TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 359, Number 3, March 2007, Pages 937–964
S 0002-9947(06)04018-9
Article electronically published on October 16, 2006

UNIQUE RANGE SETS AND UNIQUENESS POLYNOMIALS FOR ALGEBRAIC CURVES

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Abstract. We study unique range sets and uniqueness polynomials for algebraic functions on a smooth projective algebraic curve over an algebraically closed field of characteristic zero.

1. Introduction

Nevanlinna proved in 1926 that any two non-constant meromorphic functions f, g sharing five distinct values (i.e. $f^{-1}(a_i) = g^{-1}(a_i)$, for $i = 1, ..., 5$) must be the same. In [11] Sauer proved that any two different meromorphic functions on a compact Riemann surface of genus $\mathfrak{g} > 0$ cannot share more than $2 + 2\sqrt{\mathfrak{g}}$ values. This number was recently sharpened to $2+\sqrt{2\mathfrak{g}+2}$, and bounds in terms of gonality were also given by Schweizer in [12]. In [9] Gross introduced the concept of unique range sets for functions which asked when two different functions can share a set instead of several values. This problem has attracted attention not only in the area of complex analysis, but also non-archimedean analysis and number theory as well. In the course of study of unique range sets, one is often led to the determination of strong uniqueness polynomials. The purpose of this paper is to study these problems for compact Riemann surfaces. However, we prefer to phrase it in the equivalent language of smooth algebraic curves due to the possibilities of generalizing them to positive characteristic.

Throughout the paper, we let C be a smooth projective algebraic curve defined over an algebraically closed field $\mathbf{k}$ of characteristic 0. Let $\mathbf{K}$ be its function field, i.e. $\mathbf{K} := \mathbf{k}(C)$. For each point $\mathbf{p} \in C$, we may choose a uniformizer $t_{\mathbf{p}}$ to define a normalized order function $v_{\mathbf{p}} := \mathrm{ord}_{\mathbf{p}} : \mathbf{K} \to \mathbb{R} \cup \{\infty\}$ at $\mathbf{p}$. For a non-zero element $f \in \mathbf{K}$, the height $h(f)$ counts its number of poles with multiplicities, i.e. $h(f) := \sum_{\mathbf{p}\in C} -\min\{0, v_{\mathbf{p}}(f)\}$. Let S be a finite set of points in C. We denote by $\mathcal{O}_S = \{f \in \mathbf{K} \,|\, v_{\mathbf{p}}(f) \geq 0 \text{ for all } \mathbf{p} \notin S\}$ the ring of S-integers.

1.1. **Strong uniqueness polynomial.** A polynomial P in $\mathbf{k}[X]$ is called a **strong uniqueness polynomial** for a family of functions $\mathcal{F}$, if whenever there exist two non-constant f and g of $\mathcal{F}$, and a constant c such that $P(f) = cP(g)$, then we must have $c = 1$ and $f = g$. We note that this type of problem has also been studied by number theorists and presented in different manners. For interested readers, we refer to [5] for more discussion in this direction. The study of strong uniqueness polynomials for meromorphic functions, entire functions, rational functions, polynomials, non-archimedean meromorphic functions, and non-archimedean

Received by the editors September 4, 2004.
2000 *Mathematics Subject Classification.* Primary 14H05; Secondary 30D35, 14H55.

entire functions are quite complete by now. We do not attempt to give a complete introduction on these results here, but refer to [3] and [4]. However, it is important to point out how these problems were done and how they relate to the case of algebraic functions. Let $F(X,Y,Z)$ be the homogenization of $[P(X)-P(Y)]/[X-Y]$ and $F_c(X,Y,Z)$, $c\neq 0,1\in \mathbf{k}$, be the homogenization of $P(X)-cP(Y)$. Let f and g be meromorphic functions such that $P(f)=bP(g)$ for some $b\in\mathbb{C}^*$. Then $\Phi := (f,g,1):\mathbb{C}\to\mathbb{P}^2$ gives rise to a morphism, and moreover its image is in $[F(X,Y,Z)=0]$ if $b=1$ or $[F_c(X,Y,Z)=0]$ if $b=c\neq 1$. From the Picard theorem, we know this cannot happen if none of the curves $[F(X,Y,Z)=0]$ and $[F_c(X,Y,Z)=0]$, for all $c\neq 0,1$, contains any component of genus zero or one. In [3], this was done by constructing two linearly independent regular 1-forms on these curves. For the case of rational functions or non-archimedean meromorphic functions, it suffices to construct one regular 1-form on these curves. If f and g are algebraic functions in $\mathbf{K}$, then Φ becomes a morphism from C into one of the above curves. By the Hurwitz theorem, we know this cannot happen if these curves have no component of genus $\leq \mathfrak{g}$. It does not seem possible to treat this case by constructing $\mathfrak{g}+1$ linearly independent 1-forms since $\mathfrak{g}$ can be large. When $\mathfrak{g}\geq 2$ and all of the curves $[F(X,Y,Z)=0]$ and $[F_c(X,Y,Z)=0]$, for all $c\neq 0,1$, contain only components of genus ≥ 2, we cannot expect that there is no morphism between them, however, by the theorem of de Franchis we do expect there are only finitely many such morphisms. In this case, we will get a finite bound on the height of f and g. We also note that if the coefficients of $P(X)$ are in a number field $\mathbf{K}$, then by Mordell's Conjecture (now Faltings' theorem) [7] for each $c\neq 0\in\mathbf{K}$ there are only finitely many pairs of (x,y) in $\mathbf{K}\times\mathbf{K}$ with $x\neq y$ such that $P(x)=cP(y)$ if (i) $[F(X,Y,Z)=0]$ when $c=1$ or (ii) $[F_c(X,Y,Z)=0]$ when $c\neq 0,1$ contains no component of genus zero or one. Therefore, the results in [3] also give answers for such problems in number fields.

From now we will let $P(X)$ be a polynomial of degree n in $\mathbf{k}[X]$. We will use l to denote the number of distinct roots of $P'(X)$, and we will denote those roots by $\alpha_1,\alpha_2,...,\alpha_l$. We will use $m_1,m_2,...,m_l$ to denote the multiplicities of the roots in P'. Thus,

$$P'(X)=a(X-\alpha_1)^{m_1}(X-\alpha_2)^{m_2}...(X-\alpha_l)^{m_l}, \tag{1}$$

where a is some non-zero constant. We will continually assume what we call **Hypothesis I**:

$$P(\alpha_i)\neq P(\alpha_j)\quad\text{whenever}\quad i\neq j,$$

or in other words P is injective on the roots of P'. We note that the Hypothesis I is a generic condition, and one can see later from our arguments that it makes the computation easier. For simplicity, we denote the following special cases for $P(X)$ as follows:

(1A) $l=2$ and $\min\{m_1,m_2\}=1$;
(1B) $l=2$ and $m_1=m_2=1$;
(1C) $l=2$ and $m_1=m_2=2$;
(1D) $l=3$ and $m_1=m_2=m_3=1$;
(1E) $l=3$ and $m_1=m_2=m_3=1$, and there exist a permutation ϕ of $\{1,2,3\}$ such that $\phi(i)\neq i$ for $i=1,2,3$ and w satisfying $w^2+w+1=0$ such that $w=\frac{P(\alpha_i)}{P(\alpha_{\phi(i)})}$ for $i=1,2,3$.

Recall that a subset U of $\mathbf{k}$ is **affinely rigid** if there exists no linear transformation T such that $T\mathcal{U} = \mathcal{U}$. The main results are as follows.

Theorem 1. *Let $P(X)$ be a polynomial as above that satisfies Hypothesis* I.

(I) (a) *Let $\mathfrak{g} = 0$. $P(X)$ is a strong uniqueness polynomial for $\mathbf{K}$ if and only if the zero set $\mathcal{U}$ of P is affinely rigid and P does not satisfy* (1A) *or* (1E).

(b) *Let $\mathfrak{g} = 1$. $P(X)$ is a strong uniqueness polynomial for $\mathbf{K}$ if the zero set $\mathcal{U}$ of P is affinely rigid and P does not satisfy* (1A), (1C), *or* (1D).

(c) *Let $\mathfrak{g} \geq 1$. Assume that $\mathcal{U}$ is affinely rigid. $P(X)$ is a strong uniqueness polynomial for $\mathbf{K}$ if $l \geq 2\mathfrak{g}+4$.*

(II) *If $|S| = 1$, then $P(X)$ is a strong uniqueness polynomial for $\mathcal{O}_S$ if and only if $\mathcal{U}$ is affinely rigid.*

Theorem 2. *Let $P(X)$ be a polynomial as above satisfying Hypothesis* I *and let its zero set $\mathcal{U}$ be affinely rigid. Suppose that f and g are two distinct non-constant functions in $\mathbf{K}$ such that $P(f) = cP(g)$ for some $c \neq 0 \in \mathbf{k}$. Then:*

(a) *$h(f) = h(g) \leq 8\mathfrak{g} - 8$ if P does not satisfy* (1A), (1C) *or* (1D),

(b) *$h(f) = h(g) \leq 6\mathfrak{g} - 6 + 3|S|$ if f and g are S-integers and P does not satisfy* (1B).

Remark 1. If the characteristic of $\mathbf{k}$ is $p > 0$ and $p \nmid n$, the proof for the theorems above can be carried out word to word if we assume that the multiplicity of $X - \alpha_i$ in $P(X) - P(\alpha_i)$ is $m_i + 1$, for $i = 1, ..., l$.

Remark 2. In [3], they actually showed that none of the curves $[F(X,Y,Z) = 0]$ and $[F_c(X,Y,Z) = 0]$, for all $c \neq 0, 1$, contain any component of genus zero or one (resp. zero) if and only if the zero set of P is affinely rigid and P does not satisfy (1A), (1C) or (1D) (resp. (1A) or (1E)).

As we have mentioned before, the construction of regular 1-forms does not work for general function fields. We will treat the problems by comparing height functions. Another advantage of the methods we present in this paper is that it can treat the S-integer case at the same time it is corresponding to the case of entire functions (cf. [2] and [6]). The study of these problems are included in section 2.

1.2. **Unique range sets.** For simplicity of notation, for $\eta \in \mathbf{K}^\star$ we let

$$v^0_{\mathbf{p}}(\eta) := \max\{0, v_{\mathbf{p}}(\eta)\}, \qquad \bar{v}^0_{\mathbf{p}}(\eta) := \min\{1, v^0_{\mathbf{p}}(\eta)\},$$

i.e. its order of zero at $\mathbf{p}$ and its truncated value;

$$v^\infty_{\mathbf{p}}(\eta) := -\min\{0, v_{\mathbf{p}}(\eta)\}, \qquad \bar{v}^\infty_{\mathbf{p}}(\eta) := \min\{1, v^\infty_{\mathbf{p}}(\eta)\},$$

i.e. its order of pole at $\mathbf{p}$ and its truncated value.

Let $\mathcal{U}$ be a subset of $\mathbf{k}$. We define

$$E^m_{\bar{S}}(f, \mathcal{U}) = \bigcup_{a \in \mathcal{U}} \{(\mathbf{p},\ \min\{m, v^0_{\mathbf{p}}(f-a)\}) \mid \mathbf{p} \notin S\},$$

where m is a positive integer or ∞. Let f and g be two non-constant elements of $\mathbf{K}$. We say that they **share $\mathcal{U}$ over $\bar{S}$ counting multiplicities** (CM for short) if

$$E^\infty_{\bar{S}}(f, \mathcal{U}) = E^\infty_{\bar{S}}(g, \mathcal{U}),$$

and **share $\mathcal{U}$ over $\bar{S}$ ignoring multiplicities** (IM for short) if

$$E^1_{\bar{S}}(f, \mathcal{U}) = E^1_{\bar{S}}(g, \mathcal{U}).$$

We note that our definition is slightly more general than that of Gross since S can be chosen to be any finite subset of C. A set $\mathcal{U}$ is called a **unique range set over $\bar{S}$ CM** (resp. **IM**) for a subfamily $\mathcal{F}$ of $\mathbf{K}$ (for example, take $\mathcal{F}$ to be $\mathbf{K}$ or $\mathcal{O}_S$) if whenever f and g share $\mathcal{U}$ over $\bar{S}$ CM (resp. IM), then one must have $f \equiv g$.

The main results are

Theorem 3. *Let $\mathcal{U} := \{u_1, ..., u_n\}$ be an affinely rigid subset of $\mathbf{k}$, and let $P(X) = (X - u_1) \cdots (X - u_n)$ satisfying Hypothesis* I *and $P'(X)$ be as in* (1). *Assume that P does not satisfy* (1A), (1C), *or* (1D). *Assume further that $l \geq 2\mathfrak{g} + 4$ if $\mathfrak{g} \geq 2$. Then $\mathcal{U}$ is a unique range set over $\bar{S}$:*

(a) *IM for $\mathbf{K}$, if $n > \max\{2l + 13,\ 2l + 2 + 13\mathfrak{g} + 2|S|\}$;*
(b) *CM for $\mathbf{K}$, if $n > \max\{2l + 7,\ 2l + 2 + 7\mathfrak{g} + 2|S|\}$;*
(c) *IM for $\mathcal{O}_S$, if $n > \max\{2l + 6,\ 2l - 5 + 13\mathfrak{g} + 6|S|\}$;*
(d) *CM for $\mathcal{O}_S$, if $n > \max\{2l + 3,\ 2l - 2 + 7\mathfrak{g} + 3|S|\}$.*

Once again, when none of the curves $[F(X, Y, Z) = 0]$ and $[F_c(X, Y, Z) = 0]$, for all $c \neq 0, 1$, contains any component of genus zero or one, we obtain a bound on the height of f and g in the following situations.

Theorem 4. *Let $\mathcal{U} := \{u_1, ..., u_n\}$ be an affinely rigid subset of $\mathbf{k}$, and let $P(X) = (X - u_1) \cdots (X - u_n)$ satisfying Hypothesis* I *and $P'(X)$ be as in* (1). *Assume that P does not satisfy* (1A), (1C), *or* (1D).

(I) *Suppose that f and g share $\mathcal{U}$ over $\bar{S}$*
  (a) *IM, then $h(f) + h(g) \leq 26\mathfrak{g} - 20 + 4|S|$ if $n \geq 2l + 13$;*
  (b) *CM, then $h(f) + h(g) \leq 22\mathfrak{g} - 8 + 4|S|$ if $n \geq 2l + 7$.*

(II) *Suppose that f and g are S-integers and share $\mathcal{U}$ over $\bar{S}$*
  (a) *IM, then $h(f) + h(g) \leq 26\mathfrak{g} - 20 + 12|S|$ if $n \geq 2l + 6$;*
  (b) *CM, then $h(f) + h(g) \leq 22\mathfrak{g} - 8 + 10|S|$ if $n \geq 2l + 3$.*

The study of unique range sets is somehow more difficult, and far from complete except for the case of polynomials and non-archimedean entire functions. In this paper, we adapt an approach of Fujimoto [8] where he treated the case of meromorphic functions and entire functions. In section 3, we will prove a stronger version of the truncated second main theorem of algebraic function fields and treat the sharing value set problem.

2. Strong uniqueness polynomials

In this section, we let $P(X)$ be a monic polynomial of degree n in $\mathbf{k}[X]$, and let $\mathcal{U}$ be the zero set of the polynomial P. We will use l to denote the number of distinct roots of $P'(X)$, and we will denote those roots by $\alpha_1, \alpha_2, ..., \alpha_l$. We will use $m_1, m_2, ..., m_l$ to denote the multiplicities of the roots in P'. Thus,

$$P'(X) = n(X - \alpha_1)^{m_1}(X - \alpha_2)^{m_2}...(X - \alpha_l)^{m_l}.$$

We will continually assume what we call **Hypothesis I**:

$$P(\alpha_i) \neq P(\alpha_j) \quad \text{whenever} \quad i \neq j.$$

We will also use the following expansion of P at α_i:

$$P(X) - P(\alpha_i) = b_{i,m_i+1}(X - \alpha_i)^{m_i+1} + \cdots + b_{i,n}(X - \alpha_i)^n. \tag{2.1}$$

We will study some sufficient conditions for P to be a strong uniqueness polynomial for $\mathbf{K}$ and also for $\mathcal{O}_S$, the ring of S-integers in $\mathbf{K}$. When $\mathfrak{g} \geq 1$, we will also study the height of f and g if they satisfy the equation $P(f) = cP(g)$ for some non-zero $c \in \mathbf{k}$.

We first make the following observation.

Proposition 5. *Assume that $\mathcal{U}$ is affinely rigid.*

(i) *If f and g are two distinct non-constant functions in $\mathbf{K}$ such that $P(f) = cP(g)$ for some $c \neq 0 \in \mathbf{k}$, then f and g satisfy no linear relation, i.e. $g \neq \lambda f + \beta$ for any $\lambda, \beta \in \mathbf{k}$.*

(ii) *$l \geq 2$.*

Proof. Suppose that $g = \lambda f + \beta$ for some $\lambda, \beta \in \mathbf{k}$. Then $P(f) = cP(\lambda f + \beta)$ since $P(f) = cP(g)$. Clearly, $(\lambda, \beta) \neq (1, 0)$ since $f \neq g$. Let $\mathcal{U} = \{u_1, ..., u_n\}$. Then

$$(f - u_1) \dots (f - u_n) = c(\lambda f + \beta - u_1) \dots (\lambda f + \beta - u_n).$$

Since f is non-constant in $\mathbf{K}$, this has to be a trivial relation, i.e. $\lambda^{-n} = c$ and $\lambda\mathcal{U} + \beta = \mathcal{U}$. Therefore $\mathcal{U}$ is not affinely rigid. This proves (i).

For (ii), we see that if $l = 1$, then $P(X) = (X - \alpha_1)^n + b$ for some $b \in \mathbf{k}$. Let $\epsilon \neq 1$ be an n-th root of unity, let f be any non-constant function in $\mathbf{K}$, and let $g = \epsilon f - \epsilon\alpha_1 + \alpha_1$. Then $P(f) = P(g)$, which shows that $\mathcal{U}$ is not affinely rigid by (i). $\square$

For $[f, g] \in \mathbb{P}^1(\mathbf{K})$, its height is defined by

$$h(f, g) := \sum_{\mathbf{p} \in C} - \min\{v_{\mathbf{p}}(f), v_{\mathbf{p}}(g)\}.$$

Clearly, $h(f) = h(f, 1)$.

For simplicity of notation, for $i \geq 1$, $t \in \mathbf{K} \setminus \mathbf{k}$ and $\eta \in \mathbf{K}$, we denote by

$$d_t^i \eta := \frac{d^i \eta}{dt^i}, \qquad d_{\mathbf{p}}^i \eta := \frac{d^i \eta}{dt_{\mathbf{p}}^i}.$$

We recall the following well-known properties, which follow from the Riemann-Roch theorem and the sum formula.

Proposition 6. *Let $\eta \neq 0 \in \mathbf{K}$ and $[f, g] \in \mathbb{P}^1(\mathbf{K})$. We have*

(i) *$\sum_{\mathbf{p} \in C} v_{\mathbf{p}}(d_{\mathbf{p}}\eta) = 2\mathfrak{g} - 2$ if η is not constant.*

(ii) *$\sum_{\mathbf{p} \in C} v_{\mathbf{p}}(\eta) = 0$.*

(iii) *$h(\eta f, \eta g) = h(f, g)$.*

Lemma 7. *Suppose that f and g are distinct non-constant functions in $\mathbf{K}$ and $P(f) = cP(g)$ for some non-zero constant c. Then $h(f) = h(g)$, and*

(i) *$h(P'(f), P'(g)) + \sum_{\mathbf{p} \in C} \min\{v^0_{\mathbf{p}}(d_{\mathbf{p}}f), v^0_{\mathbf{p}}(d_{\mathbf{p}}g)\} \leq 2h(f) + 2\mathfrak{g} - 2$;*

(ii) *$h(P'(f), P'(g)) \leq h(f) + |S| + 2\mathfrak{g} - 2$, if f and g are S-integers;*

(iii) *$h(f) \geq 2$, if $\mathcal{U}$ is affinely rigid;*

(iv) *$|S| \geq 2$, if $\mathcal{U}$ is affinely rigid and f and g are S-integers.*

Remark. We note that (i) implies that $h(P'(f), P'(g)) \leq 2h(f) + 2\mathfrak{g} - 2$, since $v^0_{\mathbf{p}}(d_{\mathbf{p}}f) \geq 0$ and $v^0_{\mathbf{p}}(d_{\mathbf{p}}g) \geq 0$.

Proof. Since $P(f) = cP(g)$ for any $\mathbf{p}$ such that $v_{\mathbf{p}}(f) < 0$, we have $nv_{\mathbf{p}}(f) = v_{\mathbf{p}}(P(f)) = v_{\mathbf{p}}(P(g)) = nv_{\mathbf{p}}(g)$. Hence, $h(f) = h(g)$. It also yields that

$$d_t f P'(f) = c d_t g P'(g),$$

for t in $\mathbf{K} \setminus \mathbf{k}$, and hence

$$h(P'(f), P'(g)) = h(P'(f)/P'(g)) = h(c\, d_t g/d_t f) = h(d_t f, d_t g). \tag{2.2}$$

Since $d_{\mathbf{p}} f = d_t f d_{\mathbf{p}} t$,

$$v_{\mathbf{p}}(d_t f) = v_{\mathbf{p}}(d_{\mathbf{p}} f) - v_{\mathbf{p}}(d_{\mathbf{p}} t).$$

If $v_{\mathbf{p}}(f) < 0$, then $v_{\mathbf{p}}(d_{\mathbf{p}} f) = v_{\mathbf{p}}(f) - 1$. If $v_{\mathbf{p}}(f) \geq 0$, then $v_{\mathbf{p}}(d_{\mathbf{p}} f) \geq 0$.

All together, we have

$$\begin{aligned}
&h(P'(f), P'(g)) = h(d_t f, d_t g) \\
&= \sum_{\mathbf{p} \in C} -\min\{v_{\mathbf{p}}(d_t f), v_{\mathbf{p}}(d_t g)\} \\
&= \sum_{\mathbf{p} \in C} v_{\mathbf{p}}(d_{\mathbf{p}} t) + \sum_{v_{\mathbf{p}}(f) < 0} -\min\{v_{\mathbf{p}}(d_{\mathbf{p}} f), v_{\mathbf{p}}(d_{\mathbf{p}} g)\} - \sum_{v_{\mathbf{p}}(f) \geq 0} \min\{v_{\mathbf{p}}(d_{\mathbf{p}} f), v_{\mathbf{p}}(d_{\mathbf{p}} g)\} \\
&= 2\mathfrak{g} - 2 + \sum_{v_{\mathbf{p}}(f) < 0} (-v_{\mathbf{p}}(f) + 1) - \sum_{\mathbf{p} \in C} \min\{v^0_{\mathbf{p}}(d_{\mathbf{p}} f), v^0_{\mathbf{p}}(d_{\mathbf{p}} g)\} \\
&\leq h(f) + \#\{\mathbf{p} \in C \mid v_{\mathbf{p}}(f) < 0\} + 2\mathfrak{g} - 2 - \sum_{\mathbf{p} \in C} \min\{v^0_{\mathbf{p}}(d_{\mathbf{p}} f), v^0_{\mathbf{p}}(d_{\mathbf{p}} g)\}.
\end{aligned}$$

Clearly, $\#\{\mathbf{p} \in C \mid v_{\mathbf{p}}(f) < 0\} \leq h(f)$, and $\#\{\mathbf{p} \in C \mid v_{\mathbf{p}}(f) < 0\} \leq |S|$ if f is an S-integer. Therefore, we have concluded (i) and (ii).

Since f and g are not constant, we always have $h(f) \geq 1$. If $h(f) = 1$, then f has exactly one simple pole. Moreover, g also has exactly the same simple pole since they satisfy the relation $P(f) = cP(g)$. From the Laurent expansion of f and g at this simple pole, we can find a constant λ such that $f - \lambda g$ has no pole and hence it is a constant. By Proposition 5, this is impossible since $\mathcal{U}$ is assumed to be affinely rigid. Therefore, $h(f) \geq 2$.

We now prove (iv). Suppose that S contains only one point, say $\mathbf{q} \in C$, and $f \neq g$ are two S-integers such that $P(f) = cP(g)$ for some $c \neq 0 \in \mathbf{k}$. Let Φ be the morphism defined by $[f, g, 1] : C \to \mathbb{P}^2$. Then the image $\Phi(C)$ is one of the components of $[F(X, Y, Z) = 0]$ if $c = 1$ or $[F_c(X, Y, Z) = 0]$ if $c \neq 1$, since a morphism between two irreducible curves is surjective. Moreover, $\Phi(C) \cap [Z = 0] = \Phi(\mathbf{q})$ since $\mathbf{q}$ is the only pole for f and g. On the other hand, $F(X, Y, 0) = (X^n - Y^n)/(X - Y)$ and $F_c(X, Y, 0) = X^n - cY^n$ split into $n - 1$ and n distinct linear factors, respectively. If $\Phi(C)$ is not a line, then there will be at least two points in $\Phi(C) \cap [Z = 0]$, which is impossible. $\square$

The basic idea in this section is as follows. Suppose there are two distinct non-constant functions f and g in $\mathbf{K}$ such that $P(f) = cP(g)$, $c \neq 0 \in \mathbf{k}$. Lemma 7 then gives an upper bound for $h(P'(f), P'(g))$. On the other hand, to find a lower bound for $h(P'(f), P'(g))$, we will need to find an element G in $\mathbf{K}$ such that the height of G is not too big and the order of zero of G at each point of the curve is at least equal to the minimum of the order of zeros of $P'(f)$ and $P'(g)$. To construct such functions, we will use the following.

Proposition 8. *Suppose there are non-constant functions f and g in $\mathbf{K}$ such that $P(f) = cP(g)$ for some $0 \neq c \in \mathbf{k}$. If $v_{\mathbf{p}}(f - \alpha_i) > 0$ and $v_{\mathbf{p}}(g - \alpha_j) > 0$ for $\mathbf{p} \in C$, then*

(i) $(m_i + 1)v_{\mathbf{p}}(f - \alpha_i) = (m_j + 1)v_{\mathbf{p}}(g - \alpha_j)$;

(ii) $v_{\mathbf{p}}(b_{i,m_i+1}(f-\alpha_i)^{m_i+1}-cb_{j,m_j+1}(g-\alpha_j)^{m_i+1}) \geq (m_i+2)v_{\mathbf{p}}(f-\alpha_i)$, *when* $m_i = m_j$.

In particular, if $c=1$ *and* $j=i$, *then* $v_{\mathbf{p}}(g-\alpha_i) = v_{\mathbf{p}}(f-\alpha_i)$, *and*

(iii) $v_{\mathbf{p}}(f-g) \geq v_{\mathbf{p}}(f-\alpha_i)$;

(iv) $v_{\mathbf{p}}((f-\alpha_i)^{m_i+1}-(g-\alpha_i)^{m_i+1}) \geq (m_i+2)v_{\mathbf{p}}(f-\alpha_i)$.

Proof. If $v_{\mathbf{p}}(f-\alpha_i) > 0$ and $v_{\mathbf{p}}(g-\alpha_j) > 0$, then $P(\alpha_i) = cP(\alpha_j)$ since $P(f) = cP(g)$. Then the expansions of $P(X)$ at α_i and α_j in (2.1) yields

$$\begin{aligned} 0 &= P(f) - cP(g) \\ &= [b_{i,m_i+1}(f-\alpha_i)^{m_i+1} + \{\text{Higher order terms in } f-\alpha_i\ \}] \\ &\quad - c[b_{j,m_j+1}(g-\alpha_j)^{m_j+1} + \{\text{Higher order terms in } g-\alpha_j\}]. \end{aligned}$$

The assertions (i), (ii) and (iv) follow easily from this equality. (iii) follows from representing $f-g = (f-\alpha_i)-(g-\alpha_i)$. □

2.1. **On the equation $\mathbf{P(f) = P(g)}$.** Recall the following cases for $P(X)$:

(1A) $l = 2$ and $\min\{m_1, m_2\} = 1$;

(1B) $l = 2$ and $m_1 = m_2 = 1$;

(1C) $l = 2$ and $m_1 = m_2 = 2$;

(1D) $l = 3$ and $m_1 = m_2 = m_3 = 1$.

Lemma 9. *Assume that $P(X)$ is a polynomial as above satisfying Hypothesis I and let $\mathcal{U}$ be its zero set. Suppose that f and g are two distinct non-constant functions in $\mathbf{K}$ such that $P(f) = P(g)$.*

(I) *Let $\mathfrak{g} = 0$. Then either $l = 1$ or P satisfies* (1A).

(II) *Let $\mathfrak{g} \geq 1$ and $\mathcal{U}$ is affinely rigid. Then*

(a) $l \leq \mathfrak{g}+2$;

(b) $h(f) = h(g) \leq 8\mathfrak{g}-8$ *if P does not satisfy* (1A), (1C), *and* (1D);

(c) $h(f) = h(g) \leq 6\mathfrak{g}-6+3|S|$ *if f and g are S-integers and P does not satisfy* (1B).

Proof. From Lemma 7, $h(f) = h(g)$,

$$h(P'(f), P'(g)) \leq 2h(f) + 2\mathfrak{g} - 2 \tag{2.1.1}$$

and

$$h(P'(f), P'(g)) \leq h(f) + |S| + 2\mathfrak{g} - 2, \tag{2.1.2}$$

if f and g are S-integers. Since P satisfies Hypothesis I, the only common zeros of $P'(f)$ and $P'(g)$ are those $\mathbf{p} \in C$ such that $(f(\mathbf{p}), g(\mathbf{p})) = (\alpha_i, \alpha_i)$, $1 \leq i \leq l$. Therefore, $\min\{v_{\mathbf{p}}(P'(f)), v_{\mathbf{p}}(P'(g))\} > 0$ only if $v_{\mathbf{p}}(f-\alpha_i) > 0$ and $v_{\mathbf{p}}(g-\alpha_i) > 0$ for some $i = 1, ..., l$. To obtain a lower bound of $h(P'(f), P'(g))$, we will need to construct a non-zero element G in $\mathbf{K}$ with high order of zeros at each $\mathbf{p}$ such that $v_{\mathbf{p}}(f-\alpha_i) > 0$ and $v_{\mathbf{p}}(g-\alpha_i) > 0$ for $i = 1, ..., l$. By rearranging α_i, we may assume that $m_1 \geq m_2 \geq \cdots \geq m_l$. We first take

$$G := (f-g)^{m_1}$$

which is not zero since $f \neq g$. Suppose $v_{\mathbf{p}}(f-\alpha_i) > 0$ and $v_{\mathbf{p}}(g-\alpha_i) > 0$. Clearly, $v_{\mathbf{p}}(f-\alpha_j) = v_{\mathbf{p}}(g-\alpha_j) = 0$ if $j \neq i$. By Proposition 8, $v_{\mathbf{p}}(f-\alpha_i) = v_{\mathbf{p}}(g-\alpha_i)$ and

$$v_{\mathbf{p}}(G) = m_1 v_{\mathbf{p}}(f-g) \geq m_i v_{\mathbf{p}}(f-\alpha_i) = v_{\mathbf{p}}(P'(f)) = v_{\mathbf{p}}(P'(g)).$$

Therefore, we may conclude that

$$\min\{v_{\mathbf{p}}(P'(f)), v_{\mathbf{p}}(P'(g))\} - v_{\mathbf{p}}(G) \le 0 \qquad \text{if } v_{\mathbf{p}}(f) \ge 0.$$

On the other hand, if $v_{\mathbf{p}}(f) < 0$, then $v_{\mathbf{p}}(f) = v_{\mathbf{p}}(g)$, $v_{\mathbf{p}}(P'(f)) = v_{\mathbf{p}}(P'(g)) = (n-1)v_{\mathbf{p}}(f)$ and $v_{\mathbf{p}}(f-g) \ge v_{\mathbf{p}}(f)$. We have

$$\min\{v_{\mathbf{p}}(P'(f)), v_{\mathbf{p}}(P'(g))\} - v_{\mathbf{p}}(G) \le \sum_{i=2}^{l} m_i v_{\mathbf{p}}(f).$$

Therefore,

$$\begin{aligned} h(P'(f), P'(g)) &= h(P'(f)/G, P'(g)/G) \\ &= -\sum_{v_{\mathbf{p}}(f)<0} (\min\{v_{\mathbf{p}}(P'(f)), v_{\mathbf{p}}(P'(g))\} - v_{\mathbf{p}}(G)) \\ &\quad - \sum_{v_{\mathbf{p}}(f)\ge 0} (\min\{v_{\mathbf{p}}(P'(f)), v_{\mathbf{p}}(P'(g))\} - v_{\mathbf{p}}(G)) \\ &\ge -\sum_{v_{\mathbf{p}}(f)<0} \sum_{i=2}^{l} m_i v_{\mathbf{p}}(f) = \sum_{i=2}^{l} m_i h(f). \end{aligned}$$

Together with (2.1.1) we have

$$(\sum_{i=2}^{l} m_i - 2)h(f) \le 2\mathfrak{g} - 2, \tag{2.1.3}$$

and (2.1.2) gives

$$(\sum_{i=2}^{l} m_i - 1)h(f) \le |S| + 2\mathfrak{g} - 2 \tag{2.1.4}$$

if f and g are S-integers

If $\mathfrak{g} = 0$, then (2.1.3) implies that $l = 1$ or $l = 2$ and $m_2 = 1$. This completes the proof of (I).

From now we assume that $\mathfrak{g} \ge 1$. Since f is assumed to be non-constant and $\mathcal{U}$ is affinely rigid, we have $h(f) \ge 2$. Then (2.1.3) implies that $\sum_{i=2}^{l} m_i \le \mathfrak{g} + 1$, and therefore, $l \le \mathfrak{g} + 2$. This completes the proof of (II)(a).

Equation (2.1.3) also implies that $h(f) \le 2\mathfrak{g} - 2$ if $\sum_{i=2}^{l} m_i \ge 3$ which holds in the following cases: (i) $l \ge 4$, (ii) $l = 3$ except when $m_2 = m_3 = 1$, or (iii) $l = 2$ except when $m_2 \le 2$. For (II)(b), it then remains to consider when (1) $l = 3$, $m_2 = m_3 = 1$ and $m_1 \ge 2$ and (2) $l = 2$, $m_2 = 2$ and $m_1 \ge 3$.

Case 1. $l = 3$, $m_2 = m_3 = 1$ and $m_1 \ge 2$:

Let

$$G := (f-g)^2((f-\alpha_1)^{m_1+1} - (g-\alpha_1)^{m_1+1})^{m_1-1}.$$

We claim

$$(m_1+1)\min\{v_{\mathbf{p}}(P'(f)), v_{\mathbf{p}}(P'(g))\} - v_{\mathbf{p}}(G) \le 0, \qquad \text{if } v_{\mathbf{p}}(f) \ge 0.$$

From our previous discussion, it suffices to consider those $\mathbf{p}$ for which $v_{\mathbf{p}}(f-\alpha_i) > 0$ and $v_{\mathbf{p}}(g-\alpha_i) > 0$ for some $i = 1, ..., l$. If $v_{\mathbf{p}}(f-\alpha_1) > 0$ and $v_{\mathbf{p}}(g-\alpha_1) > 0$, then from Proposition 8, $v_{\mathbf{p}}(f-\alpha_1) = v_{\mathbf{p}}(g-\alpha_1) \le v_{\mathbf{p}}(f-g)$, and

$$v_{\mathbf{p}}((f-\alpha_1)^{m_1+1} - (g-\alpha_1)^{m_1+1}) \ge (m_1+2)v_{\mathbf{p}}(f-\alpha_1).$$

Therefore,

$$v_{\mathbf{p}}(G) \geq m_1(m_1+1)v_{\mathbf{p}}(f-\alpha_1) = (m_1+1)\min\{v_{\mathbf{p}}(P'(f)), v_{\mathbf{p}}(P'(g))\}.$$

If $v_{\mathbf{p}}(f-\alpha_i) > 0$ and $v_{\mathbf{p}}(g-\alpha_i) > 0$, $i = 2, 3$, from the fact that $(f-g)^{m_1+1}$ is a factor of $(f-\alpha_1)^{m_1+1} - (g-\alpha_1)^{m_1+1}$ and $v_{\mathbf{p}}(f-g) \geq v_{\mathbf{p}}(f-\alpha_i)$, we have

$$v_{\mathbf{p}}(G) \geq (m_1+1)v_{\mathbf{p}}(f-\alpha_i) = (m_1+1)\min\{v_{\mathbf{p}}(P'(f)), v_{\mathbf{p}}(P'(g))\}.$$

Therefore, the claim is valid. It is easy to see that

$$\min\{v_{\mathbf{p}}(P'(f))^{m_1+1}, v_{\mathbf{p}}(P'(g))^{m_1+1}\} - v_{\mathbf{p}}(G) \leq (3m_1+1)v_{\mathbf{p}}(f), \quad \text{ if } v_{\mathbf{p}}(f) < 0.$$

Therefore

$$\begin{aligned}(m_1+1)h(P'(f), P'(g)) &= h(P'(f)^{m_1+1}, P'(g)^{m_1+1}) \\ &= h(P'(f)^{m_1+1}/G, P'(g)^{m_1+1}/G) \\ &\leq (3m_1+1)h(f).\end{aligned}$$

Together with (2.1.1), we have

$$\frac{m_1-1}{m_1+1}h(f) \leq 2\mathfrak{g} - 2.$$

Since $m_1 \geq 2$, it yields

$$h(f) \leq 3(2\mathfrak{g} - 2).$$

Case 2. $l = 2$, $m_2 = 2$ and $m_1 \geq 3$:
Let

$$G := (f-g)^{m_1+4}((f-\alpha_1)^{m_1+1} - (g-\alpha_1)^{m_1+1})^{m_1-2}.$$

Repeat the arguments as in Case 1 to get

$$\frac{m_1-2}{m_1+1}h(f) \leq 2\mathfrak{g} - 2.$$

If $m_1 \geq 3$, then

$$h(f) \leq 4(2\mathfrak{g} - 2).$$

This completes the proof for (II)(b).

Assume further that f and g are S-integers. Then the previous arguments imply that

$$h(f) = h(g) \leq 3(2\mathfrak{g} - 2),$$

except for (1) $l = 3$ and $m_1 = m_2 = m_3 = 1$, and (2) $l = 2$, $m_2 = 1$. For (1), we simply take $G = f - g$. Then by (2.1.2), we have $h(f) \leq 2\mathfrak{g} - 2 + |S|$. For (2), we only need to consider when $l = 2$, $m_2 = 1$, and $m_1 \geq 2$. Let

$$G := (f-g)^2((f-\alpha_1)^{m_1+1} - (g-\alpha_1)^{m_1+1})^{m_1-1}.$$

Replacing (2.1.1) in the previous arguments by (2.1.2), we have

$$\frac{m_1-1}{m_1+1}h(f) \leq 2\mathfrak{g} - 2 + |S|.$$

Since $m_1 \geq 2$, we have $h(f) \leq 3(2\mathfrak{g} - 2 + |S|)$. □

2.2. **On the equation $\mathbf{P(f) = cP(g)}$, $\mathbf{c \neq 0, 1}$.** Recall the following special cases for $P(X)$:

(1A) $l = 2$ and $\min\{m_1, m_2\} = 1$;
(1B) $l = 2$ and $m_1 = m_2 = 1$;
(1C) $l = 2$ and $m_1 = m_2 = 2$;
(1D) $l = 3$ and $m_1 = m_2 = m_3 = 1$;
(1E) $l = 3$ and $m_1 = m_2 = m_3 = 1$ and there exist a permutation ϕ of $\{1, 2, 3\}$ such that $\phi(i) \neq i$ for $i = 1, 2, 3$ and w satisfying $w^2 + w + 1 = 0$ such that $w = \frac{P(\alpha_i)}{P(\alpha_{\phi(i)})}$ for $i = 1, 2, 3$.

We will establish the following results in this subsection.

Lemma 10. *Assume that $P(X)$ is a polynomial as above satisfying Hypothesis* I *and the zero set $\mathcal{U}$ of P is affinely rigid. Suppose that f and g are two distinct non-constant functions in $\mathbf{K}$ such that $P(f) = cP(g)$ for some $c \neq 0, 1 \in \mathbf{k}$. We have the following:*

(I) *Let $\mathfrak{g} = 0$. Then P satisfies* (1A) *or* (1E).
(II) *Let $\mathfrak{g} \geq 1$. Then $l \leq 2\mathfrak{g} + 3$.*

Lemma 11. *Let $P(X)$ be a polynomial as above satisfying Hypothesis* I. *Assume the zero set $\mathcal{U}$ of P is affinely rigid. Suppose that f and g are two distinct non-constant functions in $\mathbf{K}$ such that $P(f) = cP(g)$ for some $c \neq 0, 1 \in \mathbf{k}$. We have the following:*

(I) *Let $\mathfrak{g} \geq 1$. Then $h(f) = h(g) \leq 4\mathfrak{g} - 4$ if P does not satisfy* (1A) *or* (1D).
(II) *$h(f) = h(g) \leq 4\mathfrak{g} - 4 + 2|S|$, if f and g are S-integers and it does not satisfy* (1B).

The proof of the above lemmas is similar to the previous subsection. We first state some facts that will be used throughout this subsection. Let

$$l_0 := \#\{(i, j) \mid P(\alpha_i) = cP(\alpha_j)\}.$$

Since $P(X)$ satisfies Hypothesis I, it is easy to see that $0 \leq l_0 \leq l$ and $l_0 = l$ if and only if there exists a permutation ϕ of $\{1, 2, ..., l\}$ such that $(\alpha_i, \alpha_{\phi(i)}, 1) \in C_c$ for any $i = 1, ..., l$, i.e.

$$\frac{P(\alpha_1)}{P(\alpha_{\phi(1)})} = \frac{P(\alpha_2)}{P(\alpha_{\phi(2)})} = ... = \frac{P(\alpha_l)}{P(\alpha_{\phi(l)})} = c.$$

For simplicity of notation, in what follows ϕ **will always be a permutation of** $\mathbf{(1, 2, ..., l)}$ **such that** $\mathbf{\phi(i) = j}$ **if** $\mathbf{P(\alpha_i) = cP(\alpha_j)}$**.** For a fixed permutation ϕ and $1 \leq i \neq j \leq l$ we define $L^{\phi}_{i,j}(f, g) = L_{i,j}$ as follows:

$$L_{i,j} := (g - \alpha_{\phi(i)}) - \frac{\alpha_{\phi(i)} - \alpha_{\phi(j)}}{\alpha_i - \alpha_j}(f - \alpha_i) \tag{2.2.1}$$

which can also be expressed as

$$L_{i,j} = (g - \alpha_{\phi(j)}) - \frac{\alpha_{\phi(i)} - \alpha_{\phi(j)}}{\alpha_i - \alpha_j}(f - \alpha_j). \tag{2.2.2}$$

At each point $\mathbf{p} \in C$ we infer from (2.2.1) and (2.2.2) that

$$\begin{aligned} v_{\mathbf{p}}(L_{i,j}) &\geq \min\{v_{\mathbf{p}}(f - \alpha_i), v_{\mathbf{p}}(g - \alpha_{\phi(i)})\}, \qquad \text{and} \\ v_{\mathbf{p}}(L_{i,j}) &\geq \min\{v_{\mathbf{p}}(f - \alpha_j), v_{\mathbf{p}}(g - \alpha_{\phi(j)})\}. \end{aligned} \tag{2.2.3}$$

Proof of Lemma 10. Since $P(f) = cP(g)$, it follows from Lemma 7 that $h(f) = h(g)$,

$$h(P'(f), P'(g)) \leq 2h(f) + 2\mathfrak{g} - 2, \tag{2.2.4}$$

and

$$h(P'(f), P'(g)) \leq h(f) + |S| + 2\mathfrak{g} - 2, \tag{2.2.5}$$

if f and g are S-integers. Let

$$\begin{aligned} A_0 &:= \{i,\, 1 \leq i \leq l \mid P(\alpha_i) = cP(\alpha_{\phi(i)})\}, \quad l_0 = \#A_0, \\ A_1 &:= \{i \in A_0 \mid m_i = m_{\phi(i)}\}, \quad l_1 = \#A_1, \\ A_2 &:= \{i \in A_0 \mid m_i > m_{\phi(i)}\}, \quad l_2 = \#A_2, \\ A_3 &:= \{i \in A_0 \mid m_i < m_{\phi(i)}\}, \quad l_3 = \#A_3. \end{aligned}$$

Without loss of generality, we let $A_1 = \{1, 2, ..., l_1\}$, and $m_1 \geq m_2 \geq \cdots \geq m_{l_1}$. Let

$$G := (f - \alpha_{l_1})^{m_{l_1}(l_1 - 2[\frac{l_1}{2}])} \prod_{i=1}^{[\frac{l_1}{2}]} (L_{2i-1,2i})^{m_{2i-1}} \prod_{i \in A_2} (g - \alpha_{\phi(i)})^{m_{\phi(i)}} \prod_{i \in A_3} (f - \alpha_i)^{m_i},$$

which is not zero since we assume that $\mathcal{U}$ is affinely rigid. Similar to the proof of Lemma 9, we claim that

$$\min\{v_{\mathbf{p}}(P'(f)),\ v_{\mathbf{p}}(P'(g))\} - v_{\mathbf{p}}(G) \leq 0, \qquad \text{if} \quad v_{\mathbf{p}}(f) \geq 0.$$

For this, it suffices to verify for those $\mathbf{p}$ such that $v_{\mathbf{p}}(f - \alpha_i) > 0$ and $v_{\mathbf{p}}(g - \alpha_{\phi(i)}) > 0$ for some $i \in A_0$. If $i \in A_1$, then $v_{\mathbf{p}}(f - \alpha_i) = v_{\mathbf{p}}(g - \alpha_{\phi(i)})$ and the assertion follows from (2.2.3). The assertion is also clear when $i \in A_2$ or A_3 since $\min\{v_{\mathbf{p}}(P'(f)),\ v_{\mathbf{p}}(P'(g))\} = \min\{m_i v_{\mathbf{p}}(f - \alpha_i), m_{\phi(i)} v_{\mathbf{p}}(g - \alpha_{\phi(i)})\}$.

If $v_{\mathbf{p}}(f) < 0$, then $v_{\mathbf{p}}(f) = v_{\mathbf{p}}(g) = v_{\mathbf{p}}(L_{2i-1,2i}) = v_{\mathbf{p}}(g - \alpha_j) = v_{\mathbf{p}}(f - \alpha_{l_1})$. Hence,

$$\begin{aligned} &\min\{v_{\mathbf{p}}(P'(f)),\ v_{\mathbf{p}}(P'(g))\} - v_{\mathbf{p}}(G) \\ &\leq [\sum_{i=1}^{[\frac{l_1}{2}]} m_{2i} + \sum_{i \in A_2} (m_i - m_{\phi(i)}) + \sum_{i \notin A_0} m_i] v_{\mathbf{p}}(f). \end{aligned}$$

In conclusion, we have

$$\begin{aligned} h(P'(f), P'(g)) &= h(P'(f)/G, P'(g)/G) \\ &\geq [\sum_{i \notin A_0} m_i + \sum_{i=1}^{[\frac{l_1}{2}]} m_{2i} + \sum_{i \in A_2} (m_i - m_{\phi(i)})] h(f). \end{aligned}$$

Together with (2.2.4) we have

$$[-2 + \sum_{i \notin A_0} m_i + \sum_{i=1}^{[\frac{l_1}{2}]} m_{2i} + \sum_{i \in A_2} (m_i - m_{\phi(i)})]\, h(f) \leq 2\mathfrak{g} - 2, \tag{2.2.6}$$

and, similarly

$$[-2 + \sum_{i \notin A_0} m_i + \sum_{i=1}^{[\frac{l_1}{2}]} m_{2i} + \sum_{i \in A_3} (m_{\phi(i)} - m_i)]\, h(f) \leq 2\mathfrak{g} - 2. \tag{2.2.7}$$

If f and g are S-integers, then (2.2.5) implies that

$$[-1+\sum_{i\notin A_0} m_i+\sum_{i=1}^{[\frac{l_1}{2}]} m_{2i}+\sum_{i\in A_2}(m_i-m_{\phi(i)})]\,h(f)\le 2\mathfrak{g}-2+|S| \tag{2.2.8}$$

and

$$[-1+\sum_{i\notin A_0} m_i+\sum_{i=1}^{[\frac{l_1}{2}]} m_{2i}+\sum_{i\in A_3}(m_{\phi(i)}-m_i)]\,h(f)\le 2\mathfrak{g}-2+|S|. \tag{2.2.9}$$

We first consider the case $\mathfrak{g}=0$. The right-hand sides of (2.2.6) and (2.2.7) are negative, therefore

$$\sum_{i\notin A_0} m_i+\sum_{i=1}^{[\frac{l_1}{2}]} m_{2i}+\sum_{i\in A_2}(m_i-m_{\phi(i)})\le 1,$$
$$\sum_{i\notin A_0} m_i+\sum_{i=1}^{[\frac{l_1}{2}]} m_{2i}+\sum_{i\in A_3}(m_{\phi(i)}-m_i)\le 1.$$

From these two inequalities, we see that $l_0\ge l-1$. Moreover, if $l_0=l-1$, then $l_2=l_3=0$, $l_1=1$ and $m_2=1$ which implies that $l=2$ and $\min\{m_1,m_2\}=1$. Suppose now that $l_0=l$. Then it is clear that $l_1\le 3$. If $l_1=3$, then $m_2=m_3=1$ and $l_2=l_3=0$, which also implies that $l=3$ and hence $m_1=1$. If $l_1=2$, then $m_1=m_2=1$ and $l_2=l_3=0$, which implies that $l=2$. One can also easily check that it is impossible to have $l_1=0$. Finally, if $l_1=1$, then the only possibility is $l_2=l_3=1$ and $m_2-m_{\phi(2)}=1$, $m_3-m_{\phi(3)}=-1$ if we assume without loss of generality that $\{2\}=A_2$ and $\{3\}=A_3$. In this case, we replace G by $L_{1,3}^{m_1}(g-\alpha_{\phi(2)})^{m_{\phi(2)}}$ and repeat the same process to get $m_2=1$, which contradicts the fact that $m_2-m_{\phi(2)}=1$. Therefore, this case is eliminated, and the proof for assertion (I) is complete.

We now consider generally when $\mathfrak{g}\ge 1$. Since $l_0=l_1+l_2+l_3$, we deduce the following:

$$\begin{aligned}&\sum_{i\notin A_0} m_i+\sum_{i=1}^{[\frac{l_1}{2}]} m_{2i}+\sum_{i\in A_2}(m_i-m_{\phi(i)})+\sum_{i\notin A_0} m_i+\sum_{i=1}^{[\frac{l_1}{2}]} m_{2i}+\sum_{i\in A_3}(m_{\phi(i)}-m_i)\\ &\ge 2(l-l_0)+(l_1-1)+l_2+l_3=(l-l_0)+l-1\ge l-1.\end{aligned}$$

Hence, from (2.2.6) and (2.2.7) we have

$$(-5+l)h(f)\le 4\mathfrak{g}-4. \tag{2.2.10}$$

Since f and g are assumed not to be constant, $h(f)\ge 2$ if $\mathcal{U}$ is affinely rigid. (2.2.10) implies $l\le 2\mathfrak{g}+3$. $\square$

As consequences of (2.2.6), (2.2.7), (2.2.8), and (2.2.9), we have the following.

Proposition 12. *Let $P(X)$ be a polynomial as above satisfying Hypothesis* I. *Assume the zero set $\mathcal{U}$ of P is affinely rigid. Let ϕ be a permutation of $\{1,2,...,l\}$ such that $\phi(i)=j$ if $P(\alpha_i)=cP(\alpha_j)$. Suppose that there are two distinct functions f and g in $\mathbf{K}$ such that $P(f)=cP(g)$ for some $c\ne 0,1\in\mathbf{k}$. Then*

(i) $h(f)=h(g)\le 2\mathfrak{g}-2$, *if there exists i such that $|m_i-m_{\phi(i)}|\ge 3$.*

(ii) $h(f) = h(g) \le 2\mathfrak{g} - 2 + |S|$ *if* f *and* g *are* S*-integers and there exists* i *such that* $|m_i - m_{\phi(i)}| \ge 2$.

Proof of Lemma 11. We note that if $v_{\mathbf{p}}(f) \ge 0$, then

$$\min\{v_{\mathbf{p}}(P'(f)), v_{\mathbf{p}}(P'(g))\} = \min\{m_i v_{\mathbf{p}}(f - \alpha_i), m_{\phi(i)} v_{\mathbf{p}}(g - \alpha_{\phi(i)})\}$$

if $v_{\mathbf{p}}(f - \alpha_i) > 0$ and $v_{\mathbf{p}}(g - \alpha_{\phi(i)}) > 0$ for some $1 \le i \le l$; the value is zero otherwise.

Assume that $m_1 \ge m_2 \ge \cdots \ge m_l$. The proof will be split into several cases.

Case 1. $m_2 \ge 3$.

Let

$$G := L_{1,2}^{m_1} \prod_{i=3}^{l} (f - \alpha_i)^{m_i}.$$

Then

$$\min\{v_{\mathbf{p}}(P'(f)),\ v_{\mathbf{p}}(P'(g))\} - v_{\mathbf{p}}(G) = m_2 v_{\mathbf{p}}(f), \qquad \text{if } v_{\mathbf{p}}(f) < 0.$$

Similarly, we claim that

$$\min\{v_{\mathbf{p}}(P'(f)),\ v_{\mathbf{p}}(P'(g))\} - v_{\mathbf{p}}(G) \le 0, \qquad \text{if } v_{\mathbf{p}}(f) \ge 0.$$

From the choice of G, we only need to verify the claim when $v_{\mathbf{p}}(f - \alpha_i) > 0$ and $v_{\mathbf{p}}(g - \alpha_{\phi(i)}) > 0$ for $i = 1, 2$. In these cases, the claim is an implication of (2.2.3). Therefore, we have

$$h(P'(f), P'(g)) = h(P'(f)/G, P'(g)/G) \ge m_2 h(f).$$

Together with Lemma 7, it gives

$$h(f) \le (m_2 - 2)h(f) \le 2\mathfrak{g} - 2.$$

Case 2. $m_2 = 2$, $m_1 \ge 3$.

If $m_1 \ge 5$, then $m_1 - m_{\phi(1)} \ge 3$. Therefore, the assertion is concluded in this case by Proposition 12. It is left to consider when $m_1 = 3, 4$. Let

$$G := L_{1,2}^{m_1 - 1} \prod_{i=3}^{l} (f - \alpha_i)^{m_i}.$$

If $v_{\mathbf{p}}(f) < 0$, then

$$\min\{v_{\mathbf{p}}(P'(f)), v_{\mathbf{p}}(P'(f))\} - v_{\mathbf{p}}(G) = 3 v_{\mathbf{p}}(f).$$

Similarly, if $v_{\mathbf{p}}(f) \ge 0$, it is easy to check that

$$\min\{v_{\mathbf{p}}(P'(f)),\ v_{\mathbf{p}}(P'(g))\} - v_{\mathbf{p}}(G) \le 0$$

except for $\mathbf{p}$ such that $v_{\mathbf{p}}(f - \alpha_i) > 0$ and $v_{\mathbf{p}}(g - \alpha_{\phi(i)}) > 0$ for $i = 1$ or 2. For these exceptional cases, we will show that

$$\min\{v_{\mathbf{p}}(P'(f)),\ v_{\mathbf{p}}(P'(g))\} - v_{\mathbf{p}}(G) \le \min\{v^0_{\mathbf{p}}(d_{\mathbf{p}} f), v^0_{\mathbf{p}}(d_{\mathbf{p}} g)\}. \tag{2.2.11}$$

We then have

$$h(P'(f), P'(g)) = h(P'(f)/G, P'(g)/G) \le 3h(f) - \sum_{\mathbf{p} \in C} \min\{v^0_{\mathbf{p}}(d_{\mathbf{p}} f), v^0_{\mathbf{p}}(d_{\mathbf{p}} g)\}.$$

Together with Lemma 7, it yields $h(f) \le 2\mathfrak{g} - 2$.

We now prove (2.2.11) for each $\mathbf{p}$ such that $v_{\mathbf{p}}(f-\alpha_i)>0$ and $v_{\mathbf{p}}(g-\alpha_{\phi(i)})>0$ with $i=1$ or 2. For $i=1$, we have by Proposition 8 that

$$(m_1+1)v_{\mathbf{p}}(f-\alpha_1)=(m_{\phi(1)}+1)v_{\mathbf{p}}(g-\alpha_{\phi(1)}). \tag{2.2.12}$$

It then implies that

$$\min\{v_{\mathbf{p}}(P'(f)),v_{\mathbf{p}}(P'(g))\}-v_{\mathbf{p}}(G)\le v_{\mathbf{p}}(f-\alpha_1)-\frac{m_1-m_{\phi(1)}}{m_{\phi(1)}+1}v_{\mathbf{p}}(f-\alpha_1).$$

Since $v^0_{\mathbf{p}}(d_{\mathbf{p}}f)=v_{\mathbf{p}}(f-\alpha_1)-1$, it remains to show that $\frac{m_1-m_{\phi(1)}}{m_{\phi(1)}+1}v_{\mathbf{p}}(f-\alpha_1)\ge 1$. This can be done by the following observation. If $(m_1,m_{\phi(1)})=(3,2)$ or $(4,2)$, then (2.2.12) implies that $v_{\mathbf{p}}(f-\alpha_i)\ge 3$. If $(m_1,m_{\phi(1)})=(3,1)$, then $(m_1-m_{\phi(1)})/(m_{\phi(1)}+1)=1$. We note that the case $(m_1,m_{\phi(1)})=(4,1)$ can be covered by Proposition 12. Similarly, for $i=2$ we have

$$3v_{\mathbf{p}}(f-\alpha_2)=(m_{\phi(2)}+1)v_{\mathbf{p}}(g-\alpha_{\phi(2)}). \tag{2.2.13}$$

If $m_{\phi(2)}\le 2$, then $\min\{v_{\mathbf{p}}(P'(f)),v_{\mathbf{p}}(P'(g))\}\le 2v_{\mathbf{p}}(f-\alpha_2)\le (m_1-1)v_{\mathbf{p}}(f-\alpha_2)=v_{\mathbf{p}}(G)$ by (2.2.3) and $m_1\ge 3$. If $m_{\phi(2)}>2$, then $m_{\phi(2)}=m_1$ and $v_{\mathbf{p}}(L_{1,2})=v_{\mathbf{p}}(g-\alpha_{\phi(2)})<v_{\mathbf{p}}(f-\alpha_2)$. Therefore

$$\min\{v_{\mathbf{p}}(P'(f)),v_{\mathbf{p}}(P'(f))\}-v_{\mathbf{p}}(G)=v_{\mathbf{p}}(g-\alpha_{\phi(2)})-\frac{m_1-2}{3}v_{\mathbf{p}}(g-\alpha_{\phi(2)}).$$

Since $v_{\mathbf{p}}(d^0_{\mathbf{p}}g)=v_{\mathbf{p}}(g-\alpha_{\phi(2)})-1$, it remains to show that $\frac{m_1-2}{3}v_{\mathbf{p}}(g-\alpha_{\phi(2)})\ge 1$. This follows easily from (2.2.13) that $v_{\mathbf{p}}(g-\alpha_{\phi(2)})\ge 3$ if $m_1=3$ or 4.

For the rest of the proof, we will give G in each case and skip the proof if it is similar to one of the previous cases.

Case 3. $m_2=2$, $m_1=2$ and $l\ge 3$.

If $m_3=2$, then we take

$$G:=L_{1,2}L_{1,3}L_{2,3}\prod_{i=4}^{l}(f-\alpha_i)^{m_i}.$$

It then remains to consider when $m_3=1$. If $m_{\phi(1)}=m_{\phi(2)}=2$, then either (i) $l=3$ and $P(\alpha_3)\ne cP(\alpha_i)$ for any i, (i.e. $l_0\le 2$), or (ii) $l\ge 4$ and $m_{\phi(3)}=m_{\phi(4)}=1$. For (i), we let

$$G:=L^2_{1,2}\prod_{i=4}^{l}(f-\alpha_i)^{m_i};$$

for (ii), we let

$$G:=L^2_{1,2}L_{3,4}\prod_{i=5}^{l}(f-\alpha_i)^{m_i}.$$

Otherwise, we may assume without loss of generality that $m_{\phi(2)}=1$.

$$G:=L_{1,2}L_{1,3}\prod_{i=4}^{l}(f-\alpha_i)^{m_i}.$$

We will omit the proof for this case, since it is similar to the proof of Case 2.

Case 4. $m_2=2$, $m_1=2$ and $l=2$.

If $l_0=2$, then once can easy to see that $c=-1$ and $X+Y-\alpha_1-\alpha_2$ is a linear factor of $P(X)+P(Y)$ which implies that $\mathcal{U}$ is not affinely rigid. Hence we only need to consider $l_0 < 2$. If $l_0 = 0$, simply take $G = 1$. If $l_0 = 1$, we may assume that $P(\alpha_2) \neq cP(\alpha_1)$ and $P(\alpha_1) = cP(\alpha_2)$. We let

$$G := b_{1,3}(f - \alpha_1)^3 - cb_{2,3}(g - \alpha_2)^3.$$

From Proposition 8(ii) we have

$$v_{\mathbf{p}}(G) \geq 4 \min\{v_{\mathbf{p}}(f - \alpha_1), v_{\mathbf{p}}(g - \alpha_2)\}, \qquad \text{if } v_{\mathbf{p}}(f) \geq 0.$$

It is then easy to verify that

$$h(P'(f), P'(g)) = \frac{1}{2} h((f - \alpha_1)^4 (f - \alpha_2)^4 / G, (g - \alpha_1)^4 (g - \alpha_2)^4)/G) \geq \frac{5}{2} h(f),$$

and $h(f) \leq 4\mathfrak{g} - 4$, from Lemma 7.

Case 5. $m_2 = 1$.

In this case, $m_i = 1$ for all $i \geq 2$ and, under the assumption of the lemma, $l \geq 3$. By Proposition 12 we only need to consider when $m_1 \leq 3$. Without loss of generality we may assume that $m_{\phi(2)} = m_1$. Then $m_{\phi(i)} = 1$ if $i \neq 2$. If $m_1 = 3$, we let

$$G := (g - \alpha_{\phi(1)})(f - \alpha_2) L_{1,2} L_{2,3} L_{3,1} \prod_{i=4}^{l} (f - \alpha_i)^2.$$

Then

$$2 \min\{v_{\mathbf{p}}(P'(f)), v_{\mathbf{p}}(P'(g))\} - v_{\mathbf{p}}(G) = 5 v_{\mathbf{p}}(f), \qquad \text{if } v_{\mathbf{p}}(f) < 0.$$

We can also verify similarly that

$$2 \min\{v_{\mathbf{p}}(P'(f)), v_{\mathbf{p}}(P'(g))\} - v_{\mathbf{p}}(G) \leq 0, \qquad \text{if } v_{\mathbf{p}}(f) \geq 0.$$

Therefore

$$h(P'(f), P'(g)) = \frac{1}{2} h((P'(f))^2 / G, (P'(g))^2 / G) \geq \frac{5}{2} h(f)$$

and then $h(f) \leq 4\mathfrak{g} - 4$.

If $m_1 = 2$, we let

$$G := L_{1,2} L_{2,3} L_{3,1} \prod_{i=4}^{l} (f - \alpha_i)^2.$$

Similarly, if $v_{\mathbf{p}}(f) < 0$, then

$$2 \min\{v_{\mathbf{p}}(P'(f)), v_{\mathbf{p}}(P'(g))\} - v_{\mathbf{p}}(G) = 5 v_{\mathbf{p}}(f).$$

If $v_{\mathbf{p}}(f) \geq 0$, one can similarly verify Case 2 where

$$\min\{v_{\mathbf{p}}(P'(f)),\ v_{\mathbf{p}}(P'(g))\} - \frac{1}{2} v_{\mathbf{p}}(G) \leq \min\{v_{\mathbf{p}}^0(d_{\mathbf{p}} f), v_{\mathbf{p}}^0(d_{\mathbf{p}}) g\}.$$

Then, we will get $h(f) \leq 4\mathfrak{g} - 4$.

If $m_1 = 1$ and $l \geq 5$, we take

$$G := L_{1,2} L_{2,3} L_{3,4} L_{4,5} L_{5,1} \prod_{i=6}^{l} (f - \alpha_i)^2$$

which gives $h(f) \leq 4\mathfrak{g} - 4$.

If $m_1 = 1$ and $l = 4$ and $l_0 \leq l - 1$, we may assume that $P(\alpha_4) \neq cP(\alpha_j)$ for any j. We take

$$G := L_{1,2} L_{2,3} L_{3,1}$$

which then gives $h(f) \leq 4\mathfrak{g} - 4$.

If $m_1 = 1$ and $l = 4$ and $l_0 = l$, then $P(X) - cP(Y)$ has a linear factor which implies that $\mathcal{U}$ is not affinely rigid, contradicting our assumption. The proof will be given in the next proposition, which will complete the proof for (I).

(II) can be obtained from the arguments for (I) by replacing the use of (2.2.4) by (2.2.5). Therefore, it remains to treat the cases when (1) $l = 2$, $m_2 = 1$ and $m_1 \geq 2$ or (2) $l = 3$ and $m_1 = m_2 = m_3 = 1$.

Case 1. $l = 2$, $m_2 = 1$ and $m_1 \geq 2$.

In this case, we take

$$G := L_{1,2}.$$

If $v_{\mathbf{p}}(f - \alpha_1) > 0, v_{\mathbf{p}}(g - \alpha_2) > 0$, then

$$2v_{\mathbf{p}}(g - \alpha_2) = 3v_{\mathbf{p}}(f - \alpha_1) = 3v_{\mathbf{p}}(L_{1,2}).$$

Hence

$$\min\{v_{\mathbf{p}}(P'(f)), v_{\mathbf{p}}(P'(g))\} - v_{\mathbf{p}}(G) = \frac{1}{2}v_{\mathbf{p}}(f - \alpha_1) \leq v_{\mathbf{p}}(f - \alpha_1) - 1 = v^0_{\mathbf{p}}(d_{\mathbf{p}}f).$$

Similarly,

$$\min\{v_{\mathbf{p}}(P'(f)), v_{\mathbf{p}}(P'(g))\} - v_{\mathbf{p}}(G) \leq v^0_{\mathbf{p}}(d_{\mathbf{p}}g),$$

if $v_{\mathbf{p}}(f - \alpha_2) > 0, v_{\mathbf{p}}(g - \alpha_{\phi(2)}) > 0$. Therefore,

$$h(P'(f), P'(g)) = h(P'(f)/G, P'(g)/G) \leq 2h(f) - \sum_{\mathbf{p}\in C} \min\{v^0_{\mathbf{p}}(d_{\mathbf{p}}f, v^0_{\mathbf{p}}(d_{\mathbf{p}})g\}.$$

Then $h(f) \leq 2\mathfrak{g} - 2 + |S|$, by Lemma 7.

Case 2. $l = 3$ and $m_1 = m_2 = m_3 = 1$.

Let $G := L_{1,2}L_{2,3}L_{3,1}$. Then

$$\begin{aligned} h(P'(f), P'(g)) &= \frac{1}{2}h(P'(f)^2/G, P'(g)^2/G) \\ &\geq \frac{3}{2}h(f), \end{aligned}$$

which implies that $h(f) \leq 4\mathfrak{g} - 4 + 2|S|$ and hence completes the proof for (II). $\square$

Proposition 13. *Let $P(X)$ be a polynomial as above satisfying Hypothesis* I. *Suppose that $l = 4$, $m_i = 1$, and there is a permutation ϕ of $\{1, 2, ..., l\}$ such that $P(\alpha_i) = cP(\alpha_{\phi(i)})$ for each $1 \leq i \leq 4$. Then $\mathcal{U}$ is not affinely rigid.*

Proof. We will prove this by contradiction. Assume that $\mathcal{U}$ is affinely rigid. Consider the curve $C_c = [F_c(X, Y, Z) = 0]$ in $\mathbb{P}^2$ defined by the homogenization of $P(X) - cP(Y)$. We first recall from [13] that (i) F_c has no linear factor if $\mathcal{U}$ is affinely rigid; (ii) the curve C_c has only 4 ordinary singularities of multiplicities 2, and hence, its deficiency $\delta_{F_c} = 2$. Then either C_c is an irreducible curve of genus 2 or $F_c = AB$, where A and B are irreducible with degree 2 and 3, respectively.

On the other hand, we can construct 3 regular 1-forms on C_c as follows:

$$\omega_1 := \eta H_{1,2}H_{3,4}, \quad \omega_2 := \eta H_{1,3}H_{2,4}, \text{ and } \omega_3 := \eta H_{1,4}H_{2,3},$$

where

$$\eta := \frac{YdZ - ZdY}{\prod_{i=1}^4 (X - \alpha_i Z)}, \qquad \text{and } H_{i,j} := (Y - \alpha_{\phi(i)}Z) - \frac{\alpha_{\phi(i)} - \alpha_{\phi(j)}}{\alpha_i - \alpha_j}(X - \alpha_i Z).$$

Moreover, these 1-forms are non-trivial on each component of C_c since C_c has no linear components. (We refer readers to [3] or [4] for more details.) The existence of 1-forms implies that C_c has no component of genus zero. Together with the previous discussion, we see that C_c is irreducible of genus 2. This again will imply that these three 1-forms have to be linearly dependent over $\mathbf{k}$. Therefore, we have a linear relation on C_c:

$$\alpha H_{1,2}H_{3,4} + \beta H_{1,3}H_{2,4} + \gamma H_{1,4}H_{2,3} = 0,$$

where $\alpha, \beta, \gamma \in \mathbf{k}$. This equation defines a quadratic curve, and the equation is satisfied for all points in C_c. This would give a contradiction again, since we have shown that C_c is an irreducible curve of degree 4. Therefore, $\mathcal{U}$ cannot be affinely rigid. □

2.3. **Proof of Theorem 1 and Theorem 2.**

Proposition 14. *If there exists $(\lambda, \beta) \neq (1, 0) \in \mathbf{k} \times \mathbf{k}$, such that $\mathcal{U} = \lambda\mathcal{U} + \beta$, i.e. $\mathcal{U}$ is not affinely rigid, then $P(f) = \lambda^n P(\lambda^{-1}(f - \beta))$ for any $f \in \mathbf{K}$.*

Proof. Let $\mathcal{U} = \{u_1, ..., u_n\}$. Then $\mathcal{U} = \lambda\mathcal{U} + \beta$ implies that

$$P(X) = (X-u_1)\ldots(X-u_n) = (X-\lambda u_1-\beta)\ldots(X-\lambda u_n-\beta) = \lambda^n P(\lambda^{-1}(X-\beta)).$$

Therefore, $P(f) = \lambda^n P(\lambda^{-1}(f - \beta))$ for any $f \in \mathbf{K}$. □

Proof of Theorem 1 *and Theorem* 2. Theorem 2 clearly follows from Lemmas 9 and 11. Theorem 1(I)(b) follows from Theorem 2(a); Theorem 1(I)(c) from Lemmas 9 and 10; and Theorem 1 (II) from Lemma 7 and Proposition 14. When $\mathfrak{g} = 0$, we have shown in Lemmas 9 and 10 that $P(X)$ is a strong uniqueness polynomial if $\mathcal{U}$ is affinely rigid and P does not satisfy (1A) or (1E). On the other hand, if $\mathcal{U}$ is not affinely rigid, then Proposition 14 shows that $P(X)$ is not a strong uniqueness polynomial. Moreover, it is also easy to check that if P satisfies (1A), then $[F(X, Y, Z) = 0]$ is an irreducible curve of genus zero, and if P satisfies (1E), then $[F_w(X, Y, Z) = 0]$ is also an irreducible curve of genus zero. Therefore, there must exist two distinct algebraic functions f and g in $\mathbf{K}$ giving rise to a morphism from C to $[F(X, Y, Z) = 0]$ or $[F_w(X, Y, Z) = 0]$, respectively. Therefore, $P(f) = P(g)$ or $P(f) = wP(g)$, respectively. This shows that P is not a strong uniqueness polynomial if P satisfies (1A) or (1E), and hence completes the proof of Theorem 1(I)(a). □

3. Unique range sets

3.1. **Proof of Theorem 3 and Theorem 4.**

Let $\mathcal{U} := \{u_1, ..., u_n\}$ be a subset of $\mathbf{k}$, and let f and g be two distinct non-constant functions in $\mathbf{K}$. We say f and g satisfy $(C_{m_0,\bar{S}})$ if

$$(C_{m_0,\bar{S}}) \quad \sum_{i=1}^{n} \min\{m_0,\ v^0_{\mathbf{p}}(f - u_i)\} = \sum_{i=1}^{n} \min\{m_0,\ v^0_{\mathbf{p}}(g - u_i)\}, \quad \text{for all } \mathbf{p} \notin S.$$

Here m_0 is a positive integer or $m_0 = \infty$. When $m_0 = 1$, this is equivalent to saying that f and g share $\mathcal{U}$ outside of S ignoring multiplicities; and when $m_0 = \infty$, this is equivalent to saying that f and g share $\mathcal{U}$ outside of S counting multiplicities. Let

$$P(X) = (X - u_1)\cdots(X - u_n) \tag{3.1}$$

be the associate polynomial of $\mathcal{U}$, and let

$$P'(X) = n(X-\alpha_1)^{m_1}\cdots(X-\alpha_l)^{m_l}, \tag{3.2}$$

where the α_i's are distinct. In this section, we will prove the following theorem which implies Theorems 3 and 4.

Theorem 15. *Let $\mathcal{U} := \{u_1, ..., u_n\}$ be an affinely rigid subset of $\mathbf{k}$, let $P(X) = (X-u_1)\cdots(X-u_n)$ satisfy Hypothesis* I *and let $P'(X)$ be as* (1) *and* (i) *$l \geq 4$,* (ii) *$l = 3$ and $\max\{m_1, m_2, m_3\} \geq 2$ or* (iii) *$l = 2$ and $\min\{m_1, m_2\} \geq 2$. Suppose that f and g are two distinct non-constant functions in $\mathbf{K}$ satisfying $(C_{m_0,\bar{S}})$.*

(I) (a) *If $m_0 = 1$ and $n \geq 2l + 13$, then $h(f) + h(g) \leq 26\mathfrak{g} - 20 + 4|S|$.*
 (b) *If $m_0 \geq 2$ and $n \geq 2l + 7 + \frac{4}{m_0-1}$, then $h(f) + h(g) \leq 22\mathfrak{g} - 8 + 4|S|$.*
 (c) *If $m_0 = 1$, f and g are S-integers, and if $n \geq 2l+6$, then $h(f)+h(g) \leq 26\mathfrak{g} - 20 + 12|S|$.*
 (d) *If $m_0 \geq 2$, f and g are S-integers, and if $n \geq 2l + 3 + \frac{2}{m_0-1}$, then $h(f) + h(g) \leq 22\mathfrak{g} - 8 + 10|S|$.*

(II) *Assume further that $l \geq 2\mathfrak{g} + 4$ if $\mathfrak{g} \geq 2$.*
 (a) *If $m_0 = 1$, then $n \leq \max\{2l + 13,\ 2l + 2 + 13\mathfrak{g} + 2|S|\}$.*
 (b) *If $m_0 \geq 2$, then $n \leq \max\{2l + 7 + \frac{4}{m_0-1},\ 2l + 2 + (7 + \frac{4}{m_0-1})\mathfrak{g} + 2|S|\}$.*
 (c) *If $m_0 = 1$ and f and g are S-integers, then*

$$n \leq \max\{2l + 6,\ 2l - 5 + 13\mathfrak{g} + 6|S|\}.$$

 (d) *If $m_0 \geq 2$ and f and g are S-integers, then*

$$n \leq \max\{2l + 3 + \frac{2}{m_0-1},\ 2l - 2 - \frac{2}{m_0-1} + (7 + \frac{4}{m_0-1})\mathfrak{g} + (3 + \frac{2}{m_0-1})|S|\}.$$

The proof of Theorem 15 will follow the ideas in [8] and use some tools for function fields.

3.2. **Some basic results on function fields.** Throughout this section, we assume further that t ($\in \mathbf{K}$) is obtained as follows. Choose a point $\mathbf{q} \in S$, and consider the $\mathbf{k}$-vector space

$$\mathcal{L}((\mathfrak{g}+1)\mathbf{q}) := \{\eta \in \mathbf{K} \mid (\eta)_0 - (\eta)_\infty \geq -(\mathfrak{g}+1)\mathbf{q}\}.$$

From the Riemann-Roch theorem, the dimension of this vector space is at least 2. Therefore, there exists a non-constant function t in this vector space. Moreover, from this construction t has only one pole at $\mathbf{q}$ with order at most $\mathfrak{g}+1$. With this choice, we have

Proposition 16. (i) $h(t) \leq \mathfrak{g} + 1$,
 (ii) $\sum_{\mathbf{p}\notin S} v_{\mathbf{p}}(d_{\mathbf{p}}t) \leq 3\mathfrak{g}$.

Proof. (i) is clear from the construction of t. For (ii), from the Riemann-Roch theorem we have

$$\begin{aligned}\sum_{\mathbf{p}\notin S} v_{\mathbf{p}}(d_{\mathbf{p}}t) &= 2\mathfrak{g} - 2 - \sum_{\mathbf{p}\in S} v_{\mathbf{p}}(d_{\mathbf{p}}t)\\ &\leq 2\mathfrak{g} - 2 - (v_{\mathbf{q}}(t) - 1)\\ &\leq 2\mathfrak{g} - 1 + h(t) \leq 3\mathfrak{g}.\end{aligned}$$

□

Proposition 17. *Let η be a non-constant function in* $\mathbf{K}$. *Then*

$$\sum_{\mathbf{p}\in C} v^0_{\mathbf{p}}(d_{\mathbf{p}}\eta) \le 2h(\eta) + 2\mathfrak{g} - 2.$$

Moreover, if η is an S-integer, then

$$\sum_{\mathbf{p}\in C} v^0_{\mathbf{p}}(d_{\mathbf{p}}\eta) \le h(\eta) + 2\mathfrak{g} - 2 + |S|.$$

Proof. The first assertion follows from the following computation:

$$\begin{aligned}\sum_{\mathbf{p}\in C} v^0_{\mathbf{p}}(d_{\mathbf{p}}\eta) &= \sum_{v_{\mathbf{p}}(\eta)\ge 0} v_{\mathbf{p}}(d_{\mathbf{p}}\eta) = \sum_{v_{\mathbf{p}}(\eta)\ge 0} v_{\mathbf{p}}(d_t\eta) + \sum_{v_{\mathbf{p}}(\eta)\ge 0} v_{\mathbf{p}}(d_{\mathbf{p}}t)\\ &= -\sum_{v_{\mathbf{p}}(\eta)<0} v_{\mathbf{p}}(d_t\eta) + \sum_{v_{\mathbf{p}}(\eta)\ge 0} v_{\mathbf{p}}(d_{\mathbf{p}}t)\\ &= -\sum_{v_{\mathbf{p}}(\eta)<0} v_{\mathbf{p}}(d_{\mathbf{p}}\eta) + \sum_{\mathbf{p}\in C} v_{\mathbf{p}}(d_{\mathbf{p}}t)\\ &\le -\sum_{v_{\mathbf{p}}(\eta)<0} (v_{\mathbf{p}}(\eta)-1) + 2\mathfrak{g} - 2\\ &\le 2h(f) + 2\mathfrak{g} - 2.\end{aligned}$$

If η is an S-integer, then the number of $\mathbf{p}$ such that $v_{\mathbf{p}}(\eta) < 0$ is at most $|S|$. Hence,

$$-\sum_{v_{\mathbf{p}}(\eta)<0} (v_{\mathbf{p}}(\eta)-1) \le h(f) + |S|,$$

which concludes the second assertion. □

Let $\eta \ne 0 \in \mathbf{K}$, the truncated counting function over $\bar{S}$, be defined by

$$\bar{N}_{\bar{S}}(\eta) := \sum_{\mathbf{p}\notin S} \min\{1,\ v^0_{\mathbf{p}}(\eta)\}.$$

We will need a stronger version of the truncated second main theorem as follows.

Lemma 18 (The truncated second main theorem)**.** *Let f be non-constant in* $\mathbf{K}$ *and let $u_1, ..., u_n$ be n distinct elements in* $\mathbf{k}$. *Then*

$$(n-1)h(f) \le \sum_{i=1}^{n} \bar{N}_{\bar{S}}(f-u_i) + \bar{N}_{\bar{S}}(f^{-1}) - \sum_{\mathbf{p}\in C\mathcal{U}_{\bar{S}}} v_{\mathbf{p}}(d_{\mathbf{p}}f) + 2\mathfrak{g} - 2 + |S|.$$

Here, $C\mathcal{U}_{\bar{S}} = \{\mathbf{p} \notin S \mid v_{\mathbf{p}}(f-u_i) = 0$ for each $1 \le i \le n\}$.

Proof. Since $u_i \in \mathbf{k}$,

$$d_t f = d_t(f-u_i) \tag{3.3}$$

and

$$d_t f = -f^2 d_t f^{-1}. \tag{3.4}$$

We first consider when $v_{\mathbf{p}}(f) < 0$. In this case $v_{\mathbf{p}}(f^{-1}) = -v_{\mathbf{p}}(f) > 0$ min and $v_{\mathbf{p}}(f-u_i) = v_{\mathbf{p}}(f)$. From (3.4),

(3.5)
$$\frac{(f-u_1)\cdots(f-u_n)}{d_t f} = \frac{f^{-1}(f-u_1)\cdots(f-u_n)}{-f d_t f^{-1}} = \frac{f^{-1}(f-u_1)\cdots(f-u_n)}{-f d_{\mathbf{p}} f^{-1}} d_{\mathbf{p}} t.$$

It is easy to verify that

$$v_{\mathbf{p}}(fd_{\mathbf{p}}f^{-1}) = v_{\mathbf{p}}(d_{\mathbf{p}}f^{-1}/f^{-1}) \geq -\min\{1,\ v_{\mathbf{p}}(f^{-1})\}.$$

Therefore, (3.5) yields

$$\sum_{i=1}^{n} v_{\mathbf{p}}(f-u_i) - v_{\mathbf{p}}(d_t f) \leq (n-1)v_{\mathbf{p}}(f) + \min\{1,\ v_{\mathbf{p}}(f^{-1})\} + v_{\mathbf{p}}(d_{\mathbf{p}}t).$$

Hence,

(3.6)

$$-(n-1)\min\{0, v_{\mathbf{p}}(f)\} \leq -\sum_{i=1}^{n} v_{\mathbf{p}}(f-u_i) + v_{\mathbf{p}}(d_t f) + \min\{1, v_{\mathbf{p}}(f^{-1})\} + v_{\mathbf{p}}(d_{\mathbf{p}}t).$$

We now consider when $v_{\mathbf{p}}(f) \geq 0$. Let $\{\pi(1), ..., \pi(n)\} = \{1, ..., n\}$ and

$$v_{\mathbf{p}}(f-u_{\pi(1)}) \geq v_{\mathbf{p}}(f-u_{\pi(2)}) \geq \cdots \geq v_{\mathbf{p}}(f-u_{\pi(n)}) \geq 0.$$

It is easy to check that $v_{\mathbf{p}}(f-u_{\pi(2)}) = \cdots = v_{\mathbf{p}}(f-u_{\pi(n)}) = 0$. From (3.3)

$$\begin{aligned}\frac{(f-u_1)\cdots(f-u_n)}{d_t f} &= \frac{(f-u_1)\cdots(f-u_n)}{d_t(f-u_{\pi(1)})} \\ &= \frac{(f-u_{\pi(2)})\cdots(f-u_{\pi(n)})}{(f-u_{\pi(1)})^{-1}d_{\mathbf{p}}(f-u_{\pi(1)})} d_{\mathbf{p}}t.\end{aligned}$$

Similarly,

$$\sum_{i=1}^{n} v_{\mathbf{p}}(f-u_i) - v_{\mathbf{p}}(d_t f) \leq \min\{1,\ v_{\mathbf{p}}(f-u_{\pi(1)})\} + v_{\mathbf{p}}(d_{\mathbf{p}}t).$$

Therefore, we have the following two inequalities:

$$0 \leq -\sum_{i=1}^{n} v_{\mathbf{p}}(f-u_i) + v_{\mathbf{p}}(d_t f) + \sum_{i=1}^{n}\min\{1,\ v_{\mathbf{p}}(f-u_i)\} + v_{\mathbf{p}}(d_{\mathbf{p}}t), \tag{3.7}$$

$$0 \leq -\sum_{i=1}^{n} v_{\mathbf{p}}(f-u_i) + v_{\mathbf{p}}(d_t f) + 1 + v_{\mathbf{p}}(d_{\mathbf{p}}t). \tag{3.8}$$

Gathering (3.6) over all $\mathbf{p}$ such that $v_{\mathbf{p}}(f) < 0$, (3.7) over all other $\mathbf{p}$ such that $\mathbf{p} \notin S$ and $v_{\mathbf{p}}(f-u_i) > 0$ for some $1 \leq i \leq n$, and (3.8) over $\mathbf{p}$ such that $\mathbf{p} \in S$ and $v_{\mathbf{p}}(f) \geq 0$, it then yields

$$\begin{aligned}(n-1)h(f) \leq &-\sum_{i=1}^{n}\sum_{\mathbf{p}\in C} v_{\mathbf{p}}(f-u_i) + \sum_{\mathbf{p}\notin C\mathcal{U}_{\bar{S}}} v_{\mathbf{p}}(d_t f) + \sum_{i=1}^{n} \bar{N}_{\bar{S}}(f-u_i) \\ &+ \bar{N}_{\bar{S}}(f^{-1}) + |S| + \sum_{\mathbf{p}\notin C\mathcal{U}_{\bar{S}}} v_{\mathbf{p}}(d_{\mathbf{p}}t).\end{aligned} \tag{3.9}$$

Since any non-zero element in a function field has the same number of zeros and poles counting multiplicity, $\sum_{\mathbf{p}\in C} v_{\mathbf{p}}(f-u_i) = 0$ and

$$\sum_{\mathbf{p}\notin C\mathcal{U}_{\bar{S}}} v_{\mathbf{p}}(d_t f) = -\sum_{\mathbf{p}\in C\mathcal{U}_{\bar{S}}} v_{\mathbf{p}}(d_t f) = -\sum_{\mathbf{p}\in C\mathcal{U}_{\bar{S}}} v_{\mathbf{p}}(d_{\mathbf{p}}f) + \sum_{\mathbf{p}\in C\mathcal{U}_{\bar{S}}} v_{\mathbf{p}}(d_{\mathbf{p}}t).$$

Moreover, since $\sum_{\mathbf{p}\in C} v_{\mathbf{p}}(d_{\mathbf{p}}t) = 2\mathfrak{g}-2$, (3.9) yields

$$(n-1)h(f) \le \sum_{i=1}^{n} \bar{N}_{\bar{S}}(f-u_i) + \bar{N}_{\bar{S}}(f^{-1}) - \sum_{\mathbf{p}\in C\mathcal{U}_{\bar{S}}} v_{\mathbf{p}}(d_{\mathbf{p}}f) + |S| + 2\mathfrak{g} - 2.$$

□

3.3. **Proof of Theorem 15.** Suppose from now on that f and g are two distinct non-constant functions in $\mathbf{K}$ satisfying $(C_{m_0,\bar{S}})$. Let $P(X)$ be the polynomial defined in Theorem 15,

$$F := \frac{1}{P(f)}, \qquad G := \frac{1}{P(g)}$$

and

$$H := \frac{d_t^2 F}{d_t F} - \frac{d_t^2 G}{d_t G}.$$

We will conclude Theorem 15 by the following two lemmas.

Lemma 19. *Suppose $P(X)$ satisfies Hypothesis* I. *Let f and g be two distinct non-constant functions in $\mathbf{K}$ satisfying $(C_{m_0,\bar{S}})$, and let H be as above. If $H \equiv 0$, then*

$$\frac{1}{P(f)} = \frac{c_0}{P(g)} + c_1,$$

for some $c_0 \ne 0$, and $c_1 \in \mathbf{k}$, and $h(f) = h(g)$. Furthermore, suppose that (i) *$l \ge 4$,* (ii) *$l = 3$ and $\max\{m_1, m_2, m_3\} \ge 2$, or* (iii) *$l = 2$ and $\min\{m_1, m_2\} \ge 2$. Then either $c_1 = 0$ or the following hold:*

(i) $h(f) \le 10(2\mathfrak{g} - 2 + |S|)$;
(ii) $n \le \max\{5, 4\mathfrak{g} + 2|S|\}$.

Lemma 20. *Let $P(X)$, f, g, and H be as in the previous lemma. Assume that $H \not\equiv 0$.*

(I) (a) *If $m_0 = 1$, then*
  (i) $n \le \max\{2l + 13, 2l + 2 + 13\mathfrak{g} + 2|S|\}$;
  (ii) $h(f) + h(g) \le 26\mathfrak{g} - 20 + 4|S|$, *if* $n \ge 2l + 13$.
 (b) *If $m_0 \ge 2$, then*
  (i) $n \le \max\{2l + 7 + \frac{4}{m_0-1}, 2l + 2 + (7 + \frac{4}{m_0-1})\mathfrak{g} + 2|S|\}$.
  (ii) $h(f) + h(g) \le (14 + \frac{8}{m_0-1})\mathfrak{g} - (8 + \frac{8}{m_0-1}) + 4|S|$, *if* $n \ge 2l + 7 + \frac{4}{m_0-1}$.

(II) *If we assume furthermore that f and g are S-integers, then*
 (a) *If $m_0 = 1$, then*
  (i) $n \le \max\{2l + 6, 2l - 5 + 13\mathfrak{g} + 6|S|\}$.
  (ii) $h(f) + h(g) \le 26\mathfrak{g} - 20 + 12|S|$, *if* $n \ge 2l + 6$.
 (b) *If $m_0 \ge 2$, then*
  (i) $n \le \max\{2l + 3 + \frac{2}{m_0-1}, 2l - 2 - \frac{2}{m_0-1} + (7 + \frac{4}{m_0-1})\mathfrak{g} + (3 + \frac{2}{m_0-1})|S|\}$.
  (ii) $h(f) + h(g) \le (14 + \frac{8}{m_0-1})\mathfrak{g} - (8 + \frac{8}{m_0-1}) + (6 + \frac{4}{m_0-1})|S|$, *if* $n \ge 2l + 3 + \frac{2}{m_0-1}$.

Proof of Theorem 15. If $H \equiv 0$, then Lemma 19 implies

$$\frac{1}{P(f)} = \frac{c_0}{P(g)} + c_1,$$

for some $c_0 \neq 0$, and $c_1 \in \mathbf{k}$, and $h(f) = h(g)$. If $c_1 \neq 0$, then (i) $h(f) \leq 10(2\mathfrak{g} - 2 + |S|)$, and (ii) $n \leq \max\{5, 4\mathfrak{g} + 2|S|\}$. If $c_1 = 0$, then $P(g) = c_0 P(f)$. From Theorem 2, we have $h(f) \leq 8\mathfrak{g} - 8$, and $h(f) \leq 6\mathfrak{g} - 6 + 3|S|$ if f and g are S-integers. Moreover, if we assume further that $l \geq 2\mathfrak{g} + 4$ when $\mathfrak{g} \geq 2$, then a contradiction results by Theorem 1. It is then left to consider when $H \not\equiv 0$. In this case, the theorem follows directly from Lemma 20. ☐

3.4. **Proof of Lemma 19.** Recall that

$$F := \frac{1}{P(f)}, \qquad G := \frac{1}{P(g)}, \qquad \text{and} \quad H := \frac{d_t^2 F}{d_t F} - \frac{d_t^2 G}{d_t G}.$$

Let

$$H_{\mathbf{p}} := \frac{d_{\mathbf{p}}^2 F}{d_{\mathbf{p}} F} - \frac{d_{\mathbf{p}}^2 G}{d_{\mathbf{p}} G}.$$

By the chain rule,

$$H_{\mathbf{p}} = H \cdot d_{\mathbf{p}} t; \tag{3.10}$$

by direct computation

$$\begin{aligned} H_{\mathbf{p}} &= \Big\{\frac{P''(f)}{P'(f)} d_{\mathbf{p}} f - \frac{P''(g)}{P'(g)} d_{\mathbf{p}} g\Big\} \\ &\quad - 2\Big\{\frac{P'(f)}{P(f)} d_{\mathbf{p}} f - \frac{P'(g)}{P(g)} d_{\mathbf{p}} g\Big\} + \Big\{\frac{d_{\mathbf{p}}^2 f}{d_{\mathbf{p}} f} - \frac{d_{\mathbf{p}}^2 g}{d_{\mathbf{p}} g}\Big\}. \end{aligned} \tag{3.11}$$

Proof of Lemma 19. If $H \equiv 0$, then $d_t^2 F d_t G = d_t^2 G d_t F$ which implies that $d_t(d_t F / d_t G) \equiv 0$. Hence, $d_t F = c_0 d_t G$ for some $c_0 \in \mathbf{k}$ and then we have $F = c_0 G + c_1$ for some $c_1 \in \mathbf{k}$. Therefore, the first assertion is clear. We also note that $c_0 \neq 0$ since f and g are not constant. The assertion $h(f) = h(g)$ follows from the following computation:

$$nh(f) = h(P(f)) = h(\frac{1}{P(f)}) = h(\frac{c_0}{P(g)} + c_1) = h(\frac{c_0}{P(g)}) = h(P(g)) = nh(g).$$

Assume that $c_1 \neq 0$. We consider the polynomial

$$Q(X) := P(X) + \frac{c_0}{c_1}$$

and let

$$Q(X) = (X - e_1)^{n_1} (X - e_2)^{n_2} \dots (X - e_r)^{n_r}$$

be its prime decomposition. Then $Q'(X) = P'(X)$ and

$$\frac{P(g)}{c_1 P(f)} = (g - e_1)^{n_1} (g - e_2)^{n_2} \dots (g - e_r)^{n_r}. \tag{3.12}$$

Therefore, it is clear that $r \neq 1$, otherwise $l = 1$. Since $P(e_i) = \frac{c_0}{c_1}$, Hypothesis I implies that there exists at most one such that e_i that is a zero of $P'(X) = Q'(X)$. Clearly, if e_i is not a zero of $Q'(X)$, then $n_i = 1$. After arranging the index, we may assume that

$$n_2 = n_3 = \cdots = n_r = 1.$$

We claim that $r \neq 2$. If $r = 2$, then

$$Q(X) = (X - e_1)^{n-1}(X - e_2).$$

This implies that

$$P'(X) = Q'(X) = (X - e_1)^{n-2}(nX - e_1 - (n-1)e_2)$$

which implies that $l = 2$ and $\min\{m_1, m_2\} = 1$, contradicting our assumptions.

Therefore, we only need to consider when $r \geq 3$. Observing form (3.12) and using the fact that $P(e_i) \neq 0$, we see that if $v_{\mathbf{p}}(g - e_i) > 0$, then $v_{\mathbf{p}}(P(f)) = -n_i v_{\mathbf{p}}(g - e_i) < 0$. Hence, $v_{\mathbf{p}}(f) < 0$ and $v_{\mathbf{p}}(P(f)) = n v_{\mathbf{p}}(f)$. This gives

$$v_{\mathbf{p}}(g - e_i) = -\frac{n}{n_i} v_{\mathbf{p}}(f).$$

Therefore,

$$v^0_{\mathbf{p}}(g - e_i) \geq n \bar{v}^0_{\mathbf{p}}(g - e_i), \qquad \text{for } 2 \leq i \leq r$$

since $n_2 = n_3 = \cdots = n_r = 1$. Since $n_1 + (r - 1) = n$ and $r \geq 3$, it is easy to check that $n/n_1 \geq n/(n-2) > 1$. Therefore,

$$v_{\mathbf{p}}(g - e_1) = -\frac{n}{n_1} v_{\mathbf{p}}(f) \geq 2.$$

Then

$$v^0_{\mathbf{p}}(g - e_1) \geq 2\bar{v}^0_{\mathbf{p}}(g - e_1).$$

We now apply the truncated second main theorem:

$$\begin{aligned}
(r-2)h(g) &\leq \sum_{i=1}^{r} \bar{N}_{\bar{S}}(g - e_i) + 2\mathfrak{g} - 2 + |S| \\
&\leq \frac{1}{2} N_{\bar{S}}(g - e_1) + \frac{1}{n} \sum_{i=2}^{r} N_{\bar{S}}(g - e_i) + 2\mathfrak{g} - 2 + |S| \\
&\leq (\frac{1}{2} + \frac{r-1}{n}) h(g) + 2\mathfrak{g} - 2 + |S|.
\end{aligned}$$

Then

$$(\frac{n-1}{n} r - \frac{5}{2} + \frac{1}{n}) h(g) \leq 2\mathfrak{g} - 2 + |S|.$$

Since $r \geq 3$, it implies that

$$\frac{n-4}{2n} h(g) \leq 2\mathfrak{g} - 2 + |S|. \tag{3.13}$$

Since under the assumption $n \geq 5$, this gives $h(g) \leq 10(2\mathfrak{g} - 2 + |S|)$. Since $g - e_2$ is not constant, $v_{\mathbf{p}}(g - e_2) > 0$ for some $\mathbf{p}$. From the previous discussion, we see that $v_{\mathbf{p}}(g - e_2) \geq n$. Then $h(g - e_2) \geq n$, since it equals the number of zeros counting multiplicity. Therefore $h(g) = h(g - e_2) \geq n$. Then (3.13) implies that if $n \geq 5$, then

$$n \leq 4\mathfrak{g} + 2|S|.$$

□

3.5. **Proof of Lemma 20.**

Proposition 21. *If $v_{\mathbf{p}}(F) = v_{\mathbf{p}}(G) = -1$, then $v_{\mathbf{p}}(H_{\mathbf{p}}) > 0$.*

Proof. We can write $F = t_{\mathbf{p}}^{-1}\tilde{F}$, $G = t_{\mathbf{p}}^{-1}\tilde{G}$ with $v_{\mathbf{p}}(\tilde{F}) = v_{\mathbf{p}}(\tilde{G}) = 0$. Then

$$H_{\mathbf{p}} = \frac{t_p d_{\mathbf{p}}^2 \tilde{F}}{t_{\mathbf{p}} d_{\mathbf{p}} \tilde{F} - \tilde{F}} - \frac{t_p d_{\mathbf{p}}^2 \tilde{G}}{t_{\mathbf{p}} d_{\mathbf{p}} \tilde{G} - \tilde{G}},$$

which has positive order at $\mathbf{p}$. □

Proposition 22. $v_{\mathbf{p}}^{\infty}(H_{\mathbf{p}}) \le 1$, *for every $\mathbf{p} \in C$.*

Proof. This follows from the general fact that

$$v_{\mathbf{p}}\Big(\frac{d_{\mathbf{p}}\eta}{\eta}\Big) \ge -1, \qquad \text{for any } \eta \ne 0 \in \mathbf{K}.$$

□

Recall that $C\mathcal{U}_{\bar{S}} = \{\mathbf{p} \notin S \mid v_{\mathbf{p}}(f - u_i) = 0 \text{ for each } 1 \le i \le n\}$. Let $\mathcal{U}_{\bar{S}} = \{\mathbf{p} \notin S \mid v_{\mathbf{p}}(f - u_i) > 0 \text{ for some } 1 \le i \le n\}$. Let

$$\epsilon_{\mathbf{p}}^{C\mathcal{U}_{\bar{S}}} := \begin{cases} 1 & \text{if } \mathbf{p} \in C\mathcal{U}_{\bar{S}}, \\ 0 & \text{otherwise,} \end{cases} \qquad \text{and} \qquad \epsilon_{\mathbf{p}}^{\mathcal{U}_{\bar{S}}} := \begin{cases} 1 & \text{if } \mathbf{p} \in \mathcal{U}_{\bar{S}}, \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 23. *Suppose that f, g are two distinct non-constant functions satisfying $(C_{m_0,\bar{S}})$. Let $\mathbf{p} \notin S$.*

(i) *If $m_0 = \infty$, then*

$$v_{\mathbf{p}}^{\infty}(H_{\mathbf{p}}) \le \sum_{j=1}^{l} \big(\bar{v}_{\mathbf{p}}^{0}(f - \alpha_j) + \bar{v}_{\mathbf{p}}^{0}(g - \alpha_j)\big) + \big\{\bar{v}_{\mathbf{p}}^{\infty}(f) + \bar{v}_{\mathbf{p}}^{\infty}(g)\big\} + \epsilon_{\mathbf{p}}^{C\mathcal{U}_{\bar{S}}}\big\{\bar{v}_{\mathbf{p}}(d_{\mathbf{p}}f) + \bar{v}_{\mathbf{p}}(d_{\mathbf{p}}g)\big\}.$$

(ii) *If $1 \le m_0 < \infty$, then*

$$v_{\mathbf{p}}^{\infty}(H_{\mathbf{p}}) \le \sum_{j=1}^{l} \big(\bar{v}_{\mathbf{p}}^{0}(f - \alpha_j) + \bar{v}_{\mathbf{p}}^{0}(g - \alpha_j)\big) + \big\{\bar{v}_{\mathbf{p}}^{\infty}(f) + \bar{v}_{\mathbf{p}}^{\infty}(g)\big\} + \epsilon_{\mathbf{p}}^{C\mathcal{U}_{\bar{S}}}\big\{\bar{v}_{\mathbf{p}}(d_{\mathbf{p}}f) + \bar{v}_{\mathbf{p}}(d_{\mathbf{p}}g)\big\} + \frac{\epsilon_{\mathbf{p}}^{\mathcal{U}_{\bar{S}}}}{m_0'}\big\{v_{\mathbf{p}}(d_{\mathbf{p}}f) + v_{\mathbf{p}}(d_{\mathbf{p}}g)\big\},$$

where $m_0' := \max\{1, m_0 - 1\}$.

Proof. From Proposition 22, $v_{\mathbf{p}}^{\infty}(H_{\mathbf{p}}) = 1$ if $v_{\mathbf{p}}(H_{\mathbf{p}}) < 0$. From (3.11), we see that $v_{\mathbf{p}}(H_{\mathbf{p}}) < 0$ only if (a) $v_{\mathbf{p}}(f) < 0$, $v_{\mathbf{p}}(g) < 0$, (b) $v_{\mathbf{p}}(P'(f)) > 0$, $v_{\mathbf{p}}(P'(g)) > 0$, (c) $v_{\mathbf{p}}(P(f)) > 0$, $v_{\mathbf{p}}(P(g)) > 0$, or (d) $v_{\mathbf{p}}(d_{\mathbf{p}}f) > 0$, or $v_{\mathbf{p}}(d_{\mathbf{p}}g) > 0$. Therefore, the first two terms in the inequalities of (i) and (ii) come out naturally from (a) and (b). Note that

$$\frac{P'(f)}{P(f)} d_{\mathbf{p}}f - \frac{P'(g)}{P(g)} d_{\mathbf{p}}g = d_{\mathbf{p}}\Big(\frac{P(f)}{P(g)}\Big) \Big/ \frac{P(f)}{P(g)}$$

and

$$\frac{d_{\mathbf{p}}^2 f}{d_{\mathbf{p}}f} - \frac{d_{\mathbf{p}}^2 g}{d_{\mathbf{p}}g} = d_{\mathbf{p}}\Big(\frac{d_{\mathbf{p}}f}{d_{\mathbf{p}}g}\Big) \Big/ \frac{d_{\mathbf{p}}f}{d_{\mathbf{p}}g}.$$

If $m_0 = \infty$, then $(C_{m_0,\bar{S}})$ implies that $v_{\mathbf{p}}(\frac{P(f)}{P(g)}) = 0$ and hence

$$v_{\mathbf{p}}(\frac{P'(f)}{P(f)}d_{\mathbf{p}}f - \frac{P'(g)}{P(g)}d_{\mathbf{p}}g) \geq 0.$$

Moreover, if $v_{\mathbf{p}}(f-u_i) > 0$, then $v_{\mathbf{p}}(g-u_j) = v_{\mathbf{p}}(f-u_i) > 0$ for some j. Therefore, $v_{\mathbf{p}}(d_{\mathbf{p}}f) = v_{\mathbf{p}}(d_{\mathbf{p}}g)$ and hence

$$v_{\mathbf{p}}(\frac{d_{\mathbf{p}}^2 f}{d_{\mathbf{p}}f} - \frac{d_{\mathbf{p}}^2 g}{d_{\mathbf{p}}g}) \geq 0.$$

Therefore, the last term of the inequality in (i) is only counted when $\mathbf{p} \in C\mathcal{U}_{\bar{S}}$.

We now consider when m_0 is a positive integer. If $v_{\mathbf{p}}(P(f)) > 0$, then $v_{\mathbf{p}}(f-u_i) > 0$ for some i and $v_{\mathbf{p}}(g-u_j) > 0$ for some j by $(C_{m_0,\bar{S}})$. If $v_{\mathbf{p}}(d_{\mathbf{p}}f) = v_{\mathbf{p}}(d_{\mathbf{p}}g) = 0$, then $v_{\mathbf{p}}(f-u_i) = 1$ and $v_{\mathbf{p}}(g-u_j) = 1$, which implies that $v_{\mathbf{p}}(H_{\mathbf{p}}) > 0$ by Proposition 21. Therefore, we only need to consider when $v_{\mathbf{p}}(d_{\mathbf{p}}f) > 1$ or $v_{\mathbf{p}}(d_{\mathbf{p}}g) = 0$. Without loss of generality, we only consider when $v_{\mathbf{p}}(f-u_i) \geq 2$. In this case, $v_{\mathbf{p}}(d_{\mathbf{p}}f) \geq 1$, and therefore (ii) holds automatically if $m_0 = 1, 2$. Assume now that $m_0 \geq 3$. If $2 \leq v_{\mathbf{p}}(f-u_i) \leq m_0$ and $0 < v_{\mathbf{p}}(g-u_j) \leq m_0$, then the situation is the same as $m_0 = \infty$. Otherwise, either $v_{\mathbf{p}}(d_{\mathbf{p}}f) \geq m_0 - 1$ or $v_{\mathbf{p}}(d_{\mathbf{p}}g) \geq m_0 - 1$. This completes the proof of (ii). □

Proof of Lemma 20. By the second main theorem, we have

$$(n-1)h(f) \leq \sum_{i=1}^{n} \bar{N}_{\bar{S}}(f-u_i) + \bar{N}_{\bar{S}}(f^{-1}) - \sum_{\mathbf{p}\in C\mathcal{U}_{\bar{S}}} v_{\mathbf{p}}(d_{\mathbf{p}}f) + 2\mathfrak{g} - 2 + |S|. \tag{3.14}$$

We first claim that

$$\sum_{i=1}^{n} \bar{N}_{\bar{S}}(f-u_i) \leq \sum_{\mathbf{p}\notin S} \bar{v}_{\mathbf{p}}^0(H_{\mathbf{p}}) + \sum_{p\in\mathcal{U}_{\bar{S}}} v_{\mathbf{p}}(d_{\mathbf{p}}f) + \delta \sum_{p\in\mathcal{U}_{\bar{S}}} v_{\mathbf{p}}(d_{\mathbf{p}}g), \tag{3.15}$$

where $\delta = 1$ if $m_0 = 1$ and $\delta = 0$ if $m_0 \geq 2$. We first consider the case when $m_0 \geq 2$. Suppose that $\mathbf{p} \notin S$, and $v_{\mathbf{p}}(f-u_i) > 0$ for some $1 \leq i \leq n$ which clearly implies that $v_{\mathbf{p}}(f-u_m) = 0$ for all $m \neq i$. If $v_{\mathbf{p}}(f-u_i) = 1$, then $v_{\mathbf{p}}(g-u_j) = 1$ for some j since f and g satisfy $(C_{m_0,\bar{S}})$ and $m_0 \geq 2$. Hence, $v_{\mathbf{p}}(P(f)) = v_{\mathbf{p}}(P(g)) = 1$. Proposition 21 implies that $v_{\mathbf{p}}(H_{\mathbf{p}}) > 0$. If $v_{\mathbf{p}}(f-u_i) \geq 2$, then $v_{\mathbf{p}}(d_{\mathbf{p}}f) \geq 1$. Therefore, the assertion (3.15) holds. For the case $m_0 = 1$, if $v_{\mathbf{p}}(f-u_i) > 0$, then we can only conclude that (a) $v_{\mathbf{p}}(d_{\mathbf{p}}f) > 0$ or $v_{\mathbf{p}}(d_{\mathbf{p}}g) > 0$, or (b) $v_{\mathbf{p}}(d_{\mathbf{p}}f) = v_{\mathbf{p}}(d_{\mathbf{p}}g) = 0$. Similarly, the later case implies that $v_{\mathbf{p}}(H_{\mathbf{p}}) > 0$. The assertion is now clear.

Second, we claim that

$$\begin{aligned}\sum_{\mathbf{p}\notin S} \bar{v}_{\mathbf{p}}^0(H_{\mathbf{p}}) \leq{}& l(h(f)+h(g)) + \bar{N}_{\bar{S}}(f^{-1}) + \bar{N}_{\bar{S}}(g^{-1}) \\ &+ \sum_{\mathbf{p}\in C\mathcal{U}_{\bar{S}}} \left\{v_{\mathbf{p}}(d_{\mathbf{p}}f) + v_{\mathbf{p}}(d_{\mathbf{p}}g)\right\} \\ &+ \frac{1}{m_0'} \sum_{\mathbf{p}\in\mathcal{U}_{\bar{S}}} \left\{v_{\mathbf{p}}(d_{\mathbf{p}}f) + v_{\mathbf{p}}(d_{\mathbf{p}}g)\right\} + |S| + 3\mathfrak{g}.\end{aligned} \tag{3.16}$$

We have

$$
\begin{aligned}
\sum_{\mathbf{p}\notin S}\bar{v}^0_{\mathbf{p}}(H_{\mathbf{p}}) &\le \sum_{\mathbf{p}\notin S}(v^0_{\mathbf{p}}(H)+v_{\mathbf{p}}(d_{\mathbf{p}}t))\\
&\le \sum_{\mathbf{p}\in C}v^{\infty}_{\mathbf{p}}(H)+\sum_{\mathbf{p}\notin S}v_{\mathbf{p}}(d_{\mathbf{p}}t)\\
&\le \sum_{\mathbf{p}\notin S}v^{\infty}_{\mathbf{p}}(H)+|S|+3\mathfrak{g} \qquad \text{(by Propositions 16 and 22)}\\
&\le \sum_{j=1}^{l}(\bar{N}_{\bar{S}}(f-\alpha_j)+\bar{N}_{\bar{S}}(g-\alpha_j))+\bar{N}_{\bar{S}}(f^{-1})+\bar{N}_{\bar{S}}(g^{-1})\\
&\qquad +\sum_{\mathbf{p}\in C\mathcal{U}_{\bar{S}}}\big\{\bar{v}_{\mathbf{p}}(d_{\mathbf{p}}f)+\bar{v}_{\mathbf{p}}(d_{\mathbf{p}}g)\big\}\\
&\qquad +\frac{1}{m_0'}\sum_{\mathbf{p}\in\mathcal{U}_{\bar{S}}}\big\{v_{\mathbf{p}}(d_{\mathbf{p}}f)+v_{\mathbf{p}}(d_{\mathbf{p}}g)\big\}+|S|+3\mathfrak{g} \qquad \text{(by Proposition 23)}\\
&\le l(h(f)+h(g))+\bar{N}_{\bar{S}}(f^{-1})+\bar{N}_{\bar{S}}(g^{-1})+\sum_{\mathbf{p}\in C\mathcal{U}_{\bar{S}}}\big\{\bar{v}_{\mathbf{p}}(d_{\mathbf{p}}f)+\bar{v}_{\mathbf{p}}(d_{\mathbf{p}}g)\big\}\\
&\qquad +\frac{1}{m_0'}\sum_{\mathbf{p}\in\mathcal{U}_{\bar{S}}}\big\{v_{\mathbf{p}}(d_{\mathbf{p}}f)+v_{\mathbf{p}}(d_{\mathbf{p}}g)\big\}+|S|+3\mathfrak{g}.
\end{aligned}
$$

By (3.14), (3.15) and (3.16), we have

$$
\begin{aligned}
(n-1)h(f) &\le l(h(f)+h(g))+2\bar{N}_{\bar{S}}(f^{-1})+\bar{N}_{\bar{S}}(g^{-1})\\
&\quad +\sum_{\mathbf{p}\in C\mathcal{U}_{\bar{S}}}\big\{\bar{v}_{\mathbf{p}}(d_{\mathbf{p}}f)+\bar{v}_{\mathbf{p}}(d_{\mathbf{p}}g)\big\}+\delta\sum_{\mathbf{p}\in\mathcal{U}_{\bar{S}}}v_{\mathbf{p}}(d_{\mathbf{p}}g)\\
&\quad +\frac{1}{m_0'}\big\{\sum_{\mathbf{p}\in\mathcal{U}_{\bar{S}}}v_{\mathbf{p}}(d_{\mathbf{p}}f)+\sum_{\mathbf{p}\in\mathcal{U}_{\bar{S}}}v_{\mathbf{p}}(d_{\mathbf{p}}g)\big\}+5\mathfrak{g}-2+2|S|.
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
(n-1)h(g) &\le l(h(f)+h(g))+\bar{N}_{\bar{S}}(f^{-1})+2\bar{N}_{\bar{S}}(g^{-1})\\
&\quad +\sum_{\mathbf{p}\in C\mathcal{U}_{\bar{S}}}\big\{\bar{v}_{\mathbf{p}}(d_{\mathbf{p}}f)+\bar{v}_{\mathbf{p}}(d_{\mathbf{p}}g)\big\}+\delta\sum_{\mathbf{p}\in\mathcal{U}_{\bar{S}}}v_{\mathbf{p}}(d_{\mathbf{p}}f)\\
&\quad +\frac{1}{m_0'}\big\{\sum_{\mathbf{p}\in\mathcal{U}_{\bar{S}}}v_{\mathbf{p}}(d_{\mathbf{p}}f)+\sum_{\mathbf{p}\in\mathcal{U}_{\bar{S}}}v_{\mathbf{p}}(d_{\mathbf{p}}g)\big\}+5\mathfrak{g}-2+2|S|.
\end{aligned}
$$

Adding the two inequalities, we have

(3.17)

$$
\begin{aligned}
(n-1)(h(f)+h(g)) &\le 2l(h(f)+h(g))+3(\bar{N}_{\bar{S}}(f^{-1})+\bar{N}_{\bar{S}}(g^{-1}))\\
&\quad +(\frac{2}{m_0'}+1+\delta)\big\{\sum_{\mathbf{p}\notin S}v^0_{\mathbf{p}}(d_{\mathbf{p}}f)+\sum_{\mathbf{p}\notin S}v^0_{\mathbf{p}}(d_{\mathbf{p}}g)\big\}+10\mathfrak{g}-4+4|S|\\
&\le (2l+5+\frac{4}{m_0'}+2\delta)(h(f)+h(g))\\
&\qquad +(14+\frac{8}{m_0'}+4\delta)\mathfrak{g}-(8+\frac{8}{m_0'}+4\delta)+4|S|,
\end{aligned}
$$

by Proposition 17. Therefore,

$$(n-2l-6-\frac{4}{m_0'}-2\delta)(h(f)+h(g)) \le (14+\frac{8}{m_0'}+4\delta)\mathfrak{g}-(8+\frac{8}{m_0'}+4\delta)+4|S|.$$

If $n \ge 2l+7+\frac{4}{m_0'}+2\delta$, then

$$h(f)+h(g) \le (14+\frac{8}{m_0'}+4\delta)\mathfrak{g}-(8+\frac{8}{m_0'}+4\delta)+4|S|.$$

Since f and g are not constant, $h(f)+h(g)\ge 2$. This inequality also gives

$$n \le 2l+2+(7+\frac{4}{m_0'}+2\delta)\mathfrak{g}+2|S|.$$

For the case that f and g are S-integers, $\bar{N}_{\bar{S}}(f^{-1}) = \bar{N}_{\bar{S}}(g^{-1}) \le |S|$, and by Proposition 17 we have

$$\sum_{\mathbf{p}\notin S} v^0_{\mathbf{p}}(d_{\mathbf{p}}f)+\sum_{\mathbf{p}\notin S} v^0_{\mathbf{p}}(d_{\mathbf{p}}g) \le h(f)+h(g)+4\mathfrak{g}-4+2|S|.$$

Then (3.17) yields

$$\begin{aligned}&(n-2l-2-\frac{2}{m_0'}-\delta)(h(f)+h(g))\\ &\qquad\le (14+\frac{8}{m_0'}+4\delta)\mathfrak{g}-(8+\frac{8}{m_0'}+4\delta)+(6+\frac{4}{m_0'}+2\delta)|S|.\end{aligned}$$

If $n \ge 2l+3+\frac{2}{m_0'}+\delta$, then

$$h(f)+h(g) \le (14+\frac{8}{m_0'}+4\delta)\mathfrak{g}-(8+\frac{8}{m_0'}+4\delta)+(6+\frac{4}{m_0'}+2\delta)|S|.$$

Since f and g are not constant, $h(f)+h(g)\ge 2$. This inequality also gives

$$n \le 2l-2-\frac{2}{m_0'}-\delta+(7+\frac{4}{m_0'}+2\delta)\mathfrak{g}+(3+\frac{2}{m_0'}+\delta)|S|.$$

$\square$

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