n)}) \nearrow \mu_{\tau \circ \pi \circ \Phi}(\Omega_i),$$

$i = 1, 2, ..., K$. Since $\mu_{\tau \circ \pi \circ \Phi}(X \setminus \bigcup_i^m C_{r_i}) = 0$,

$$\tau(\sum_{i=1}^{K} e_i^{(n)}) \nearrow 1$$

as $n \to \infty$. So

$$\tau(1 - \sum_{i=1}^{K} e_i^{(n)}) \to 0 \ \text{ as } \ n \to \infty.$$

Since $\mu_{\tau \circ \pi \circ \Phi}(\Omega_i) > 0$ for $i = 1, 2, ..., K$, we obtain projections q_j ($= e_j(n)$ for some large n) so that

$$(4KL + 5)\tau(1 - \sum_{k=1}^{K} q_k) < \tau(q_i), \ \ i = 1, 2, ..., K.$$

By Lemma 2.5 of [34] and by the choice of η, one obtains

$$\|\pi \circ \Phi(f) - [(1-q)\pi \circ \Phi(f)(1-q) + \sum_{k=1}^{K} f(\zeta_k) q_k]\| < \varepsilon/2 \quad \text{and}$$
$$\|q\pi \circ \Phi(f) - \pi \circ \Phi(f) q\| < \varepsilon/2$$

for all $f \in \mathcal{F}_0$, where $q = \sum_{k=1}^{K} q_k$. Thus, for all sufficiently large n, there are non-zero mutually orthogonal projections $p(1, n), p(2, n), ..., p(K, n) \in B_n$ such that

$$\|\phi_n(f) - [(1-p_n)\phi_n(f)(1-p_n) + \sum_{k=1}^{K} f(\zeta_k) p(k, n)]\| < \varepsilon/2,$$
$$\|p_n \phi_n(f) - \phi_n(f) p_n\| < \varepsilon/2 \ \text{ for all } \ f \in \mathcal{F}_0 \quad \text{and}$$
$$(4KL + 4)\tau_n(1 - p_n) < \tau_n(p(k, n)),$$

where $p_n = \sum_{k=1}^{K} p(i, n)$. Note that $O_i \subset \Omega_i$, $i = 1, 2, ..., m$. Since $\{x_1, x_2, ..., x_m\}$ is $\eta/2$-dense in X, by the choice of η and by replacing some of points ξ_j which close to x_i within $\eta/2$ by x_i and $p(k, n)$ by sum of some $p(k', n)$'s, we may assume that $K = m$ and $\xi_i = x_i$. Furthermore, it is also clear that we may assume that

$$\tau(p_k) > \frac{\sigma \cdot \eta}{2}, \ \ k = 1, 2, ..., m.$$

This is a contradiction. $\square$

Lemma 4.4. *Let X be a compact metric space, let $\varepsilon > 0$ and let $\mathcal{F} \subset C(X)$ be a finite subset. Let $L > 0$ be an integer and let $\eta > 0$ be such that $|f(x) - f(x')| < \varepsilon/8$ if $\mathrm{dist}(x, x') < \eta$. Then, for any integer $s > 0$, any finite $\eta/2$-dense subset of X for which $O_i \cap O_j = \emptyset$ for $i \neq j$, where*

$$O_i = \{\xi \in X : \mathrm{dist}(\xi, x_i) < \eta/2s\}$$

and any $1/2s > \sigma > 0$, there exist $\gamma > 0$, a finite subset $\mathcal{G} \subset C(X)$ and $\delta > 0$ satisfying the following:

For any unital separable stably finite C^*-algebra A with real rank zero, $\tau \in T(A)$ and any unital $\mathcal{G}$-δ-multiplicative contractive completely positive linear map $\phi, \psi : C(X) \to A$ with

$$|\tau \circ \phi(g) - \tau \circ \psi(g)| < \gamma \quad \text{for all } g \in \mathcal{G},$$

and if

$$\mu_{\tau\circ\phi}(O_i),\ \ \mu_{\tau\circ\psi}(O_i) > \sigma \cdot \eta,$$

for all i, then there are mutually orthogonal projections $p_1, p_2, ..., p_m$ and $q_1, q_2, ...,$ q_m in A such that

$$\|\phi(f) - ((1-p)\phi(f)(1-p) + \sum_{i=1}^{m} f(x_i)p_i)\| < \varepsilon \ \text{ for all } f \in \mathcal{F}$$
$$\text{and} \quad \|(1-p)\phi(f) - \phi(f)(1-p)\| < \varepsilon/2,$$

where $p = \sum_{i=1}^m p_i$,

$$\tau(p_k) > (3mL+3)\tau(1 - \sum_{i=1}^{m} p_i) \quad \text{and} \quad \tau(p_k) > \frac{\sigma \cdot \eta}{2}, k = 1, 2, ..., m,$$

and

$$\|\psi(f) - ((1-q)\psi(f)(1-q) + \sum_{i=1}^{m} f(x_i)q_i)\| < \varepsilon \ \text{ for all } f \in \mathcal{F}$$
$$\text{and} \quad \|(1-q)\psi(f) - \psi(f)(1-q)\| < \varepsilon,$$

where $q = \sum_{i=1}^m q_i$,

$$\tau(q_k) > (3mL+3)\tau(1 - \sum_{i=1}^{m} q_i) \quad \text{and} \quad \tau(q_k) > \sigma \cdot \eta/2, \ \ k = 1, 2, ..., m,$$

and

$$|\tau(q_k) - \tau(p_k)| < \frac{\min_k\{\tau(p_k)\}}{3mL}, \ \ k = 1, 2, ..., m. \tag{e 4.7}$$

(Here γ depends on m, which depends on η as well as σ. But η does not depend on σ.)

Proof. First we note that only (e 4.7) needs a proof. The proof is similar to that of Lemma 4.3. But we need to "dig" out the projections simultaneously

Let B_n be any sequence of unital separable C^*-algebras of real rank zero, let $\phi_n, \psi_n : C(X) \to B_n$ be any unital contractive completely positive linear maps and let $\tau_n \in T(B_n)$ such that

$$\lim_{n\to\infty} \|\phi_n(ab) - \phi_n(a)\phi_n(b)\| = 0, \lim_{n\to\infty} \|\psi_n(ab) - \psi_n(a)\psi_n(b)\| = 0 \quad \text{and}$$
$$\lim_{n\to\infty} |\tau_n \circ \phi_n(f) - \tau_n \circ \psi_n(f)\| = 0$$

for all $a, b, f \in C(X)$. Let $\Phi, \Psi : C(X) \to l^\infty(\{B_n\})$ be defined by $\Phi(f) = \{\phi_n(f)\}$ and $\Psi_n(f) = \{\psi_n(f)\}$, respectively. Let $\pi : l^\infty(\{B_n\}) \to q_\infty(\{B_n\})$ be the quotient map. As in the proof of Lemma 4.3, $\pi \circ \Phi, \pi \circ \Psi : C(X) \to q_\infty(\{B_n\})$ are unital homomorphisms. By the assumption, borrowing the notation in the proof of Lemma 4.3, $\tau \circ \pi \circ \Phi = \tau \circ \pi \circ \Psi$. Thus, in the proof of Lemma 4.3, one has

$$\mu_{\tau,\pi\circ\Phi}(\Omega_i) = \mu_{\tau,\pi\circ\Psi}(\Omega_i),\ i = 1, 2, ..., K.$$

Note that we assume that $\mu_{\tau,\pi\circ\Phi}(\Omega_i) > 0$, $i = 1, 2, ..., K$. Note also that $m \le K \le m^m$. Let $\{e'_j(n)\}$ be an approximate identity for $Her(\pi \circ \Psi_n(\Omega_j))$. Put

$$r = \inf\{\mu_{\tau,\pi\circ\Phi}(\Omega_i) : i = 1, 2, ..., K\} > 0.$$

So one can choose n so that

$$|\tau(e'_j(n)) - \tau(e_j(n))| < r/(4KL + 4) \ \text{ as well as}$$

$$\tau(e'_j(n)) > (4KL + 4)\tau(1 - \sum_{j=1}^{K} e'_j(n)), j = 1, 2, ..., K.$$

We then apply the last argument of the proof of Lemma 4.3. It is then clear from the proof of Lemma 4.3 that, by matching the size of projections, one may further require (e 4.7) to be held. □

The following is taken from Theorem 3.1 in [20] (see also Remark 1.1 in [20]). Some special cases of this can be found in [31], [15]. A different form, but similar in nature, of the following was proved in [8].

Theorem 4.5. *Let X be a compact metric space. For any $\varepsilon > 0$ and any finite subset $\mathcal{F} \subset C(X)$, there exist $\delta > 0$, $\eta > 0$, an integer $N > 0$, a finite subset $\mathcal{G} \subset C(X)$ and a finite subset $\mathcal{P} \subset \mathbf{P}(C(X))$ satisfy the following:*

For any unital C^*-algebra A with real rank zero, stable rank one and weakly unperforated $K_0(A)$ and any unital $\mathcal{G}$-δ-multiplicative contractive completely positive linear maps $\phi, \psi : C(X) \to A$, if

$$[\phi]|_{\mathcal{P}} = [\psi]|_{\mathcal{P}},$$

then there exists a unitary $u \in M_{Nk+1}(A)$ such that

$$\phi(f) \oplus f(x_1) \cdot 1_N \oplus f(x_2) \cdot 1_N \oplus \cdots \oplus f(x_k) \cdot 1_N \overset{u}{\sim}_{\varepsilon} \psi(f),$$
$$f(x_1) \cdot \oplus 1_N \oplus f(x_2) \cdot 1_N \oplus \cdots \oplus f(x_k) \cdot 1_N$$

for all $f \in \mathcal{F}$, and for any η-dense set $\{x_1, x_2, ..., x_k\}$ in X.

Theorem 4.6. *Let X be a compact metric space, $\varepsilon > 0$ and $\mathcal{F} \subset C(X)$ be a finite subset. Let $\eta > 0$ be such that $|f(x) - f(x')| < \varepsilon/8$ if $\mathrm{dist}(x, x') < \eta$. Then, for any integer $s > 0$, any finite $\eta/2$-dense subset $\{x_1, x_2, ..., x_m\}$ of X for which $O_i \cap O_j = \emptyset$, where*

$$O_i = \{x \in X : \mathrm{dist}(x, x_i) < \eta/2s\}$$

and any $1/2s > \sigma > 0$, there exist $\gamma > 0$, a finite subset $\mathcal{G} \subset C(X)$, $\delta > 0$, and a finite subset $\mathcal{P} \subset \mathbf{P}(C(X))$ satisfying the following:

For any unital separable simple C^*-algebra A with $TR(A) = 0$, any $\mathcal{G}$-δ-multiplicative unital contractive completely positive linear maps $\phi, \psi : C(X) \to A$ with

$$|\tau \circ \phi(g) - \tau \circ \psi(g)| < \gamma \ \textit{for all}\ g \in \mathcal{G} \quad \textit{and} \quad \textit{for all}\ \tau \in T(A),\ \textit{if}$$

$$\mu_{\tau\circ\phi}(O_i),\ \ \mu_{\tau\circ\psi}(O_i) > \sigma \cdot \eta,$$

for all i and for all $\tau \in T(A)$, and, if

$$[\phi]|_{\mathcal{P}} = [\psi]|_{\mathcal{P}},$$

then there exists a unitary $u \in A$ such that

$$\phi \overset{u}{\sim}_{\varepsilon} \psi \text{ on } \mathcal{F}.$$

(Here η does not depend on σ. But γ does depend on σ as well as η.)

Proof. Fix $\varepsilon > 0$ and a finite subset $\mathcal{F} \subset C(X)$. Let δ_1, $\mathcal{G}_1$, $\mathcal{P}_1$, η_1 and integer N be required by Theorem 4.5 for $\varepsilon/8$ and $\mathcal{F}$. Let $L = 2N$. There is a finite subset $\mathcal{F}_1 \subset C(X)$ and $\delta_2 > 0$ satisfying the following: For any $\mathcal{F}_1$-δ_2-multiplicative contractive completely positive linear maps $H_1, H_2 : C(X) \to B$ (for any unital B) if $H_1 \approx_{\delta_2} H_2$ on $\mathcal{F}_1$ implies that

$$[H_1]|_{\mathcal{P}_1} = [H_2]|_{\mathcal{P}_1}. \tag{e 4.8}$$

Without loss of generality, to simplify notation, we may assume that $\mathcal{F}_1 \supset \mathcal{F}$ and $\delta_2 < \delta_1$. Set $\mathcal{F}_2 = \mathcal{G}_1 \cup \mathcal{F}_1$. Let $\eta_2 > 0$, $\sigma > 0$, $\delta_3 > 0$, $\gamma_1 > 0$ and a finite subset $\mathcal{G}_2$ be required in Lemma 4.4 corresponding to $\varepsilon/16$, $\mathcal{F}_2$, L. We now let $\mathcal{G} = \mathcal{F}_2 \cup \mathcal{G}_2$ and $\eta = \min\{\eta_1, \eta_2\}$. We may assume that $\delta_3 < \delta_2$. Let $x_1, x_2, ..., x_m \in X$ be an $\eta/2$-dense subset. Suppose that

$$O_i \cap O_j = \emptyset, \text{ if } i \neq j,$$

where $O_i = \{x \in X : \mathrm{dist}(x, x_i) < \eta/2s\}$, and suppose that

$$\mu_{\tau\circ\phi}(O_i),\ \mu_{\tau\circ\psi}(O_i) > \eta \cdot \sigma \text{ for all } \tau \in T(A), 1 \leq i \leq m.$$

We also assume that

$$|\tau \circ \phi(f) - \tau \circ \psi(f)| < \gamma/2 \text{ for all } f \in \mathcal{G} \quad \text{and} \quad \text{for all } \tau \in T(A). \tag{e 4.9}$$

Since $TR(A) = 0$, there exists a sequence of finite-dimensional C^*-subalgebras B_n with $e_n = 1_{B_n}$ and a sequence of contractive completely positive linear maps $\phi_n : A \to B_n$ such that

(1) $\lim_{n\to\infty} \|e_n a - a e_n\| = 0$ for all $a \in A$,
(2) $\lim_{n\to\infty} \|\phi_n(a) - e_n a e_n\| = 0$ for all $a \in A$ and $\phi_n(1) = e_n$,
(3) $\lim_{n\to\infty} \|\phi_n(ab) - \phi_n(a)\phi_n(b)\| = 0$ for all $a, b \in A$, and
(4) $\tau(1 - e_n) \to 0$ uniformly on $T(A)$.

In what follows, we may assume that

$$\tau(1 - e_n) < \frac{\sigma \cdot \eta}{8mL + 1} \text{ for all } \tau \in T(A). \tag{e 4.10}$$

We write $B_n = \bigoplus_{i=1}^{r(n)} D(i,n)$, where each $D(i,n)$ is a simple finite-dimensional C^*-algebra, a full matrix algebra. Denote by $\Phi(i,n) : A \to D(i,n)$ the map which is the composition of the projection map from B_n onto $D(i,n)$ with ϕ_n. Denote by $\tau(i,n)$ the standard normalized trace on $D(i,n)$. Note that any weak limit of $\tau(i,n)\circ\Phi(i,n)$ gives a tracial state of A. By (e 4.9) and (4), we have, for all sufficiently large n,

$$|\tau(i,n) \circ \Phi(i,n) \circ \phi(g) - \tau(i,n) \circ \Phi(i,n) \circ \psi(g)| < \gamma \text{ for all } g \in \mathcal{G}.$$

Put $\phi_{(i,n)} = \Phi(i,n) \circ \phi$ and $\psi_{(i,n)} = \Phi(i,n) \circ \psi$. By (4), for all sufficiently large n, we have

$$\mu_{\tau(i,n)\circ\phi_{(i,n)}}(O_k) \geq \sigma \cdot \eta/2 \quad \text{and} \quad \mu_{\tau(i,n)\circ\psi_{(i,n)}}(O_k) \geq \sigma \cdot \eta/2, \ \ k = 1, 2, ..., m, \tag{e 4.11}$$

for each i. From (1), (2), (3) above and (e 4.11), by applying Lemma 4.4, for each i and all sufficiently large n, we have

(e 4.12)
$$\|\phi_{(i,n)}(f)-[(1_{D(i,n)}-p_{i,n})\phi_{(i,n)}(f)(1_{D(i,n)}-p_{i,n})+\sum_{j=1}^m f(x_j)p(j,i,n)]\|<\varepsilon/8,$$

(e 4.13)
$$\|\psi_{(i,n)}(f)-[(1_{D(i,n)}-q_{i,n})\psi_{(i,n)}(f)(1_{D(i,n)}-q_{i,n})+\sum_{j=1}^m f(x_j)q(j,i,n)]\|<\varepsilon/8,$$

$$\|p_{i,n}\phi_{i,n}(f)-\phi_{i,n}(f)p_{i,n}\|<\varepsilon/8 \quad\text{and}\quad \|q_n\psi_{i,n}(f)-\psi_{i,n}n(f)q_{i,n}\|<\varepsilon/8$$

for all $f\in\mathcal{G}$, where $p(1,i,n),p(2,i,n),...,p(m,i,n)\in D(i,n)$ are non-zero mutually orthogonal projections, $p_{i,n}=1-\sum_{j=1}^m p(j,i,n)$ and $q_{i,n}=1-\sum_{j=1}^m q(j,i,n)$. Moreover,

$$(3mL+3)\tau(i,n)(1_{D(i,n)}-p_{i,n})<\tau(i,n)(p(k,i,n)), \tag{e 4.14}$$

$$(3mL+3)\tau(i,n)(1_{D(i,n)}-q_{i,n})<\tau(i,n)(q(k,i,n)), \tag{e 4.15}$$

(e 4.16)
$$|\tau(i,n)(p(k,i,n))-\tau(i,n)(q(k,i,n))|<\min_k\{\tau(i,n)(p(k,i,n))\}/3mL \quad\text{and}$$

$$\tau(i,n)(p(k,i,n)),\ \tau(i,n)(q(k,i,n))>\sigma\cdot\eta/2,\ 1\le k\le m. \tag{e 4.17}$$

For convenience, we may assume that

$$\tau(i,n)(p(k,i,n))\ge\tau(i,n)(q(k,i,n)),\ \ 1\le k\le m_1<m,\quad\text{and}$$
$$\tau(i,n)(q(k,i,n))\ge\tau(i,n)(p(k,i,n)),\quad m_1+1\le k\le m.$$

In $D(i,n)$, there are projections $p(k,i,n)'\le p(k,i,n)$, $k=1,2,...,m_1$, and $q(k,i,n)'\le q(k,i,n)$, $k=m_1+1,m_1+2,...,m$, such that

$$\tau(i,n)(p(k,i,n)')=\tau(i,n)(q(k,i,n)),k=1,2,...,m_1,\quad\text{and}$$
$$\tau(i,n)(q(k,i,n)')=\tau(i,n)(p(k,i,n)),k=m_1+1,m_1+2,...,m.$$

Put $p_{i,n}'=\sum_{k=1}^{m_1}p(k,i,n)'+\sum_{k=m_1+1}^m p(k,i,n)$ and

$$q_{i,n}'=\sum_{k=1}^{m_1}q(k,i,n)+\sum_{k=m_1+1}^m q(k,i,n)'.$$

One computes, by (e 4.14), (e 4.15) and (e 4.16), that

$$(2mL+1)\tau(i,n)(1_{D(i,n)}-p_{i,n}')<\tau(i,n)(p(k,i,n)),\ \ \tau(i,n)(p(k,i,n)')\quad\text{and}$$
$$(2mL+1)\tau(i,n)(1_{D(i,n)}-q_{i,n}')<\tau(i,n)(q(k,i,n)),\ \ \tau(i,n)(q(k,i,n)')\quad\text{and}$$
$$\tau(i,n)(p(k,i,n)')>(\sigma\cdot\eta)/2\ \text{ for }\ 1\le k\le m_1,\quad\text{and}$$
$$\tau(i,n)(q(k,i,n))>(\sigma\cdot\eta)/2\ \text{ for }\ m_1<k\le m.$$

One also has

$$\|\phi_{(i,n)}(f)-[(1_{D(i,n)}-p_{i,n}')\phi_{(i,n)}(f)(1_{D(i,n)}-p_{i,n}')$$
$$+\sum_{j=1}^{m_1}f(x_j)p(j,i,n)'+\sum_{j=m_1+1}^m f(x_j)p(j,i,n)]\|<\varepsilon/8$$

and

$$\|\psi_{(i,n)}(f) - [(1_{D(i,n)} - q'_{i,n})\psi_{(i,n)}(f)(1_{D(i,n)} - q'_{i,n}) + \sum_{j=1}^{m_1} f(x_j)q(j,i,n) + \sum_{j=m_1+1}^{m} f(x_j)q(j,i,n)']\| < \varepsilon/8$$

for all $f \in \mathcal{G}$.

Note that in $D(i,n)$, there is a unitary $u_{(i,n)}$ such that

$$u^*_{(i,n)}p(j,i,n)'u_{(i,n)} = q(j,i,n), 1 \le j \le m_1, \quad \text{and}$$
$$u^*_{(i,n)}p(j,i,n)u_{(i,n)} = q(j,i,n)', m_1 < j \le m.$$

Thus without loss of generality, we may assume that $p(j,i,n)' = q(j,i,n)$ and $p(j,i,n)' = q(j,i,n)$. Furthermore, by changing notation if necessary, we may assume that (e 4.12) and (e 4.13) hold as well as

$$(2mL+1)\tau(i,n)(1-p_n) < \tau(i,n)(p(j,i,n)) \tag{e 4.18}$$

and $p_{i,n} = q_{i,n}$, $p(j,i,n) = q(j,i,n)$. By combining all i, without loss generality, we may write that

(e 4.19)
$$\|\phi_n(f) - [(1_{B_n} - P_n)\phi_n(f)(1_{B_n} - P_n) + \sum_{k=1}^{m} f(x_k)P(k,n)]\| < \varepsilon/4 \quad \text{and}$$

$$\|\psi_n(f) - [(1_{B_n} - P_n)\psi_n(f)(1|_{B_n} - P_n) + \sum_{k=1}^{m} f(x_k)P(k,n)]\| < \varepsilon/4 \tag{e 4.20}$$

for all $f \in \mathcal{F}$. Here $P(k,n) = \sum_{i=1}^{r(n)} p(k,i,n)$ and $P_n = 1_{B_n} - \sum_{k=1}^{m} P(k,n)$.

Furthermore, we also have

(e 4.21)
$$(2mL+1)[1_{B_n} - P_n] \le [P(k,n)] \text{ in } B_n \quad \text{and} \quad t(P(k,n)) \ge \frac{\sigma \cdot \eta}{4} \text{ for all } t \in T(A).$$

It follows from (e 4.21) and (e 4.10) that ($TR(A) = 0$)

$$(mL+1)[1_A - P_n] \le [P(k,n)] \text{ in } A, \quad k = 1,2,...,m. \tag{e 4.22}$$

Put

$$H_1(f) = (1_A - P_n)h_1(f)(1_A - P_n) \quad \text{and} \quad H_2(f) = (1_A - P_n)h_2(f)(1_A - P_n)$$

for $f \in C(X)$. It follows from (1), (2) above and by (e 4.19) and (e 4.20), for all sufficiently large n, one estimates that

$$\phi \sim_{\varepsilon/4} H_1 \oplus (\bigoplus_{k=1}^{m} f(x_i)P(k,n)) \quad \text{and} \tag{e 4.23}$$

$$\psi \sim_{\varepsilon/4} H_2 \oplus (\bigoplus_{k=1}^{m} f(x_i)P(k,n)) \tag{e 4.24}$$

on $\mathcal{G}$. By the choice of $\mathcal{F}_1$ and δ_2, one obtains that

$$[H_1 \oplus \bigoplus_{k=1}^{m} f(x_i)P(k,n)]|_{\mathcal{P}_1} = [H_2 \oplus \bigoplus_{k=1}^{m} f(x_i)P(k,n)]|_{\mathcal{P}_1}.$$

It follows that (working in each group $K_i(A\otimes C_n)$)

$$[H_1]|_{\mathcal{P}_1}=[H_2]|_{\mathcal{P}_1}.$$

Denote $E=1_A-P_n$ and define $H_0':C(X)\to M_m(EAE)$ by

$$H_0'(f)=\mathrm{diag}(f(x_1),f(x_2),...,f(x_m))\ \text{ for all }\ f\in C(X)$$

and define $H_0=H_0'\oplus H_0'\oplus\cdots\oplus H_0':C(X)\to M_{mN}(EAE)$. Then, by the choice of N, η_1, δ_1, $\mathcal{G}_1$ and $\mathcal{P}_1$ and applying Theorem 4.5, we obtain a unitary $u\in M_{mN+1}(EAE)$ such that

$$H_1\oplus H_0\overset{u}{\approx}_{\varepsilon/2}H_2\oplus H_0\ \text{ on }\ \mathcal{F}. \tag{e 4.25}$$

Rewrite $H_0(f)=\sum_{k=1}^m f(x_k)E_k'$, where $E_1',E_2',...,E_m'$ are mutually orthogonal projections, for $f\in C(X)$. By (e 4.22), there is a unitary $W\in A$ such that $W^*E_k'W\le P(k,n)$, $k=1,2,...,m$. Put $Q_k=P(k,n)-W^*E_k'W$ and define $H_{00}(f)=\sum_{k=1}^m f(x_k)Q_k$ for $f\in C(X)$. Then one has

$$H_1\oplus H_0\oplus H_{00}\sim H_1\oplus(\bigoplus_{k=1}^m f(x_k)P(k,n))\quad\text{and} \tag{e 4.26}$$

$$H_2\oplus H_0\oplus H_{00}\sim H_2\oplus(\bigoplus_{k=1}^m f(x_i)P(k,n)) \tag{e 4.27}$$

on $\mathcal{F}$. Finally, by (e 4.25), (e 4.26), (e 4.27), (e 4.23) and (e 4.24), one obtains

$$\phi\sim_\varepsilon\psi\ \text{ on }\ \mathcal{F}.$$

$\square$

Now we are ready to prove Theorem 3.3.

Proof of Theorem 3.3. Let $\eta>0$ and let $x_1,x_2,...,x_m\in X$ be an $\eta/4$-dense subset. Choose an integer $s>0$ such that

$$O_i\cap O_j=\emptyset,\ \text{ if }\ i\neq j,$$

where $O_i=\{x\in X:\mathrm{dist}(x,x_i)<\eta/2s\}$. Let $g_i\in C(X)$ such that $0\le g_i\le 1$, $g_i(x)=1$ if $\mathrm{dist}(x,x_i)<\eta/8s$ and $g_i(x)=0$ if $\mathrm{dist}(x,x_i)\ge\eta/4s$, $i=1,2,...,m$.

Since A is simple and h_1 is a monomorphism,

$$\inf\{\tau(g_i):\tau\in T(A)\}=a_i>0, i=1,2,...,m.$$

Thus $\mu_{\tau\circ h_1}(O_i)>a_i$ for all $\tau\in T(A)$, $i=1,2,...,m$. Choose $d=\inf\{a_i:1\le i\le m\}/\eta$. Then

$$\mu_{\tau\circ h_1}(O_i)>d\cdot\eta, i=1,2,...,m.$$

Choose $\sigma=\min\{d/2,1/2s\}$. Let $\mathcal{G}=\mathcal{G}\cup\{g_i:1\le i\le m\}$. By choosing small γ, if

$$|\tau\circ h_1(g)-\tau\circ h_2(g)|<\gamma\quad\text{for all }\ g\in\mathcal{G},$$

one also has

$$\mu_{\tau\circ h_2}(O_i)>\sigma\circ\eta,\ \ i=1,2,...,m.$$

We see then Theorem 3.3 follows from Theorem 4.6. $\square$

Proof of Theorem 3.4. It is an immediate consequence of Theorem 3.3. $\square$

Definition 4.7. Let X be a compact metric space and let A be a stably finite C^*-algebra. Let $C = PM_k(C(X))P$. Suppose that $h : C \to A$ is a unital homomorphism and $\tau \in T(A)$. Define $\phi : C(X) \to C$ by $\phi(f) = f \cdot P$ for $f \in C(X)$. Define $\tilde{\tau} = \tau \circ \phi : C(X) \to \mathbb{C}$. We use $\tilde{\mu}_\tau$ for the probability measure induced by $\tilde{\tau}$. This notation will be used below and in the proof of Lemma 6.3. Note also that if X is connected and has finite dimension, then C is a full hereditary C^*-subalgebra of $M_k(C(X))$, consequently $K_i(C) = K_i(C(X))$ $(i = 0, 1)$.

Corollary 4.8. *Let X be a finite-dimensional compact metric space and let $C = PM_k(C(X))P$, where $P \in M_k(C(X))$ is a projection. Let $\varepsilon > 0$, $\mathcal{F} \subset C$ be a finite subset. Let $\eta > 0$ be such that $|f(x) - f(x')| < \varepsilon/8$. Then, for any integer $s > 0$, any finite $\eta/2$-dense subset $\{x_1, x_2, ..., x_m\} \subset X$ such that $O_i \cap O_j = \emptyset$ if $i \neq j$, where*

$$O_j = \{x \in X : \operatorname{dist}(x, x_i) < \eta/2s\}$$

and any $1/2s > \sigma > 0$, there exist $\gamma > 0$, $\delta > 0$, a finite subset $\mathcal{G} \subset C$ and a finite subset $\mathcal{P} \subset \mathbf{P}(C)$ satisfying the following:

For any unital simple C^*-algebra A with $TR(A) = 0$, any unital $\mathcal{G}$-δ-multiplicative contractive completely positive linear maps, $\phi, \psi : C \to A$ with

$$|\tau \circ \phi(a) - \tau \circ \psi(a)| < \gamma \;\; for \;\; all \;\; g \in \mathcal{G} \;\; and \;\; for \;\; all \;\; \tau \in T(A),$$

if

$$\tilde{\mu}_{\tau \circ \phi}(O_i), \tilde{\mu}_{\tau \circ \psi}(O_i) > \sigma \cdot \eta$$

for all i and $\tau \in T(A)$, and if

$$[\phi]|_{\mathcal{P}} = [\psi]|_{\mathcal{P}},$$

then there exists a unitary in A such that

$$\phi \overset{u}{\sim}_\varepsilon \psi \quad on \;\; \mathcal{F}.$$

(Here η does not depend on σ. But γ does depend on η and σ.)

Proof. First let us assume that $C = M_k(C(X))$. Let $\{e_{ij}\}$ be a system of matrix unitis. Suppose that $\{\phi_n\} : C \to A$ is a unital sequentially asymptotic morphism. There is a sequence of projections $p_n \in A$ such that

$$\lim_{n\to\infty} \|\phi_n(e_{11}) - p_n\| = 0.$$

One obtains a sequence of elements $a_n \in A$ such that $a_n \phi_n(e_{11}) a_n = p_n$ and

$$\lim_{n\to\infty} \|a_n - p_n\| = 0. \tag{e 4.28}$$

Let $\phi'_n(c) = a_n \phi_n(c) a_n$ for $c \in e_{11} C e_{11}$ ($\cong C(X)$). Then ϕ'_n is a completely positive linear map. Since $\phi'_n(1_C) = p_n$, it is a contractive completely positive linear map. Since $\sum_{i=1}^k e_{ii} = 1_C$, it is easy to check that, for all large n, there are elements $\{a_{ij}^{(n)}\} \subset A$ such that $\{a_{ij}^{(n)}\}$ forms a system of matrix unit (with size k) such that $a_{11}^{(n)} = e_n$ and $\sum_{i=1}^k e_{ii} = 1_A$. In other words, we may write $A = M_k(e_n A e_n)$.

Define $\phi''_n : C \to M_k(e_n A e_n)$ by $\phi''_n = \phi'_n \otimes \mathrm{id}_{M_k}$. By (e 4.28), we have

$$\lim_{n\to\infty} \|\phi_n - \phi'_n\| = 0.$$

It follows that

$$\lim_{n\to\infty} \|\phi_n - \phi''_n\| = 0.$$

Therefore we may assume that $\phi(e_{11})$ is a projection. Note that $\tau(\phi(f \cdot e_{11})) = (1/k)\tilde{\tau} \circ \phi(f \cdot P))$ for $f \in C(X)$. It is then clear, by identifying $e_{11}Ce_{11}$ with $C(X)$, that we reduce the case that $C = M_k(C(X))$ to the case that $C = C(X)$ which has been proved in Theorem 4.6.

Now we consider the general case. Suppose that $C = PM_k(C(X))P$ and $\dim X = d$. It follows from 8.12 of [21] (see also 6.10.3 of [1]) that there exists an integer $K \geq 1$ ($K \leq 2dk$) such that there is a projection $Q \in M_K(C)$ and partial isometry $z \in M_k(C)$ such that $z^*z = P$ and $zz^* \leq Q$, and $QM_K(C)Q \cong M_l(C(X))$ for some integer $Kk \geq l > 0$. Suppose that $\{\phi_n\}$ and $\{\psi_n\}$ are two unital sequentially asymptotic morphisms from C into A. Define $\Phi_n = \phi_n \otimes \mathrm{id}_{M_K}$ and $\Psi_n = \psi_n \otimes \mathrm{id}_{M_K}$. Then $\{\Phi_n\}$ and $\{\Psi_n\}$ are two unital sequentially asymptotic morphisms from $M_K(C)$ into $M_K(A)$. Use Φ_n and Ψ_n again for the restriction of Φ_n and Ψ_n on $QM_K(C)Q$. There exist projections $e_n, e_n' \in M_K(A)$ such that

$$\lim_{n\to\infty} \|\Phi_n(Q) - e_n\| = 0 \quad \text{and} \quad \lim_{n\to\infty} \|\Psi_n(Q) - e_n'\| = 0.$$

One obtains $a_n, a_n' \in M_K(A)$ such that

$$a_n\Phi_n(Q)a_n = e_n, a_n'\Psi_n(Q)a_n' = e_n \quad \text{and} \quad \lim_{n\to\infty} \|a_n - e_n\| = 0.$$

Thus, by replacing Φ_n by $a_n\Phi_n a_n$ and Ψ_n by $a_n'\Psi a_n'$, we may assume that Φ_n and Ψ_n map into e_nAe_n and $e_n'Ae_n'$, respectively. By the assumption, we may also assume that e_n and e_n' are unitarily equivalent. Without loss of generality, therefore, we may assume that $e_n = e_n'$. Note that we have proved the case that $C = M_k(C(X))$. Since C is a C^*-subalgebra of $QM_K(C)Q \cong M_l(C(X))$, one easily concludes that this corollary holds. □

5. Approximately conjugate

Proposition 5.1. *Let (X, α) and (X, β) be minimal systems such that $TR(A_\alpha) = TR(A_\beta) = 0$. Then (X, α) and (X, β) are C^*-strongly approximately flip conjugate if and only if there exists an isomorphism $\phi : A_\alpha \to A_\beta$, a sequence of unitaries $\{u_n\}$ in A_β and a sequence of isomorphisms $\chi_n : C(X) \to C(X)$ such that*

$$\lim_{n\to\infty} \|\mathrm{ad}\, u_n \circ \phi \circ j_\alpha(f) - j_\beta \circ \chi_n(f)\| = 0 \ \textit{for all} \ f \in C(X).$$

Proof. The "if part" is obvious. Suppose that (X, α) and (X, β) are C^*-strongly approximately flip conjugate. Suppose that $\phi_n : A_\alpha \to A_\beta$ are isomorphisms such that $[\phi_n] = [\phi_1]$ in $KL(A_\alpha, A_\beta)$ for all n and

$$\lim_{n\to\infty} \|\phi_n \circ j_\alpha(f) - j_\beta \circ \chi_n(f)\| = 0 \ \text{for all} \ f \in C(X).$$

Let $\{\mathcal{F}_n\}$ be an increasing sequence of finite subsets of A_α for which the union $\bigcup_n \mathcal{F}_n$ is dense in A. Since $[\phi_n] = [\phi_1]$ in $KL(A_\alpha, A_\beta)$, by Theorem 2.3 in [37], there is a unitary $u_n \in A_\beta$ such that

$$\mathrm{ad}\, u_n \circ \phi_1 \sim_{1/2^n} \phi_n \ \text{on} \ \mathcal{F}_n.$$

It follows immediately that

$$\lim_{n\to\infty} \|\mathrm{ad}\, u_n \circ \phi_1 \circ j_\alpha(f) - j_\beta \circ \chi_n(f)\| = 0 \ \text{for all} \ f \in C(X).$$

□

Proof of Theorem 3.8. To see the "only if " part, suppose that there exists a sequence of isomorphisms $\phi_n : A_\alpha \to A_\beta$ and there exists a sequence of isomorphisms $\chi_n : C(X) \to C(X)$ such that $[\phi_n] = [\phi_1]$ in $KL(A_\alpha, A_\beta)$ for all n and

$$\lim_{n\to\infty} \|\phi_n(j_\alpha(f)) - j_\beta \circ \chi_n(f)\| = 0 \text{ for all } f \in C(X). \tag{e 5.29}$$

Put $\theta = [\phi_n]$. It follows from [53] and (e 5.29) that

$$[j_\alpha] \times \theta = [j_\beta \circ \chi_n] \text{ for all } n.$$

Since $[\phi_n] = \theta$, it follows that

$$(\phi_n)_\rho = \theta_\rho.$$

Therefore

$$\lim_{n\to\infty} \|\rho_{A_\beta} \circ j_\beta \circ \chi_n - \theta_\rho \circ \rho_{A_\alpha} \circ j_\alpha(f)\| = 0$$

for all $f \in C(X)$. This proves the "only if " part.

To prove the "if" part, we apply Theorem 3.3. Since both A_α and A_β are simple amenable separable C^*-algebras which satisfy the UCT, if (X, α) and (X, β) satisfy the condition of the theorem, then, by [38], there is an isomorphism $h : A_\alpha \to A_\beta$ such that $[h] = \theta$. Moreover, for any finite subset $\mathcal{G} \subset C(X)$ and any $\gamma > 0$, there exists $N > 0$ such that, for all $n \geq N$,

$$|\tau \circ h \circ j_\alpha(f) - \tau \circ j_\beta \circ \chi_n(f)| < \gamma \text{ for all } f \in C(X)$$

and all $\tau \in T(A_\beta)$. Since

$$[h \circ j_\alpha] = [j_\beta \circ \chi_n] \text{ in } KL(C(X), A_\beta),$$

it follows from Theorem 3.3 that there are unitaries $u_n \in A_\beta$ such that

$$\lim_{n\to\infty} \|u_n^*(h \circ j_\alpha(f))u_n - j_\beta \circ \chi_n(f)\| = 0 \text{ for all } f \in C(X).$$

Let $h_n = \mathrm{ad}u_n \circ h$. We conclude that α and β are C^*-strongly approximately flip conjugate. $\square$

Definition 5.2 (3.1 of [45])**.** Let X be a compact metric space and let $\alpha, \beta : X \to X$ be minimal homeomorphisms. We say that α and β are *weakly approximately conjugate* if there exists two sequences of homeomorphisms $\sigma_n, \gamma_n : X \to X$ such that

$$\lim_{n\to\infty} (\sigma_n \circ \alpha \circ \sigma_n^{-1})(f) = \beta(f) \quad \text{and}$$
$$\lim_{n\to\infty} (\gamma_n \circ \beta \circ \gamma_n^{-1})(f) = \alpha(f) \text{ for all } f \in C(X).$$

It easy to see (as in 3.2 of [45]) that there exists a sequentially asymptotic morphism $\{\phi_n\} : A_\alpha \to A_\beta$ and a sequentially asymptotic morphism $\{\psi_n\} : A_\beta \to A_\alpha$ such that

$$\lim_{n\to\infty} \|\phi_n(u_\alpha) - u_\beta\| = 0, \quad \lim_{n\to\infty} \|\psi_n(u_\beta) - u_\alpha\| = 0,$$
$$\lim_{n\to\infty} \|\phi_n(j_\alpha(f)) - j_\beta(f \circ \gamma_n)\| = 0 \quad \text{and} \quad \lim_{n\to\infty} \|\psi_n(j_\beta(f)) - j_\alpha(f \circ \sigma_n)\| = 0$$

for all $f \in C(X)$.

It is proved in [45] that α and β are weakly approximate conjugate if α and β have the same period spectrum, i.e., $D(K_0(A_\alpha), 1) = D(K_0(A_\beta), 1)$, where

$$D(K_0(C), 1) = \{n \in \mathbf{N} : nx = [1_C] \text{ for some } x \in K_0(C)_+\}.$$

A much stronger version of it, called "approximately K-conjugacy" was introduced in [45] for the Cantor set.

Definition 5.3. Let X be a compact metric space and let $\alpha, \beta : X \to X$ be minimal homeomorphisms. We say α and β are *approximately K-conjugate*, if α and β are weakly approximately conjugate with the conjugate maps $\{\sigma_n\}$ and $\{\gamma_n\}$ such that the induced sequentially asymptotic morphisms $\{\phi_n\}$ and $\{\psi_n\}$ induce two elements in $KL(A_\alpha, A_\beta)$ and unit preserving order isomorphisms in $Hom(K_*(A_\alpha), K_*(A_\beta))$. Furthermore, we require that $[\{\phi_n\}] \times [\{\psi_n\}] = [\mathrm{id}_{A_\alpha}]$ and $[\{\psi_n\}] \times [\{\phi_n\}] = [\mathrm{id}_{A_\beta}]$.

We say α and β are *approximately flip K-conjugate*, if α and β, or α and β^{-1}, or α^{-1} and β are approximately K-conjugate.

It should be noted that if α and β are actually flip conjugate, then the conjugate map σ gives an isomorphism between A_α and A_β. In particular, σ induces an element in $KL(A_\alpha, A_\beta)$ which gives a unit preserving order isomorphism.

Theorem 5.4. *Let X be a compact metric space and let $\alpha, \beta : X \to X$ be minimal homeomorphisms. Suppose that $TR(A_\alpha) = TR(A_\beta) = 0$. Suppose that α and β are approximately flip K-conjugate. Then α and β are C^*-strongly approximately flip conjugate.*

Proof. It is clear that it suffices to show that α and β are approximately K-conjugate implies that they are C^*-strongly approximately flip conjugate.

Suppose that $\sigma_n, \gamma_n : X \to X$ are homeomorphisms such that

$$\lim_{n\to\infty} (\sigma_n \circ \alpha \circ \sigma_n^{-1})(f) = \beta(f) \quad \text{and} \quad \lim_{n\to\infty} (\gamma_n \circ \beta \circ \gamma_n^{-1})(f) = \alpha(f)$$

for all $f \in C(X)$. Let $\{\phi_n\} : A_\alpha \to A_\beta$ and $\{\psi_n\} : A_\beta \to A_\alpha$ be the sequentially asymptotic morphisms induced by $\{\sigma_n\}$ and $\{\gamma_n\}$. By the assumption, there is $z \in KL(A_\alpha, A_\beta)$ and $\zeta \in KL(A_\beta, A_\alpha)$ such that

$$[\phi_n]|_{\mathcal{P}} = z|_{\mathcal{P}}$$

for any finite subset $\mathcal{P} \subset \mathbf{P}(A_\alpha)$ and all sufficiently large n, and

$$[\psi_n]|_{\mathcal{Q}} = \zeta|_{\mathcal{Q}}$$

for any finite subset $\mathcal{Q} \subset \mathbf{P}(A_\beta)$ and all sufficiently large n. By the assumption, z gives a unit preserving order isomorphism from $K_i(A_\alpha)$ to $K_i(A_\beta)$ $(i = 0, 1)$. From

$$\lim_{n\to\infty} \|\phi_n \circ j_\alpha(f) - j_\beta \circ \gamma_n(f)\| = 0 \ \text{ for all } \ f \in C(X), \tag{e 5.30}$$

one concludes that

$$[j_\alpha] \times z = [j_\beta \circ \gamma_n] \ \text{ in } \ KL(C(X), A_\beta).$$

Moreover, by (e 5.30),

$$\lim_{n\to\infty} \|z_\rho \circ \rho_{A_\alpha} \circ j_\alpha(f) - \rho_{A_\beta} \circ j_\beta \circ \gamma_n(f)\| = 0$$

for all $f \in C(X)$. It follows from Theorem 3.8 that α and β are C^*-strongly approximately conjugate. $\square$

The converse is also true, at least for the case when X is the Cantor set. It is not known if the converse of Theorem 5.4 is true in general.

Proof of Theorem 3.11. Most of the proof was given in [45]. In particular, the equivalence of (iii), (iv) and (v) are given there. The equivalence of (iii) and (vi) was given in [16]. That (ii) implies (i) follows from Theorem 5.4. Moreover, it is obvious that (i) implies (iv). It follows from Lemma 3 of [56] that if $\sigma \in [[\beta]]$, then there is a unitary $u \in A_\beta$ such that $u^* j_\beta(f) u = j_\beta(f \circ \sigma))$ for all $f \in C(X)$. Then an easy computation of ordered K-theory of A_α and A_β shows that (v) implies (ii). □

6. The Rokhlin property

Lemma 6.1. *Let A be a unital simple separable C^*-algebra with stable rank one and real rank zero and let $\alpha \in Aut(A)$ such that $\alpha_{*0}|_G = \mathrm{id}_G$ for some subgroup $G \subset K_0(A)$ for which $\rho_A(G) = \rho_A(K_0(A))$. Then $\tau(a) = \tau \circ \alpha(a)$ for all $\tau \in T(A)$ and for all $a \in A$.*

Proof. First we claim that $\alpha_{*0}(\ker\rho_A) = \ker\rho_A$. Since α is an automorphism, it suffices to show that $\alpha_{*0}(\ker\rho_A) \subset \ker\rho_A$. Let $z \in \ker\rho_A$. Take $x \in K_0(A)_+ \setminus \{0\}$. Then $x \pm nz \geq 0$ for all positive integer n. It follows that $\alpha_{*0}(x \pm nz) \geq 0$. Thus $\rho_A(\alpha_{*0}(x \pm nz)) \geq 0$. It is then easy to see that $\rho_A(\alpha_{*0}(z)) = 0$.

For any projection $p \in A$, there is $x \in G$ such that $x - [p] \in \ker\rho_A$. It follows that $x \geq 0$. Since A has stable rank one, there is a projection $q \in A$ such that $[q] = x$. Thus $\tau(p) = \tau(q)$ for all $\tau \in T(A)$. Since $[q] \in G$, by the above assumptions, $[\alpha(q)] = [q]$. Again since A has stable rank one, there is a partial isometry $v \in A$ such that

$$v^* v = q \quad \text{and} \quad vv^* = \alpha(q).$$

By the first part of the proof, $\alpha_{*0}([p] - [q]) \in \ker\rho_A$. In particular, $\tau(\alpha(p)) = \tau(\alpha(q))$ for all $\tau \in T(A)$. It follows that

$$\tau(p) = \tau(q) = \tau(\alpha(q)) = \tau(\alpha(p)). \tag{e 6.31}$$

In other words, $[p] - [\alpha(p)] \in \ker\rho_A$.

Suppose that $a = \sum_{i=1}^n \lambda_i p_i$, where λ_i are scalars and $p_1, p_2, ..., p_n$ are mutually orthogonal projections. Then $\alpha(a) = \sum_{i=1}^n \lambda_i \alpha(p_i)$. It follows from (e 6.31) that $\tau(a) = \tau \circ \alpha(a)$ for all $\tau \in T(A)$. Since A has real rank zero, it follows from [6] that every self-adjoint element is a norm-limit of self-adjoint elements with finite spectrum. It follows that $\tau(a) = \tau \circ \alpha(a)$ for all self-adjoint elements. The lemma then follows. □

Lemma 6.2. *Let A be a unital simple separable C^*-algebra with $TR(A) = 0$ and let $\alpha \in Aut(A)$ such that $\alpha_{*0}|_G = \mathrm{id}_G$ for some subgroup G of $K_0(A)$ for which $\rho_A(G) = \rho_A(K_0(A))$. Suppose that $\{p_j\}$ is a central sequence of projections such that $[p_j] \in G$ and define $\phi_j(a) = p_j a p_j$ and $\psi_n(a) = \alpha(p_j) a \alpha(p_j)$, $j = 1, 2, ...$ Then $\{\phi_n\}$ and $\{\psi_n\}$ are two sequentially asymptotic morphisms. Suppose also that there are finite-dimensional C^*-subalgebras B_j and $C_j = \alpha(B_j)$ with $1_{B_j} = p_j$ and $1_{C_j} = \alpha(p_j)$ such that $[p_{j,i}] \in G$ for each minimal central projection $p_{j,i}$ of B_j $(1 \leq i \leq k(j))$ and there are sequentially asymptotic morphisms $\{\phi'_j\}$ and $\{\psi'_j\}$ such that*

$$\phi'_j(a) \subset B_j, \quad \psi'_j(a) \subset C_j,$$
$$\lim_{j\to\infty} \|\phi_j(a) - \phi'_j(a)\| = 0 \;\; \text{and} \;\; \lim_{j\to\infty} \|\psi_j(a) - \psi'_j(a)\| = 0$$

for all $a \in A$.

Then, for any $\varepsilon > 0$ and for any finite subset $\mathcal{G} \subset A$ and a finite subset of projections $\mathcal{P}_0 \subset M_k(A)$ for which $[p] \in G$ for all $p \in \mathcal{P}_0$, there exists an integer $J > 0$ such that

$$|\tau \circ \phi_j(a) - \tau \circ \psi_j(a)| < \varepsilon/\tau(p_j) \ \text{ for all } \ a \in \mathcal{G}$$

and for all $\tau \in T(A)$, and, for all $j > J$,

$$[\phi_j(p)] = [\psi_j(p)] \ \text{ in } \ K_0(A).$$

Proof. Let $q \in M_k(A)$ be a projection such that $[q] \in G$. Put $\beta = \alpha \otimes \mathrm{id}_{M_k}$. By replacing A by $M_k(A)$, α by β, ϕ_j by $\phi_j \otimes \mathrm{id}_{M_k}$ and ψ_j by $\psi \otimes \mathrm{id}_{M_k}$, respectively, to simplify notation, we may assume that $q \in A$. By the assumption, there is a partial isometry $v \in A$ such that $vv^* = q$ and $vv^* = \alpha^{-1}(q)$. Define $B_{j,i} = p_{j,i}B_j$, $i = 1, 2, ..., k(j)$. Keep in mind that $B_{j,i}$ is a simple finite-dimensional C^*-subalgebra. Define

$$\phi_{j,i} = p_{j,i}\phi_j p_{j,i}, \ \ \phi'_{j,i} = p_{j,i}\phi'_j, \ \ \psi_{j,i} = \alpha(p_{j,i})\psi_j\alpha(p_{j,i}) \quad \text{and} \quad \psi'_{j,i} = \alpha(p_{j,i})\psi'_j,$$

$i = 1, 2, ..., k(j)$.

By replacing ψ_j by $\mathrm{ad}\, w_j \circ \psi_j$ for some suitable unitaries w_j, if necessary, we may assume that B_j is orthogonal to C_j. There is a partial isometry $z_{j,i} \in A$ such that

$$z_{j,i}^* z_{j,i} = p_{j,i} \quad \text{and} \quad z_{j,i}z_{j,i}^* = \alpha(p_{j,i}).$$

Then $z_{j,i}, B_{j,i}, C_{j,i}$ generate a C^*-subalgebra $D_{j,i} \cong M_2(B_{j,i})$ which is a simple finite-dimensional C^*-algebra.

There are projections $e_{j,i} \in B_{j,i}$ and $e'_{j,i} \in C_{j,i}$ such that

$$\lim_{j\to\infty} \|e_{j,i} - \phi_{j,i}(q)\| = 0 \quad \text{and} \quad \lim_{j\to\infty} \|e'_{j,i} - \psi_{j,i}(q)\| = 0.$$

Moreover,

$$\lim_{j\to\infty} \|p_{j,i}v^*p_{j,i}vp_{j,i} - e_{j,i}\| = 0 \quad \text{and} \quad \lim_{j\to\infty} \|p_{j,i}vp_{j,i}v^*p_{j,i} - \psi_{j,i}(\alpha^{-1}(q))\| = 0.$$

Thus

$$[e_{j,i}] = [\psi_{j,i}(\alpha^{-1}(q))]$$

for all large j. On the other hand,

$$\alpha(p_{j,i})q\alpha(p_{j,i}) = \alpha(p_{j,i}\alpha^{-1}(q)p_{j,i}).$$

Therefore

$$[e'_{j,i}] = [\alpha(e_{j,i})]$$

for all large j. But $[\alpha(e_{j,i})] - [e_{j,i}] \in \ker\rho_A$. So, for any $\tau \in T(A)$,

$$\tau(e_{j,i}) = \tau(\alpha(e_{j,i})) \ \text{ for all } \ \text{ large } j.$$

However, for each trace $\tau \in T(A)$, there is $\lambda_{j,i} > 0$ such that $\tau|_{D_{j,i}} = \lambda_{j,i}Tr$, where Tr is the standard trace on $D_{j,i}$. Hence

$$Tr(e_{j,i}) = Tr(e'_{j,i}) \ \text{ for all } \ \text{ large } j.$$

It follows that

$$[e_{j,i}] = [e'_{j,i}] \ \text{ in } K_0(D_{j,i}) \ \text{ for all } \ \text{ large } j.$$

Thus

$$[\phi_j(q)] = [\psi_j(q)] \ \text{ in } K_0(A) \ \text{ for all } \ \text{ large } j. \tag{e 6.32}$$

Since $[p_j] \in G$, there is a unitary Z_j such that

$$Z_j^* \alpha(p_j) Z_j = p_j.$$

Suppose that $a = \sum_{k=1}^m \lambda_k e_k$, where λ_k are scalars and $e_1, e_2, ..., e_m$ are mutually orthogonal projections. Then, since A has stable rank one (since $TR(A) = 0$), by (e 6.32), there is a sequence of unitaries $\{U_j\}$ in $p_j A p_j$ such that

$$\lim_{j\to\infty} \|U_j^* \phi_j(a) U_j - \mathrm{ad}\, Z_j \circ \psi_j(a)\| = 0.$$

It follows that there is an integer $J_0 > 0$ such that

$$|\tau(\phi_j(a)) - \tau(\psi_j(a))| < \varepsilon$$

for all $\tau \in T(p_j A p_j)$ and for all $j \geq J$. Since $TR(A) = 0$, the set of self-adjoint elements with finite spectrum is dense in $A_{s.a.}$. The lemma follows. $\square$

Lemma 6.3. *Let A be a unital separable simple amenable C^*-algebra with $TR(A) = 0$ satisfying the UCT, and let $\alpha \in Aut(A)$ be such that $\alpha_{*0}|_G = \mathrm{id}_G$ for some subgroup G of $K_0(A)$ for which $\rho_A(G) = \rho_A(K_0(A))$. Suppose also that $\{p_j(l)\}$, $l = 0, 1, 2, ..., L$, are central sequences of projections in A such that*

$$p_j(l)p_j(l') = 0 \ \text{ if } l \neq l' \quad \text{and} \quad \lim_{j\to\infty} \|p_j(l) - \alpha^l(p_j(0))\| = 0, \ \ 1 \leq l \leq L.$$

Then there exist central sequences of projections $\{q_j(l)\}$ and central sequences of partial isometries $\{u_j(l)\}$ such that $q_j(l) \leq p_j(l)$,

$$u_j(l)^* u_j(l) = q_j(l), \ \ u_j(l) u_j^*(l) = \alpha^l(q_j(0)),$$

for all large j,

$$\lim_{j\to\infty} \|\alpha^l(q_j(0)) - q_j(l)\| = 0$$

and

$$\lim_{j\to\infty} \tau(p_j(l) - q_j(l)) = 0$$

uniformly on $T(A)$.

Proof. It follows from Theorem 5.2 of [39] and Theorem 4.18 of [13] that we may assume that $A = \overline{\bigcup_n A_n}$, where each A_n has the form $P_n M_{k(n)}(C(X))P_n$, where each X is a finite-dimensional compact metric space and $P_n \in M_{k(n)}(C(X))$ is a projection. We may further assume that ϕ_n are injective. Fix a finite subset $\mathcal{F} \subset A$ and $\varepsilon > 0$. Let $\mathcal{F}_1 = \bigcup_{l=0}^L \alpha^{-l}(\mathcal{F})$. Without loss of generality, we may assume that $\mathcal{F}_1 \subset C$, where $C = P_n M_{k(n)}(C(X))P_n$.

Note that $\{\alpha(p_j)\}$ is also a central sequence. Since $\lim_{j\to\infty} \|p_j(l) - \alpha^l(p_j(0))\| = 0$, there are unitaries $w_j \in A$ such that

$$\lim_{j\to\infty} \|w_j - 1\| = 0 \quad \text{and} \quad w_j^* \alpha^l(p_j(0)) w_j = p_j(l). \tag{e 6.33}$$

Let $\beta_j = \mathrm{ad} w_j \circ \alpha$.

Let $\mathcal{P} \subset \mathbf{P}(C)$ be a finite subset. Choose an integer $k_0 > 0$ such that

$$\mathcal{P} \cap K_i(C, \mathbb{Z}/j\mathbb{Z}) = \{0\}, \ \ j \geq k_0. \tag{e 6.34}$$

Fix $\eta > 0$, and let $\{x_1, x_2, ..., x_K\}$ be an $\eta/2$-net in X. Suppose that $s > 0$ is an integer so that $O_i \cap O_j = \emptyset$, if $i \neq j$, $i, j, = 1, 2, ..., K$, where

$$O_i = \{x \in X : \mathrm{dist}(x, x_i) < \eta/2s\}, \ \ i = 1, 2, ..., K.$$

Let f_i be a non-negative continuous function in $C(X)$ such that $0 \le f_i \le 1$, $f_i(y) = 1$ if $\mathrm{dist}(y, x_i) \le \varepsilon/4$ and $f_i(y) = 0$ if $\mathrm{dist}(y, x_i) \ge \eta/2$. Let $a_i = f_i P_n \in C$. Since A is simple, there are $b_{i,k} \in A$ such that

$$\sum_{k=1}^{m(i)} b_{ik}^* a_i b_{ik} = 1_A.$$

Let $c = \max\{b_{ik}^* b_{ik} : 1 \le k \le K\}$ and $M_0 = \max\{m(k) : k = 1, 2, ..., K\}$. Put $\sigma = 1/4cM_0\eta$.

Since $\{p_j(l)\}$ is central,

$$\lim_{j\to\infty} \|\sum_{k=1}^{m(k)} p_j(l) b_{ik}^* p_j(l) a_i p_j(l) b_{ik} p_j(l) - p_j(l)\| = 0, \tag{e 6.35}$$

$i = 1, 2, ..., K$.

Since $TR(A) = 0$, there is a central sequence of projections $\{e_n\}$ and a sequence of finite-dimensional C^*-subalgebra $B_n \subset A$ with $1_{B_n} = e_n$, such that

(i) $e_n x e_n \subset_{\varepsilon_n} B_n$ for all $x \in A$, where $\varepsilon_n > 0$ and $\sum_{n=1}^\infty \varepsilon_n < \infty$, and

(ii) $\lim_{n\to\infty} \tau(1 - e_n) = 0$ uniformly on $T(A)$.

There is a subsequence $\{n(j)\}$ and projections $Q_j(l) \le p_j(l)$ $(l = 0, 1, ..., L)$ such that

$$\lim_{j\to\infty} \|Q_j(l) - p_j(l) e_{j(n)}\| = 0.$$

Note also that, by (i),

$$\lim_{j\to\infty} \mathrm{dist}(p_j e_{n(j)}, B_{n(j)}) = 0.$$

There are also sequences of projections $E_{n(j),l} \in B_{n(j)}$ such that $E_{n(j),l} E_{n(j),l'} = 0$, if $l \ne l'$,

$$\lim_{j\to\infty} \|E_{n(j),l} - Q_j(l)\| = 0.$$

Let z_n be a sequence of unitaries such that

$$\lim_{n\to\infty} \|z_j - 1\| = 0,\ z_j^* E_{n(j),l} z_j = Q_j(l) \quad \text{and} \quad z_j^* E'_{n(j),l} z_j = Q_j(l), 0 \le l \le L.$$

Set $D_{n(j)} = Q_j(0) z_n^* B_{n(j)} z_n Q_j(0)$, $n = 1, 2, ...$ Note $D_{n(j)}$ is of finite dimension. Write $D_{n(j)} = \bigoplus_t^{M(j,n)} D_{n(j),t}$, where each $D_{n(j),t}$ is simple and has rank $R(n(j), t)$. Let $d_{n(j),t}$ be a minimal projection in $D_{n(j),t}$. Since $TR(A) = 0$, by Theorem 7.1 of [36], A has real rank zero, stable rank one and weakly unperforated $K_0(A)$. It follows from [2] that $\rho_A(K_0(A))$ is dense in $Aff(T(A))$. Consequently, one has $(k_0)!$ many mutually orthogonal and mutually equivalent projections $d_{n(j),t,s} \le d_{n(j),t}$ $(s = 1, 2, ..., (k_0)!)$ such that $[d_{n(j),t,s}] \in G$ and

$$\tau(d_{n(j),t} - \sum_{s=1}^{(k_0)!} d_{n(j),t,s}) < \frac{\tau(d_{n(j),t})}{2^j M(j,n)} \ \text{for all} \ \tau \in T(A). \tag{e 6.36}$$

Put

$$g_{n(j),s,t} = \mathrm{diag}(\overbrace{d_{n(j),t,s}, d_{n(j),t,s}, ..., d_{n(j),t,s}}^{R(n(j),t)}),$$

$$g_{n(j),s} = \bigoplus_{t=1}^{M(j,n)} g_{n(j),s,t} \quad \text{and} \quad g_{n(j)} = \sum_{s=1}^{(k_0)!} g_{n(j),s}.$$

Note that $g_{n(j),s}$ commutes with every element in $D_{n(j)}$. Put $C_{n(j),s} = g_{n(j),s}D_{n(j)}$ and $C_{n(j)} = g_{n(j)}D_{n(j)}$. Then $C_{n(j),s}$ and $C_{n(j)}$ are finite-dimensional C^*-subalgebras, and the image of each minimal central projection of $C_{n(j),s}$ and $C_{n(j)}$ in $K_0(A)$ are in G.

Combining (i) above, we note that $\{g_{n(j),s}\}$ and $\{g_{n(j)}\}$ are central (as $j \to \infty$) and

(1) $\mathrm{dist}(g_{n(j),s}xg_{n(j),s}, C_{n(j),s}) \to 0$ and $\mathrm{dist}(g_{n(j)}xg_{n(j)}, C_{n(j)}) \to 0$ as $j \to \infty$,

(2) $\tau(Q_j - g_{n(j)}) \to 0$ as $j \to \infty$ (by (e 6.36)).

For each $x \in \mathcal{F}$ and $l = 0, 1, ..., L$, there are $y_j(l) \in C_{n(j)}$ such that

$$\|g_{n(j)}\alpha^{-l}(x)g_{n(j)} - y_j(l)\| \to \text{ as } j \to \infty.$$

Thus

$$\|\beta_j^l(g_{n(j)})x\beta_j^l(g_{n(j)}) - \beta_j^l(y_j(l))\| \to 0 \text{ as } j \to \infty.$$

Therefore

$$\mathrm{dist}(\beta_j^l(d_{n(j)})x\beta_j^l(d_{n(j)}), \beta_j^l(C_{n(j)})) \to 0, \;\; l = 1, 2, ..., L,$$

as $j \to \infty$. Define

$$\Phi'_{j,s}(x) = g_{n(j),s}xg_{n(j),s}, \;\; \Phi'_j(x) = g_{n(j)}xg_{n(j)},$$
$$\Psi'_{j,s,l}(x) = \beta_j^l(g_{n(j),s})x\beta_j^l(g_{n(j),s}) \quad \text{and} \quad \Psi'_{j,l}(x) = \beta_j^l(g_{n(j)})x\beta_j^l(g_{n(j)})$$

for $x \in A$. Let

$$L_{j,s,0} : g_{n(j),s}Ag_{n(j),s} \to C_{n(j),s}, \;\; L_j : g_{n(j)}Ag_{n(j)} \to C_{n(j)},$$
$$L_{j,s,l} : \beta_j^l(g_{n(j),s})A\beta_j^l(g_{n(j),s}) \to \beta_j^l(C_{n(j),s}) \quad \text{and}$$
$$L_{j,l} : \beta_j^l(g_{n(j)})A\beta_j^l(g_{n(j)}) \to \beta_j^l(C_{n(j)})$$

be contractive completely positive linear maps which are extensions of $\mathrm{id}_{C_{n(j),s}}$, $\mathrm{id}_{C_{n(j)}}$, $\mathrm{id}_{\beta_j^l(C_{n(j),s})}$ and $\mathrm{id}_{\beta_j^l(C_{n(j)})}$, $l = 1, 2, ..., L$, respectively. Put

$$\Phi_{j,s} = L_{j,s,0} \circ \Phi_{j,s}, \;\; \Phi_j = L_j \circ \Phi_j, \;\; \Psi_{j,s,l} = L_{j,s,l} \circ \Psi_{j,s,l} \quad \text{and} \quad \Psi_{j,l} = L_{j,l} \circ \Psi'_{j,l}.$$

Note that $\{\Phi_{j,s}\}$, $\{\Phi_j\}$, $\{\Psi_{j,s,l}\}$ and $\{\Psi_{j,l}\}$ are sequentially asymptotic morphisms $(1 \le s \le (k_0)!)$. We also have

$$\Phi_j = \bigoplus_s^{(k_0)!} \Phi_{j,s} \quad \text{and} \quad \Psi_{j,l} = \bigoplus_s^{(k_0)!} \Psi_{j,s,l}, \;\; l = 1, 2, ..., L. \tag{e 6.37}$$

Let $H_j = \imath_j \circ \Phi_j \circ \imath$, where $\imath : C \to A$ and $\imath_j : C_{n(j)} \to g_{n(j)}Ag_{n(j)}$ are embeddings (we may also omit $\imath$ and $\imath_j$ when there will be no confusion). There is a unitary $Z_{j,l} \in A$ such that $Z_{j,l}^* g_{n(j)} Z_{j,l} = \beta_j^l(g_{n(j)})$, $l = 1, 2, ..., L$. Define $H_{j,l} = \mathrm{ad}\, Z_{j,l} \circ \imath_{j,l} \circ \Psi_{j,l} \circ \imath$, where $\imath_{j,l} : \beta_j^l(C_{j(n)} \to \beta_j^l(g_{n(j)})A\beta_j^l(g_{n(j)})$ is an embedding.

Therefore, for all sufficiently large j, by (e 6.37),

$$[H_j]|_{\mathcal{P} \cap K_i(C(X), \mathbb{Z}/j\mathbb{Z})} = 0 \quad \text{and} \quad [H'_j]|_{\mathcal{P} \cap K_i(C(X), \mathbb{Z}/j\mathbb{Z})} = 0 \tag{e 6.38}$$

for $i = 0, 1, 0 < j \le k_0$. Since both H_j and $H_{j,l}$ factor through a finite-dimensional C^*-subalgebra one has

$$[H_j]|_{\mathcal{P} \cap K_1(C(X)} = 0 \quad \text{and} \quad [H_{j,l}]|_{\mathcal{P} \cap K_1(C(X))} = 0. \tag{e 6.39}$$

By applying Lemma 6.2, one computes that, for all sufficiently large j,

$$[H_j]|_{\mathcal{P} \cap K_0(C(X))} = [H_{j,l}]|_{\mathcal{P} \cap K_0(C(X))}. \tag{e 6.40}$$

Combining (e 6.40), (e 6.37), (e 6.39) and (e 6.34), one has, if j is sufficiently large,

$$[H_j]|_{\mathcal{P}} = [H_{j,l}]|_{\mathcal{P}}. \tag{e 6.41}$$

It follows from 6.2 that

$$\lim_{j\to\infty}(\sup\{|\tau\circ\Phi_j(a) - \tau\circ \mathrm{ad}\, Z_j\circ\Psi_{j,l}(a)| : \tau\in T(g_{n(j)}Ag_{n(j)})\}) = 0 \tag{e 6.42}$$

for all $a\in A$. Hence

$$\lim_{j\to\infty}(\sup\{|\tau\circ H_j(x) - \tau\circ H_{j,l}(x)| : \tau\in T(g_{n(j)}Ag_{n(j)})\}) = 0 \text{ for all } x\in C.$$

It follows from (e 6.35) that

$$\lim_{j\to\infty}\|\sum_{k=1}^{m(i)} g_{n(j)}b_{ik}^* g_{n(j)}a_i g_{n(j)}b_{ik}^* g_{n(j)} - g_{n(j)}\| = 0 \quad \text{and} \tag{e 6.43}$$

$$\lim_{j\to\infty}\|\sum_{k=1}^{m(i)} \beta_j^l(g_{n(j)})b_{ik}^*\beta_j^l(g_{n(j)})a_i\beta_j^l(g_{n(j)})b_{ik}\beta_j^l(g_{n(j)}) - \beta_j^l(g_{n(j)})\| = 0. \tag{e 6.44}$$

It follows from (e 6.44), when j is sufficiently large, that

$$\mu_\tau(O_k)\ge \sigma\eta/2 \text{ for all } \tau\in T(g_{n(j)}Ag_{n(j)}),\ \ k=1,2,...,K. \tag{e 6.45}$$

Therefore, by 4.8, there is unitary $v_{j,l}\in g_{n(j)}Ag_{n(j)}$ such that

$$\lim_{j\to\infty}\|\mathrm{ad} v_{j,l}\circ H_{j,l}(x) - H_j(x)\| = 0 \text{ for all } x\in\mathcal{F}. \tag{e 6.46}$$

Define $z_{j,l} = g_{n(j)}v_{j,l}^*Z_{j,l}^*\beta_j^l(g_{n(j)})$. Then

$$z_{j,l}(H_j(x)+\Psi_{j,l}(x)) = z_{j,l}\Psi_{j,l}(x) \quad \text{and} \quad (H_j(x)+\Psi_{j,l}(x))z_j = H_j(x)z_j. \tag{e 6.47}$$

Thus, by (e 6.46),

$$\begin{aligned}
&\lim_{j\to\infty}\|z_{j,l}\Psi_{j,l}(x) - H_j(x)z_{j,l}\|\\
&\quad = \lim_{j\to\infty}\|z_{j,l}\Psi_{j,l}(x) - H_j(x)v_{j,l}^*Z_{j,l}^*\beta_j^l(g_{n(j)})\|\\
&\quad = \lim_{j\to\infty}\|z_{j,l}\Psi_{j,l}(x) - v_{j,l}^*H_{j,l}(x)Z_{j,l}^*\beta_j^l(g_{n(j)})\|\\
&\quad = \lim_{j\to\infty}\|z_{j,l}\Psi_{j,l}(x) - v_{j,l}^*Z_{j,l}^*(Z_{j,l}H_{j,l}(x)Z_{j,l}^*\beta_j^l(g_{n(j)}))\|\\
&\quad = \lim_{j\to\infty}\|z_{j,l}\Psi_{j,l}(x) - v_{j,l}^*Z_{j,l}^*\Psi_{j,l}(x)\| = 0
\end{aligned}$$

for all $x\in\mathcal{F}$. In other words,

$$\lim_{j\to\infty}\|z_{j,l}\Psi_{j,l}(x) - H_j(x)z_{j,l}\| = 0 \text{ for all } x\in\mathcal{F}. \tag{e 6.48}$$

Note that, for all $x\in A$,

$$z_{j,l}x = z_{j,l}\beta_j^l(g_{n(j)})x \quad \text{and} \quad xz_{j,l} = xg_{n(j)}z_{j,l}.$$

Therefore, since $\{g_{n(j)}\}$ and $\{\beta_j^l(g_{n(j)})\}$ are central, we have, for $x\in\mathcal{F}$,

$$\begin{aligned}
\lim_{j\to\infty}\|z_{j,l}x - xz_{j,l}\| &= \lim_{j\to\infty}\|z_{j,l}\beta_j^l(g_{n(j)})x - xg_{n(j)}z_{j,l}\|\\
&= \lim_{j\to\infty}\|z_{j,l}\beta_j^l(g_{n(j)})x\beta_j^l(g_{n(j)}) - g_{n(j)}xg_{n(j)}z_{j,l}\|\\
&= \lim_{j\to\infty}\|z_{j,l}\Psi_{j,l}(x) - H_j(x)z_{j,l})\| = 0.
\end{aligned}$$

On the other hand, we have

$$z_{j,l}^*z_{j,l} = g_{n(j)} \quad \text{and} \quad z_{j,l}z_{j,l}^* = \beta_j^l(g_{n(j)}). \tag{e 6.49}$$

Recall that $\beta_j^l(g_{n(j)}) = w_j^*\alpha^l(g_{n(j)})w_j$. Define $u_j(l) = w_j^* z_{j,l}$. Then

$$(u_j(l))^* u_j(l) = z_{j,l}^* w_j w_j^* z_{j,l} = g_{n(j)} \quad \text{and} \quad u_j(l)(u_j(l))^* = \alpha^l(g_{n(j)}). \tag{e 6.50}$$

Denote $q_j(0) = g_{n(j)}$ and $q_j(l) = \beta_j^l(g_{n(j)})$, $l = 1, 2, ..., L$. We also have, by (i) and (2) above,

$$q_j(l) \le Q_j(l) \le p_j(l) \quad \text{and} \quad \tau(p_j(l) - q_j(l)) \to 0 \quad \text{as } j \to \infty \text{ uniformly on } T(A). \tag{e 6.51}$$

Finally, since (e 6.33),

$$\lim_{j\to\infty} \|u_j x - x u_j\| = 0 \text{ for all } x \in \mathcal{F}. \tag{e 6.52}$$

□

The following lemma is taken from [47] that has its origin in [24].

Lemma 6.4. *Let A be a unital separable simple C^*-algebra with $TR(A) = 0$ and with unique tracial state and let $\alpha \in \mathrm{Aut}(A)$ such that $\alpha_{*0}^r|G = \mathrm{id}_G$ for some subgroup $G \subset K_0(A)$ for which $\rho_A(G) = \rho_A(K_0(A))$ and for some integer $r \ge 1$. Let $m \in \mathbb{N}$, $m_0 \ge m$, be the smallest integer such that $m_0 = 0 \bmod r$ and $l = m + (r-1)(m_0+1)$.*

Suppose that $\{e_i^{(n)}\}$, $i = 0, 1, ..., l$, $n = 1, 2, ...$, are $l+1$ sequences of projections in A satisfying the following:

$$\|\alpha(e_i^{(n)}) - e_{i+1}^{(n)}\| < \delta_n, \lim_{n\to\infty} \delta_n = 0,$$
$$e_i^{(n)} e_j^{(n)} = 0, \text{ if } i \ne j,$$

and for each i, $\{e_i^{(n)}\}$ is a central sequence. Then for each $i = 0, 1, 2, \dots, m$, there are central sequences of projections $\{p_i^{(n)}\}$ $(i = 0, 1, 2, ..., m)$ and a central sequence of partial isometries $\{w_i^{(n)}\}$ such that

$$(w_i^{(n)})^* w_i^{(n)} = p_i^{(n)} \quad \text{and} \quad w_i^{(n)}(w_i^{(n)})^* = p_{i+1}^{(n)}, \ i = 0, 1, ..., m-1,$$

where $p_i^{(n)} \le \sum_{j=0}^{r-1} e_{i+j(m_0+1)}^{(n)}$ and

$$\tau(\sum_{i=0}^{l} e_i^{(n)} - \sum_{i=0}^{m} p_i^{(n)}) \to 0$$

as $n \to \infty$ uniformly on $T(A)$. Moreover, for each i,

$$\lim_{n\to\infty} \|\alpha(p_i^{(n)}) - p_{i+1}^{(n)}\| = 0.$$

Proof. Since $\alpha_{*0}^r|_G = \mathrm{id}_G$, it follows from Lemma 6.2 that there is a central sequence of projections $\{q_0^{(n)}\}$ and a central sequence of partial isometries $\{z(0, j, n)\}$ and $q_0^{(n)} \le e_0^{(n)}$,

$$z(0, j, n)^* = q_0^{(n)}, \ z(0, j, n)z(0, j, n)^* = \alpha^{rj}(q_0^{(n)})$$

and

$$\tau(e_0^{(n)} - q_0^{(n)}) \to 0.$$

Moreover, there are central sequences of projections $\{q_{rj}^{(n)}\}$ such that $q_{rj}^{(n)} \le e_{rj}^{(n)}$ and

$$\lim_{n\to\infty} \|q_{rj}^{(n)} - \alpha^{rj}(q_0^{(n)})\| = 0, \;\; j = 1, 2, ..., [l/r].$$

Since $q_0^{(n)} \le e_0^{(n)}$ and $\|\alpha(e_0^{(n)}) - e_1^{(n)}\| < \delta_n$, there exists a projection $q_1^{(n)} \le e_1^{(n)}$ such that

$$\|\alpha(q_0^{(n)}) - q_1^{(n)}\| < 2\delta_n.$$

Similarly, we obtain projections $q_i^{(n)} \le e_i^{(n)}$ such that

$$\lim_{n\to\infty} \|\alpha(q_{i-1}^{(n)}) - q_i^{(n)}\| = 0, \;\; i = 2, 3, ..., r-1.$$

Moreover, it is easy to check that

$$\lim_{n\to\infty} \|\alpha^i(q_0^{(n)}) - q_i^{(n)}\| = 0, i = 1, 2, ..., r-1.$$

Since $\{q_0^{(n)}\}$ is central, $\{\alpha^i(q_0^{(n)})\}$ is also central for each i. It follows that $\{q_i^{(n)}\}$ are central for $i = 1, 2, ..., r-1$ and for $i = rj$, $j = 1, 2, ..., [l/r]$. Again, there are projections $q_{i+rj}^{(n)} \le e_{i+rj}^{(n)}$ such that

$$\lim_{n\to\infty} \|\alpha^{i+rj}(q_0^{(n)}) - q_{i+rj}^{(n)}\| = 0, \; i = 1, 2, ..., r-1.$$

Define $z'_{i,j,n} = \alpha^i(z'_{0,j,n})$. Then

$$\begin{aligned} (z'_{i,j,n})^* z'_{i,j,n} &= \alpha^i(z^*_{0,j,n} z_{0,j,n}) = \alpha^i(q_0^{(n)}) \quad \text{and} \\ z'_{i,j,n}(z'_{i,j,n})^* &= \alpha^i(z_{0,j,n} z^*_{0,j,n}) = \alpha^{i+rj}(q_0^{(n)}). \end{aligned}$$

In particular, $\{z_{i,j,n}\}$ is central, since $\{z_{0,j,n}\}$ is central. Because

$$\lim_{n\to\infty} \|\alpha^i(q_0^{(n)}) - q_i^{(n)}\| = 0 \quad \text{and} \quad \lim_{n\to\infty} \|\alpha^{i+rj}(q_0^{(n)}) - \alpha^{rj}(q_i^{(n)})\| = 0$$

for $i = 1, 2, ..., r-1$, one obtains sequences of unitaries $\{Z_j^{(n)}\}$ and $\{U_j^{(n)}\}$ such that

$$\begin{aligned} &\lim_{n\to\infty} \|Z_j^{(n)} - 1\| = 0, \;\; (Z_j^n)^* \alpha^i(q_0^{(n)}) Z_j^{(n)} = q_j^{(n)}, \quad \text{and} \\ &\lim_{n\to\infty} \|U_j^{(n)} - 1\| = 0, \;\; U_j^{(n)} \alpha^{i+rj}(q_0^{(n)}) (U_j^{(n)})^* = \alpha^{rj}(q_i^{(n)}), \end{aligned}$$

$j = 1, 2, ..., r-1$. Define $z_{i,j,n} = U_j^{(n)} z'_{i+1,j,n} Z_j^{(n)}$. Then $\{z_{i,j,n}\}$ is central and

$$z^*_{i,j,n} z_{i,j,n} = q_i^{(n)} \quad \text{and} \quad z_{i,j,n} z^*_{i,j,n} = \alpha^{rj}(q_i^{(n)}).$$

Now put

$$p_i^{(n)} = \sum_{j=0}^{r-1} q_{i+j(m_0+1)}^{(n)}, i = 0, 1, 2, ..., m.$$

Note that

$$\tau(\sum_{i=0}^{l} e_i^{(n)} - \sum_{i=0}^{m} p_i^{(n)}) < \sum_{i=0}^{l} \tau(e_i^{(n)} - q_i^{(n)}) = l \cdot \tau(e_0^{(n)} - q_0^{(n)}) \to 0.$$

To prove that the so defined $p_i^{(n)}$ meets the requirements, one checks exactly the same way as in the proof of Lemma 3.2 of [47]. $\square$

Let $\{E_{i,j}\}$ be a system of matrix units and let $\mathcal{K}$ be the compact operators on $\ell^2(\mathbb{Z})$ where we identify $E_{i,i}$ with the one-dimensional projection onto the functions supported by $\{i\} \subset \mathbb{Z}$. Let S be the canonical shift operator on $\ell^2(\mathbb{Z})$. Define an automorphism σ of $\mathcal{K}$ by $\sigma(x) = SxS^*$ for all $x \in \mathcal{K}$. Then $\sigma(E_{i,j}) = E_{i+1,j+1}$. For any $N \in \mathbb{N}$ let $P_N = \sum_{i=0}^{N-1} E_{i,i}$.

To prove Theorem 3.14, we quote the following lemma.

Lemma 6.5 (Kishimoto, 2.1 of [24])**.** *For any $\eta > 0$ and $n \in \mathbb{N}$ there exist $N \in \mathbb{N}$ and projections $e_0, e_1, \ldots, e_{n-1}$ in $\mathcal{K}$ such that*

$$\sum_{i=0}^{n-1} e_i \le P_N,$$
$$\|\sigma(e_i) - e_{i+1}\| < \eta,\ i = 0, \ldots, n-1,\ e_n = e_0,$$
$$\frac{n \dim e_0}{N} > 1 - \eta.$$

Proof of Theorem 3.16. We use a modified argument of Theorem 3.4 in [47] by applying Lemma 6.4. We proceed as follows.

Let $\varepsilon > 0$. Let $\varepsilon/2 > \eta > 0$ and $m \in \mathbb{N}$ be given. Choose N which satisfies the conclusion of Lemma 6.5 (with this η and $n = m$). Identify $P_N\mathcal{K}P_N$ with M_N. Let $\mathcal{G} = \{E_{i+1,i} : i = 0, 1, ..., N-1\}$ be a set of generators of M_N. Let $e_0, e_1, ..., e_{m-1}$ be as in the conclusion of Lemma 6.5. For any $\varepsilon > 0$, there is $\delta > 0$ that depends only on N, such that if $\|ag - ga\| < \delta$ for $g \in \mathcal{G}$, then

$$\|ae_i - e_ia\| < \varepsilon/2,\ \ i = 0, 1, ..., n.$$

We assume that $\delta < \eta$. Fix a finite subset $\mathcal{F}_0 \subset A$. Choose $m_0 \in \mathbb{N}$ such that $m_0 \ge m$ is the smallest integer with $m_0 = 0 \bmod r$. Let $L = N + (r-1)(m_0+1)$.

Since α has the tracial Rokhlin property, there exists a sequence of projections $\{e_i^{(k)} : i = 0, 1, ..., L\}$ satisfying the following:

$$\|\alpha(e_i^{(k)}) - e_{i+1}^{(k)}\| < \frac{\delta}{(2^k)4N},\ \ e_i^{(k)}e_j^{(k)} = 0,\ \text{ if }\ i \ne j,$$
$$\lim_{k\to\infty} \|e_i^{(k)}a - ae_i^{(k)}\| = 0\ \text{ for all }\ a \in A,\ i = 0, 1, ..., L, \quad \text{and}$$
$$\tau(1 - \sum_{i=0}^{L-1} e_i^{(k)}) < \eta/2\ \text{ for all }\ \tau \in T(A),\ k = 1, 2, ...$$

By applying Lemma 6.4, we obtain a central sequence $\{w_i^{(k)}\}$ in A such that

$$(w_i^{(k)})^* w_i^{(k)} = P_0^{(k)} \quad \text{and}$$
$$w_i^{(k)}(w_i^{(k)})^* = P_i^{(k)},\ k = 0, 1, ...,\ i = 0, 1, \ldots, N,$$
$$P_i^{(k)}P_j^{(k)} = 0,\ i \ne j,$$
$$\|\alpha(P_i^{(k)}) - P_{i+1}^{(k)}\| < \frac{\delta}{4L},\ k = 0, 1, ...,\ i = 0, 1, \ldots, N-1,$$
$$\tau(\sum_{i=0}^{L-1} e_i^{(k)} - \sum_{i=0}^{N-1} P_i^{(k)}) < \eta/2, \quad \text{for all }\ \tau \in T(A),$$

where $P_i^{(k)} \le \sum_{j=0}^{r-1} e_{i+(m_0+1)j}^{(k)}$ for $i = 0, 1, \ldots, N$.

It follows that $\{\alpha^l(w^{(k)})\}$, $l=0,1,...,N$, are all central sequences. As in the same argument in Lemma 6.4 there is a unitary $u_k \in U(A)$ with $\|u_k - 1\| < \delta/2N$ such that $\mathrm{ad}\, u_k \circ \alpha(P_i^{(k)}) = P_{i+1}^{(k)}$, $i=0,1,...,N-1$. Put $\beta_k = \mathrm{ad}\, u_k \circ \alpha$, and $w^{(k)} = w_0^{(k)}$. Choose a large k, such that

$$\|\beta_k^l(w^{(k)})a - a\beta_k^l(w^{(k)})\| < \delta \text{ for all } a \in \mathcal{F}_0,$$

$l=0,1,...,N$.

Now let C_1 and C_2 be the C^*-algebras generated by $w^{(k)}, \beta_k^1(w^{(k)}),...,\beta_k^{N-1}(w^{(k)})$ and by $w^{(k)}, \beta_k^1(w^{(k)}),...,\beta_k^N(w^{(k)})$, respectively. Note that $C_1 \cong M_N$, $C_2 \cong M_{N+1}$. Define a homomorphism $\Phi : C_1 \to \mathcal{K}$ by

$$\Phi(\beta_k^i(w^{(k)})) = E_{i+1,i}, \ i=0,1,...,N-1$$

(see Lemma 6.5). Then one has $\sigma \circ \Phi|_{C_1} = \Phi \circ \beta_k|_{C_1}$ and $\Phi(C_1) = P_N\mathcal{K}P_N$. Now we apply Lemma 6.5 to obtain mutually orthogonal projections $e_0, e_1, ..., e_{m-1}$ in M_N such that

$$\|\sigma(e_i) - e_{i-1}\| < \eta \quad \text{and} \quad \frac{m \dim e_0}{N} > 1-\eta.$$

Let $p_i = \Phi^{-1}(e_i)$, $i=0,1,...,m-1$. One estimates that

$$\tau(\sum_{i=0}^{N-1} P_i^{(k)} - \sum_{i=0}^{m-1} p_i) < 1 - \sum_{i=0}^{m-1} \frac{\dim(e_0)}{N} = 1 - \frac{m\dim(e_0)}{N} < \eta < \frac{\varepsilon}{2}$$

for all $\tau \in T(A)$. So one has mutually orthogonal projections $p_0, p_1, p_2, ..., p_{m-1}$ such that

$$\|\beta_k(p_i) - p_{i+1}\| < \frac{\varepsilon}{2}, i=0,1,2,...,m-1, \ p_m = p_0.$$

By the choice of δ, one also has

$$\|ap_i - p_ia\| < \varepsilon, \ i=0,1,...,m-1, \quad \text{for all } a \in \mathcal{F}_0 \quad \text{and}$$

$$\tau(1-\sum_{i=0}^{m-1} p_i) < \tau(1-\sum_{i=0}^{L-1} e_i^{(k)}) + \tau(\sum_{i=0}^{L-1} e_i^{(k)} - \sum_{i=0}^{N-1} P_i^{(k)}) + \frac{\varepsilon}{2} < \eta/2+\eta/2+\frac{\varepsilon}{2} < \varepsilon,$$

for all $\tau \in T(A)$. Since

$$\|\beta_k - \alpha\| < \delta/2 < \varepsilon/2,$$

one finally has

$$\|\alpha(p_i) - p_{i+1}\| < \varepsilon, \ \ i=0,1,...,m-1, \ \ p_m = p_0.$$

In other words, α has the tracial cyclic Rokhlin property. □

Proof of Theorem 3.15. This follows from Theorem 3.14 and Theorem 3.4 of [47]. □

Proof of Theorem 3.16. The fact that (i), (ii) and (iii) are equivalent (without assuming that $\alpha_{*0}^r|_G = \mathrm{id}_G$) is established in [50].

That (iv) ⇒ (v) is given by Theorem 2.9 in [47] (see also [44]). It is known that (v) ⇒ (iv).

Thus we have shown that (i), (ii), (iii), (iv) and (v) are equivalent.

To see these imply (vi), we apply the classification theorem in [39]. It follows that $A \rtimes_\alpha \mathbb{Z}$ is a unital simple AH-algebra with no dimension growth and with real rank zero. By (ii), it has a unique tracial state.

It is obvious that (vi) implies (iii). □

Proof of Theorem 3.18. It follows from Theorem 8.3 in [50] that $\alpha_{\theta,\gamma,d,f}$ has the tracial Rokhlin property. It also follows from Corollary 8.4 in [50] that $A_\theta \rtimes_\alpha \mathbb{Z}$ is a unital simple C^*-algebra with real rank zero and with unique tracial state. It follows from Theorem 3.16 that $A_\theta \rtimes_\alpha \mathbb{Z}$ is an AH-algebra with no dimension growth and with real rank zero. This proves (1). To see (2), since $d = 0$, by Lemma 1.7 of [50], $K_0(A_\theta \rtimes_\alpha \mathbb{Z})$ is torsion free. It follows that it is an A$\mathbb{T}$-algebra. □

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