TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 359, Number 2, February 2007, Pages 605–618
S 0002-9947(06)03958-4
Article electronically published on June 13, 2006

GROWTH AND ERGODICITY OF CONTEXT-FREE LANGUAGES II: THE LINEAR CASE

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To Wolfgang Woess on his 50th anniversary

Abstract. A language L over a finite alphabet $\mathbf{\Sigma}$ is called growth-sensitive if forbidding any non-empty set F of subwords yields a sub-language L^F whose exponential growth rate is smaller than that of L. Say that a context-free grammar (and associated language) is ergodic if its dependency di-graph is strongly connected. It is known that regular and unambiguous non-linear context-free languages which are ergodic are growth-sensitive. In this note it is shown that ergodic unambiguous linear languags are growth-sensitive, closing the gap that remained open.

1. Introduction

Let L be a language over the alphabet $\mathbf{\Sigma}$, that is, a subset of the free monoid $\mathbf{\Sigma}^*$ of all finite words over $\mathbf{\Sigma}$. We write ε for the empty word and $\mathbf{\Sigma}^+ = \mathbf{\Sigma}^* \setminus \{\varepsilon\}$. For a word $w \in \mathbf{\Sigma}^*$, its *length* (number of letters) is denoted by $|w|$. The *growth rate* of $L \subseteq \Sigma^*$ is the number

$$\gamma(L) = \limsup_{n\to\infty} |\{w \in L : |w| \leq n\}|^{\frac{1}{n}}. \tag{1.1}$$

Also recall that L has *exponential growth* if $\gamma(L) > 1$ and it has *sub-exponential growth* otherwise, namely, if $\gamma(L) = 1$.

A language L over $\mathbf{\Sigma}$ is *growth-sensitive* if

$$\gamma(L^F) < \gamma(L)$$

for any non-empty $F \subset \mathbf{\Sigma}^*$ consisting of subwords of elements of L, where

$$L^F = \{w \in L : \text{no } \ v \in F \ \text{ is a subword of } \ w\}.$$

A *context-free grammar* is a quadruple $\mathcal{C} = (\mathbf{V}, \mathbf{\Sigma}, \mathbf{P}, S)$, where $\mathbf{V}$ is a finite set of *variables,* disjoint from the finite alphabet $\mathbf{\Sigma}$, the variable S is the *start symbol,* and $\mathbf{P} \subset \mathbf{V} \times (\mathbf{V} \cup \mathbf{\Sigma})^*$ is a finite set of *production rules.* We write $T \vdash u$ or $(T \vdash u) \in \mathbf{P}$ if $(T, u) \in \mathbf{P}$. For $v, w \in (\mathbf{V} \cup \mathbf{\Sigma})^*$, we write $v \implies w$ if $v = v_1 T v_2$ and $w = v_1 u v_2$, where $T \vdash u$, $v_1 \in (\mathbf{V} \cup \mathbf{\Sigma})^*$ and $v_2 \in \mathbf{\Sigma}^*$. A *rightmost derivation* is a sequence $v = w_0, w_1, \ldots, w_k = w \in (\mathbf{V} \cup \mathbf{\Sigma})^*$ such that $w_{i-1} \implies w_i$ (the w_i's are called *sentential forms*); we then write $v \stackrel{*}{\implies} w$. For $T \in \mathbf{V}$, we consider the language $L_T = \{w \in \mathbf{\Sigma}^* : T \stackrel{*}{\implies} w\}$. The *language generated by* $\mathcal{C}$ is $L(\mathcal{C}) = L_S$.

Received by the editors November 4, 2004.
2000 *Mathematics Subject Classification.* Primary 68Q45; Secondary 05A16, 05C20, 68Q42.
Key words and phrases. Context-free grammar, linear language, dependency di-graph, ergodicity, ambiguity, growth, higher block languages, automaton, bilateral automaton.

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A *context-free language* is a language generated by a context-free grammar.

If in addition the production rules $T \vdash u$ are such that u contains at most one variable in $\mathbf{V}$, then the grammar and the corresponding language are termed *linear*. In particular, if $u \in \Sigma^* \cup \Sigma^*\mathbf{V}$ (resp. $\Sigma^* \cup \mathbf{V}\Sigma^*$), then the language is *left* (resp. *right*) *linear*. In both cases, language and grammar are also called *regular*.

A general context-free grammar $\mathcal{C}$ is called *unambiguous* if for any $w \in L(\mathcal{C})$ there is a unique rightmost derivation $S \overset{*}{\Longrightarrow} w$. A context-free language is *unambiguous* if it is generated by some unambiguous grammar. Note that there are context-free languages that cannot be generated by unambiguous grammars: these are called *inherently ambiguous* languages. In our setting, the language

$$L = \{a^n b^n c^m : n, m > 0\} \cup \{a^n b^m c^m : n, m > 0\}$$

is linear and inherently ambiguous (using Ogden's iteration lemma [17] (see also Chapter 6 in [9]) it can be deduced that one always has two different derivations for the words of the form $a^n b^n c^n$). Thus there exist inherently ambiguous linear languages.

We shall always assume to have a *reduced* grammar $\mathcal{C}$, that is, each variable is used in some rightmost derivation of a word in $L(\mathcal{C})$ and, in particular, $L_T \neq \emptyset$ for each variable T.

The *dependency di-graph* $\mathcal{D} = \mathcal{D}(\mathcal{C})$ of a context-free grammar $\mathcal{C} = (\mathbf{V}, \mathbf{\Sigma}, \mathbf{P}, S)$ is the oriented graph with vertex set $\mathbf{V}$, with an edge from T to U (notation $T \to U$) if in $\mathbf{P}$ there is a production $T \vdash u$ with u containing U. (Compare e.g. with [14].) Slightly differing from [14] and [5], [6] we stress that we allow *multiple edges* (several edges may go from one vertex to another) and *loops* (a vertex is connected to itself); see Definition 3.2 for more details. We write $T \overset{*}{\to} U$ if in $\mathcal{D}$ there is an oriented path of length ≥ 0 from T to U.

Consider the equivalence relation on $\mathbf{V}$ where $T \sim U$ if $T \overset{*}{\to} U$ and $U \overset{*}{\to} T$. The equivalence classes, denoted $\mathbf{V}_j$, $j = 0, \ldots, N$ (with $S \in \mathbf{V}_0$), are called the *strong components* of $\mathcal{D}(\mathcal{C})$. The strong components are partially ordered: $\mathbf{V}_j \prec \mathbf{V}_k$ if there is an oriented path from $T \in \mathbf{V}_j$ to U in $\mathbf{V}_k$ (independent of the choice of representatives).

Definition 1.1. A context-free grammar $\mathcal{C}$ is called *ergodic* if the dependency di-graph $\mathcal{D}(\mathcal{C})$ is strongly connected, i.e., it consists of a single strong component.

If $\mathcal{C}$ is linear, then we require in addition that every word $w \in L(\mathcal{C})$ occurs in a non-terminal sentential form (see also condition (ii) in Definition 1.4 below).

A context-free (resp. linear/regular) language L is *ergodic* if it is generated by an ergodic, reduced context-free (resp. linear/right linear) grammar. If this grammar is also unambiguous, we say that L is an ergodic, unambiguous language of the respective type.

It is well known that ergodic regular languages of exponential growth are growth-sensitive (see for instance [4]). This statement also has several analogues in different setups and terminologies, for instance in the context of Symbolic Dynamics [16, Cor. 4.4.9], [3] and Asymptotic Group Theory [10].

Recently it was shown in [5, 6] that unambiguous ergodic *non-linear* context-free languages of exponential growth are also growth-sensitive. Thus, a small gap remained open, concerning (unambiguous) *linear* languages that are not regular: the approach in [5, 6] did not work for this subclass of context-free languages. In this note we close this gap by showing the following.

Theorem 1.2. *Ergodic unambiguous linear languages of exponential growth are growth-sensitive.*

As regular languages are always unambiguous, in combination with [4, 5, 6] we thus have:

Corollary 1.3. *Ergodic unambiguous context-free languages of exponential growth are growth-sensitive.*

The strategy (used in [4, 5, 6] and that we shall make use of here) for proving that every L in a given class $\mathcal{L}$ of languages is growth-sensitive is the following.

Step 1. Consider the set $(\mathbf{\Sigma}^2)^*$ of all words over $\mathbf{\Sigma}^2$. Its letters are of the form (ab), where $a, b \in \mathbf{\Sigma}$. Define $\phi : \mathbf{\Sigma}^* \to (\mathbf{\Sigma}^2)^*$ by $\phi(w) = \varepsilon$ if $|w| \le 1$, and

$$\phi(a_1 \cdots a_n) = (a_1a_2)(a_2a_3)\cdots(a_{n-2}a_{n-1})(a_{n-1}a_n) \quad \text{if } n \ge 2\,. \tag{1.2}$$

For any language $L \subset \mathbf{\Sigma}^*$, consider the associated 2-*block-language* $\phi(L)$ over the alphabet

$$\mathbf{\Sigma}_{(2)}(L) = \{(ab) : a, b \in \mathbf{\Sigma}\,,\ ab \text{ is a subword of } w \text{ for some } w \in L\}\,.$$

Then $\gamma(L) = \gamma\big(\phi(L)\big)$. Step 1 is the following: prove that $L \in \mathcal{L}$ implies $\phi(L) \in \mathcal{L}$.

Step 2. Show that each $L \in \mathcal{L}$ is growth-sensitive to forbidding one (or more) elements a of its alphabet $\mathbf{\Sigma}$; in the case that L is linear we restrict this condition to those a's which are involved in some non-terminal production (see Definition 1.1).

Then each $L \in \mathcal{L}$ will be growth-sensitive to forbidding any $F \subset \mathbf{\Sigma}^*$ (with the usual restrictions from Definition 1.1 when L is linear). Indeed, it is enough to prove this when $F = \{v_1\}$ consists of a single word v_1. If $m = |v_1|$, then after $m-1$ iterations, $v_m = \phi^{(m-1)}(v_1)$ is a *letter* in the alphabet of $\phi^{(m-1)}(L)$. Therefore Steps 1 and 2 imply

$$\gamma(L^{\{v_1\}}) = \gamma\Big(\big(\phi^{(m-1)}(L)\big)^{\{v_m\}}\Big) < \gamma\big(\phi^{(m-1)}(L)\big) = \gamma(L).$$

As pointed out in [5], it turns out that when passing from a grammar $\mathcal{C}$ generating the language L to the grammar $\mathcal{C}'$ generating the corresponding 2-block language $\phi(L)$, the ergodicity property is not preserved if $\mathcal{C}$ is non-regular. Again, we can overcome this problem by defining a weak form of ergodicity, namely *essential ergodicity* which now will be preserved and shall make everything work out perfectly.

First recall that the start symbol S of $\mathcal{C}$ is *isolated* if it does not occur on the right-hand side of any production rule (equivalently if $\mathbf{V}_0 = \{S\}$ as a strong component in $\mathcal{D}(\mathcal{C})$). For any language $L \subset \mathbf{\Sigma}^*$, we define its *proper subword closure* as $SUB(L) = \{v \in \Sigma^* : v \text{ is a proper subword of } w \text{ for some } w \in L\}$.

Definition 1.4. Let $\mathcal{C}$ be a reduced linear grammar with $|L(\mathcal{C})| = \infty$.

A strong component $\mathbf{V}_j$ of $\mathcal{D}(\mathcal{C})$ is called *left*, resp. *right linear* (both called *regular* for short), if for any $T \in \mathbf{V}_j$ the existence of a production of the form $T \vdash T'w$, resp. $T \vdash wT'$, with $w \in \mathbf{\Sigma}^+$ infers $T' \notin \mathbf{V}_j$. In this case, every variable in $\mathbf{V}_j$ is also called regular. We denote by $\mathbf{V_{ess}}$ the set of all *essential*, i.e., non-regular variables.

The grammar $\mathcal{C}$ is called *essentially ergodic* if

(i) the set $\mathbf{V_{ess}}$ forms a single strong component,

(ii) for each $w \in L(\mathcal{C})$, there is $T \in \mathbf{V_{ess}}$ such that $w \in SUB(L_T)$ and

(iii) $\mathbf{V}_j \preceq \mathbf{V_{ess}}$ for all j's.

As remarked in [6] this definition is useful only when L is a non-regular language. In fact, as a consequence of the Substitution Theorem for regular languages (see [11, §3.4]), if all variables are regular, then $L(\mathcal{C})$ is a regular language.

It is also clear that an ergodic linear grammar is essentially ergodic, so that Theorem 1.2 follows from the more general

Theorem 1.5. *Essentially ergodic unambiguous linear languages of exponential growth are growth-sensitive.*

The paper is organized as follows.

In Section 2 we present a canonical form for a linear grammar $\mathcal{C}$ (Section 2.1) from which a linear grammar $\mathcal{C}'$ generating the 2-block language $L(\mathcal{C}') = L_2(\mathcal{C})$ is derived (Section 2.2). Once verified that essentially ergodicity is preserved while passing to the canonical form and then to $\mathcal{C}'$, Step 1 of the strategy is completely achieved.

Regarding Step 2, the harmonic-analytical approach in [5] which, as already mentioned there, did not work for linear languages, is now replaced in Section 3 by a purely combinatorial graph theoretical argument based on an entropic inequality for oriented graphs. This inequality was originally proved by Scarabotti [18]—where some ideas of Gromov were used and the usual Perron-Frobenius theory was avoided—was then extended to oriented graphs with specified initial and terminal vertices in [4] where it was applied to the (regular) ergodic case, and it is now generalized for essential ergodicity.

In Section 5 we discuss the more general situation, also involving ambiguous grammars and their generated languages, by considering a notion of weighted-growth-sensitivity, and we prove an analogue of the main theorem in this setting.

In the Appendix, for the sake of completeness and the convenience of the reader, the proof (from [4]) of the entropic inequality in the ergodic case is presented.

2. Linear grammars and 2-block (linear) languages

2.1. **A canonical form for linear grammars.** Recall that two grammars are called *equivalent* if they generate the same language.

Now we present an algorithm which transforms any linear grammar into an equivalent one which is in a *canonical form*. Recall that a variable $A \in \mathbf{P}$ is *superfluous* if either there is no derivation $S \stackrel{*}{\Longrightarrow} w$ with $w \in (\mathbf{V} \cup \mathbf{\Sigma})^*$ containing A or $L_A = \emptyset$, and that a grammar $\mathcal{C}$ is *reduced* if it has no superfluous variables.

Proposition 2.1 (Canonical form for linear grammars)**.** *Let $\mathcal{C} = (\mathbf{V}, \mathbf{\Sigma}, \mathbf{P}, S)$ be a linear grammar. Then $\mathcal{C}$ is equivalent to a reduced grammar $\overline{\mathcal{C}} = (\overline{\mathbf{V}}, \mathbf{\Sigma}, \overline{\mathbf{P}}, S)$ where the productions are only of the form*

$$\begin{aligned} &A_1 \vdash a_1 B_1, \\ &A_2 \vdash B_2 a_2, \\ &A_3 \vdash a_3, \end{aligned} \tag{2.1}$$

where $A_1, A_2, A_3, B_1, B_2 \in \overline{\mathbf{V}}$ and $a_1, a_2, a_3 \in \mathbf{\Sigma}$.

If $\varepsilon \in L(\mathcal{C})$, in addition to the previous productions one also has $(S \vdash \varepsilon) \in \overline{\mathbf{P}}$.

Moreover, if $\mathcal{C}$ is unambiguous/(essentially) ergodic, then also $\overline{\mathcal{C}}$ is unambiguous/(essentially) ergodic.

Proof. We can suppose that $\mathcal{C}$ is reduced, otherwise we eliminate the superfluous variables and all productions involving them.

Next we transform the grammar $\mathcal{C}$ into a grammar $\mathcal{C}'$ which is ε-free, that is, there is no rule of the form $A \vdash \varepsilon$. There is a simple algorithm for passing from $\mathcal{C}$ to $\mathcal{C}'$ that generates $L \setminus \{\varepsilon\}$; see e.g. [11, Section 4.3].

Similarly one eliminates the *chain rules*, i.e. productions of the form $A \vdash B$, where $A, B \in \mathbf{V}$. Again there is a simple algorithm that transforms a reduced grammar $\mathcal{C}'$ into an equivalent reduced grammar $\mathcal{C}''$ without chain rules; see e.g. [11, Section 4.3] or [15, Corollary 5.3].

Note that these transformations preserve unambiguity and (essential) ergodicity, namely $\mathcal{C}''$ is unambiguous (resp. (essentially) ergodic) if the original grammar $\mathcal{C}$ is such.

The generic production in $\mathcal{C}''$ is then of the form

$$A \vdash a_1 a_2 \cdots a_m B b_1 b_2 \cdots b_n \tag{2.2}$$

or

$$C \vdash c_1 c_2 \cdots c_s, \tag{2.3}$$

where $A, B, C \in \mathbf{V}''$, $a_i, b_j, c_k \in \mathbf{\Sigma}$ and $m+n, s \geq 1$. Call the *length* of a production (2.2) or (2.3) the quantity $\ell = m + n + 1$ and $\ell = s$, respectively.

Suppose that in (2.2) one has $m \geq 2$. Then the length of the production (2.2) can be shortened by enlarging $\mathbf{V}''$ to $\mathbf{V}'' \cup \{B_1\}$, where $B_1 \notin \mathbf{V}''$ and then substituting (2.2) in $\mathbf{P}''$ with the two productions

$$\begin{aligned} A &\vdash \quad a_1 a_2 \cdots a_{m-1} B_1 b_1 b_2 \cdots b_n, \\ B_1 &\vdash \quad a_m B. \end{aligned} \tag{2.4}$$

Similarly one acts on the right part of (2.2) whenever $n \geq 2$.

If $m = n = 1$, then (2.4) is replaced by

$$\begin{aligned} A &\vdash a_1 B_1, \\ B_1 &\vdash B b_1. \end{aligned} \tag{2.5}$$

Finally, given a production of the form (2.3), one enlarges $\mathbf{V}''$ to $\mathbf{V}'' \cup \{C_1\}$, where $C_1 \notin \mathbf{V}''$, replaces it with

$$\begin{aligned} C &\vdash c_1 c_2 \cdots c_{s-1} C_1, \\ C_1 &\vdash c_s, \end{aligned} \tag{2.6}$$

and then reduces to the previous cases. By recurrence one shortens the production lengths to minimality obtaining a grammar $\overline{\mathcal{C}}$ in the desired canonical form.

Again, it is immediate that both unambiguity and (essential) ergodicity are preserved when passing from $\mathcal{C}''$ to $\overline{\mathcal{C}}$. □

2.2. **The 2-block language.** Suppose that L is generated by an unambiguous, (essentially) ergodic linear grammar $\mathcal{C}$ in standard form. We now present an algorithm for passing from $\mathcal{C}$ to a new linear grammar $\mathcal{C}'$ which generates the 2-block language $L_2 = \phi(L)$.

For $T \in \mathbf{V}$, a *T-sentential form* is any element $w \in (\mathbf{\Sigma} \cup \mathbf{V})^*$ such that there is a rightmost derivation $T \overset{*}{\Longrightarrow} w$. A *sentential form* is a T-sentential form for some T. By linearity, a straightforward induction argument shows that a sentential form always looks as follows:

$$a_1 a_2 \cdots a_n T b_1 b_2 \cdots b_m, \tag{2.7}$$

where $a_i, b_j \in \mathbf{\Sigma}$, $T \in \mathbf{V}$ and $n, m \geq 0$. We now transform each sentential form w as in (2.7) into a new expression $\Phi(w)$ by using ϕ (as defined in (1.2)) and inserting brackets as follows:

$$\phi(a_1 a_2 \cdots a_n)[a_n T b_1]\phi(b_1 b_2 \cdots b_m) \tag{2.8}$$

if $n, m \geq 1$ and

$$\phi(a_1 a_2 \cdots a_n)[a_n T] \text{ or } [T b_1]\phi(b_1 b_2 \cdots b_m) \tag{2.9}$$

if $m = 0$ or $n = 0$, respectively. The resulting expressions in the square brackets will become our new variables. In principle, $[aTb]$ stands for "anything that be derived from T, with a leading a and a final b added", or, more precisely, its image under ϕ. The meaning of $[aT]$ amd $[Tb]$ is similar.

We thus define $\mathbf{V}'$ as the set consisting of $[S]$ and all expressions $[aT], [Tb]$ and $[aTb]$ that occur in some $\Phi(w)$, where w is a sentential form of $\mathcal{C}$.

We now exhibit the new grammar $\mathcal{C}' = (\mathbf{V}', \mathbf{\Sigma}_2, \mathbf{P}', [S])$ for L_2. The next list displays the rules in $\mathbf{P}$ followed by the corresponding new rules in $\mathbf{P}'$.

$$\begin{aligned}
&\text{If } S \vdash bT : [S] \vdash [bT];\\
&\text{if } S \vdash Ta : [S] \vdash [Ta];\\
&\text{if } T \vdash c \quad : [Ta] \vdash (ca), [bT] \vdash (bc), [bTa] \vdash (bc)(ca);\\
&\text{if } T \vdash cU : [Ta] \vdash [cUa], [bT] \vdash (bc)[cU], [bTa] \vdash (bc)[cUa];\\
&\text{if } T \vdash Uc : [Ta] \vdash [Uc](ca), [bT] \vdash [bUc], [bTa] \vdash [bUc](ca).
\end{aligned} \tag{2.10}$$

Here, $T, U \in \mathbf{V}$ and $a, b, c \in \mathbf{\Sigma}$ have to be such that the occuring expressions in brackets belong to $\mathbf{V}'$. By the construction of $\mathcal{C}'$, for any sequence of sentential forms $w_1, w_2, \ldots, w_n$ with respect to $\mathcal{C}$, we have

$$S \implies w_1 \implies w_2 \implies \ldots \implies w_n \qquad \text{in } \mathcal{C}$$

if and only if

$$[S] \implies \Phi(w_1) \implies \Phi(w_2) \implies \ldots \implies \Phi(w_n) \qquad \text{in } \mathcal{C}'.$$

Therefore $\mathcal{C}'$ generates $\phi(L) \setminus \{\varepsilon\} = L_2$, as required.

Before proceeding, let us note that $\mathcal{C}'$ has chain rules, which can be eliminated by the algorithm of Proposition 2.1 but are "harmless" anyway (they cannot be concatenated into an infinite loop—Harrison [11] uses "cycle-free" for such grammars). Thus, we continue to work with $\mathcal{C}'$.

We say that $[aTb]$ is an *interior* variable of $\mathcal{C}'$ if $T \in \mathbf{V_{ess}}$, and the string aTb occurs in a sentential form of $\mathcal{C}$ that derives from some element of $\mathbf{V_{ess}}$.

Proposition 2.2. *If $\mathcal{C}$ (in canonical form) is unambiguous/essentially ergodic, then so is $\mathcal{C}'$. Moreover $\mathbf{V}'_{\mathbf{ess}}$ consists of the interior variables.*

Proof. As in the proof of Proposition 3 in [5], we have that $[S]$ is isolated and consequently regular; also, the only variables $[aT]$, $[Tb]$ or $[aTb]$ in $\mathbf{V}'$ that may be non-regular are those induced by some element T in $\mathbf{V_{ess}}$. In particular, as a variable of the form $[Tb]$ can only occur at the leftmost point of a production in $\mathbf{P}'$, it must be regular (in fact its strong component is right-linear). The same holds for variables of the form $[aT]$. Thus we are left with the variables $[aTb] \in \mathbf{V}'$ with $T \in \mathbf{V_{ess}}$. We have

Claim 1. *Let $[aTb], [cUd] \in \mathbf{V}'$ with $T, U \in \mathbf{V_{ess}}$ and $[cUd]$ interior. Then $[aTb] \xrightarrow{*} [cUd]$ in $\mathcal{D}(\mathcal{C}')$. In particular, the interior variables of $\mathcal{C}'$ form a strong component.*

Claim 2. *If $T_0 \in \mathbf{V_{ess}}$ and $[aT_0b] \in \mathbf{V}'$ is non-interior, then it is regular.*

Claim 3. *Every $T \in \mathbf{V_{ess}}$ occurs in some inner variable of $\mathcal{C}'$. In particular, the component formed by the interior variables in $\mathbf{V}'$ is non-regular.*

Claim 4. *$\forall w' \in L(\mathcal{C}')$ there exists $T' \in \mathbf{V}'_{\mathbf{ess}}$ such that $w' \in SUB(L_{T'})$.*

Claim 5. *$\mathbf{V}'_j \preceq \mathbf{V}'_{\mathbf{ess}}$ for all j's.*

The proofs of Claims 1-3 are identical to the corresponding ones in [5], pp. 4608-4609, and we omit them. As for Claim 4, let $w' \in L(\mathcal{C}')$ and set $w = \phi^{-1}(w') \in L(\mathcal{C})$; by essential ergodicity (point (ii) in Definition 1.4) of $\mathcal{C}$ we have that there exists $T \in \mathbf{V_{ess}}$ such that $w \in SUB(L_T)$. By Claim 3, there exist a and $b \in \mathbf{\Sigma}$ such that $[aTb] \in \mathbf{V}'_{\mathbf{ess}}$ so that $w' \in SUB(L_{[aTb]})$. Claim 5 is straightforward. Finally, unambiguity is preserved because of the one-to-one correspondence between the sentential forms of $\mathcal{C}$ and those of $\mathcal{C}'$ as indicated above. □

Corollary 2.3. *Suppose that L is an unambiguous/(essentially) ergodic linear language. Then the 2-block language $L_2 = \phi(L)$ is also unambiguous/(essentially) ergodic and linear.*

3. On the growth of (essentially) ergodic automata

In order to have essential ergodicity of $\mathcal{C}$ *encoded* inside its associated dependency di-graph $\mathcal{D}(\mathcal{C})$, we enrich the latter with some additional structure.

Definition 3.1. An *automaton* is a 5-tuple $\mathcal{A} = (\mathcal{G}, X, \mathcal{I}, \mathcal{F}, \omega)$, where $\mathcal{G} = (V, E)$ is a finite oriented graph with vertex set V and edge set E, X is a finite set, $\mathcal{I}, \mathcal{F} \subseteq V$ are the set of *initial* and *terminal* vertices, respectively, and $\omega : E \to X$ is a *labelling* of the edges.

If in addition $X = X_\ell \coprod X_r$ and $\omega(E) \cap X_\ell \neq \emptyset \neq \omega(E) \cap X_r$, then $\mathcal{A}$ is called a *bilateral* automaton.

An *(oriented) path* in $\mathcal{G}$ is a sequence $p = e_1e_2 \cdots e_n$ of edges such that the terminal vertex of the i-th is the initial vertex of the $(i+1)$-st, $i = 1, 2, \ldots, n-1$; the number $|p| = n$ and the word $\omega(p) = \omega(e_1)\omega(e_2) \cdots \omega(e_n) \in X^*$ are the *length* and the *label* of the path p, respectively. Also denote by $p(i) \in V$ the i-th vertex of the path, so that $p(0)$ and $p(|p|)$ are the *start* and *final* vertex of p, respectively. Finally let $\mathcal{P}$ denote the set of all paths in $\mathcal{G}$.

A path with $p(0) \in \mathcal{I}$ and $p(|p|) \in \mathcal{F}$ is termed *admissible*: denote by $\mathcal{P}^a$ the set of admissible paths in $\mathcal{A}$. The number

$$\gamma(\mathcal{A}) = \limsup_{n\to\infty} |\{p \in \mathcal{P}^a : |p| \leq n\}|^{\frac{1}{n}}$$

is the *growth (rate)* of $\mathcal{A}$.

The automaton $\mathcal{A}$ is termed *unambiguous* if $p \neq p' \in \mathcal{P}^a$ infers $\omega(p) \neq \omega(p')$.

Denote by $V_0, V_1, \ldots, V_N \subset V$ the strong components (for all i and for any $v_1, v_2 \in V_i$ there exists an oriented path $p \in \mathcal{P}$ starting at v_1 and terminating at v_2) of $\mathcal{G}$ (with $V_0 \ni S$). The set $\{V_0, \ldots, V_N\}$ is partially ordered: $V_i \preceq V_j$ if there exist $v_i \in V_i$ and $v_j \in V_j$ such that $v_i \to v_j$.

Definition 3.2. The automaton $\mathcal{A}$ is *ergodic* if $\mathcal{G}$ is strongly connected, i.e. there exists a unique strong component: $V_0 = V$.

Suppose that $\mathcal{A}$ is a bilateral automaton; a strong component V_i is called left (resp. right) linear, briefly *regular*, if given two vertices $v_1, v_2 \in V_i$ such that $v_1 \to v_2 \in E$, then $\omega(v_1 \to v_2) \in X_\ell$ (resp. X_r). Then $\mathcal{A}$ is *essentially ergodic* if it is unambiguous and

(i) there exists a unique component which is non-regular, call it the *essential* component and denote it by $V_{\mathbf{ess}}$,

(ii) for each $p \in \mathcal{P}^a$, there is a $q \in \mathcal{P}$ with $q(0) \in V_{\mathbf{ess}}$ such that $\omega(p) = \omega(q)$,

(iii) $V_j \preceq V_{\mathbf{ess}}$ for all $j = 0, 1, \ldots, N$.

If $\mathcal{A}$ is essentially ergodic, denote by $\mathcal{G}_{\mathbf{ess}} = (V_{\mathbf{ess}}, E_{\mathbf{ess}})$ the strong component endowed with its induced graph structure, $\mathcal{I}_{\mathbf{ess}} = V_{\mathbf{ess}}$, $\mathcal{F}_{\mathbf{ess}} = \mathcal{F} \cap V_{\mathbf{ess}}$ (note that by (iii) above the latter is non-empty), $\omega_{\mathbf{ess}} = \omega|_{E_{\mathbf{ess}}}$ and by $\mathcal{A}_{\mathbf{ess}} = (\mathcal{G}_{\mathbf{ess}}, X, \mathcal{I}_{\mathbf{ess}}, \mathcal{F}_{\mathbf{ess}}, \omega_{\mathbf{ess}})$ the *essential sub-automaton* of $\mathcal{A}$ (which is clearly ergodic).

Let $\mathcal{C}$ be a linear grammar in canonical form. We introduce an equivalence relation on $\mathbf{P}$ by declaring $(T_1 \vdash aU_1) \sim (T_2 \vdash aU_2)$, $(T_1 \vdash U_1 b) \sim (T_2 \vdash U_2 b)$ and $(T_1 \vdash c) \sim (T_2 \vdash c)$ for $T_i, U_j \in \mathbf{V}$ and $a, b, c \in \mathbf{\Sigma}$. We denote by $[T \vdash aU], [T \vdash Ub]$ and $[T \vdash c]$ the corresponding equivalence classes and by $[\mathbf{P}]$ the set of such classes.

The bilateral automaton associated with a linear grammar $\mathcal{C}$ in canonical form is $\mathcal{A} = (\mathcal{D}(\mathcal{C}), [\mathbf{P}], \mathcal{I}, \mathcal{F}, \omega)$, where $\mathbf{P} = \mathbf{P}_\ell \coprod \mathbf{P}_r$ consists of the left and right productions (we assume, by convention, that a production of the form $T \vdash a$, where $a \in \mathbf{\Sigma}$, is a left production), $\mathcal{I} = \{S\}$, $\mathcal{F} = \{T \in V \equiv \mathbf{V} : \exists a \in \mathbf{\Sigma}, \exists (T \vdash a) \in \mathbf{P}\}$ and $\omega : E \to [\mathbf{P}]$ is the map which assigns to each edge $T \to U$ in $\mathcal{D}$ the equivalence class of the production from which it was originated.

An instant of thought gives that $\mathcal{C}$ is unambiguous/(essentially) ergodic if and only if its associated bilateral automaton is unambiguous/(essentially) ergodic. Moreover, if $\mathcal{C}$ is unambiguous, the growth rate of $\mathcal{A}$, $\gamma(\mathcal{A})$ equals $\gamma(L)$, the growth rate of the language L generated by the grammar $\mathcal{C}$.

We are thus led to consider the growth of (admissible) paths inside $\mathcal{A}$ for which the following new result holds true.

Theorem 3.3. *Let $\mathcal{A} = (\mathcal{G}(V, E), X, \mathcal{I}, \mathcal{F}, \omega)$ be an (essentially) ergodic automaton. Let $e \in E$ be an edge in $\mathcal{A}_{\mathbf{ess}}$ and denote by $\mathcal{A}^e$ the automaton obtained from $\mathcal{A}$ by removing the edge e. Then $\gamma(\mathcal{A}^e) < \gamma(\mathcal{A})$.*

For the ergodic case the proof is more or less well known: the standard argument is based on Perron-Frobenius theory. The growth of a graph (or of its associated language, shift, etc.) is given by the Perron-Frobenius eigenvalue of its adjacency matrix. The deletion of an edge between vertices i and j corresponds to replacing the entry $a_{i,j} = |\{e \in E : e(0) = i, e(1) = j\}|$ of the matrix by $a_{i,j} - 1$, so that the corresponding Perron-Frobenius eigenvalue decreases [19], [16, 4.4.7]. For the sake of completeness and for the reader's convenience, in the Appendix we present a proof of the ergodic case—based on purely combinatorial arguments (inspired by a method of Gromov [18]) and which does not make use of the Perron-Frobenius theory—from [4, Proposition 3].

The crucial step in the proof of the above theorem is that for an essentially ergodic automaton $\mathcal{A}$, its growth $\gamma(\mathcal{A})$ equals the growth $\gamma(\mathcal{A}_{\mathbf{ess}})$ of its essential sub-automaton (see Lemma 3.5 below), and one is thus reduced to the ergodic case. However, we need a little more work.

Let $\mathcal{A} = (\mathcal{G}(V,E), X, \mathcal{I}, \mathcal{F}, \omega)$ be an automaton with $V = \{v_1, v_2, \ldots, v_K\}$ and set

$$\begin{aligned}
&\gamma(n) = |\{p \in \mathcal{P} : |p| \leq n\}|, && \gamma = \limsup_{n\to\infty} \gamma(n)^{\frac{1}{n}},\\
&\gamma_i(n) = |\{p \in \mathcal{P} : p(0) = v_i, |p| \leq n\}|, && \gamma_i = \limsup_{n\to\infty} \gamma_i(n)^{\frac{1}{n}},\\
&\gamma_{i,\mathcal{F}}(n) = |\{p \in \mathcal{P} : p(0) = v_i, p(|p|) \in \mathcal{F}, |p| \leq n\}|, && \gamma_{i,\mathcal{F}} = \limsup_{n\to\infty} \gamma_{i,\mathcal{F}}(n)^{\frac{1}{n}},\\
&\gamma_{\mathcal{I},\mathcal{F}}(n) = |\{p \in \mathcal{P} : p(0) \in \mathcal{I}, p(|p|) \in \mathcal{F}, |p| \leq n\}|, && \gamma_{\mathcal{I},\mathcal{F}} = \limsup_{n\to\infty} \gamma_{\mathcal{I},\mathcal{F}}(n)^{\frac{1}{n}},
\end{aligned}$$

so that, by definition, $\gamma_{\mathcal{I},\mathcal{F}} = \gamma(\mathcal{A})$. Then

Lemma 3.4. *Suppose the automaton $\mathcal{A}$ is ergodic. Then for all $i = 1, 2, \ldots, K$*

$$\gamma = \gamma_i = \gamma_{i,\mathcal{F}} = \gamma_{\mathcal{I},\mathcal{F}}.$$

Proof. For all $i \neq j$ fix a (minimal) oriented path p_{ij} starting at v_i and ending at v_j and set $M = \max_{i,j} |p_{ij}|$. Given a path p with $p(0) = v_j$ the composition $q = p_{ij}p$ starts at v_i and $|q| \leq |p| + M$: this gives an injection of $\{p \in \mathcal{P} : p(0) = v_j, |p| \leq n\}$ into $\{q \in \mathcal{P} : q(0) = v_i, |q| \leq n + M\}$ so that $\gamma_j \leq \gamma_i$ which, by symmetry, gives the invariance of γ_i w.r. to i. Similarly, for each j fix p_j a (minimal) oriented path starting at v_j and ending at some terminal vertex; set $M' = \max_j |p_j|$. With each oriented path p starting at v_i and terminating at some v_j, we then associate the path $q = pp_j$ (with $q(0) = v_i$ and ending at some terminal) of length $|q| \leq |p| + M'$: the correspondence $p \mapsto q$ is no more injective, but it is almost $(K + M')$-to-one (by minimality). We thus have

$$\gamma_i(n) \leq (K + M')\gamma_{i,\mathcal{F}}(n + M')$$

which gives $\gamma_i \leq \gamma_{i,\mathcal{F}}$. On the other hand, $\gamma_{i,\mathcal{F}}(n) \leq \gamma_{\mathcal{I},\mathcal{F}}(n) \equiv \sum_{j:v_j\in\mathcal{I}} \gamma_{j,\mathcal{F}}(n) \leq |\mathcal{I}|\gamma_i(n)$ and one gets $\gamma_{i,\mathcal{F}} \leq \gamma_{\mathcal{I},\mathcal{F}} \leq \gamma_i$, which ends the poof. □

Lemma 3.5. *The growth of an essentially ergodic automaton is the same as that of its essential sub-automaton: $\gamma_{\mathcal{A}} = \gamma_{\mathcal{A}_{\mathbf{ess}}}$.*

Proof. Let $v_1, v_2, \ldots, v_K$ denote the vertices in the essential component $\mathcal{A}_{\mathbf{ess}}$ of $\mathcal{A}$ and order lexicographically the $(n+1)$-tuples $(i_0, i_1, \ldots, i_n)$, where $1 \leq i_j \leq K$. For each $i = 1, 2, \ldots, K$ let $p_i \in \mathcal{P}$ be a (minimal) oriented path starting from an initial vertex and ending at v_i and set $M = \max_i |\gamma_i|$; the same argument in the proof of the equalities $\gamma_i = \gamma_j$ in the previous lemma yields $\gamma_{\mathcal{A}_{\mathbf{ess}}} \leq \gamma_{\mathcal{A}}$. As for the converse implication, let $p \in \mathcal{P}^a$; by essential ergodicity there exists an oriented path p' in $\mathcal{A}_{\mathbf{ess}}$, not necessarily admissible, such that $|p| = |p'| =: n$ and $\omega(p') = \omega(p)$; such a path p' can be chosen uniquely by chosing the $(n+1)$-tuple $(i_0, i_1, \ldots, i_n)$ to be minimal, where $p(j) = v_{i_j}$, $1 \leq j \leq n$. By unambiguity and the previous lemma we have $\gamma_{\mathcal{A}} \leq \gamma_{\mathcal{A}_{\mathbf{ess}}}$, which ends the proof. □

As remarked above, the above lemma gives Theorem 3.3 (modulo its purely ergodic version whose proof is presented in the Appendix).

3.1. **End of the proof of Theorem 1.5.** By virtue of Corollary 2.3 we only need to pass over Step 2. Let L be a linear unambiguous (essentially) ergodic language generated by an unambiguous (essentially) ergodic linear grammar $\mathcal{C}$ in canonical form and denote by $\mathcal{A} = \mathcal{A}(\mathcal{C})$ the associated unambiguous (essentially) ergodic automaton.

Suppose $a \in \Sigma$ is involved in some non-terminal poduction. The language L^a obtained from L by forbidding the letter a is still linear: it is generated by the linear grammar $\mathcal{C}^a = (\mathbf{V}^a, \mathbf{\Sigma}, \mathbf{P}^a, S)$, where $\mathbf{P}^a$ are all productions not involving a

and $\mathbf{V}^a$ are the corresponding non-superfluous variables; note that the associated automaton $\mathcal{A}^a = \mathcal{A}(\mathcal{C}^a)$ need not be essentially ergodic any more.

Let $e \in E$ be an edge in $\mathcal{A}$ with $\omega(e) \in [\mathbf{P}] \setminus [\mathbf{P}^a]$, in other words an edge corresponding to a non-terminal left or right production involving a, and denote by $\mathcal{A}^e$ the automaton obtained from $\mathcal{A}$ by removing e.

We then have

$$\gamma(L^a) = \gamma(\mathcal{A}^a) \leq \gamma(\mathcal{A}^e) < \gamma(\mathcal{A}) = \gamma(L)$$

(where $<$ follows from Theorem 3.3), and the proof is now complete. □

4. An example

In this section we present an example of an unambiguous ergodic linear language of exponential growth which, by virtue of Theorem 1.2, is growth-sensitive. Let $\mathcal{C} = (\mathbf{V}, \mathbf{\Sigma}, \mathbf{P}, S)$ be the grammar defined by $\mathbf{V} = \{S\}, \mathbf{\Sigma} = \{a, b, c, d\}$ and $\mathbf{P} = \{S \vdash aSb \mid cSd \mid ab \mid cd\}$. It is clearly linear and the (linear) language generated by it is $L = \{w(a, c)w(b, d)^t\} \subseteq \{a, c\}^+\{b, d\}^+$, where w^t denotes the *transposed* of w. Note that the map $L \ni w(a, c)w(b, d)^t \mapsto w(a, b) \in \{a, b\}^+$ is a bijective correspondence between L and the free monoid of rank two. Denote by $\gamma_L(n) = |\{w \in L : |w| \leq n\}|$ the growth function of L (and similarly define $\gamma_{\{a,b\}^+}(n)$). As $|(w(a, c)w(b, d)^t| = 2|w(a, b)|$ one immediately derives that $\gamma_L(2n) = \gamma_L(2n + 1)$ and that $\gamma_L(2n) = \gamma_{\{a,b\}^+}(n)$. Therefore the growth rate of L is

$$\gamma(L) = \limsup_{n\to\infty} \sqrt[2n]{\gamma_L(2n)} = \sqrt[2]{\gamma(\{a, b\}^+)} = \sqrt[2]{2} > 1,$$

and L has exponential growth.

A canonical form for $\mathcal{C}$ is $\mathcal{C}' = (\mathbf{V}', \mathbf{\Sigma}, \mathbf{P}', S)$, where $\mathbf{V}' = \mathbf{V} \coprod \{T, U\}$ and $\mathbf{P}' = \{S \vdash aT \mid cU; T \vdash Sb \mid b; U \vdash Sd \mid d\}$. The corresponding bilateral (3-state) automaton, clearly ergodic and unambiguous, is pictured in Figure 1.

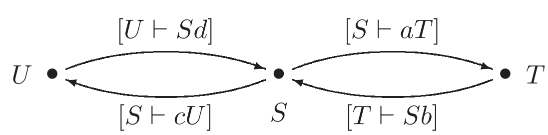

Figure 1. The bilateral automaton $\mathcal{A}(\mathcal{C}')$.

We now show that L is *non-regular*. For this purpose we recall the following well-known *pumping lemma* for regular languages (see, for instance, [8, 12]).

Lemma 4.1. *Let L' be a regular language. Then there exists a postive integer K such that for all $w \in L'$ of length $|w| \geq K$ there exist a decomposition $w = xyz$ with $|y| \geq 1$ (i.e. y is non-trivial) such that for all integers $n \geq 0$ one has $xy^nz \in L'$.*

Suppose, by contradiction, that our language L is regular. Let K be the corresponding positive integer as indicated in the previous lemma and fix a word $W = w(a, c)w(b, d)^t \in L$ of length $|W| \geq K$. If the corresponding subword y is a subword of $w(a, c)$ (a similar argument for y a subword of $w(b, d)^t$ applies), then for $n > |W|$ we would then have $xy^nz \notin L$ as $xy^n \in \{a, c\}^+$ and $|xy^n| > |z|$, contradicting the fact that, for any word in L, the length of the $\{a, c\}$-part equals the length of the $\{b, d\}$-one. Similarly, if y contains at least a letter in $\{a, c\}$ and one in $\{b, d\}$, as soon as $n > 1$ one has $xy^nz \notin \{a, c\}^+\{b, d\}^+$.

We end this section by mentioning that in [2] an algorithm is presented which determines whether a given unambiguous linear language has polynomial or exponential growth (it is a more general result due to Trofimov [20] and, independently, to Incitti [13] and to Bridson and Gilman [1], that context-free languages have either polynomial or exponential growth, i.e. there exists no context-free language of intermediate growth). It basically consists in applying Ufnarowskii's criterion [21] for the growth of oriented graphs (that he used for determining the growth of affine algebras) to the bilateral automaton associated with a linear grammar in canonical form generating the given language. Roughly speaking, the language has exponential growth if and only if in the bilateral automaton there exist "doubly cyclic" vertices (in the case of the example above, a doubly cyclic vertex is the start symbol S in Figure 1). When the language has polynomial growth, the degree d can be recovered again from the automaton: d is the maximal number of "simple cycles" (of any length) intervealed by "simple chains" (of any length) in some "circuit" in $\mathcal{A}$; for the terminology and for more details we directly refer to [2, 21].

5. Weighted-growth-sensitivity of linear languages

Let $\mathcal{C} = (\mathbf{V}, \mathbf{\Sigma}, \mathbf{P}, S)$ be a context-free grammar. For a variable $T \in \mathbf{V}$ we define the *ambiguity degree* $d_T(w)$ of a word $w \in \mathbf{\Sigma}^*$ as the number of all different righmost derivations $T \overset{*}{\Longrightarrow} w$. We have $d_T(w) > 0$ if and only if $w \in L_T$. If the grammar is reduced, without chain rules, i.e. productions of the form $T \vdash U$, where $T, U \in \mathbf{V}$, and ε-free, then $d_T(w) < \infty$ always. We remark that there is a simple algorithm that transforms a (reduced) grammar into an equivalent (reduced) grammar without chain rules (see, for instance [11], Section 4.3, or [15]). Thus, with this terminology, the grammar is unambiguous if and only if $d_S(w) = 1$ for all $w \in L(\mathcal{C})$.

From now on all context-free grammars considered shall be reduced, with no chain rules and ε-free.

Let $L = L(\mathcal{C})$ be a context-free language generated by a context-free grammar $\mathcal{C}$. The *weighted growth function* and the *weighted growth rate* of L with respect to $\mathcal{C}$ are the function

$$\gamma_{L,\mathcal{C}}(n) = \sum_{w \in L : |w| \leq n} d_S(w) \tag{5.1}$$

and the number

$$\gamma_{L,\mathcal{C}} = \limsup_{n \to \infty} \sqrt[n]{\gamma_{L,\mathcal{C}}(n)}, \tag{5.2}$$

respectively.

Note that if $\mathcal{C}$ is unambiguous, then $\gamma_{L,\mathcal{C}} = \gamma_L$ the (standard) growth rate of L defined in the Introduction.

We then say that a linear language is *weighted-growth-sensitive* if, given any context-free grammar $\mathcal{C}$ generating it and any non-empty $F \subset \mathbf{\Sigma}^*$, one has

$$\limsup_{n \to \infty} \sqrt[n]{\sum_{w \in L^F : |w| \leq n} d_S(w)} < \gamma_{L,\mathcal{C}}.$$

With this terminology we can state the following version of Theorem 1.5 for (possibly inherently ambiguous) languages.

Proposition 5.1. (*Essentially*) *ergodic linear languages of exponential growth are weighted-growth-sensitive.*

Proof. The proof is the same as for Theorem 1.5. We only observe that in the present setting there is a one-to-one length-preserving correspondence between the set of different derivations $S \overset{*}{\Longrightarrow} w$ for all the words w in L and all admissible paths in $\mathcal{A}$, so that we have $\gamma_{L,\mathcal{C}} \equiv \gamma(\mathcal{A})$. □

6. Appendix: Proof of Theorem 3.3 (the ergodic case)

Let $\mathcal{A} = (\mathcal{G}(V,E), X, \mathcal{I}, \mathcal{F}, \omega)$ be an ergodic automaton. Denote by $\mathcal{H} = \mathcal{G}^e$ the subgraph obtained by removing the edge $e \in E$.

In what follows it will convenient to associate with a path $p = e_1 e_2 \cdots e_n \in \mathcal{P}$ the sequence of its vertices $\overline{p} = v_0 v_1 \cdots v_n$, where $v_i = p(i)$ (note that if $\mathcal{G}$ is simple, i.e. $E \subseteq V \times V$, then $\overline{p}$ determines p uniquely). We then say that p is *simple* if all the vertices are distinct; it is a *cycle* when $v_0 = v_n$ and $v_i \neq v_j$ if $\{i,j\} \neq \{0,n\}$.

Given an arbitrary path $p = e_1 e_2 \cdots e_n$, we can form its *decomposition into cycles* as follows. If $\overline{p} = v_0 v_1 \cdots v_n$, let i_1 be the largest index such that the vertices $v_0, v_1, \ldots, v_{i_1 - 1}$ are all distinct. Then $v_{i_1} = v_{j_1}$ for a suitable $j_1 < i_1$ and $c_1 := e_{j_1+1} e_{j_1+2} \cdots e_{i_1}$ is the *first cycle* of p; also set $r_1 := e_1 e_2 \cdots e_{j_1}$. Consider further the largest index $i_2 > i_1$, such that the vertices $v_{i_1}, v_{i_1+1}, \ldots, v_{i_2-1}$ are all distinct. Then $V_{i_2} = v_{j_2}$ for a suitable $j_2 \in \{i_1, i_1+1, \ldots i_2 - 1\}$ and $c_2 := e_{j_2+1} e_{j_2+2} \cdots e_{i_2}$ is the *second cycle* of the path; set $r_2 := e_{i_1+1} e_{i_1+2} \cdots e_{j_2}$. Continuing this way we obtain the canonical decomposition $p \equiv r_1 c_1 r_2 c_2 \cdots r_k c_k r_{k+1}$, where $c_1, c_2, \ldots, c_k$ are the cycles and $r_1, r_2, \ldots, r_{k+1}$ are simple (possibly empty) paths. With this notation we say that

$$p_s := e_1 e_2 \cdots e_{j_s} e_{i_s+1} e_{i_s+2} \cdots e_n \equiv r_1 c_1 r_2 c_2 \cdots r_s r_{s+1} \cdots c_k r_k$$

is obtained from p by *collapsing* the s-th cycle c_s.

The following is a sort of *pumping lemma*. It is proved in [18], but we include the easy proof for the convenience of the reader.

Lemma 6.1. *Let $\mathcal{G} = (V,E)$ be a strongly connected oriented graph and e an edge in E. Then there exists n such that if p is any path in $\mathcal{G}$ of length n, then there exists a path p' of length n, with the same extremities of p and containing e.*

Proof. From the strong connectedness of $\mathcal{G}$ there follows the existence of a path q starting at p_+, terminating in p_+ and containing e. Also, if n is large enough, in the canonical decomposition $p \equiv r_1 c_1 r_2 c_2 \cdots r_k c_k r_{k+1}$ of p there exists a cycle c repeated many times. If the length of c is ℓ, the length of q is m and the cycle c is repeated at least m times, we may collapse the first m copies of c and add ℓ copies of q at the beginning, and obtain the desired path p'. □

Coming back to the proof of the theorem, we first note that when $\mathcal{G}$ is simple and $\mathcal{I} = \mathcal{F} = V$ the statement reduces to the main result of [18]. We also fix some notation. Let $\mathcal{P}_n(\mathcal{G})$ and $\mathcal{P}_n^a(\mathcal{G})$ denote the set of all paths (resp. admissible paths) of length n in $\mathcal{G}$. $\mathcal{P}_{m,k}^+(\mathcal{G})$ denotes the set of all paths $p = e_1 e_2 \cdots e_k$ of length k which are the *initial part* of an admissible path of length m (i.e. there exists $p' = e_1' e_2' \cdots e_m' \in \mathcal{P}_m^a(\mathcal{G})$ s.t. $e_i = e_i' : \ i = 1, \ldots, k$). Similarly, $\mathcal{P}_{m,k}^-(\mathcal{G})$ denotes the set of all paths $p = e_1 e_2 \cdots e_k$ of length k which are the *terminal part* of an admissible path. Use the same notation and meaning for $\mathcal{H}$ instead of $\mathcal{G}$.

Let n be the integer given by Lemma 6.1 and set

$$\alpha = \frac{1}{|\mathcal{P}_n(\mathcal{H})|}.$$

Fix $k \in \mathbb{N}$ and $r \in \{0, 1, \dots, n-1\}$. We show that

$$|\mathcal{P}^a_{kn+r}(\mathcal{H})| \leq (1-\alpha)^k |\mathcal{P}^a_{kn+r}(\mathcal{G})|. \tag{6.1}$$

From the definition of α one clearly has

$$|\mathcal{P}^-_{kn+r,(k-1)n+r}(\mathcal{G})| \geq \alpha |\mathcal{P}^a_{kn+r}(\mathcal{G})|. \tag{6.2}$$

In what follows a path $p \in \mathcal{P}^a_{kn+r}(\mathcal{G})$ is regarded as a concatenation $p = p_1 p_2 \cdots p_k p_{k+1}$, where $p_1, \dots, p_k \in \mathcal{P}_n(\mathcal{G})$ and $p_{k+1} \in \mathcal{P}_r(\mathcal{G})$.

For $h = 1, 2, \dots, k$, define π_h as the set of all $p \in \mathcal{P}^a_{kn+r}(\mathcal{G})$ such that p_h contains the edge e. By Lemma 6.1, for any $p' \in \mathcal{P}^-_{kn+r,(k-1)n+r}(\mathcal{G})$ there exists a path $q \in \mathcal{P}^+_{kn+r,n}(\mathcal{G})$ containing e and such that $qp' \in \mathcal{P}^a_{kn+r}(\mathcal{G})$. Then $|\pi_1| \geq |\mathcal{P}^-_{kn+r,(k-1)n+r}(\mathcal{G})|$ which, together with (6.2), gives

$$|\mathcal{P}^a_{kn+r}(\mathcal{G}) \setminus \pi_1| \leq (1-\alpha)|\mathcal{P}^a_{kn+r}(\mathcal{G})|. \tag{6.3}$$

Now set $\mathcal{P}^h_{kn+r}(\mathcal{G}) = \mathcal{P}^a_{kn+r}(\mathcal{G}) \setminus \bigcup_{\ell=1}^h \pi_\ell$, $C^h = \{p \in \mathcal{P}^h_{kn+r}(\mathcal{G}) : p_{h+1} \text{ contains } e\}$ and let D^h be the set of all pairs $(q_1, q_2) \in \mathcal{P}^+_{kn+r,hn}(\mathcal{G}) \times \mathcal{P}^-_{kn+r,(k-h-1)n+r}(\mathcal{G})$ such that there exists $q \in \mathcal{P}_n(\mathcal{G})$ with $q_1 q q_2 \in \mathcal{P}^h_{kn+r}(\mathcal{G})$. With this terminology, (6.3) becomes $|\mathcal{P}^1_{kn+r}(\mathcal{G})| \leq (1-\alpha)|\mathcal{P}^a_{kn+r}(\mathcal{G})|$.

By Lemma 6.1, for any $(q_1, q_2) \in D^h$ there exists $p = q_1 q' q_2 \in \mathcal{P}^a_{kn+r}(\mathcal{G})$ with q' containing e; in other words $p \in C^h$.

Recalling the definition of α this gives $|C^h| \geq |D^h| \geq \alpha |\mathcal{P}^h_{kn+r}(\mathcal{G})|$. Thus

$$\begin{aligned} |\mathcal{P}^h_{kn+r}(\mathcal{G})| = |\mathcal{P}^a_{kn+r}(\mathcal{G}) \setminus \textstyle\bigcup_{\ell=1}^h \pi_\ell| = |\mathcal{P}^h_{kn+r}(\mathcal{G}) \setminus \pi_h| = |\mathcal{P}^{h-1}_{kn+r}(\mathcal{G}) \setminus C^{h-1}| \\ \leq (1-\alpha)|\mathcal{P}^{h-1}_{kn+r}(\mathcal{G})| \leq (1-\alpha)^h |\mathcal{P}^a_{kn+r}(\mathcal{G})|, \end{aligned}$$

where the last inequality follows by induction on h. Since $\mathcal{P}^a_{kn+r}(\mathcal{H}) \subseteq \mathcal{P}^k_{kn+r}(\mathcal{G})$, setting $h = k$ in the above inequalities, we obtain (6.1).

Taking the $(kn+r)$-th roots in (6.1) we get

$$|\mathcal{P}^a_{kn+r}(\mathcal{H})|^{\frac{1}{kn+r}} \leq (1-\alpha)^{\frac{k}{nk+r}} \cdot |\mathcal{P}^a_{kn+r}(\mathcal{G})|^{\frac{1}{kn+r}}$$

and observing that

$$\gamma(\mathcal{A}^e) \equiv \limsup_{m\to\infty} |\mathcal{P}^a_m(\mathcal{H})|^{\frac{1}{m}} = \max_{r=0,1,\dots,n-1} \limsup_{k\to\infty} |\mathcal{P}^a_{kn+r}(\mathcal{H})|^{\frac{1}{kn+r}}$$

(and similarly for $\gamma(\mathcal{A})$), one finally obtains

$$\gamma(\mathcal{A}^e) \leq (1-\alpha)^{\frac{1}{n}} \cdot \gamma(\mathcal{A}) < \gamma(\mathcal{A}).$$

□

Acknowledgments

This paper was written during my visit at the Mathematics Department of the Texas A & M University in the Fall 2004. I warmly thank Gilles Pisier and Slava Grigorchuk for their invitation and kindest hospitality and Zoran Sunik for stimulating discussions. I also express my gratitude to Luc Boasson for his precious help with some references.

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