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MARTINGALE PROPERTY OF EMPIRICAL PROCESSES

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ABSTRACT. It is shown that for a large collection of independent martingales, the martingale property is preserved on the empirical processes. Under the assumptions of independence and identical finite-dimensional distributions, it is proved that a large collection of stochastic processes are martingales essentially if and only if the empirical processes are also martingales. These two results have implications on the testability of the martingale property in scientific modeling. Extensions to submartingales and supermartingales are given.

1. INTRODUCTION

A martingale is a fundamental concept in probability theory. The theory of martingales has important applications not only in many areas in mathematics, such as in harmonic analysis, potential theory and partial differential equations ([3], [6], [7] and [10]), but also in finance and other fields. For example, the so-called martingale measures play a key role in the pricing of various financial instruments ([11] and [14]).

This paper considers a process consisting of a large collection of independent martingales. We study martingale property of the empirical process for a given random realization, where the empirical process is a stochastic process whose sample space is the space indexing the collection of independent martingales. Since the empirical process is observable after a random realization, its properties will have implications on the testability of assumptions on the underlying stochastic processes in scientific modeling.

However, there is a technical mathematical problem with a large collection of independent martingales indexed by a non-atomic probability measure space. Proposition 1.1 of [25] shows that independence and joint measurability with respect to the usual measure-theoretic product are never compatible with each other except for some trivial cases. Earlier observations of such a measurability problem can be traced back to the work of Doob in [8] and [9, p. 67]. This means that a richer product measure-theoretic framework that extends the usual measure-theoretic product is needed for the study of processes with independent random variables or independent stochastic processes. As demonstrated in [24] and [25], the Loeb product probability spaces ([19] and [21]) do provide such a rich framework for working with processes with independent random variables. Here we show that the Loeb

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product framework also works well for processes consisting of a large collection of independent martingales.

Theorem 3.2 below shows that for a continuum of (discrete or continuous time) martingales that are essentially independent of each other, the empirical processes are essentially martingales. In addition, for a continuum of stochastic processes that are essentially independent of each other and have essentially the same finite-dimensional distributions, Theorem 3.4 shows that the stochastic processes are essentially martingales if and only if the empirical processes are also essentially martingales. These two results have implications on the testability of the martingale property in scientific modeling. Propositions 5.1 and 5.2 extend these results to the case of submartingales and supermartingales.

The rest of the paper is organized as follows. Section 2 introduces some basic notations and definitions as well as a statement of Keisler's Fubini theorem ([16]). The statement of the main results are given in Section 3, and their proofs in Section 4. Some simple extensions to submartingales and supermartingales are presented in Section 5. The Appendix includes the precise statements of the exact law of large numbers and a duality result in [24], which play a crucial role in the proofs of the main results in Section 4.

Before moving to the next section, we provide two remarks. First, in order to express the martingale property of empirical processes in general, one cannot consider a countable collection of stochastic processes. Second, a meta-theorem in mathematical logic (such as the transfer principle in [21]) guarantees that exact properties on Loeb spaces correspond to approximate properties on a sequence of finite probability spaces. In particular, by applying the routine (but tedious) procedure of lifting, pushing-down and transfer as in Section 9 of [24], one can claim martingale property in some approximate sense for a large but finite collection of independent martingales.

2. Preliminaries

As noted in the Introduction, the Loeb product probability spaces provide a suitable framework for the study of processes with independent random variables. We shall use this framework to study a large collection of stochastic processes as in Section 8 in [24]. The reader is referred to [1] and [21] for details on Loeb spaces. However, one can read this paper simply by assuming that the Loeb product space is a rich extension of the usual product retaining the Fubini-type property. We also note that Loeb space techniques have been applied fruitfully to mathematical physics (Chapters 5–7 in [1]), probability theory (for example, [2], [17], [22] and [23]), and other fields.

Let $(I, \mathcal{I}, \lambda)$ and $(\Omega, \mathcal{F}, P)$ be two atomless Loeb probability spaces. Their usual measure-theoretic product is denoted by $(I \times \Omega, \mathcal{I} \otimes \mathcal{F}, \lambda \otimes P)$. The completion of this usual product is also denoted by the same notation. What makes the Loeb spaces crucial in our context is that they have another kind of product, called the Loeb product, denoted by $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$. Note that the Loeb product is indeed uniquely determined by its factor Loeb spaces as shown in [18]. As noted in [2], the Loeb product extends the usual product.

Since $(I, \mathcal{I}, \lambda)$ and $(\Omega, \mathcal{F}, P)$ are assumed to be atomless, Theorem 6.2 in [24] shows that the Loeb product space $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$ is very rich in the sense that it can be endowed with independent processes that are not measurable with respect to the usual product σ-algebra $\mathcal{I} \otimes \mathcal{F}$ but have almost independent random variables with any variety of distributions. Thus, $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$ is always a proper extension of $(I \times \Omega, \mathcal{I} \otimes \mathcal{F}, \lambda \otimes P)$ (see Proposition 6.6 of [24]). Many examples of Loeb product measurable sets that are not measurable in $\mathcal{I} \otimes \mathcal{F}$ can be found in [4].

The following Fubini-type property of the Loeb product space is crucial (see [16], [20] and [21]).

Proposition 2.1 (Keisler's Fubini theorem). *Let f be a real-valued integrable function on $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$. Then* (1) *for λ-almost all $i \in I$, $f(i, \cdot)$ is an integrable function on $(\Omega, \mathcal{F}, P)$;* (2) *the function $\int_\Omega f(i, \omega) dP(\omega)$ on I is integrable on $(I, \mathcal{I}, \lambda)$; and* (3)

$$\int_I \int_\Omega f(i, \omega) dP(\omega) d\lambda(i) = \int_{I \times \Omega} f(i, \omega) d\lambda \boxtimes P(i, \omega).$$

Similar properties hold for the functions $f(\cdot, \omega)$ on I and the function $\int_I f(i, \omega) d\lambda(i)$ on Ω.

Let T be a set of time parameters, which is assumed to be the set of $\mathbb{Z}_+$ of positive integers or an interval (starting from 0) in the set $\mathbb{R}_+$ of non-negative real numbers. Let $\mathcal{B}(T)$ be the power set of T when T is the countable set $\mathbb{Z}_+$, and the Borel σ-algebra on T when T is an interval. Let X be a real-valued measurable function on the mixed product measurable space $((I \times \Omega) \times T, (\mathcal{I} \boxtimes \mathcal{F}) \otimes \mathcal{B}(T))$. A Fubini-type result can also be obtained for real-valued functions on the mixed product $((I \times \Omega) \times T, (\mathcal{I} \boxtimes \mathcal{F}) \otimes \mathcal{B}(T))$ ([23], p. 164). Throughout the paper, we assume that for each $t \in T$, $X(\cdot, \cdot, t)$ is integrable on the Loeb product space $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$, i.e., $\int_{I \times \Omega} |X(i, \omega, t)| d\lambda \boxtimes P(i, \omega) < \infty$.

For any $i \in I$, let $X^i(\cdot, \cdot) := X(i, \cdot, \cdot)$ be the corresponding function on $\Omega \times T$, and for any $\omega \in \Omega$, let $X^\omega(\cdot, \cdot) := X(\cdot, \omega, \cdot)$ be the corresponding function on $I \times T$. The Fubini-type property for the mixed product space implies that X^i is a measurable process on $(\Omega \times T, \mathcal{F} \otimes \mathcal{B}(T))$ for λ-almost all $i \in I$, and X^ω is a measurable process on $(I \times T, \mathcal{I} \otimes \mathcal{B}(T))$ for P-almost all $\omega \in \Omega$. Thus, X can be viewed as a family of stochastic processes, $X^i, i \in I$, with a sample space $(\Omega, \mathcal{F}, P)$ and a time parameter space T. For $\omega \in \Omega$, X^ω is called an empirical process with the index space $(I, \mathcal{I}, \lambda)$ as the sample space.

Note that we can take I to be a hyperfinite set in an ultrapower construction on the set of natural numbers ([1] and [21]), where I is simply an equivalence class of a sequence of finite sets. The cardinality of the set I in the usual sense is indeed the cardinality of the continuum. This means that $X^i, i \in I$, is indeed a continuum collection of stochastic processes.

For $i \in I$, let $\{\mathcal{F}_t^i\}_{t \in T}$ be a filtration on $(\Omega, \mathcal{F}, P)$. That is, it is a non-decreasing family of sub-σ-algebras of $\mathcal{F}$, and each of them contains all the P-null sets in $\mathcal{F}$. The stochastic process X^i is said to be $\{\mathcal{F}_t^i\}_{t \in T}$-adapted if the random variable $X_t^i := X(i, \cdot, t)$ is $\mathcal{F}_t^i$-measurable for all $t \in T$. The X^i is said to be an $\{\mathcal{F}_t^i\}_{t \in T}$-martingale if it is $\{\mathcal{F}_t^i\}_{t \in T}$-adapted and

$$\mathbf{E}\left(X_t^i | \mathcal{F}_s^i\right) = X_s^i, \quad s, t \in T, s \leq t.$$

For more details on martingales, the reader is referred to [7] and [9].

Let $\{\tilde{\mathcal{F}}_t^i\}_{t\in T}$ be the natural filtration generated by the stochastic process X^i as follows:

$$\tilde{\mathcal{F}}_t^i := \sigma(\{X_s^i : s \in T, s \le t\}),\ t \in T,$$

where $\sigma(\{X(i,\cdot,s) : s \in T, s \le t\})$ is the smallest σ-algebra containing all the P-null sets and with respect to $\mathcal{F}$ in which the random variables $\{X_s^i : s \in T, s \le t\}$ are measurable.

Now, for $\omega \in \Omega$, let $\{\mathcal{G}_t^\omega\}_{t\in T}$ be the natural filtration generated by the empirical process X^ω, i.e.,

$$\mathcal{G}_t^\omega := \sigma(\{X_s^\omega : s \in T, s \le t\}),\ t \in T,$$

where $X_s^\omega := X(\cdot,\omega,s)$. It is obvious that the empirical process X^ω is $\{\mathcal{G}_t^\omega\}$-adapted.

Note that X can be viewed as a stochastic process itself with the time parameter space T and the sample space $(I\times\Omega, \mathcal{I}\boxtimes\mathcal{F}, \lambda\boxtimes P)$. It thus also generates a natural filtration on the Loeb product space, which is denoted by

$$\mathcal{H}_t := \sigma(\{X_s : s \in T, s \le t\}),\quad t \in T,$$

where $X_s := X(\cdot,\cdot,s)$. Of course, $\{X_t\}_{t\in T}$ is $\{\mathcal{H}_t\}_{t\in T}$-adapted.

It is clear that martingales with respect to the above three natural filtrations can be defined as in the case of $\{\mathcal{F}_t^i\}_{t\in T}$. In the next section, we present two theorems on the martingale property of the empirical processes X^ω.

3. The main results

Before stating the first main result, we need the following definition on the independence of stochastic processes.

Definition 3.1. (1) Two real-valued stochastic processes φ and ψ on the same sample space with time parameter space T are said to be independent, if, for any positive integers m, n, and for any $t_1^1, \cdots, t_m^1$ in T and $t_1^2, \cdots, t_n^2$ in T, the random vectors $(\varphi_{t_1^1}, \cdots, \varphi_{t_m^1})$ and $(\psi_{t_1^2}, \cdots, \psi_{t_n^2})$ are independent.

(2) We say that the stochastic processes $\{X^i, i \in I\}$ are essentially independent if, for $\lambda\boxtimes\lambda$-almost all $(i_1,i_2) \in I\times I$, the stochastic processes X^{i_1} and X^{i_2} are independent.

Note that the essential independence of the stochastic processes $\{X^i, i \in I\}$ as defined above only uses pairwise independence. Though pairwise independence and mutual independence are quite different for a countable collection of random variables (the first being, in general, weaker than the second, [13], p. 126), they are essentially equivalent for a continuum collection of random variables/stochastic processes (see Theorem 3 and Remark 4.16 in [25]). We also note that if, for all $(i_1,i_2) \in I\times I$ with $i_1 \ne i_2$, X^{i_1} and X^{i_2} are independent, then the non-atomic property of λ implies that the stochastic processes $\{X^i, i \in I\}$ are essentially independent.

We are now ready to state the first main result of this paper. It says that for a large collection of essentially independent martingales, the martingale property is essentially preserved on the empirical processes.

Theorem 3.2. *Assume that the stochastic processes $\{X^i, i \in I\}$ are essentially independent. If, for λ-almost all $i \in I$, the stochastic process X^i is an $\{\mathcal{F}_t^i\}_{t\in T}$-martingale on $(\Omega,\mathcal{F},P)$, then, for P-almost all $\omega \in \Omega$, the empirical process X^ω is a $\{\mathcal{G}_t^\omega\}_{t\in T}$-martingale on $(I,\mathcal{I},\lambda)$.*

To state the next theorem, we need the following definition.

Definition 3.3. (1) Two real-valued stochastic processes φ and ψ on some (possibly different) sample spaces with time parameter space T are said to have the same finite-dimensional distributions if, for any $t_1, \cdots, t_n \in T$, the random vectors $(\varphi_{t_1}, \cdots, \varphi_{t_n})$ and $(\psi_{t_1}, \cdots, \psi_{t_n})$ have the same distribution.

(2) We say that the stochastic processes $\{X^i, i \in I\}$ have essentially the same finite-dimensional distributions if there is a real-valued stochastic process Y with time parameter space T such that for λ-almost all $i \in I$, the stochastic processes X^i and Y have the same finite-dimensional distributions.

Under the assumptions of essential independence and essentially identical finite-dimensional distributions, the next main result of this paper shows that a large collection of stochastic processes are essentially martingales with respect to the natural filtration if and only if the empirical processes are also essentially martingales.

Theorem 3.4. *Assume that the stochastic processes $\{X^i, i \in I\}$ are essentially independent and have essentially the same finite-dimensional distributions. Then, the following are equivalent:*

(1) *For λ-almost all $i \in I$, the stochastic process X^i is an $\{\tilde{\mathcal{F}}^i_t\}_{t\in T}$-martingale on $(\Omega, \mathcal{F}, P)$.*

(2) *For P-almost all $\omega \in \Omega$, the empirical process X^ω is a $\{\mathcal{G}^\omega_t\}_{t\in T}$-martingale on $(I, \mathcal{I}, \lambda)$.*

4. Proof of the main results

This section is divided into three subsections. The first subsection provides a proof of Theorem 3.2 in the case of discrete time, while the second subsection considers the case of continuous time. The proof of Theorem 3.4 is given in the last subsection.

4.1. **Proof of Theorem 3.2 with discrete time.**

Proposition 4.1. *Theorem* 3.2 *holds for $T = \mathbb{Z}_+$.*

Proof. As in the proof of Theorem 5.8 in [24], we define an $\mathbb{R}^\infty$-valued process g on $I \times \Omega$ by letting $g(i, \omega) = \{X(i, \omega, n)\}_{n=1}^\infty$. Then it is clear that the essential independence of the real-valued discrete parameter stochastic processes $\{X^i, i \in I\}$ is equivalent to the essential independence of the $\mathbb{R}^\infty$-valued random variables $\{g_i, i \in I\}$. Note that $\mathbb{R}^\infty$ can be given a complete metric compatible with the product topology.

By the exact law of large numbers in [24] (which is included in this paper as Proposition 6.1 in the appendix), there exists a set $A \in \mathcal{F}$ with $P(A) = 1$ such that for any $\omega \in A$, the distribution μ_ω on $\mathbb{R}^\infty$ induced by the sample function g_ω on I is equal to the distribution μ on $\mathbb{R}^\infty$ induced by g viewed as a random variable on $I \times \Omega$.

Now let k, m be any positive integers with $k < m$, and let h be any bounded Borel function from $\mathbb{R}^k$ to $\mathbb{R}$. Define a Borel function H from $\mathbb{R}^\infty$ to $\mathbb{R}$ by setting $H(\{x_n\}_{n=1}^\infty) = h(x_1, \ldots, x_k) \cdot (x_m - x_k)$. Then, it is clear that

$$H(g)(i, \omega) = h\left(X(i, \omega, 1), \ldots, X(i, \omega, k)\right)\left[X(i, \omega, m) - X(i, \omega, k)\right].$$

Since h is a bounded Borel function, $H(g)$ is thus integrable on $(I\times\Omega, \mathcal{I}\boxtimes\mathcal{F}, \lambda\boxtimes P)$. It is obvious that

$$H(g_\omega)(i) = h(X_1^\omega(i), \dots, X_k^\omega(i))\,[X_m^\omega(i) - X_k^\omega(i)]\,.$$

For any $\omega \in A$, we thus obtain

$$\int_{I\times\Omega} H(g)(i,\omega)d\lambda \boxtimes P(i,\omega) = \int_{\mathbb{R}^\infty} Hd\mu = \int_{\mathbb{R}^\infty} Hd\mu_\omega = \int_I H(g_\omega)d\lambda(i).$$

This means that

$$\begin{aligned}&\int_I h(X_1^\omega(i), \dots, X_k^\omega(i))\,[X_m^\omega(i) - X_k^\omega(i)]\,d\lambda \\ &= \int_{I\times\Omega} h\left(X_1(i,\omega), \dots, X_k(i,\omega)\right)[X_m(i,\omega) - X_k(i,\omega)]\,d\lambda \boxtimes P.\end{aligned} \tag{1}$$

Next, since X^i is an $\{\mathcal{F}_t^i\}_{t\in T}$-martingale for λ-almost all $i \in I$, we have, for λ-almost all $i \in I$, $\mathbf{E}_P\left(\left[X_m^i - X_k^i\right]|\mathcal{F}_k^i\right) = 0$ for all positive integers $k < m$. Since $X_1^i, \dots, X_k^i$ are all $\mathcal{F}_k^i$-measurable, we have, for λ-almost all $i \in I$,

$$\mathbf{E}_P\left(h\left(X_1^i(\omega), \dots, X_k^i(\omega)\right)\left[X_m^i - X_k^i\right]|\mathcal{F}_k^i\right) = 0. \tag{2}$$

Thus, for λ-almost all $i \in I$,

$$\int_\Omega h\left(X_1^i(\omega), \dots, X_k^i(\omega)\right)\left[X_m^i - X_k^i\right]dP(\omega) = 0. \tag{3}$$

By taking the integration on I on both sides and by using Keisler's Fubini Theorem (which is included in this paper as Proposition 2.1), we obtain

$$\int_{I\times\Omega} h\left(X_1(i,\omega), \dots, X_k(i,\omega)\right)[X_m(i,\omega) - X_k(i,\omega)]\,d\lambda \boxtimes P = 0. \tag{4}$$

By (1), we know that, for any $\omega \in A$,

$$\int_I h\left(X_1^\omega(i), \dots, X_k^\omega(i)\right)[X_m^\omega(i) - X_k^\omega(i)]\,d\lambda = 0 \tag{5}$$

holds for any positive integers with $k < m$, and for any bounded Borel function h from $\mathbb{R}^k$ to $\mathbb{R}$. Since $(X_1^\omega, \dots, X_k^\omega)$ generates $\mathcal{G}_k^\omega$ (with the addition of the λ-null sets in $\mathcal{I}$), the arbitrary choice of h implies

$$\mathbf{E}_\lambda\left([X_m^\omega - X_k^\omega]\,|\mathcal{G}_k^\omega\right) = 0. \tag{6}$$

Hence, we know that the empirical process X^ω is a $\{\mathcal{G}_t^\omega\}_{t\in T}$-martingale on $(I, \mathcal{I}, \lambda)$ for P-almost all $\omega \in \Omega$. □

4.2. **Proof of Theorem 3.2 with continuous time.** For the case where T is an interval (starting from 0) in $\mathbb{R}_+$, we remark that even if $T = \mathbb{R}_+$, the martingale property we want to check only depends upon the closed time interval $[0, t]$ for $t \in \mathbb{R}_+$. Thus, without loss of generality, we assume that $T = [0, 1]$. Before proceeding further, let us present some auxiliary results.

The following lemma is a version of Hoover and Keisler's lemma (see [15], p. 172, and [17], which is also stated as Lemma 8.1 in [24]) in a way convenient for our present situation. It suffices to take $\Lambda = I \times \Omega$, the separable metric space to be $\mathbb{R}$, and to denote the composition $\phi \circ \psi$ by ψ in the statement of Lemma 8.1 of [24].

Lemma 4.2. *There is a sequence $\{t_n\}_{n=1}^{\infty}$ in $[0,1]$ and a Borel function $\psi : \mathbb{R}^{\infty} \times [0,1] \to \mathbb{R}$, such that for any $t \in [0,1]$*

$$X(i,\omega,t) = \psi(\{X(i,\omega,t_n)\}_{n=1}^{\infty}, t)$$

holds for $\lambda \boxtimes P$-almost all $(i,\omega) \in I \times \Omega$.

Now we have the following.

Proposition 4.3. *Theorem* 3.2 *is true for $T = [0,1]$.*

Proof. We follow the same idea as in the proof of Proposition 4.1.

Define an $\mathbb{R}^{\infty}$-valued process Y on $I \times \Omega$ by setting

$$Y(i,\omega) = \{X(i,\omega,t_n)\}_{n=1}^{\infty}.$$

Thus, Lemma 4.2 says that for any $t \in [0,1]$, $X(i,\omega,t) = \psi(Y(i,\omega),t)$ holds for $\lambda \boxtimes P$-almost all $(i,\omega) \in I \times \Omega$.

First, the almost independence of the real-valued stochastic processes $\{X^i, i \in I\}$ implies the almost independence of the $\mathbb{R}^{\infty}$-valued random variables $\{Y_i, i \in I\}$. By the exact law of large numbers in [24] (see Proposition 6.1 in the Appendix), there exists a set $B \in \mathcal{F}$ with $P(B) = 1$ such that for any $\omega \in B$, the distribution ν_ω on $\mathbb{R}^{\infty}$ induced by the sample function Y_ω on I is equal to the distribution ν on $\mathbb{R}^{\infty}$ induced by Y viewed as a random variable on $I \times \Omega$.

Now let s, t be any numbers in $[0,1]$ with $s < t$, $\{r_m\}_{m=1}^{\infty}$ any sequence in the interval $[0,s]$, and let f be any bounded Borel function from $\mathbb{R}^{\infty}$ to $\mathbb{R}$. Define a Borel function F from $\mathbb{R}^{\infty}$ to $\mathbb{R}$ by setting

$$F(x) = f\left(\{\psi(x,r_m)\}_{m=1}^{\infty}\right)[\psi(x,t) - \psi(x,s)]$$

for each $x = \{x_k\}_{k=1}^{\infty} \in \mathbb{R}^{\infty}$. Then, it is clear that

$$\begin{aligned} F(Y)(i,\omega) &= f\left(\{\psi(Y(i,\omega),r_m)\}_{m=1}^{\infty}\right)[\psi(Y(i,\omega),t) - \psi(Y(i,\omega),s)] \\ &= f\left(\{X_{r_m}(i,\omega)\}_{m=1}^{\infty}\right)[X_t(i,\omega) - X_s(i,\omega)]. \end{aligned}$$

Since f is a bounded Borel function, $F(Y)$ is thus integrable on $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$. It is obvious that

$$F(Y_\omega)(i) = f\left(\{X^{\omega}_{r_m}(i)\}_{m=1}^{\infty}\right)[X^{\omega}_t(i) - X^{\omega}_s(i)].$$

For any $\omega \in B$, we thus obtain

$$\int_{I \times \Omega} F(Y)(i,\omega) d\lambda \boxtimes P(i,\omega) = \int_{\mathbb{R}^{\infty}} F d\nu = \int_{\mathbb{R}^{\infty}} F d\nu_\omega = \int_I F(Y_\omega) d\lambda(i).$$

This means that

$$\begin{aligned} &\int_I f\left(\{X^{\omega}_{r_m}(i)\}_{m=1}^{\infty}\right)[X^{\omega}_t(i) - X^{\omega}_s(i)]\, d\lambda \\ &= \int_{I \times \Omega} f\left(\{X_{r_m}(i,\omega)\}_{m=1}^{\infty}\right)[X_t(i,\omega) - X_s(i,\omega)]\, d\lambda \boxtimes P. \end{aligned} \tag{7}$$

Next, since X^i is an $\{\mathcal{F}_t^i\}_{t\in T}$-martingale for λ-almost all $i \in I$, we have, for λ-almost all $i \in I$, $\mathbf{E}_P\left(\left[X_t^i - X_s^i\right] | \mathcal{F}_s^i\right) = 0$ for all $s,t \in [0,1]$ with $s < t$. For λ-almost all $i \in I$, the $\mathcal{F}_s^i$-measurability of $f\left(\{X_{r_m}^i(\cdot)\}_{m=1}^\infty\right)$ implies

$$\mathbf{E}_P\left(f\left(\{X_{r_m}^i(\cdot)\}_{m=1}^\infty\right)\left[X_t^i(\cdot) - X_s^i(\cdot)\right] | \mathcal{F}_s^i\right) = 0. \tag{8}$$

Thus, for λ-almost all $i \in I$,

$$\int_\Omega f\left(\{X_{r_m}^i(\omega)\}_{m=1}^\infty\right)\left[X_t^i(\omega) - X_s^i(\omega)\right] dP(\omega) = 0. \tag{9}$$

By taking the integration on I on both sides and by using Keisler's Fubini Theorem (Proposition 2.1), we obtain

$$\int_{I\times\Omega} f\left(\{X_{r_m}(i,\omega)\}_{m=1}^\infty\right)\left[X_t^i - X_s^i\right] d\lambda \boxtimes P = 0. \tag{10}$$

By (7), we know that, for any $\omega \in B$,

$$\int_I f\left(\{X_{r_m}^\omega(i)\}_{m=1}^\infty\right)\left[X_t^\omega(i) - X_s^\omega(i)\right] d\lambda = 0 \tag{11}$$

holds for any $s,t \in [0,1]$ with $s < t$, any sequence $\{r_m\}_{m=1}^\infty$ in the interval $[0,s]$, and for any bounded Borel function f from $\mathbb{R}^\infty$ to $\mathbb{R}$. Since $\mathcal{G}_s^\omega = \sigma(\{X(\cdot,\omega,r) : r \le s, r \in [0,1]\})$, it is easy to verify that

$$\mathcal{G}_s^\omega = \bigcup_{\{r_m\}_{m=1}^\infty \in [0,s]^\infty} \sigma(\{X(\cdot,\omega,r_m) : 1 \le m \le \infty\}),$$

which simply means that for any $C \in \mathcal{G}_s^\omega$, there is $\{r_m\}_{m=1}^\infty$ in $[0,s]^\infty$ such that $C \in \sigma(\{X(\cdot,\omega,r_m) : 1 \le m \le \infty\})$. This fact together with (11) implies that for any $s,t \in [0,1]$ with $s < t$,

$$\mathbf{E}_\lambda\left(\left[X_t^\omega - X_s^\omega\right] | \mathcal{G}_s^\omega\right) = 0. \tag{12}$$

Hence, we know that the empirical process X^ω is a $\{\mathcal{G}_t^\omega\}_{t\in T}$-martingale on $(I, \mathcal{I}, \lambda)$ for P-almost all $\omega \in \Omega$. □

4.3. **Proof of Theorem 3.4.** We are now ready to prove Theorem 3.4.

Proof. (1) $\Longrightarrow$ (2) is already covered by Theorem 3.2. We only need to show (2) $\Longrightarrow$ (1). Assume that for P-almost all $\omega \in \Omega$, the empirical process X^ω is a $\{\mathcal{G}_t^\omega\}_{t\in T}$-martingale on $(I, \mathcal{I}, \lambda)$. We need to show that for λ-almost all $i \in I$, the stochastic process X^i is an $\{\tilde{\mathcal{F}}_t^i\}_{t\in T}$-martingale on $(\Omega, \mathcal{F}, P)$.

Since the stochastic processes $\{X^i, i \in I\}$ are essentially independent and have essentially the same finite-dimensional distributions, the duality result in Theorem 8.12 of [24] (which is included in this paper as Proposition 6.2 in the Appendix) implies that the empirical processes $\{X^\omega, \omega \in \Omega\}$ are also essentially independent. Since the empirical process X^ω is also assumed to be a $\{\mathcal{G}_t^\omega\}_{t\in T}$-martingale for P-almost all $\omega \in \Omega$, we can use Theorem 3.2 to conclude that for λ-almost all $i \in I$, the stochastic process X^i is an $\{\tilde{\mathcal{F}}_t^i\}_{t\in T}$-martingale on $(\Omega, \mathcal{F}, P)$ by exchanging the notations of the two variables i and ω. □

5. Extensions

It is easy to generalize Theorems 3.2 and 3.4 to the cases of submartingales and supermartingales. One can simply prove the corresponding results for submartingales and supermartingales by replacing the equalities in Equations 2 - 6 and 8 - 12 with suitable inequalities in the proofs of Propositions 4.1 and 4.3. The proof of Theorem 3.4 also works for the new cases. Thus, we state the following two results without proofs.

Proposition 5.1. *Assume that the stochastic processes $\{X^i, i \in I\}$ are essentially independent. If, for λ-almost all $i \in I$, the stochastic process X^i is an $\{\mathcal{F}_t^i\}_{t\in T}$-submartingale (supermartingale) on $(\Omega, \mathcal{F}, P)$, then, for P-almost all $\omega \in \Omega$, the empirical process X^ω is a $\{\mathcal{G}_t^\omega\}_{t\in T}$-submartingale (supermartingale) on $(I, \mathcal{I}, \lambda)$.*

Proposition 5.2. *Assume that the stochastic processes $\{X^i, i \in I\}$ are essentially independent and have essentially the same finite-dimensional distributions. Then, the following are equivalent:*

(1) *For λ-almost all $i \in I$, the stochastic process X^i is an $\{\tilde{\mathcal{F}}_t^i\}_{t\in T}$-submartingale (supermartingale) on $(\Omega, \mathcal{F}, P)$.*

(2) *For P-almost all $\omega \in \Omega$, the empirical process X^ω is a $\{\mathcal{G}_t^\omega\}_{t\in T}$-submartingale (supermartingale) on $(I, \mathcal{I}, \lambda)$.*

Based on the proof of Theorem 3.2 (especially, (2)-(4) and (8)-(10), the following remark is obvious.

Remark 5.3. If the stochastic process X^i is an $\{\mathcal{F}_t^i\}_{t\in T}$-martingale (submartingale or supermartingale) on $(\Omega, \mathcal{F}, P)$ for λ-almost all $i \in I$, then the stochastic process $\{X_t\}_{t\in T}$ is an $\{\mathcal{H}_t\}_{t\in T}$-martingale (submartingale or supermartingale) on $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$. That is, the essential independence of the stochastic processes $\{X^i, i \in I\}$ is not required to claim the martingale (submartingale or supermartingale) property for $\{X_t\}_{t\in T}$ with the natural filtration $\{\mathcal{H}_t\}_{t\in T}$.

6. Appendix

For the convenience of the reader, we include in this Appendix the statements of two results in [24], which are used crucially in the proofs of Theorems 3.2 and 3.4.

The following is an exact law of large numbers for a continuum of essentially independent random variables, which can be found in Theorem 5.2 in [24] (also see Proposition 3.1 in [25]).

Proposition 6.1. *Let g be a process from the Loeb product space $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$ to a separable metric space S. If the random variables $g_i := f(i, \cdot)$ are essentially independent, i.e., for $\lambda \boxtimes \lambda$-almost all $(i_1, i_2) \in I \times I$, g_{i_1} and g_{i_2} are independent, then, for P-almost all $\omega \in \Omega$, the distribution μ_ω on S induced by the sample functions $g_\omega := f(\cdot, \omega)$ on I equals the distribution μ on S induced by g viewed as a random variable on $I \times \Omega$.*

Let $\{X^i, i \in I\}$ be the family of stochastic processes as introduced in Section 2. The next result shows a duality between the stochastic processes $\{X^i, i \in I\}$ and the empirical processes $\{X^\omega, \omega \in \Omega\}$. The case of continuous time is Theorem 8.12 in [24], while the discrete time case follows trivially from Theorem 7.16, as noted in Remark 5.9 in [24].

Proposition 6.2. *The stochastic processes* $\{X^i, i \in I\}$ *are essentially independent and have essentially the same finite-dimensional distributions if and only if so are the empirical processes* $\{X^\omega, \omega \in \Omega\}$.

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