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NULLIFICATION AND CELLULARIZATION OF CLASSIFYING SPACES OF FINITE GROUPS

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Abstract. In this note we discuss the effect of the $\mathrm{B}\mathbb{Z}/p$-nullification $\mathbf{P}_{\mathrm{B}\mathbb{Z}/p}$ and the $\mathrm{B}\mathbb{Z}/p$-cellularization $\mathbf{CW}_{\mathrm{B}\mathbb{Z}/p}$ over classifying spaces of finite groups, and we relate them with the corresponding functors with respect to Moore spaces that have been intensively studied in the last years. We describe $\mathbf{P}_{\mathrm{B}\mathbb{Z}/p}\mathrm{B}G$ by means of a covering fibration, and we classify all finite groups G for which $\mathrm{B}G$ is $\mathrm{B}\mathbb{Z}/p$-cellular. We also carefully study the analogous functors in the category of groups, and their relationship with the fundamental groups of $\mathbf{P}_{\mathrm{B}\mathbb{Z}/p}\mathrm{B}G$ and $\mathbf{CW}_{\mathrm{B}\mathbb{Z}/p}\mathrm{B}G$.

1. Introduction

Let A be a pointed connected space. A.K. Bousfield defined in [Bou94] the A-nullification functor $\mathbf{P}_A$ as the localization L_f with respect to the constant map $f : A \longrightarrow *$. Roughly speaking, if X is another pointed space, $\mathbf{P}_A(X)$ is the biggest "quotient" of X that has no essential map from any suspension of A. A little later, Dror-Farjoun defined ([DF95]) the somewhat dual notion of A-cellularization of a space X as the largest space $\mathbf{CW}_A X$ endowed with a universal map $\mathbf{CW}_A X \xrightarrow{cw} X$ that induces a weak equivalence between the mapping spaces $\mathrm{map}_*(A, \mathbf{CW}_A X)$ and $\mathrm{map}_*(A, X)$. The A-cellularization of X can also be viewed as the closest approximation of X that can be built out of A taking pointed homotopy colimits. The close relationship that exists between these two functors was clarified by the work of Chachólski ([Cha96]), where the description of each functor in terms of the other is presented, and the crucial concept of closed class is introduced. Moreover, these functors have been widely studied and used since they appeared; see for example [Dwy94], [Bou97], and [CDI02].

Dror-Farjoun also defines the A-homotopy theory of a space, where A and its suspensions play the role that S^0 and its suspensions play in classical homotopy theory. For example, he defines the A-homotopy groups $\pi_i(X; A)$ as the homotopy classes of pointed maps $[\Sigma^i A, X]_*$. In this framework, the functors $\mathbf{P}_{\Sigma^i A}(X)$ and $\mathbf{CW}_{\Sigma^i A}(X)$ can be viewed, respectively, as the i-th Postnikov section and the i-connective cover of X. Moreover the spaces for which $X \simeq \mathbf{P}_A(X)$ (A-null spaces) play the role of weakly contractible spaces, and the spaces such that $A \simeq \mathbf{CW}_A X$ (A-cellular spaces) are the analogues of the CW-complexes. Indeed, the A-cellularization is nothing but the A-cellular approximation of X.

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Our main interest is in $\mathrm{B}\mathbb{Z}/p$-homotopy theory. After the positive resolution by Miller ([Mil84]) of the Sullivan conjecture, thorough research has been conducted which has provided information about the space $\mathrm{map}_*(\mathrm{B}\mathbb{Z}/p, X)$ when the target is nilpotent, and hence it has given clues about the value of the $\mathrm{B}\mathbb{Z}/p$-nullification functor on X; a good survey on this topic can be found in the book [Sch94]. For example, we have the result of Lannes-Schwartz ([LS89]) which points out that if moreover the $\mathbb{Z}/p$-cohomology of X has finite type and $\pi_1(X)$ is finite, then X is $\mathrm{B}\mathbb{Z}/p$-null if and only if the cohomology algebra of X is locally finite with respect to the action of the Steenrod algebra. Unfortunately, not much is known if we do not impose the nilpotence hypothesis.

We have focused our study on the $\mathrm{B}\mathbb{Z}/p$-nullification and $\mathrm{B}\mathbb{Z}/p$-cellularization of classifying spaces of finite groups. Even if the group G is not nilpotent (and hence $\mathrm{B}G$ is not nilpotent as a space, and we cannot apply the aforementioned results), this seems to be an accessible case, because we have a precise description of the space $\mathrm{map}_*(\mathrm{B}\mathbb{Z}/p, \mathrm{B}G)$ as the space of group homomorphisms $\mathrm{Hom}(\mathbb{Z}/p, G)$, which is homotopically discrete. In particular, this gives a feeling that the value of these functors on $\mathrm{B}G$ will depend strongly on the group theory of G.

One of the classical invariants that measure the p-primary part of the homotopy structure of a space X are the homotopy groups with coefficients in $\mathbb{Z}/p$. Recall that the $(n+1)$-st homotopy group of X with coefficients in $\mathbb{Z}/p$ is defined to be the group of homotopy classes of pointed maps $[M(\mathbb{Z}/p, n), X]_*$, where $M(\mathbb{Z}/p, n)$ is the corresponding Moore space and $n \geq 1$. Thus, understanding these homotopy groups amounts to describing the $M(\mathbb{Z}/p, 1)$-homotopy theory of X. This problem has been attacked succesfully by Bousfield ([Bou97]) and Rodríguez-Scherer ([RS98]), who describe precisely the $M(\mathbb{Z}/p, 1)$-nullification and $M(\mathbb{Z}/p, 1)$-cellularization of $\mathrm{B}G$. Then it seems natural to ask if there exists any similar description of the $\mathrm{B}\mathbb{Z}/p$-homotopy theory of $\mathrm{B}G$ and, if so, what relationship it has with the above-mentioned p-primary part of the classical homotopy structure of $\mathrm{B}G$.

Our first result gives a partial answer to this question by characterizing the $\mathrm{B}\mathbb{Z}/p$-nullification of the classifying space of a finite group G by means of a covering fibration. Recall that Bousfield defines in [Bou97, 7.1] the $\mathbb{Z}/p$-radical $\mathrm{T}_{\mathbb{Z}/p}G$ of G (sometimes denoted by O_pG in the framework of group theory) as the smallest normal subgroup of G that contains all the p-torsion.

Theorem 3.5. *Let G be a finite group and p a prime number. Then, the $\mathrm{B}\mathbb{Z}/p$-nullification $\mathbf{P}_{\mathrm{B}\mathbb{Z}/p}\mathrm{B}G$ of $\mathrm{B}G$ fits into a fibration sequence:*

$$\prod_{q\neq p} \mathrm{B}(\mathrm{T}_{\mathbb{Z}/p}G)^\wedge_q \longrightarrow \mathbf{P}_{\mathrm{B}\mathbb{Z}/p}\mathrm{B}G \longrightarrow \mathrm{B}(G/\mathrm{T}_{\mathbb{Z}/p}G),$$

where $\mathrm{B}(\mathrm{T}_{\mathbb{Z}/p}G)^\wedge_q$ denotes the Bousfield-Kan $\mathbb{Z}/q$-completion of $\mathrm{BT}_{\mathbb{Z}/p}G$ (see [BK72] *and the end of the preliminaries for the definition and main properties of this functor).*

It is easy to see from this result that the only groups G for which $\mathrm{B}G$ is $\mathrm{B}\mathbb{Z}/p$-acyclic are the p-groups, and, according to [Lev95], $\mathbf{P}_{\mathrm{B}\mathbb{Z}/p}\mathrm{B}G$ has nonzero homotopy groups in arbitrarily high dimensions if and only if the $\mathbb{Z}/p$-radical of G is not a p-group. Moreover, if G is simple and contains p-torsion, the $\mathrm{B}\mathbb{Z}/p$-nullification of $\mathrm{B}G$ is simply connected, and it is not hard to prove that if G is nilpotent, $\mathbf{P}_{\mathrm{B}\mathbb{Z}/p}\mathrm{B}G$ is an Eilenberg-Mac Lane space that is nilpotent too.

If $\mathrm{M}(\mathbb{Z}/p,1)$ is a Moore space of dimension 2, the results of [Bou97, section 7] and the previous theorem guarantee that the $\mathrm{B}\mathbb{Z}/p$-nullification of $\mathrm{B}G$ is homotopy equivalent to its $\mathrm{M}(\mathbb{Z}/p,1)$-nullification (the map $\mathbf{P}_{\mathrm{B}\mathbb{Z}/p}\mathrm{B}G \to \mathbf{P}_{M(\mathbb{Z}/p,1)}\mathrm{B}G$ coming from the fact that $\mathrm{B}\mathbb{Z}/p$ is $M(\mathbb{Z}/p,1)$-acyclic). Hence, these functors coincide on classifying spaces of finite groups, and this proves that the aforementioned relationship between the p-primary part of $\mathrm{B}G$ and its $\mathrm{B}\mathbb{Z}/p$-homotopy theory is really close. On the other hand, it is easy to see that these two functors do not coincide over every X: take for example $p=2$ and $X=\mathbb{R}\mathbf{P}^2$, which is a model for $M(\mathbb{Z}/2,1)$ and is $\mathrm{B}\mathbb{Z}/2$-null by Miller's theorem ([Mil84]).

An important consequence of the previous observation is that a great part of the results of [RS98] relative to Moore spaces remain valid for the case of $\mathrm{B}\mathbb{Z}/p$-nullification, and in particular they allow us to obtain a very precise description of the value of the acyclic functor $\bar{\mathbf{P}}_{\mathrm{B}\mathbb{Z}/p}$ over $\mathrm{B}G$. We recall that the acyclic functor is the colocalization associated to the nullification.

We finish the study of $\mathbf{P}_{\mathrm{B}\mathbb{Z}/p}\mathrm{B}G$ using our description of it for proving Proposition 3.9, that states that the nullification functors $\mathbf{P}_{\mathrm{B}\mathbb{Z}/p}$ and $\mathbf{P}_{\mathrm{B}\mathbb{Z}/q}$ commute over $\mathrm{B}G$ for distinct primes primes p and q (see [RS00] for an overview of the problem of commutation in localization theory), and establishing some relations (Propositions 3.12 and 3.14) between the $\mathrm{B}\mathbb{Z}/p$-nullification functor and the Bousfield-Kan $\mathbb{Z}/p$-completion and $\mathbb{Z}[1/p]$-completion.

The second part of this note is devoted to the analysis of the $\mathrm{B}\mathbb{Z}/p$-cellularization of $\mathrm{B}G$. Our main result on this topic has been the characterization of the class of finite groups G such that $\mathrm{B}G$ is $\mathrm{B}\mathbb{Z}/p$-cellular.

Theorem 4.14. *Let G be a finite $\mathbb{Z}/p$-cellular group. Then $\mathrm{B}G$ is $\mathrm{B}\mathbb{Z}/p$-cellular if and only if G is a p-group generated by order p elements.*

The proof relies essentially on a result of Chachólski (Proposition 2.5) about preservation of cellularity under fibrations.

In general the $\mathrm{B}\mathbb{Z}/p$-cellularization of $\mathrm{B}G$ is related very closely with the $\mathbb{Z}/p$-cellularization $\mathrm{CW}_{\mathbb{Z}/p}G$ of G in the category of groups (see section 4 for the precise definition and [RS98] for the original reference), and hence we devote a great part of our efforts to clarifying this relation. So, after describing the kernel of the central extension that determines the $\mathbb{Z}/p$-cellularization

$$0 \longrightarrow A \longrightarrow \mathrm{CW}_{\mathbb{Z}/p}G \longrightarrow \mathrm{S}_{\mathbb{Z}/p}G \longrightarrow 0$$

(where $\mathrm{S}_{\mathbb{Z}/p}G$ is the the $\mathbb{Z}/p$-*socle* of G, that is, the (normal) subgroup of G generated by order p elements), we prove that A can provide a large amount of information on some group-theoretic invariants of G, such as the Schur multiplier or the universal central extension with coefficients in $\mathbb{Z}$ or $\mathbb{Z}[1/p]$ (if G is respectively perfect or $\mathbb{Z}[1/p]$-perfect). In fact, the latter is in particular the fundamental group of $\overline{\mathbf{P}}_{\mathrm{B}\mathbb{Z}/p}\mathrm{B}G$, and this is used for proving that the $\mathrm{B}\mathbb{Z}/p$-acyclic functor "commutes" with the fundamental group. The section concludes with a description of $\pi_1(\mathbf{CW}_{\mathrm{B}\mathbb{Z}/p}\mathrm{B}G)$.

In the last section we apply the results of the two previous sections for analyzing the effect of nullification and cellularization on (classifying spaces of) some families of discrete groups. For the first of them, the output is usually an aspherical space or a Bousfield-Kan completion; for the second, we compute $\mathbf{CW}_{\mathrm{B}\mathbb{Z}/p}\mathrm{B}G$ in some examples (Q_n, SD_n, $M_n(p)$, some concrete cases of the other families), and for all

the groups we study we obtain the $\mathbb{Z}/p$-cellularization of G and we decide if its classifying space is $\mathrm{B}\mathbb{Z}/p$-cellular or not.

Notation. Throughout this note the word "space" will stand for "CW-complex", and usually we will suppose that these spaces are pointed. The notation $X_p^\wedge$ will denote Bousfield-Kan $\mathbb{Z}/p$-completion of X, whereas $\mathbb{Z}[1/p]_\infty X$ will stand for the $\mathbb{Z}[1/p]$-completion.

2. Preliminaries

A space X is called *A-null* if the natural inclusion $X \hookrightarrow \mathrm{map}(A, X)$ (as constant maps) is a weak equivalence. The *A-nullification* $\mathbf{P}_A X$ (sometimes called *A-periodization*) is the only A-null space, up to homotopy equivalence, endowed with a map $X \longrightarrow \mathbf{P}_A X$ which induces, for every A-null space Y, a weak homotopy equivalence

$$\mathrm{map}(\mathbf{P}_A X, Y) \simeq \mathrm{map}(X, Y).$$

In this way a functor $\mathbf{P}_A : \mathbf{Spaces} \longrightarrow \mathbf{Spaces}$ is defined, which is coaugmented and idempotent. We remark that for connected spaces there is no problem in defining the nullification functor in the pointed category; in this case, X is A-null if $\mathrm{map}_*(A, X)$ is weakly contractible, and the criterion for a space Y to be the A-nullification of X is the same as before, but using pointed mapping spaces.

There are various constructions of $\mathbf{P}_A X$, and the easiest is probably the following: take a space X_λ for every ordinal λ; define $X_0 = X$ and by induction let $X_{\lambda+1}$ be the space obtained from X_λ gluing cones over all homotopy classes of maps from n-suspensions of A into X_λ. Then, the A-nullification of X is defined as the colimit of all these spaces. We want to point out that this construction is not functorial; the standard one that enjoys this property is developed in [DF95, 1.B], and other constructions can be found in Bousfield ([Bou94, 2.8]) and Chachólski ([Cha96, 17.1]).

The A-nullification can also be defined as the localization in the sense of Dror-Farjoun with respect to the trivial map $A \longrightarrow *$, and using the fact that the following properties are easy to prove ([DF95, 1.A.8]):

- The natural map $\mathbf{P}_A(X \times Y) \longrightarrow \mathbf{P}_A X \times \mathbf{P}_A Y$ is a homotopy equivalence.
- If X is A-null, then it is also ΣA-null, and $\Omega^n X$ is A-null, for every n.
- If $\mathbf{P}_A X \simeq *$, then $\mathbf{P}_{\Sigma A} \Sigma X \simeq *$.
- If we have a fibration where the base and the fibre are A-null, then the total space is A-null.
- If X is 1-connected, $\mathbf{P}_A X$ is 1-connected too.

One of the most important achievements of the work of Dror-Farjoun is the description of the behaviour of the localization functors with respect to fibrations. In particular, he proves the following results, which will be crucial and used repeatedly in our work.

Theorem 2.1 ([DF95, 1.H.1 and 3.D.3])**.** *Let $F \longrightarrow E \longrightarrow B$ be a fibration. Then*

(1) *If $\mathbf{P}_A(F)$ is contractible, then the induced map $\mathbf{P}_A(E) \longrightarrow \mathbf{P}_A(B)$ is a homotopy equivalence.*
(2) *If B is A-null, then the fibration is preserved under A-nullification.*

The simplest example of nullification is the Postnikov n-section, which is exactly the S^{n+1}-nullification. Other widely studied examples have been Quillen plus-construction $X \longrightarrow X^+$, which is the nullification with respect to a large space that is acyclic for a certain homology theory (see [BC99], for example), the nullification with respect to Moore spaces, or the $\mathrm{B}\mathbb{Z}/p$-nullification of classifying spaces of compact Lie groups such that the group of components is a p-group. As we will see, our work is closely related to these last two examples.

To conclude with nullification, we briefly discuss the concept of A-periodic equivalence, due to Bousfield.

Definition 2.2. A map $f : X \longrightarrow Y$ is called an *A-periodic equivalence* if for any A-null space Z and any choice of basepoints in X and Z, the map f induces a weak homotopy equivalence $\mathrm{map}_*(Y, Z) \simeq \mathrm{map}_*(X, Z)$. In particular, the coaugmentation $X \longrightarrow \mathbf{P}_A X$ is always an A-periodic equivalence, because the functor $\mathbf{P}_A$ is idempotent.

The main properties of A-periodic equivalences can be found in [Cha96, section 13].

In the same way as the A-nullification isolates "the part" of a space X that is not visible by means of $\mathrm{map}_*(A, X)$, the A-cellularization describes to what extent a space can be built using A as building blocks. We will begin our overview of cellularization by defining the concept of cellular space.

Definition 2.3. A space X is called *A-cellular* if for any choice of basepoint in X and for every pointed map $f : Y \longrightarrow Z$ such that $f_* : \mathrm{map}_*(A, Y) \longrightarrow \mathrm{map}_*(A, Z)$ is a weak equivalence, we have that the induced map of mapping spaces $f_* : \mathrm{map}_*(X, Y) \longrightarrow \mathrm{map}_*(X, Z)$ is also a weak equivalence. It can be proved that this is equivalent to saying that X can be built as an (iterated) pointed homotopy colimit of copies of A.

Hence, the *A-cellularization* of X is defined as the unique A-cellular space $\mathbf{CW}_A X$ (up to homotopy) such that there exists a canonical augmentation $cw : \mathbf{CW}_A X \longrightarrow X$ which induces a weak homotopy equivalence $\mathrm{map}_*(A, \mathbf{CW}_A X) \simeq \mathrm{map}_*(A, X)$. In particular, the augmentation cw has the following two features ([DF95, 2.E.8]):

(1) If $f : Y \longrightarrow X$ induces a weak homotopy equivalence $\mathrm{map}_*(A, Y) \simeq \mathrm{map}_*(A, X)$, there is a map $f' : \mathbf{CW}_A X \longrightarrow Y$ such that $f \circ f'$ is homotopic to cw. Moreover, f' is unique up to homotopy.
(2) If Z is A-cellular and $g : Z \longrightarrow X$ is a map, then there exists $g' : Z \longrightarrow \mathbf{CW}_A X$ such that $cw \circ g'$ is homotopic to g and g' is unique up to homotopy.

Dror Farjoun also gave the first two constructions of the functor $\mathbf{CW}_A$, a standard and functorial one ([DF95, 2.E.3]), and another more intuitive but with the disadvantage of being not functorial ([DF95, 2.E.5]). However, here we will recall the construction of Chachólski ([Cha96, section 7]), as it will be more useful for our purposes.

Proposition 2.4. *If A is connected, the A-cellularization of X has the homotopy type of the homotopy fibre of the map $\eta : X \longrightarrow LX$, where η is the composition of the inclusion $X \hookrightarrow C_f$ into the homotopy cofibre of the evaluation map $\bigvee_{[A,X]_*} A \longrightarrow X$, with the nullification $C_f \longrightarrow \mathbf{P}_{\Sigma A} C_f$.*

This can be interpreted as a definition of the functor $\mathbf{CW}_A$ in terms of $\mathbf{P}_A$, and in fact it is also possible to describe $\mathbf{P}_A$ in terms of the A-cellularization. It is worth pointing out that any construction of $\mathbf{CW}_A$ must be worked out in the *pointed* category, because it is not possible to define $\mathbf{CW}_A$ over unpointed spaces ([Cha96, 7.4]).

Chachólski also makes the key observation that A-cellular spaces constitute a *closed class*, i.e. a class of spaces that is closed under weak equivalences and pointed homotopy colimits. In particular, the class of A-cellular spaces is the smallest closed class that contains A. Next we list some important properties of A-cellular spaces, which are nothing but the translations of the corresponding properties of closed classes. The last one will be particularly important in this note, because it allows us to build new cellular spaces from old ones. The proofs can be found in [Cha96, section 4].

Proposition 2.5. *Let A be a space. Then*

- *If X is weakly contractible, it is A-cellular.*
- *If B is A-cellular and X is B-cellular, then X is A-cellular.*
- *If X is A-cellular, then the n-suspension $\Sigma^n X$ is A-cellular.*
- *If $F \longrightarrow E \longrightarrow B$ is a fibration with a section and F and B are A-cellular, then E is A-cellular. In particular, the product of two A-cellular spaces is A-cellular, and for every pair of spaces X, Y we have a weak equivalence $\mathbf{CW}_A(X \times Y) \simeq \mathbf{CW}_A X \times \mathbf{CW}_A Y$.*

In [DF95, 3.5], one can find many examples of interesting A-cellular spaces. For example, the James construction $\Omega\Sigma A$ is A-cellular, the Dold-Thom functor $SP^\infty A$ is A-cellular, the classifying space of a group $\mathrm{B}G$ is ΣG-cellular for every group G, etc. In particular, he proves that for every $n \geq 1$ the $\mathrm{B}\mathbb{Z}/p$-cellularization of $\mathrm{B}\mathbb{Z}/p^n$ is $\mathrm{B}\mathbb{Z}/p$, a fact that can be considered as a starting point for our work.

It is also worth recalling the concept of A-*cellular equivalence*.

Definition 2.6. A map $X \longrightarrow Y$ is called an A-cellular equivalence if it induces a weak equivalence $\mathrm{map}_*(A, X) \simeq \mathrm{map}_*(A, Y)$.

The main properties of the A-cellular equivalences can be found in [Cha96, section 6].

We finish this sample in localization, which certainly does not claim to be exhaustive, by commenting on the relationship between the functors $\mathbf{P}_A$ and $\mathbf{CW}_A$. For this, it is necessary to define the acyclics, which at any rate have their own interest.

Definition 2.7. A space X is called A-*acyclic* if $\mathbf{P}_A X$ is contractible. The functor $\bar{\mathbf{P}}_A : \mathbf{Spaces}_* \longrightarrow \mathbf{Spaces}_*$, which sends every space to the homotopy fiber of its A-nullification, is augmented and idempotent (in fact, it is a colocalization), and its image is the class of A-acyclic spaces.

The class of A-acyclic spaces was the crucial ingredient that Chachólski used for stating the main result of [Cha96], which strongly generalized a theorem of Dror-Farjoun, and gave an amazingly sharp description of to what extent the functors $\mathbf{P}_A$ and $\mathbf{CW}_A$ can be considered "dual".

Theorem 2.8. *Let A be a space. Then*

(1) *A space X is A-null if and only if its A-cellularization is a point.*

(2) *A space X is A-acyclic if and only if it belongs to the smallest closed class that contains A and is closed by extension by fibrations. In particular, every A-cellular space is A-acyclic.*

We recall that a closed class $\mathcal{C}$ is closed by extensions by fibrations if for every fibration $F \longrightarrow E \longrightarrow B$ such that $F \in \mathcal{C}$ and $B \in \mathcal{C}$, we have $E \in \mathcal{C}$.

This result and [DF95, 9.A.6], which show that the A-cellularization is a kind of mixing process between the functors $\bar{\mathbf{P}}_A$ and $\bar{\mathbf{P}}_{\Sigma A}$, have made much more accessible the computation of the value of the A-cellularization of a space, and have greatly stimulated the research on this field. Among the most recent works on it, we can quote [CDI02], where the authors generalize the notion of dimension of a CW-complex to the A-cellular framework, or [DGI02], where the concept of cellularization is extended to algebraic categories of R-modules.

We end this preliminary section by recalling the main features of the p-completion functor of Bousfield-Kan, especially when applied to the classifying space of a finite group. The reason for doing so is that, according to the main Theorem 3.5, our computation of $\mathbf{P}_{\mathrm{B}\mathbb{Z}/p}\mathrm{B}G$ presents the universal covering of the $\mathrm{B}\mathbb{Z}/p$-nullification of $\mathrm{B}G$ (and sometimes $\mathbf{P}_{\mathrm{B}\mathbb{Z}/p}\mathrm{B}G$ itself) as a (finite) product of $\mathbb{Z}/q$-completions.

In general, if R is a commutative ring with unity, there exists ([BK72, I]) a coaugmented endofunctor R_∞ of the category of spaces such that if $f : X \longrightarrow Y$ is a map that induces isomorphism in homology with coefficients in R, then $R_\infty f$ is a homotopy equivalence. Moreover, R_∞ preserves disjoint unions and finite products.

We focus our attention on the $\mathbb{Z}/p$-completion of X, denoted $X_p^\wedge$. The $\mathbb{Z}/p$-completion is particularly useful when applied to $\mathbb{Z}/p$-good spaces, namely spaces for which the natural coaugmentation $X \longrightarrow X_p^\wedge$ is a mod p homology equivalence. By a result of Bousfield-Kan ([BK72, VII.5]), every space with a finite fundamental group is $\mathbb{Z}/p$-good, so in particular the classifying of a finite group is also. Hence, the mod p (co)homology of $\mathrm{B}G_p^\wedge$ is nothing but the mod p (co)homology of G, and moreover, by the universal properties of the $\mathbb{Z}/p$-completion ([BK72, VII.2.1]), both the mod q homology ($q \neq p$) and the rational homology of $\mathrm{B}G$ are trivial, so that in the cases in which the (co)homology of $\mathrm{B}G$ is known we have an almost complete description of the homological structure of $\mathrm{B}G_p^\wedge$. On the other hand, the result [BK72, VII.4.2] allows us to take the opposite point of view, describing $\mathrm{B}G$ (if G is nilpotent) as a product of $\mathbb{Z}/p$-completions.

It is remarkable that we have not only information about the (co)homology of $\mathrm{B}G_p^\wedge$, but a quite sharp description of its homotopy type. We summarize the main results on this subject in the next proposition.

Proposition 2.9. *Let G be a finite group, $\mathrm{B}G_p^\wedge$ its $\mathbb{Z}/p$-completion for a prime p. Then we have:*

- ([BK72, VII.4.3]) *The homotopy groups of the $\mathbb{Z}/p$-completion of $\mathrm{B}G$ are finite p-groups, and in particular $\mathrm{B}G_p^\wedge$ is a nilpotent space.*
- ([Cas91]) *$\mathrm{B}G_p^\wedge$ is a $K(H,1)$ if and only if G is p-nilpotent (and hence the elements of G that are not of p-torsion form a group).*
- ([Lev95, 1.1.4]) *If $\pi_n(\mathrm{B}G_p^\wedge) \neq 0$ for some $n \geq 2$, then $\mathrm{B}G_p^\wedge$ has an infinite number of nonzero homotopy groups.*
- ([BK72, II.5]; see also [Lev95]) *The fundamental group of $\mathrm{B}G_p^\wedge$ is the p-profinite completion of G, that can be described as the quotient of G by its maximal normal p-perfect group.*

The conclusion is that the large amount of research undertaken in the last thirty years on this subject has greatly clarified much of the structure of the $\mathbb{Z}/p$-completion of the classifying space of a finite group (and in general a $\mathbb{Z}/p$-good space). As we said in the Introduction, little is known about $\mathrm{B}\mathbb{Z}/p$-nullification of non-nilpotent infinite-dimensional spaces (see [Dwy94] and [Nei94] for the main references on this topic), so we expect that the aforementioned background on $\mathbb{Z}/p$-completion makes our description of the $\mathrm{B}\mathbb{Z}/p$-nullification of $\mathrm{B}G$ more useful.

Remark 2.10. Bousfield-Kan $\mathbb{Z}/p$-completion is usually referred to in the literature simply as p-completion, but we have preferred to use the classical name to avoid any confusion with completion in the ring $\mathbb{Z}[1/p]$, which also appears in this note.

3. $\mathrm{B}\mathbb{Z}/p$-nullification of classifying spaces of finite groups

3.1. **Computing the nullification of $\mathbf{B}G$.** Let G be a finite group and p a prime number. As we stated in the Introduction, we have been interested in studying the p-primary part of the classifying space of G using the functors $\mathbf{P}_{\mathrm{B}\mathbb{Z}/p}$ and $\mathbf{CW}_{\mathrm{B}\mathbb{Z}/p}$; in this section we will be concerned with the former. So, our first result on this topic is a characterization of the space $\mathbf{P}_{\mathrm{B}\mathbb{Z}/p}\mathrm{B}G$ by means of a covering fibration. As far as we know, these spaces have only been described for G nilpotent, and in this case the proof is really easy: if H is the p-torsion subgroup of G, it is enough to $\mathrm{B}\mathbb{Z}/p$-nullify the fibration

$$\mathrm{B}H \longrightarrow \mathrm{B}G \longrightarrow \mathrm{B}(G/H)$$

to obtain that $\mathbf{P}_{\mathrm{B}\mathbb{Z}/p}\mathrm{B}G \simeq \mathrm{B}(G/H)$. The difficulty of the general case lies in the fact that in general the minimal subgroup that contains the p-torsion has elements that are *not* of p-torsion. We avoid this problem by first identifying the case in which the $\mathrm{B}\mathbb{Z}/p$-nullification is simply connected, and then passing to the general case.

We begin the section with a technical lemma that will be very useful.

Lemma 3.1. *Let G be a finite group, p a prime and X a $\mathrm{B}\mathbb{Z}/p$-null space. If G has no nontrivial quotients of order coprime with p and $h : X \longrightarrow \mathrm{B}\pi_1(X)$ is the covering fibration of X, then for every continuous map $f : \mathrm{B}G \longrightarrow X$ the composite $h \circ f$ is inessential.*

Proof. As X is $\mathrm{B}\mathbb{Z}/p$-null, we have that for every map $g : \mathrm{B}\mathbb{Z}/p \longrightarrow \mathrm{B}G$ the composite $h \circ f \circ g$ is inessential, and in particular $\pi_1 h \circ \pi_1 f \circ \mu$ is zero for *every* group homomorphism $\mu : \mathbb{Z}/p \longrightarrow G$. Hence, the image of $\pi_1 h \circ \pi_1 f$ is a quotient of G whose order is coprime with p, so by hypothesis it must be trivial. Thus, $\pi_1 f \circ \pi_1 h$ must be the zero homomorphism, and so $h \circ f \simeq *$, as we wanted. □

Now, as a first step in the proof of the main Theorem 3.5, the next proposition will compute an illuminating particular case of it.

Proposition 3.2. *Let G be a finite group and p prime. Suppose that G has no nontrivial quotients of order prime to p. Then we have $\mathbf{P}_{\mathrm{B}\mathbb{Z}/p}\mathrm{B}G \simeq \prod_{q\neq p} \mathrm{B}G^\wedge_q$.*

Proof. First of all, we have to prove that $\prod_{q\neq p} \mathrm{B}G^\wedge_q$ is a $\mathrm{B}\mathbb{Z}/p$-null space. But this is clear because

$$\mathrm{map}_*(\mathrm{B}\mathbb{Z}/p, \mathrm{B}G^\wedge_q) \simeq \mathrm{map}_*((\mathrm{B}\mathbb{Z}/p)^\wedge_q, \mathrm{B}G^\wedge_q) \simeq *,$$

where the first equivalence holds because $\mathrm{B}G^\wedge_q$ is $\mathbb{Z}/q$-complete.

So, now we must see that if X is another $\mathrm{B}\mathbb{Z}/p$-null space, there exists a homotopy equivalence

$$\mathrm{map}_*(\mathrm{B}G, X) \simeq \mathrm{map}_*(\prod_{q\neq p} \mathrm{B}G_q^\wedge, X).$$

We will consider two cases.

First, we will suppose that X is a simply connected space. In this case, Sullivan's arithmetic square gives us a homotopy equivalence

$$\mathrm{map}_*(\mathrm{B}G, X) \simeq \mathrm{map}_*(\mathrm{B}G, \prod_{q \text{ prime}} X_q^\wedge).$$

We shall check that $\mathrm{map}_*(\mathrm{B}G, X_p^\wedge)$ is contractible. The space $X_p^\wedge$ is $\mathbb{Z}/p$-complete, so we have an equivalence $\mathrm{map}_*(\mathrm{B}G, X_p^\wedge) \simeq \mathrm{map}_*(\mathrm{B}G_p^\wedge, X_p^\wedge)$. Using Jackowski-McClure-Oliver subgroup decomposition ([JMO92], see also [Dwy97]), we obtain that

$$\mathrm{map}_*(\mathrm{B}G_p^\wedge, X_p^\wedge) \simeq \mathrm{map}_*((\mathrm{hocolim}_{\mathbf{O}_\mathcal{C}} \beta_\mathcal{C})_p^\wedge, X_p^\wedge),$$

where $\beta_\mathcal{C}$ is a functor whose values have the homotopy type of classifying spaces of p-subgroups of G, and $\mathbf{O}_\mathcal{C}$ is a $\mathbb{Z}/p$-acyclic category. Again, because $X_p^\wedge$ is $\mathbb{Z}/p$-complete, we have

$$\mathrm{map}_*((\mathrm{hocolim}_{\mathbf{O}_\mathcal{C}} \beta_\mathcal{C})_p^\wedge, X_p^\wedge) \simeq \mathrm{map}_*(\mathrm{hocolim}_{\mathbf{O}_\mathcal{C}} \beta_\mathcal{C}, X_p^\wedge).$$

Now, if $\mathrm{B}\mathbf{O}_\mathcal{C}$ stands for the realization of the nerve of $\mathbf{O}_\mathcal{C}$, the $\mathbb{Z}/p$-acyclicity of $\mathrm{B}\mathbf{O}_\mathcal{C}$ gives homotopy equivalences of *unpointed* mapping spaces

$$\mathrm{map}(\mathrm{B}\mathbf{O}_\mathcal{C}, X_p^\wedge) \simeq \mathrm{map}((\mathrm{B}\mathbf{O}_\mathcal{C})_p^\wedge, X_p^\wedge) \simeq X_p^\wedge.$$

But according to [Dwy94, 3.8], there exists a homotopy equivalence

$$\mathrm{map}(\mathrm{B}\mathbf{O}_\mathcal{C}, X_p^\wedge) \simeq \mathrm{map}(\mathrm{hocolim}_{\mathbf{O}_\mathcal{C}} \beta_\mathcal{C}, X_p^\wedge) \simeq X_p^\wedge,$$

so joining all these equivalences, it is immediate from the classical fibration of [DF95, remark to 1.A.1] that the *pointed* mapping space $\mathrm{map}_*(\mathrm{hocolim}_{\mathbf{O}_\mathcal{C}} \beta_\mathcal{C}, X_p^\wedge)$ is contractible, and so is $\mathrm{map}_*(\mathrm{B}G, X_p^\wedge)$, as required.

We now have the following string of weak equivalences:

$$\begin{aligned}\mathrm{map}_*(\mathrm{B}G, X) &\simeq \mathrm{map}_*(\mathrm{B}G, \prod_{q\neq p} X_q^\wedge) \simeq \prod_{q\neq p} \mathrm{map}_*(\mathrm{B}G, X_q^\wedge) \\ &\overset{(*)}{\simeq} (\prod_{q\neq p} \mathrm{map}_*(\mathrm{B}G_q^\wedge, X_q^\wedge) \overset{(**)}{\simeq} \prod_{q\neq p} \mathrm{map}_*(\prod_{r\neq p} \mathrm{B}G_r^\wedge, X_q^\wedge) \simeq \mathrm{map}_*(\prod_{r\neq p} \mathrm{B}G_r^\wedge, X)\end{aligned}$$

where the equivalence $(*)$ holds because X is simply connected (and therefore $X_q^\wedge$ is $\mathbb{Z}/q$-complete for every q), and $(**)$ holds because the space $\mathrm{map}_*(\mathrm{B}G_q^\wedge, X_r^\wedge)$ is contractible if q and r are distinct primes.

Now we can attack the general case. Let X be a $\mathrm{B}\mathbb{Z}/p$-null space. If $\tilde{X}$ is the universal cover of X, we have the covering fibration

$$\tilde{X} \longrightarrow X \xrightarrow{h} \mathrm{B}\pi_1(X).$$

According to Lemma 3.1, the image of the map

$$\mathrm{map}_*(\mathrm{B}G, X) \xrightarrow{h_\natural} \mathrm{map}_*(\mathrm{B}G, \mathrm{B}\pi_1(X))$$

is contained in the component $\mathrm{map}_*(\mathrm{B}G, \mathrm{B}\pi_1(X))_c$ of the constant map. Then consider the following diagram, where the left column is a fibration, the horizontal

maps are all induced by the product of the $\mathbb{Z}/q$-completions of $\mathrm{B}G$, and $\tilde{X}$ is the universal cover of X:

$$\begin{tikzcd}
\mathrm{map}_*(\prod_{q\neq p}\mathrm{B}G_q^\wedge, \tilde{X}) \arrow[r, "\simeq"] \arrow[d, "(2)"] & \mathrm{map}_*(\mathrm{B}G, \tilde{X}) \arrow[d, "(3)"] \\
\mathrm{map}_*(\prod_{q\neq p}\mathrm{B}G_q^\wedge, X) \arrow[r, "(1)"] \arrow[d, "h_\natural"] & \mathrm{map}_*(\mathrm{B}G, X) \arrow[d, "h_\natural"] \\
\mathrm{map}_*(\prod_{q\neq p}\mathrm{B}G_q^\wedge, \mathrm{B}\pi_1(X)) \arrow[r] & \mathrm{map}_*(\mathrm{B}G, \mathrm{B}\pi_1(X))_c
\end{tikzcd}$$

The top horizontal map is an equivalence by the first case done before and [ABN94, 9.7], and the right column has been seen to take values in the component of the constant map, so it is a fibration. It is known that this component is contractible, and the same is true for $\mathrm{map}_*(\prod_{q\neq p}\mathrm{B}G_q^\wedge, \mathrm{B}\pi_1(X))$, because $\mathrm{B}\pi_1(X)$ is an Eilenberg-Mac Lane space and $\prod_{q\neq p}\mathrm{B}G_q^\wedge$ is simply connected (recall from the preliminaries that if G is a finite group and p is a prime number, it is known that the fundamental group of $\mathrm{B}G_p^\wedge$ is the quotient of G by its p-perfect maximal normal subgroup $O^p(G)$ and in particular, the fundamental group is a p-group; so, as G has no quotient of order coprime with p, $\pi_1(\prod_{q\neq p}\mathrm{B}G_q^\wedge)$ is trivial). Hence, the maps (2) and (3) are weak equivalences; by the commutativity of the diagram, this means that (1) is a weak equivalence, and we are done. □

Remark 3.3. According to [BK72, VII.4.2 and 4.3], $\prod_{q\neq p}\mathrm{B}G_q^\wedge$ is homotopy equivalent to the $\mathbb{Z}[1/p]$-completion of $\mathrm{B}G$.

For the general case of the theorem we will need to identify in some way the p-torsion of G, and this will lead us to the concept of $\mathbb{Z}/p$-radical.

Definition 3.4. Let G be a finite group and p a prime. The $\mathbb{Z}/p$-*radical* of G (sometimes called the $\mathbb{Z}/p$-isolator) is the minimal normal subgroup $\mathrm{T}_{\mathbb{Z}/p}G$ that contains all the p-torsion elements of G.

The following features of this subgroup are easy to prove:

- The index of $\mathrm{T}_{\mathbb{Z}/p}G$ in G is coprime with p, and $\mathrm{T}_{\mathbb{Z}/p}G$ is minimal among the normal subgroups of G for which this condition holds.
- $\mathrm{T}_{\mathbb{Z}/p}G$ is a characteristic subgroup of G, i.e. every automorphism of G restricts to an automorphism of $\mathrm{T}_{\mathbb{Z}/p}G$.
- $\mathrm{T}_{\mathbb{Z}/p}G$ has no normal subgroups whose index in $\mathrm{T}_{\mathbb{Z}/p}G$ is coprime with p.

Now we are prepared to prove the general case of the theorem.

Theorem 3.5. *Let G be a finite group and p a prime. Then we have that the $\mathrm{B}\mathbb{Z}/p$-nullification of $\mathrm{B}G$ fits into the following fibration sequence:*

$$\prod_{q\neq p}\mathrm{B}(\mathrm{T}_{\mathbb{Z}/p}G)_q^\wedge \longrightarrow \mathbf{P}_{\mathrm{B}\mathbb{Z}/p}\mathrm{B}G \longrightarrow \mathrm{B}(G/\mathrm{T}_{\mathbb{Z}/p}G).$$

Proof. The $\mathbb{Z}/p$-radical is normal in G, so we can consider the fibration of classifying spaces

$$\mathrm{B}\mathrm{T}_{\mathbb{Z}/p}G \longrightarrow \mathrm{B}G \longrightarrow \mathrm{B}(G/\mathrm{T}_{\mathbb{Z}/p}G).$$

The quotient group $G/\mathrm{T}_{\mathbb{Z}/p}G$ has order coprime with p, so its classifying space is $\mathrm{B}\mathbb{Z}/p$-null. Now, by Theorem 2.1, the sequence of nullifications

$$\mathbf{P}_{\mathrm{B}\mathbb{Z}/p}\mathrm{B}\mathrm{T}_{\mathbb{Z}/p}G \longrightarrow \mathbf{P}_{\mathrm{B}\mathbb{Z}/p}\mathrm{B}G \longrightarrow \mathrm{B}(G/\mathrm{T}_{\mathbb{Z}/p}G)$$

is a fibration sequence. But $\mathbf{P}_{\mathrm{B}\mathbb{Z}/p}\mathrm{B}\mathrm{T}_{\mathbb{Z}/p}G \simeq \prod_{q\neq p}\mathrm{B}(\mathrm{T}_{\mathbb{Z}/p}G)^\wedge_q$ by Proposition 3.2, so the theorem is proved. □

Note that, as $\mathrm{T}_{\mathbb{Z}/p}G$ has no quotient coprime with p, $\prod_{q\neq p}\mathrm{B}(\mathrm{T}_{\mathbb{Z}/p}G)^\wedge_q$ is a simply connected space, and thus it is the universal cover of $\mathbf{P}_{\mathrm{B}\mathbb{Z}/p}\mathrm{B}G$.

Now, if $S = \{p_1, \ldots, p_r\}$ is a finite collection of prime numbers, the S-radical of G can be defined as the minimal normal subgroup $\mathrm{T}_S G \trianglelefteq G$ that contains all of the elements whose order is a product of powers of the primes p_i. This group satisfies analogous properties to those of the $\mathbb{Z}/p$-radical, and contains $\mathrm{T}_{\mathbb{Z}/p_i}G$ for every prime $p_i \in S$. Now we can generalize the previous theorem.

Proposition 3.6. *Let G be a finite group, $S = \{p_1, \ldots, p_n\}$ a finite collection of prime numbers, and $W = \mathrm{B}\mathbb{Z}/p_1 \vee \ldots \vee \mathrm{B}\mathbb{Z}/p_n$. Then we have that the W-nullification of* $\mathrm{B}G$ *fits into the following fibration sequence:*

$$\prod_{q\notin S}\mathrm{B}(\mathrm{T}_S G)^\wedge_q \longrightarrow \mathbf{P}_W\mathrm{B}G \longrightarrow \mathrm{B}(G/\mathrm{T}_S G).$$

Following [RS00, 1.1], it is enough to consider only the case in which the primes are distinct, because there is a homotopy equivalence $\mathbf{P}_{\mathrm{B}\mathbb{Z}/p}\mathrm{B}G \simeq \mathbf{P}_{\mathrm{B}\mathbb{Z}/p\vee\mathrm{B}\mathbb{Z}/p}\mathrm{B}G$.

It is also interesting to note that the $\mathrm{B}\mathbb{Z}/p$-nullification of the classifying space of a finite simple group is nothing but a completion.

Corollary 3.7. *If p is a prime number and G is a finite simple group that has p-torsion, then we have $\mathbf{P}_{\mathrm{B}\mathbb{Z}/p}\mathrm{B}G = \mathbb{Z}[1/p]_\infty\mathrm{B}G$.*

Proof. It is a direct consequence of Proposition 3.2. □

As explained in the Introduction, it is a remarkable fact that if $M(\mathbb{Z}/p, 1)$ is a 2-dimensional Moore space, there is a natural map $f : \mathbf{P}_{\mathrm{B}\mathbb{Z}/p}\mathrm{B}G \longrightarrow \mathbf{P}_{M(\mathbb{Z}/p,1)}\mathrm{B}G$. According to Theorem 3.5 and [RS98, 2.3], f is a homotopy equivalence. In other words, this statement tells us that the $\mathrm{B}\mathbb{Z}/p$-nullification of $\mathrm{B}G$ depends only on the 2-skeleton of $\mathrm{B}G$. On the other hand, it proves the following beautiful result, that concerns localization of groups.

Corollary 3.8. *If G is a finite group and p is a prime number, $L_{\mathbb{Z}/p}\pi_1(\mathrm{B}G)$ is isomorphic to $\pi_1(\mathbf{P}_{\mathrm{B}\mathbb{Z}/p}\mathrm{B}G)$. Here $L_{\mathbb{Z}/p}$ denotes the usual localization of G with respect to the null map $\mathbb{Z}/p \longrightarrow *$.*

Proof. It needs only to be pointed out that $\pi_1(\mathbf{P}_{\mathrm{B}\mathbb{Z}/p}\mathrm{B}G) \simeq G/\mathrm{T}_{\mathbb{Z}/p}G \simeq L_{\mathbb{Z}/p}G$. See [Cas94, 3.2] for details. □

In section 4 we will use these results to give a precise description of the fundamental group of $\bar{\mathbf{P}}_{\mathrm{B}\mathbb{Z}/p}\mathrm{B}G$, a way to compute it and a characterization of the finite groups whose classifying space is $\mathrm{B}\mathbb{Z}/p$-acyclic.

We would like to finish this section by mentioning the article [Dwy94], where it is proved that the $\mathrm{B}\mathbb{Z}/p$-nullification of the classifying space of a compact Lie group G whose group of components is a p-group is homotopy equivalent to its $\mathbb{Z}[1/p]$-localization. We consider our work complementary to that, and it would be

desirable to find a way to arrange all these data to find a description of $\mathbf{P}_{\mathrm{B}\mathbb{Z}/p}\mathrm{B}G$ for every compact Lie group G.

3.2. **Commutation of the nullification functors.** It is known that localization functors usually do not commute, not even in the case of nullifications. There are several examples of this in [RS00], where the authors also try to elucidate what happens if we apply in succession two localization functors to a certain space.

We will now prove that for distinct primes p and q, the functors $\mathbf{P}_{\mathrm{B}\mathbb{Z}/p}$ and $\mathbf{P}_{\mathrm{B}\mathbb{Z}/q}$ *do* commute if we apply them over $\mathrm{B}G$. Bearing in mind what was said, this is an interesting exception to the general case.

Proposition 3.9. *Let G be a finite group, and let p and q be distinct primes. Then we have homotopy equivalences*

$$\mathbf{P}_{\mathrm{B}\mathbb{Z}/p}\mathbf{P}_{\mathrm{B}\mathbb{Z}/q}\mathrm{B}G \simeq \mathbf{P}_{\mathrm{B}\mathbb{Z}/q}\mathbf{P}_{\mathrm{B}\mathbb{Z}/p}\mathrm{B}G \simeq \mathbf{P}_{\mathrm{B}\mathbb{Z}/p\vee\mathrm{B}\mathbb{Z}/q}\mathrm{B}G.$$

Proof. The proof of this result will be divided in two cases: in the first one we will suppose that G coincides with its S-radical for $S = \{p, q\}$, and then we will pass to the general case.

So, let $G = \mathrm{T}_S G$. We will prove that $\mathbf{P}_{\mathrm{B}\mathbb{Z}/p}\mathbf{P}_{\mathrm{B}\mathbb{Z}/q}\mathrm{B}G$ is homotopy equivalent to $\mathbf{P}_{\mathrm{B}\mathbb{Z}/p\vee\mathrm{B}\mathbb{Z}/q}\mathrm{B}G$, and the other equivalence will follow interchanging the roles of p and q. As $\mathbf{P}_{\mathrm{B}\mathbb{Z}/p\vee\mathrm{B}\mathbb{Z}/q}\mathrm{B}G$ is by definition a $\mathrm{B}\mathbb{Z}/p$-null space, we only need to check that for every $\mathrm{B}\mathbb{Z}/p$-null space X there exists a homotopy equivalence

$$\mathrm{map}_*(\mathbf{P}_{\mathrm{B}\mathbb{Z}/p\vee\mathrm{B}\mathbb{Z}/q}\mathrm{B}G, X) \simeq \mathrm{map}_*(\mathbf{P}_{\mathrm{B}\mathbb{Z}/q}\mathrm{B}G, X)$$

which should be given by a coaugmentation $\mathbf{P}_{\mathrm{B}\mathbb{Z}/q}\mathrm{B}G \longrightarrow \mathbf{P}_{\mathrm{B}\mathbb{Z}/p\vee\mathrm{B}\mathbb{Z}/q}\mathrm{B}G$. So, let X be a $\mathrm{B}\mathbb{Z}/p$-null space.

First, as $\mathbf{P}_{\mathrm{B}\mathbb{Z}/p\vee\mathrm{B}\mathbb{Z}/q}\mathrm{B}G$ is $\mathrm{B}\mathbb{Z}/q$-null, we have a natural map

$$\mathbf{P}_{\mathrm{B}\mathbb{Z}/q}\mathrm{B}G \longrightarrow \mathbf{P}_{\mathrm{B}\mathbb{Z}/p\vee\mathrm{B}\mathbb{Z}/q}\mathrm{B}G$$

which induces another between the mapping spaces

$$\mathrm{map}_*(\mathbf{P}_{\mathrm{B}\mathbb{Z}/p\vee\mathrm{B}\mathbb{Z}/q}\mathrm{B}G, Y) \longrightarrow \mathrm{map}_*(\mathbf{P}_{\mathrm{B}\mathbb{Z}/q}\mathrm{B}G, Y)$$

for every space Y. So, if we consider the $\mathrm{B}\mathbb{Z}/p$-null space X and its universal cover $\tilde{X}$ we have the following commutative diagram:

$$\begin{tikzcd}
\mathrm{map}_*(\mathbf{P}_{\mathrm{B}\mathbb{Z}/p\vee\mathrm{B}\mathbb{Z}/q}\mathrm{B}G, \tilde{X}) \arrow[r] \arrow[d] & \mathrm{map}_*(\mathbf{P}_{\mathrm{B}\mathbb{Z}/q}\mathrm{B}G, \tilde{X}) \arrow[d] \\
\mathrm{map}_*(\mathbf{P}_{\mathrm{B}\mathbb{Z}/p\vee\mathrm{B}\mathbb{Z}/q}\mathrm{B}G, X) \arrow[r] \arrow[d, "\tilde{p}_{p,q}"] & \mathrm{map}_*(\mathbf{P}_{\mathrm{B}\mathbb{Z}/q}\mathrm{B}G, X) \arrow[d, "\tilde{p}_q"] \\
\mathrm{map}_*(\mathbf{P}_{\mathrm{B}\mathbb{Z}/p\vee\mathrm{B}\mathbb{Z}/q}\mathrm{B}G, \mathrm{B}\pi_1(X)) \arrow[r, "r_\natural"] & \mathrm{map}_*(\mathbf{P}_{\mathrm{B}\mathbb{Z}/q}\mathrm{B}G, \mathrm{B}\pi_1(X))
\end{tikzcd} \tag{3.2.1}$$

In order to prove that the columns are fibrations, we must check that the maps $\tilde{p}_{p,q}$ and $\tilde{p}_q$ take values in both cases in the component of the constant map. The first is trivial, because by Proposition 3.6, the space $\mathbf{P}_{\mathrm{B}\mathbb{Z}/p\vee\mathrm{B}\mathbb{Z}/q}\mathrm{B}G$ is simply connected. In the other case, we must verify that if we have a map $f : \mathbf{P}_{\mathrm{B}\mathbb{Z}/q}\mathrm{B}G \longrightarrow X$, the composition with the projection $\pi : X \longrightarrow \mathrm{B}\pi_1(X)$ is homotopic to the constant map, but this is easily deduced from Lemma 3.1, knowing that X is $\mathrm{B}\mathbb{Z}/p$-null and $\mathbf{P}_{\mathrm{B}\mathbb{Z}/q}\mathrm{B}G$ is $\mathrm{B}\mathbb{Z}/q$-null.

To finish the proof of the first case, we must see that the top horizontal map of the diagram (3.2.1) is a weak equivalence. We know ([ABN94, 9.4]) that if X is a $\mathrm{B}\mathbb{Z}/p$-null space, its universal cover is too, so we only need to check that the previously described map

$$\mathrm{map}_*(\mathbf{P}_{\mathrm{B}\mathbb{Z}/p\vee\mathrm{B}\mathbb{Z}/q}\mathrm{B}G, X) \longrightarrow \mathrm{map}_*(\mathbf{P}_{\mathrm{B}\mathbb{Z}/q}\mathrm{B}G, X)$$

is a homotopy equivalence if X is a simply connected, $\mathrm{B}\mathbb{Z}/p$-null space. As X is simply connected, it is homotopy equivalent to the pullback of the Sullivan arithmetic square ([BK72, V,6]). By Theorem 3.5 and Proposition 3.6, the rationalizations of $\mathbf{P}_{\mathrm{B}\mathbb{Z}/p\vee\mathrm{B}\mathbb{Z}/q}\mathrm{B}G$ and $\mathbf{P}_{\mathrm{B}\mathbb{Z}/q}\mathrm{B}G$ are homotopy equivalent to a point, so

$$\mathrm{map}_*(\mathbf{P}_{\mathrm{B}\mathbb{Z}/p\vee\mathrm{B}\mathbb{Z}/q}\mathrm{B}G, X) \simeq \mathrm{map}_*(\mathbf{P}_{\mathrm{B}\mathbb{Z}/p\vee\mathrm{B}\mathbb{Z}/q}\mathrm{B}G, \prod_{r \text{ prime}} X_r^\wedge),$$

and the same is true for maps which come from $\mathbf{P}_{\mathrm{B}\mathbb{Z}/q}\mathrm{B}G$.

Now, if $r \neq q$, by Proposition 3.12 we have that $(\mathbf{P}_{\mathrm{B}\mathbb{Z}/q}\mathrm{B}G)_r^\wedge \simeq \mathrm{B}G_r^\wedge$, and on the other hand, the $\mathbb{Z}/q$-completion of the fibration of Theorem 3.5 gives us that $(\mathbf{P}_{\mathrm{B}\mathbb{Z}/q}\mathrm{B}G)_q^\wedge$ is contractible (Proposition 3.14). In addition, observe that $\mathrm{map}_*(\mathrm{B}G_p^\wedge, X_p^\wedge)$ is contractible too (see the proof of Theorem 3.5), so we obtain

$$\mathrm{map}_*(\mathbf{P}_{\mathrm{B}\mathbb{Z}/q}\mathrm{B}G, X) \simeq \prod_{r\neq p,q} \mathrm{map}_*(\mathrm{B}G_r^\wedge, X_r^\wedge).$$

A similar line of reasoning proves that

$$\mathrm{map}_*(\mathbf{P}_{\mathrm{B}\mathbb{Z}/p\vee\mathrm{B}\mathbb{Z}/q}\mathrm{B}G, X) \simeq \prod_{r\neq p,q} \mathrm{map}_*(\mathrm{B}G_r^\wedge, X_r^\wedge),$$

and we have finished the proof of the first case.

Let G now be a finite group, and let us consider the fibration

$$\mathrm{BT}_S G \longrightarrow \mathrm{B}G \longrightarrow \mathrm{B}(G/\mathrm{T}_S G).$$

As the base is $\mathrm{B}\mathbb{Z}/p \vee \mathrm{B}\mathbb{Z}/q$-null, the fibration is preserved after $\mathrm{B}\mathbb{Z}/p \vee \mathrm{B}\mathbb{Z}/q$-nullification (Theorem 2.1), and also after $\mathrm{B}\mathbb{Z}/p$-nullification and $\mathrm{B}\mathbb{Z}/q$-nullification. Moreover, $\mathbf{P}_{\mathrm{B}\mathbb{Z}/p\vee\mathrm{B}\mathbb{Z}/q}\mathrm{B}G$ is $\mathrm{B}\mathbb{Z}/p$-null and $\mathrm{B}\mathbb{Z}/q$-null, and so we have a commutative diagram where the rows are fibrations:

$$\begin{tikzcd}
\mathbf{P}_{\mathrm{B}\mathbb{Z}/p\vee\mathrm{B}\mathbb{Z}/q}\mathrm{BT}_S G \arrow[r] & \mathbf{P}_{\mathrm{B}\mathbb{Z}/p\vee\mathrm{B}\mathbb{Z}/q}\mathrm{B}G \arrow[r] & \mathrm{B}(G/\mathrm{T}_S G) \\
\mathbf{P}_{\mathrm{B}\mathbb{Z}/p}\mathbf{P}_{\mathrm{B}\mathbb{Z}/q}\mathrm{BT}_S G \arrow[r] \arrow[u] & \mathbf{P}_{\mathrm{B}\mathbb{Z}/p}\mathbf{P}_{\mathrm{B}\mathbb{Z}/q}\mathrm{B}G \arrow[r] \arrow[u] & \mathrm{B}(G/\mathrm{T}_S G) \arrow[u, "\mathrm{Id}"']
\end{tikzcd}$$

Now, the left column is an equivalence by the case already proved, so the natural map $\mathbf{P}_{\mathrm{B}\mathbb{Z}/p}\mathbf{P}_{\mathrm{B}\mathbb{Z}/q}\mathrm{B}G \longrightarrow \mathbf{P}_{\mathrm{B}\mathbb{Z}/p\vee\mathrm{B}\mathbb{Z}/q}\mathrm{B}G$ is an equivalence too, and we are done. □

We finish the section by presenting a slight generalization of the previous proposition.

Corollary 3.10. *Let $\{p_1, \ldots, p_r\}$ and $\{q_1, \ldots, q_s\}$ be two families of prime numbers, and denote $W = \mathrm{B}\mathbb{Z}/p_1 \vee \ldots \vee \mathrm{B}\mathbb{Z}/p_r$ and $W' = \mathrm{B}\mathbb{Z}/q_1 \vee \ldots \vee \mathrm{B}\mathbb{Z}/q_s$. Then we have $\mathbf{P}_W\mathbf{P}'_W\mathrm{B}G \simeq \mathbf{P}_{W\vee W'}\mathrm{B}G \simeq \mathbf{P}_{W'}\mathbf{P}_W\mathrm{B}G$.*

Proof. It is proved along the lines of the previous proposition. We leave the details to the reader. □

3.3. **Relation between nullification and completion.** It seems quite natural to ask for the relation between the effect on classifying spaces of finite groups of the $\mathrm{B}\mathbb{Z}/p$-nullification and the $\mathbb{Z}[1/p]_\infty$-completion, because these two functors "kill" the p-primary part of $\mathrm{B}G$. We have already seen, for instance, that they coincide if G is a simple group, but now we will see by means of an easy example that this is not always true.

Example 3.11. Consider the dihedral group D_{15}, which is isomorphic to the semidirect product $\mathbb{Z}/15 \rtimes \mathbb{Z}/2$. It will be seen in section 5.1 that the $\mathrm{B}\mathbb{Z}/3$-nullification of $\mathrm{B}D_{15}$ is homotopy equivalent to $\mathrm{B}D_{10}$.

On the other hand, the group $\mathbb{Z}/15$ is normal in the semidirect product, and we can consider the associated fibration

$$\mathrm{B}\mathbb{Z}/15 \longrightarrow \mathrm{B}D_{15} \longrightarrow \mathrm{B}\mathbb{Z}/2.$$

The space $\mathrm{B}\mathbb{Z}/2$ is $\mathbb{Z}/2$-complete, so this fibration is preserved by $\mathbb{Z}/2$-completion, and we obtain the homotopy equivalence $(\mathrm{B}D_{15})^\wedge_2 \simeq \mathrm{B}\mathbb{Z}/2$.

Now, it is obvious that $\pi_1(\mathbf{P}_{\mathrm{B}\mathbb{Z}/3}\mathrm{B}D_{15}) = D_{10}$, while $\pi_1((\mathrm{B}D_{15})^\wedge_5 \times \mathrm{B}\mathbb{Z}/2) = \mathbb{Z}/2$, because $(\mathrm{B}D_{15})^\wedge_5$ is simply connected (D_{15} has no quotient groups that are 5-groups). So, in this case, we obtain that the $\mathrm{B}\mathbb{Z}/3$-nullification cannot be the product of the $\mathbb{Z}/2$-completion and the $\mathbb{Z}/5$-completion of $\mathrm{B}D_{15}$, and in particular $\mathbf{P}_{\mathrm{B}\mathbb{Z}/3}\mathrm{B}D_{15} \neq \mathbb{Z}[1/3]_\infty \mathrm{B}D_{15}$.

Now we will study what happens if we apply in succession the $\mathrm{B}\mathbb{Z}/p$-nullification and $\mathbb{Z}/q$-completion functors to $\mathrm{B}G$, in the two possible orders, and for primes p and q not necessarily distinct. We come to the conclusion that the functor $\mathbf{P}_{\mathrm{B}\mathbb{Z}/p}$ is quite sensitive, in the sense that it kills the p-primary structure of $\mathrm{B}G$ leaving untouched the q-primary part which is detected by the $\mathbb{Z}/q$-completion functor.

We first consider the case $p \neq q$.

Proposition 3.12. *Let G be a finite group, and let p and q be distinct primes. Then we have homotopy equivalences*

$$\mathbf{P}_{\mathrm{B}\mathbb{Z}/p}(\mathrm{B}G^\wedge_q) \simeq \mathrm{B}G^\wedge_q \simeq (\mathbf{P}_{\mathrm{B}\mathbb{Z}/p}\mathrm{B}G)^\wedge_q.$$

Proof. Every map $\mathrm{B}\mathbb{Z}/p \longrightarrow \mathrm{B}G^\wedge_q$ factors through the $\mathbb{Z}/q$-completion of $\mathrm{B}\mathbb{Z}/p$, because $\mathrm{B}G^\wedge_q$ is $\mathbb{Z}/q$-complete. But $(\mathrm{B}\mathbb{Z}/p)^\wedge_q$ is contractible, so $\mathrm{B}G^\wedge_q$ is $\mathrm{B}\mathbb{Z}/p$-null and the first equivalence is proved.

For the second, note that the map $\mathrm{B}\mathbb{Z}/p \longrightarrow *$ is a $\mathbb{Z}/q$-equivalence, and so, the canonical coaugmentation $\mathrm{B}G \longrightarrow \mathbf{P}_{\mathrm{B}\mathbb{Z}/p}\mathrm{B}G$ is again a $\mathbb{Z}/q$-equivalence. □

Remark 3.13. The arguments of this proof remain valid if we replace $\mathrm{B}G$ by any $\mathbb{Z}/q$-good space.

If we now consider the case $p = q$, we obtain the following.

Proposition 3.14. *If G is a finite group and p is a prime number, then $\mathbf{P}_{\mathrm{B}\mathbb{Z}/p}(\mathrm{B}G^\wedge_p)$ is contractible, as is $(\mathbf{P}_{\mathrm{B}\mathbb{Z}/p}\mathrm{B}G)^\wedge_p$.*

Proof. We must check that, for every $\mathrm{B}\mathbb{Z}/p$-null space, the space $\mathrm{map}_*(\mathrm{B}G^\wedge_p, X)$ is contractible. Consider the covering fibration

$$\tilde{X} \xrightarrow{h} X \xrightarrow{f} \mathrm{B}\pi_1(X).$$

Recall that if X is $\mathrm{B}\mathbb{Z}/p$-null, its universal cover $\tilde{X}$ is $\mathrm{B}\mathbb{Z}/p$-null too (see [ABN94, 9.7]). First we consider a finite group G such that $\mathrm{B}G^\wedge_p$ is a simply connected space.

In this case we have $\mathrm{map}_*(\mathrm{B}G_p^\wedge, \mathrm{B}\pi_1(X)) \simeq *$, so in order to prove that $\mathbf{P}_{\mathrm{B}\mathbb{Z}/p}\mathrm{B}G_p^\wedge$ is contractible we need only check that $\mathrm{map}_*(\mathrm{B}G_p^\wedge, \tilde{X})$ is contractible. But this is proved in exactly the same way as the proof of Proposition 3.2, noting the fact that if $q \neq p$, $\mathrm{map}_*(\mathrm{B}G_p^\wedge, \tilde{X}_q^\wedge)$ is contractible.

Now, let G be any finite group. If we denote by $O^p(G)$ the maximal p-perfect normal subgroup of G, then Proposition 2.9 and [Lev95] imply that

$$\mathrm{B}O^p(G)_p^\wedge \longrightarrow \mathrm{B}G_p^\wedge \longrightarrow \mathrm{B}(G/O^p(G))$$

is the covering fibration of $\mathrm{B}G_p^\wedge$. But both the fibre and the base of the fibration are $\mathrm{B}\mathbb{Z}/p$-acyclic, the former by the previous paragraph, the latter because $G/O^p(G)$ is a p-group, so $\mathrm{B}G_p^\wedge$ is $\mathrm{B}\mathbb{Z}/p$-acyclic too and we have finished the proof of the first statement.

To prove the second, note that the universal cover of $\mathbf{P}_{\mathrm{B}\mathbb{Z}/p}\mathrm{B}G$, that is, $\prod_{q\neq p}(\mathrm{B}G)_q^\wedge$ for q prime, is $\mathbb{Z}/p$-homology equivalent to a point. So by the fibre lemma II.5.1 of [BK72], the covering fibration

$$\prod_{q\neq p}(\mathrm{B}G)_q^\wedge \longrightarrow \mathbf{P}_{\mathrm{B}\mathbb{Z}/p}\mathrm{B}G \longrightarrow \mathrm{B}(G/\mathrm{T}_{\mathbb{Z}/p}G)$$

is preserved by $\mathbb{Z}/p$-completion. But the $\mathbb{Z}/p$-completions of the base space and the fibre are contractible, so $(\mathbf{P}_{\mathrm{B}\mathbb{Z}/p}\mathrm{B}G)_p^\wedge$ is contractible too, and we are done. □

In conclusion, we will establish the relationship between the $\mathrm{B}\mathbb{Z}/p$-nullification and the Bousfield-Kan completion with coefficients in the ring $\mathbb{Z}[1/p]$.

Proposition 3.15. *Let G be a finite group, and let p and q be two distinct primes. Then the following relations hold:*

(1) $\mathbb{Z}[1/p]_\infty\mathbf{P}_{\mathrm{B}\mathbb{Z}/p}\mathrm{B}G \simeq \mathbf{P}_{\mathrm{B}\mathbb{Z}/p}\mathbb{Z}[1/p]_\infty\mathrm{B}G \simeq \mathbb{Z}[1/p]_\infty\mathrm{B}G$.
(2) $\mathbb{Z}[1/p]_\infty\mathbf{P}_{\mathrm{B}\mathbb{Z}/q}\mathrm{B}G \simeq \mathbf{P}_{\mathrm{B}\mathbb{Z}/q}\mathbb{Z}[1/p]_\infty\mathrm{B}G \simeq \mathbb{Z}[1/p, 1/q]_\infty\mathrm{B}G$.

Proof. It is an immediate consequence of the previous results of this section, taking into account the results ([BK72, VII, 4.2 and 4.3]) that allow us to express the $\mathbb{Z}[1/p]$-completion of $\mathbf{P}_{\mathrm{B}\mathbb{Z}/p}\mathrm{B}G$ as the product of their $\mathbb{Z}/p'$-completions in the rest of the primes. □

4. Cellularization

Let G be a finite group and p a prime. In Corollary 3.8 we have seen that the $\mathrm{B}\mathbb{Z}/p$-nullification of $\mathrm{B}G$ is intimately related to the $\mathbb{Z}/p$-localization of G as a group. Likewise, it turns out to be interesting to study the $\mathbb{Z}/p$-cellularization of the group G for obtaining information about the $\mathrm{B}\mathbb{Z}/p$-cellularization of $\mathrm{B}G$. This is essentially our approach to this subject, and it is worth recalling the main definitions concerning cellularization in the category of groups. Recall that a group G is $\mathbb{Z}/p$-*cellular* if and only if it can be built from $\mathbb{Z}/p$ by (maybe iterated) colimits, and the $\mathbb{Z}/p$-*cellularization* of G is the unique $\mathbb{Z}/p$-cellular group $\mathrm{CW}_{\mathbb{Z}/p}G$ endowed with an augmentation $\mathrm{CW}_{\mathbb{Z}/p}G \longrightarrow G$ which induces an isomorphism $\mathrm{Hom}(\mathbb{Z}/p, \mathrm{CW}_{\mathbb{Z}/p}G) \simeq \mathrm{Hom}(\mathbb{Z}/p, G)$. This concept was first defined in [RS98] and mainly used for describing the cellularization with respect to Moore spaces.

We begin our study by showing that the problem of the $\mathrm{B}\mathbb{Z}/p$-cellularization of classifying spaces of finite groups can be reduced to the problem of the $\mathrm{B}\mathbb{Z}/p$-cellularization of classifying spaces of $\mathbb{Z}/p$-*cellular* groups (in fact, finite $\mathbb{Z}/p$-cellular groups; see Remark 4.6).

Proposition 4.1. *If G is a finite group, the natural map $\mathrm{CW}_{\mathbb{Z}/p}G \longrightarrow G$ induces a homotopy equivalence $\mathbf{CW}_{\mathrm{B}\mathbb{Z}/p}\mathrm{BCW}_{\mathbb{Z}/p}G \simeq \mathbf{CW}_{\mathrm{B}\mathbb{Z}/p}\mathrm{B}G$.*

Proof. The augmentation $\mathrm{CW}_{\mathbb{Z}/p}G \longrightarrow G$ induces a map

$$\mathrm{map}_*(\mathrm{B}\mathbb{Z}/p, \mathbf{CW}_{\mathrm{B}\mathbb{Z}/p}\mathrm{BCW}_{\mathbb{Z}/p}G) \longrightarrow \mathrm{map}_*(\mathrm{B}\mathbb{Z}/p, \mathrm{B}G)$$

that, by the following string of weak equivalences,

$$\begin{aligned}\mathrm{map}_*(\mathrm{B}\mathbb{Z}/p, \mathbf{CW}_{\mathrm{B}\mathbb{Z}/p}\mathrm{BCW}_{\mathbb{Z}/p}G) &\simeq \mathrm{map}_*(\mathrm{B}\mathbb{Z}/p, \mathrm{BCW}_{\mathbb{Z}/p}G)\\ \simeq \mathrm{Hom}(\mathbb{Z}/p, \mathrm{CW}_{\mathbb{Z}/p}G) &\simeq \mathrm{Hom}(\mathbb{Z}/p, G) \simeq \mathrm{map}_*(\mathrm{B}\mathbb{Z}/p, \mathrm{B}G),\end{aligned}$$

is a weak equivalence. As $\mathbf{CW}_{\mathrm{B}\mathbb{Z}/p}\mathrm{BCW}_{\mathbb{Z}/p}G$ is $\mathrm{B}\mathbb{Z}/p$-cellular, we are done. □

Hence, it becomes interesting to find appropriate tools for computing the $\mathbb{Z}/p$-cellularization of a finite group G. To this aim, we recall the following concept from group theory, that will be crucial in the sequel.

Definition 4.2. If G is a finite group, the *$\mathbb{Z}/p$-socle* $\mathrm{S}_{\mathbb{Z}/p}G$ of G is the subgroup of G generated by the order p elements of G.

It can be seen that this subgroup is always normal and characteristic, and it is contained in the $\mathbb{Z}/p$-radical $\mathrm{T}_{\mathbb{Z}/p}G$. In fact, the following holds.

Proposition 4.3. *If G is a $\mathbb{Z}/p$-cellular group, then $G = \mathrm{S}_{\mathbb{Z}/p}G = \mathrm{T}_{\mathbb{Z}/p}G$.*

Proof. If G is $\mathbb{Z}/p$-cellular, it is a colimit of $\mathbb{Z}/p$'s, and hence it is generated by order p elements. As the $\mathbb{Z}/p$-socle is the group generated by the order p elements of G, $G = \mathrm{S}_{\mathbb{Z}/p}G$, and the other equality is obvious from the inclusions $\mathrm{S}_{\mathbb{Z}/p}G \subseteq \mathrm{T}_{\mathbb{Z}/p}G \subseteq G$. □

As explained in [RS98], the most relevant property of the $\mathbb{Z}/p$-socle is the fact that the inclusion $\mathrm{S}_{\mathbb{Z}/p}G \subseteq G$ always induces an isomorphism $\mathrm{Hom}(\mathbb{Z}/p, \mathrm{S}_{\mathbb{Z}/p}G) \simeq \mathrm{Hom}(\mathbb{Z}/p, G)$, which implies that $\mathrm{CW}_{\mathbb{Z}/p}G \simeq \mathrm{CW}_{\mathbb{Z}/p}\mathrm{S}_{\mathbb{Z}/p}G$. This last assertion shows that the computation of the $\mathbb{Z}/p$-cellularization of groups can again be reduced to the case of groups generated by order p elements. Moreover, it is worth pointing out that we are interested only in finite groups, and in this case it is not hard to calculate the $\mathbb{Z}/p$-socle of a group starting from a presentation of G (using GAP, for example).

Now, the main tool we are going to use in our description of the $\mathbb{Z}/p$-cellularization is the following version of a theorem of Rodríguez-Scherer ([RS98, 3.7]).

Theorem 4.4. *For each group G, there is a central extension*

$$0 \longrightarrow A \longrightarrow \mathrm{CW}_{\mathbb{Z}/p}G \longrightarrow \mathrm{S}_{\mathbb{Z}/p}G \longrightarrow 0$$

such that A has no order p elements and is universal with respect to this property.

The key case is G finite and equal to its $\mathbb{Z}/p$-socle; thus, $G = \mathrm{S}_{\mathbb{Z}/p}G$, and the essential problem here is computing A. According to Proposition 2.4, this group is the second homotopy group of the $\Sigma M(\mathbb{Z}/p, 1)$-nullification of the cofibre C_f of the evaluation map $f : \bigvee_{[M(\mathbb{Z}/p,1),\mathrm{B}G]} M(\mathbb{Z}/p, 1) \longrightarrow BG$, where here $M(\mathbb{Z}/p, 1)$ stands for a two-dimensional Moore space. It is not hard to see that $\pi_2(\mathbf{P}_{\Sigma M(\mathbb{Z}/p,1)}C_f) = \pi_2(C_f)/\mathrm{T}_{\mathbb{Z}/p}\pi_2(C_f)$, so our problem is to describe the second homotopy group of this homotopy cofibre. As G is generated by order p elements, C_f is simply connected, and then $\pi_2(C_f) = H_2(C_f)$. Using this property, we have computed the group A in the following way.

Lemma 4.5. *Let G be a finite group generated by order p elements. Then the group A of Theorem 4.4 is isomorphic to $H_2(G)/T_{\mathbb{Z}/p}(H_2(G))$.*

Proof. The Mayer-Vietoris sequence of the cofibration f has the form

$$0 \longrightarrow H_2(G) \longrightarrow \pi_2(C_f) \longrightarrow \bigoplus \mathbb{Z}/p \longrightarrow H_1(G) \longrightarrow 0.$$

But as $A = \pi_2(C_f)/\mathrm{T}_{\mathbb{Z}/p}\pi_2(C_f)$, the result follows. □

Remark 4.6. Observe in particular that A is finite because H_2G is finite. This implies that the $\mathbb{Z}/p$-cellularization of a finite group is again a finite group.

Incidentally, the second homotopy group of the cofibre C_f can be computed in a direct way, making use of the definition.

Proposition 4.7. *Let G be a finite group generated by order p elements, denote by H the free product $*\mathbb{Z}/p$ extended over all the homomorphisms $\mathbb{Z}/p \longrightarrow G$, and let K be the kernel of the evaluation map $H \longrightarrow G$. If C_f is the cofibre previously defined, we have $\pi_2(C_f) = K/[K, H]$.*

Proof. It is an easy consequence of [BL87, 3.4], taking, in the notation there, $P = H$, $M = H$ and $N = K$. □

Note that a presentation of $\pi_2(C_f)$ can be obtained using, for example, the Reidemeister-Schreier method, and in particular the previous result implies the classical Hopf formula ([Bro82, 5.3]) for groups generated by order p elements.

A complete classification result for finite $\mathbb{Z}/p$-cellular groups can be easily deduced from Lemma 4.5.

Proposition 4.8. *Let G be a finite group. Then G is $\mathbb{Z}/p$-cellular if and only if it is generated by order p elements and $H_2(G)$ is a p-group.*

Proof. If G is $\mathbb{Z}/p$-cellular, it is a colimit of $\mathbb{Z}/p$'s, and hence is generated by order p elements. According to Theorem 4.4 and the previous proposition, H_2G must be equal to its $\mathbb{Z}/p$-radical, and hence it must be a p-group. Conversely, if G is generated by order p elements, then $G = \mathrm{S}_{\mathbb{Z}/p}G$, and the fact that $H_2(G)$ is a p-group implies that the group A of Theorem 4.4 is trivial; so G is $\mathbb{Z}/p$-cellular. □

Having computed A, the only thing that remains is to identify the extension giving the $\mathbb{Z}/p$-cellularization. This is usually not hard, because the key result (Theorem 4.4) describes that extension with great precision. Furthermore, in some favorable cases, we can go further and find explicitly the cohomology class associated to this extension. For this reason we now turn our attention to *perfect* groups. Recall that a group G is called perfect if it is equal to its commutator subgroup or, equivalently, if the first integral homology group is trivial.

Proposition 4.9. *Let G be a finite perfect group generated by order p elements, let*

$$0 \longrightarrow H_2G \longrightarrow \tilde{G} \longrightarrow G \longrightarrow 0$$

be the universal central extension of G, let B be the quotient of H_2G by the p-torsion (which is in fact a subgroup of H_2G), and let

$$0 \longrightarrow B \longrightarrow H \longrightarrow G \longrightarrow 0$$

be the central extension $\mathcal{E}$ induced by the previous one. Then, the latter is equivalent to the extension of Theorem 4.4, and in particular H is isomorphic to the $\mathbb{Z}/p$-cellularization of G.

Proof. The group B has no p-torsion, so by the universality of the extension $\mathcal{E}'$ of Theorem 4.4 that defines the $\mathbb{Z}/p$-cellularization there is a unique morphism of extensions $\mathcal{E}' \longrightarrow \mathcal{E}$ that is the identity over G.

On the other hand, it is clear that for any central extension whose kernel does not have p-torsion, the unique morphism that comes from the universal central extension to this one factors through $\mathcal{E}$. This proves that there is again a unique morphism over G from $\mathcal{E}$ to $\mathcal{E}'$, and it is easy to check, by universality, that the two morphisms that we have defined are one inverse to each other. Hence, the extensions $\mathcal{E}$ and $\mathcal{E}'$ are equivalent, and we are done. □

The methods of computation of $\mathrm{CW}_{\mathbb{Z}/p}G$ developed above can be reinterpreted in a framework of group theory as tools for describing the universal central extension of a finite perfect group G. For example, we can get the following easy and interesting consequence.

Corollary 4.10. *If G is a perfect group generated by order p elements and whose Schur multiplier has no p-torsion, then the $\mathbb{Z}/p$-cellularization of G is isomorphic to its universal covering group $\tilde{G}$.*

A somewhat similar line of reasoning can sometimes be applied to non-perfect groups, as we see in the next proposition.

Proposition 4.11. *Let p be an odd prime, and G a finite group such that the second homology group of its $\mathbb{Z}/p$-socle is $\mathbb{Z}/2$. Then the central extension of Theorem 4.4 that defines the $\mathbb{Z}/p$-cellularization of G is the one that is not trivial.*

Proof. The $\mathbb{Z}/p$-socle $\mathrm{S}_{\mathbb{Z}/p}G$ of G is generated by order p elements, so its abelianization is an elementary abelian p-group. Hence, $Ext(\mathrm{S}^{ab}_{\mathbb{Z}/p}G, \mathbb{Z}/2) = 0$ and, by the universal coefficient theorem, we have the isomorphisms

$$H^2(\mathrm{S}_{\mathbb{Z}/p}G, \mathbb{Z}/2) \cong \mathrm{Hom}(H_2(\mathrm{S}_{\mathbb{Z}/p}G), H_2(\mathrm{S}_{\mathbb{Z}/p}G)) \cong \mathbb{Z}/2.$$

This implies that there are only two central extensions of $\mathrm{S}_{\mathbb{Z}/p}G$ by $\mathbb{Z}/2$. Now we observe that the group defined by the trivial central extension $\mathbb{Z}/2 \times \mathrm{S}_{\mathbb{Z}/p}G$ is not $\mathbb{Z}/p$-cellular, because it cannot be generated by order p elements, and so the cellularization is identified by the nontrivial extension, as we required. □

This result will be very useful in certain cases, as we will see in the next section.

Having carefully studied the $\mathbb{Z}/p$-cellularization functor in the category of finite groups, it turns out to be interesting to relate it to the other group colocalization involved in this work, namely the kernel $\bar{L}_{\mathbb{Z}/p}G$ of the localization map $G \longrightarrow L_{\mathbb{Z}/p}G$ described in Lemma 3.1. This relationship is established in the next proposition.

Proposition 4.12. *If G is a finite group such that $\mathrm{T}_{\mathbb{Z}/p}G = \mathrm{S}_{\mathbb{Z}/p}G$, the $\mathbb{Z}/p$-cellularization of G is isomorphic to the fundamental group of $\bar{\mathbf{P}}_{\mathrm{B}\mathbb{Z}/p}\mathrm{B}G$, which is usually called $\mathrm{D}_{\mathbb{Z}/p}G$.*

Proof. Following [MP01], the group $\mathrm{D}_{\mathbb{Z}/p}G$ is defined by a central extension $\mathcal{D}$ given by

$$0 \longrightarrow H_2(\mathrm{T}_{\mathbb{Z}/p}G; \mathbb{Z}[1/p]) \longrightarrow \mathrm{D}_{\mathbb{Z}/p}G \longrightarrow \mathrm{T}_{\mathbb{Z}/p}G \longrightarrow 0$$

which is universal among the central extensions

$$0 \longrightarrow B \longrightarrow E \longrightarrow \mathrm{T}_{\mathbb{Z}/p}G \longrightarrow 0$$

such that $\mathrm{Hom}(\mathbb{Z}/p, B) = 0$ and $\mathrm{Ext}(\mathbb{Z}/p, B) = 0$.

Now, these conditions hold for the extension $\mathcal{E}'$ of Theorem 4.4 that defines $\mathrm{CW}_{\mathbb{Z}/p}G$, because p does not divide the order of A. So, there is just one map $f : \mathcal{D} \longrightarrow \mathcal{E}'$ that is the identity over $\mathrm{T}_{\mathbb{Z}/p}G$.

On the other hand, as $H_2(\mathrm{T}_{\mathbb{Z}/p}G; \mathbb{Z}[1/p])$ has no p-torsion there exists by Theorem 4.4 a unique map $g : \mathcal{E}' \longrightarrow \mathcal{D}$ which is again the identity over G. By universality, $\mathcal{E}' \simeq \mathcal{D}$, and the result follows. □

This proposition states that the tools we have developed above for the computation of the $\mathbb{Z}/p$-cellularization of a group G are also useful for calculating $\mathrm{D}_{\mathbb{Z}/p}G$, or in the language of group theory, the universal central extension of the $\mathbb{Z}/p$-radical of G with coefficients in $\mathbb{Z}[1/p]$. See [RS98, 4.5] and [MP01] for more details on universal central extensions with coefficients.

We can also prove an analogous statement to the commutation result, section 3.1, which appears as an easy consequence of the previous proposition.

Corollary 4.13. *If G is a finite $\mathbb{Z}/p$-cellular group, then there is an isomorphism*

$$\pi_1 \bar{\mathbf{P}}_{\mathrm{B}\mathbb{Z}/p}\mathrm{B}G \cong \bar{L}_{\mathbb{Z}/p}G,$$

which is in fact isomorphic to G.

Proof. According to the previous proposition, $\mathrm{D}_{\mathbb{Z}/p}G = G$, and G coincides with its $\mathbb{Z}/p$-radical because it is $\mathbb{Z}/p$-cellular. But the $\mathbb{Z}/p$-radical is precisely the kernel of the $\mathbb{Z}/p$-localization of G, so we are done. □

It is worth pointing out that the hypothesis of G $\mathbb{Z}/p$-cellular is essential, as one can see, taking for example $G = \mathbb{Z}/p^2$.

Now we have completed the description of the $\mathbb{Z}/p$-cellularization of finite groups, and we are prepared to present what is probably the main result of this section, a complete characterization of the finite groups G whose classifying space $\mathrm{B}G$ is $\mathrm{B}\mathbb{Z}/p$-cellular.

Proposition 4.14. *Let G be a finite $\mathbb{Z}/p$-cellular group. Then $\mathrm{B}G$ is $\mathrm{B}\mathbb{Z}/p$-cellular if and only if G is a p-group.*

Proof. If G is not a p-group, $H_n(G)$ is not p-torsion for a certain $n \geq 2$. Using the result [RS98, 7.3], its classifying space is not $\mathrm{M}(\mathbb{Z}/p, 1)$-cellular for any two-dimensional Moore space $\mathrm{M}(\mathbb{Z}/p, 1)$. Hence by Proposition 2.5 it cannot be $\mathrm{B}\mathbb{Z}/p$-cellular, because $\mathrm{B}\mathbb{Z}/p$ is itself $\mathrm{M}(\mathbb{Z}/p, 1)$-cellular.

Conversely, suppose G is a p-group. We use induction over the order of the group G. It is clear that $\mathrm{B}\mathbb{Z}/p$ is $\mathrm{B}\mathbb{Z}/p$-cellular, so we admit the hypothesis is true for every finite $\mathbb{Z}/p$-cellular p-group whose order is strictly smaller than p^k. Let G be a $\mathbb{Z}/p$ group of order p^k then, and consider a minimal system of order p generators $\{x_1, \ldots, x_r, y\}$, which exists because the group is finite and $\mathbb{Z}/p$-cellular. Denote by H the minimal subgroup of G generated by $\{x_1, \ldots, x_r\}$ and their conjugates by y^j, with $0 \leq j \leq p-1$. It is clear that the group H is generated by order p elements, so according to Proposition 4.12 and Theorem 3.5 it is $\mathbb{Z}/p$-cellular.

By definition, every element of H can be written as a juxtaposition

$$y^{j_1} x_{a_1}^{k_1} y^{j_2} \ldots x_{a_{m-1}}^{k_{m-1}} y^{j_m} x_{a_m}^{k_m},$$

where $0 \leq j_s \leq p-1$, $0 \leq k_s \leq p-1$, $1 \leq a_s \leq r$, and it is allowed that $a_s = a_{s'}$ if $s \neq s'$. By definition of the system of generators of H, we have that $\sum_{i=1}^{m} j_i = 0$,

and this implies that $y^t \notin H$ if $1 \leq t \leq p-1$. On the other hand, it is easy to see that G/H is isomorphic to $\langle y \rangle = \mathbb{Z}/p$.

The group H is normal in G, so we have an extension

$$0 \longrightarrow H \longrightarrow G \longrightarrow G/H \longrightarrow 0.$$

As $G/H \simeq \mathbb{Z}/p$, the extension splits and the associated fibration has a section. Now, H is $\mathbb{Z}/p$-cellular and strictly contained in G, so by the inductive hypothesis $\mathrm{B}H$ is $\mathrm{B}\mathbb{Z}/p$-cellular. As $\mathrm{B}\mathbb{Z}/p$ is $\mathrm{B}\mathbb{Z}/p$-cellular too, the result of Proposition 2.5 implies that G is also, as we required. □

We will conclude this section by describing the fundamental group of $\mathbf{CW}_{\mathrm{B}\mathbb{Z}/p}\mathrm{B}G$.

Proposition 4.15. *Let G be a finite $\mathbb{Z}/p$-cellular group. Let us denote by r the order of $H_2(G)$ and s the number of different homomorphisms $\mathbb{Z}/p \longrightarrow G$. Then the fundamental group π of the $\mathrm{B}\mathbb{Z}/p$-cellularization of $\mathrm{B}G$ fits into a central extension*

$$0 \longrightarrow H \longrightarrow \pi \longrightarrow G \longrightarrow 0,$$

where π is a finite group whose order is bounded by prs.

Proof. We consider the evaluation map $\bigvee_{[\mathrm{B}\mathbb{Z}/p,\mathrm{B}G]_*} \longrightarrow \mathrm{B}G$, where the wedge is extended over all the elements of $[\mathrm{B}\mathbb{Z}/p, \mathrm{B}G]_*$, and let C_g be the homotopy cofibre of this map. The Mayer-Vietoris sequence of this cofibration is as follows:

$$0 \longrightarrow H_2(\mathrm{B}G) \longrightarrow H_2(C_g) \longrightarrow \bigoplus \mathbb{Z}/p \longrightarrow G^{ab} \longrightarrow 0,$$

where the rank of the elementary abelian p-group $\bigoplus \mathbb{Z}/p$ is the number of homomorphisms $\mathbb{Z}/p \longrightarrow G$. The cofibre C_g is simply connected, so $H_2(C_g) = \pi_2(C_g)$, and its order is clearly bounded by prs. Now, the fundamental group π is characterized by a central extension (given by the description of $\mathbf{CW}_{\mathrm{B}\mathbb{Z}/p}\mathrm{B}G$ in Proposition 2.4)

$$0 \longrightarrow A \longrightarrow \pi \longrightarrow \mathrm{S}_{\mathbb{Z}/p}G \longrightarrow 0,$$

where A is the second homotopy group of the $\Sigma\mathrm{B}\mathbb{Z}/p$-nullification of C_g, which is a quotient of $\pi_2(C_g)$. We finish by observing that the $\mathbb{Z}/p$-socle of G is $\mathbb{Z}/p$-cellular and hence equal to G. □

The problem of determining exactly how many $\mathbb{Z}/p$'s appear in the kernel of this extension seems very difficult, because it depends essentially on how many maps from $\mathrm{M}(\mathbb{Z}/p, 2)$ to the succesive cofibers that appear in the construction of the $\Sigma\mathrm{B}\mathbb{Z}/p$-nullification of C_g can be lifted to $\Sigma\mathrm{B}\mathbb{Z}/p$. However, in the next section we will see some examples of groups for which it is possible to determine exactly the fundamental group of $\mathbf{CW}_{\mathrm{B}\mathbb{Z}/p}\mathrm{B}G$.

The previous result allows us to characterize the finite groups such that the $\mathrm{B}\mathbb{Z}/p$-cellularization of its classifying space is a $K(G,1)$.

Proposition 4.16. *Let G be a finite group. Then $\pi_n(\mathbf{CW}_{\mathrm{B}\mathbb{Z}/p}\mathrm{B}G) = 0$ for $n \geq 2$ if and only if the $\mathbb{Z}/p$-cellularization of G is a p-group.*

Proof. If $\mathrm{CW}_{\mathbb{Z}/p}G$ is a p-group, then by Propositions 4.14 and 4.3

$$\mathbf{CW}_{\mathrm{B}\mathbb{Z}/p}\mathrm{B}G \simeq \mathbf{CW}_{\mathrm{B}\mathbb{Z}/p}\mathrm{BCW}_{\mathbb{Z}/p}G \simeq \mathrm{BCW}_{\mathbb{Z}/p}G.$$

Conversely, if $\mathbf{CW}_{\mathrm{B}\mathbb{Z}/p}\mathrm{B}G$ is an aspherical space $\mathrm{B}G'$, the group G' is $\mathbb{Z}/p$-cellular, because taking a fundamental group "commutes" with (homotopy) colimits. The result now follows from Proposition 4.3 and the previous proposition. □

5. Examples

In this section we describe the effect of nullification and cellularization when applied to (classifying spaces of) some families of discrete groups, namely dihedral, special linear, symmetric, alternating, quaternionic, semidihedral and $M_m(p)$-groups. The omitted details of the structure of the groups involved can be found in [Wei77], [Gor80] or [Rob96], and the structure of the $\mathbb{F}_p$-cohomology of most of them (that characterizes the $\mathbb{Z}/p$-completion, as seen in the preliminaries) is described in [AM94], [FP78] and [Tho86]. We always suppose that the primes that appear divide the order of G; otherwise the classifying space of the group is $\mathrm{B}\mathbb{Z}/p$-null and hence its $\mathrm{B}\mathbb{Z}/p$-cellularization is a point.

5.1. **Dihedral groups.** Let $D_n = \{X, Y; X^n = 1, Y^2 = 1, (XY)^2 = 1\}$ be the dihedral group of order $2n$.

a) *Nullification.*

For computing $\mathbf{P}_{\mathrm{B}\mathbb{Z}/p}\mathrm{B}D_n$, we will distinguish the cases $p \neq 2$ and $p = 2$.

Suppose first that p is different from 2. Then $|D_n| = p^r q$, p coprime with q. We only need to consider the case $r > 0$. Now, the $\mathbb{Z}/p$-radical $\mathrm{T}_{\mathbb{Z}/p}D_n$ of D_n is the subgroup generated by X^{n/p^r}, which is easily seen to be the unique p-Sylow subgroup of D_n (in particular is normal). Moreover, $\langle X^{n/p^r}\rangle$ is isomorphic to $\mathbb{Z}/p^r$, and $D_n/\langle X^{n/p^r}\rangle$ is isomorphic to D_{n/p^r}. So, by the result of Theorem 3.5 we have the following covering fibration:

$$\prod_{s\neq p}(\mathrm{B}\mathbb{Z}/p^r)^\wedge_s \longrightarrow \mathbf{P}_{\mathrm{B}\mathbb{Z}/p}\mathrm{B}D_n \longrightarrow \mathrm{B}D_{n/p^r}.$$

Now, as $(\mathrm{B}\mathbb{Z}/p^r)^\wedge_s$ is contractible if $s \neq p$ we obtain the fact $\mathbf{P}_{\mathrm{B}\mathbb{Z}/p}\mathrm{B}D_n$ is homotopy equivalent to $\mathrm{B}D_{n/p^r}$.

Now we attack the case $p = 2$. It is clear by the relations that define the group that Y and XY belong to the $\mathbb{Z}/2$-radical of D_n. But this implies that X belongs too, and then $\mathrm{T}_{\mathbb{Z}/2}(D_n) = D_n$. Hence, $\mathbf{P}_{\mathrm{B}\mathbb{Z}/2}\mathrm{B}D_n = \prod_{q\neq 2}(\mathrm{B}D_n)^\wedge_q$.

b) *Cellularization.*

If $p \neq 2$, the $\mathbb{Z}/p$-socle of D_n is the cyclic group of order p whose generator is identified inside D_n with $X^{n/p}$. Hence, the $\mathbb{Z}/p$-cellularization of D_n is $\mathbb{Z}/p$, and then $\mathbf{CW}_{\mathrm{B}\mathbb{Z}/p}\mathrm{B}D_n = \mathrm{B}\mathbb{Z}/p$.

If $p = 2$, we can make the change $Z = XY$ to obtain the presentation $D_n = \{X, Z; X^2 = 1, Z^2 = 1, (XZ)^n = 1\}$. This proves that D_n is always generated by order two elements. In particular, if $n = 2^j$ for some natural number j, the corresponding dihedral group is a 2-group, and hence by Proposition 4.14 its classifying space is $\mathrm{B}\mathbb{Z}/2$-cellular.

In the case $n \neq 2^j$, we can affirm that $\mathrm{B}D_n$ is not $\mathrm{B}\mathbb{Z}/2$-cellular, because it has odd primary torsion, but we can prove that the group is $\mathbb{Z}/2$-cellular. In fact, we give an explicit construction of D_n, for every n, as a colimit of copies of $\mathbb{Z}/2$.

Consider the second presentation given above, the usual presentations $\mathbb{Z}/2 = \{A; A^2 = 1\}$, $\mathbb{Z}/2 * \mathbb{Z}/2 = \{B, C; B^2 = 1, C^2 = 1\}$, and suppose n is odd. Then D_n is the coequalizer of the homomorphisms

$$\begin{array}{ccc} \mathbb{Z}/2 & \longrightarrow & \mathbb{Z}/2 * \mathbb{Z}/2, \\ A & \longrightarrow & BCB\ldots CB, \end{array}$$

where B appears $\frac{n+1}{2}$ times, and

$$\begin{array}{ccc} \mathbb{Z}/2 & \longrightarrow & \mathbb{Z}/2 * \mathbb{Z}/2, \\ A & \longrightarrow & CB \dots BC, \end{array}$$

where now C appears $\frac{n+1}{2}$ times. The case n even is similar, with B appearing $\frac{n}{2}$ times in the first homomorphism, and C appearing $\frac{n}{2}+1$ times in the second. As every coequalizer is a colimit, we have proved that the dihedral groups are always $\mathbb{Z}/2$-cellular.

5.2. **Finite projective special linear groups and special linear groups.** Next we study the family of groups $SL(2,q)$, q prime, and their quotient groups $PSL(2,q)$. If $q=2$, then $SL(2,q)=PSL(2,q)=D_6$, and this case has already been studied in the previous section. Remember in the sequel that $PSL(2,q)$ is simple if $q \geq 5$.

a) *Nullification.*

Consider the presentation of the tetrahedral group given by

$$PSL(2,3) = \{X, Y; X^3 = 1, Y^3 = 1, (XY)^2 = 1\}.$$

The unique nontrivial normal subgroup of $PSL(2,3)$ is $H = \{1, XY, YX, XY^2X\}$, which is isomorphic to the Klein group $\mathbb{Z}/2 \times \mathbb{Z}/2$, and it is easy to see that this is precisely the $\mathbb{Z}/2$-radical of $PSL(2,3)$. The associated extension gives rise to a fibration of classifying spaces

$$\mathrm{B}\mathbb{Z}/2 \times \mathrm{B}\mathbb{Z}/2 \longrightarrow \mathrm{B}PSL(2,3) \longrightarrow \mathrm{B}\mathbb{Z}/3.$$

As the fibre is $\mathrm{B}\mathbb{Z}/2$-acyclic, we have a homotopy equivalence $\mathbf{P}_{\mathrm{B}\mathbb{Z}/2}\mathrm{B}PSL(2,3) \simeq \mathrm{B}\mathbb{Z}/3$. For $p=3$, $PSL(2,3)$ is generated by order 3 elements, so it is equal to its $\mathbb{Z}/3$-radical, and hence $\mathbf{P}_{\mathrm{B}\mathbb{Z}/3}\mathrm{B}PSL(2,3) \simeq \mathrm{B}PSL(2,3)^{\wedge}_{2}$. If $q \geq 5$, then $\mathrm{T}_{\mathbb{Z}/p}(PSL(2,q)) = PSL(2,q)$ for every prime p dividing the order of $PSL(2,q)$, because the group is simple. Then, by Theorem 3.5, the $\mathrm{B}\mathbb{Z}/p$-nullification of $\mathrm{B}PSL(2,q)$ is homotopy equivalent to the $\mathbb{Z}[1/p]$-completion of $\mathrm{B}PSL(2,q)$. We now consider the non-projective case. We always have the fibration

$$\mathrm{B}\mathbb{Z}/2 \longrightarrow \mathrm{B}SL(2,q) \longrightarrow \mathrm{B}PSL(2,q).$$

Now the fibre is $\mathrm{B}\mathbb{Z}/2$-acyclic, and we have $\mathbf{P}_{\mathrm{B}\mathbb{Z}/2}\mathrm{B}PSL(2,q) \simeq \mathbf{P}_{\mathrm{B}\mathbb{Z}/2}\mathrm{B}SL(2,q)$, which is again simply connected except for the pathological case $q=3$. If $p \neq 2$ and divides the order of $PSL(2,q)$, this group is always generated by the transvections of order p ([Rob96, 3.2.10]), and again the $\mathrm{B}\mathbb{Z}/p$-nullification of its classifying space is homotopy equivalent to its $\mathbb{Z}[1/p]$-completion.

b) *Cellularization.*

We will again first consider the case $q=3$, the tetrahedral group.

If $p=2$, we have said that the $\mathbb{Z}/2$-radical is the Klein group $\mathbb{Z}/2 \times \mathbb{Z}/2$, which is equal to its $\mathbb{Z}/2$-socle, because it is generated by order 2 elements. As the Klein group is indeed a 2-group, we have $\mathbf{CW}_{\mathrm{B}\mathbb{Z}/2}\mathrm{B}PSL(2,3) = \mathrm{B}\mathbb{Z}/2 \times \mathrm{B}\mathbb{Z}/2$.

In the case $p=3$, recall $PSL(2,3)$ is generated by the order 3 transvections, so it is equal to its $\mathbb{Z}/3$-socle. It is known that the Schur multiplier $H_2(PSL(2,3))$ is $\mathbb{Z}/2$, and then according to Theorem 4.4 and Lemma 4.5, the $\mathbb{Z}/3$-cellularization of $PSL(2,3)$ is defined by a central extension

$$0 \longrightarrow \mathbb{Z}/2 \longrightarrow \mathrm{CW}_{\mathbb{Z}/3}PSL(2,3) \longrightarrow PSL(2,q) \longrightarrow 0.$$

This extension is nontrivial by Proposition 4.11, and it is known that the only nontrivial central extension of $PSL(2,3)$ by $\mathbb{Z}/2$ is $SL(2,3)$, so the $\mathbb{Z}/3$-cellularization of $PSL(2,3)$ is $SL(2,3)$ and hence the latter group is $\mathbb{Z}/3$-cellular, because cellularization is an idempotent functor.

If $q \geq 5$, the group $PSL(2,q)$ is perfect, and its universal central extension is precisely

$$0 \longrightarrow \mathbb{Z}/2 \longrightarrow SL(2,q) \longrightarrow PSL(2,q) \longrightarrow 0.$$

Hence, the Schur multiplier of $PSL(2,q)$ is $\mathbb{Z}/2$, and by Lemma 4.5 this group is $\mathbb{Z}/2$-cellular. On the other hand, $SL(2,q)$ is the universal covering group of itself, and in particular $H_2(SL(2,q)) = 0$, so by Corollary 4.10 it is $\mathbb{Z}/2$-cellular. For p odd, the same corollary proves that if p divides the order of $PSL(2,3)$, the $\mathbb{Z}/p$-cellularization of $PSL(2,q)$ is $SL(2,q)$, and this group is $\mathbb{Z}/p$-cellular.

To conclude, observe that neither $SL(2,q)$ nor $PSL(2,q)$ are p-groups for any primes $p \geq 2$ and $q > 2$; hence, their classifying spaces cannot be $\mathrm{B}\mathbb{Z}/p$-cellular. It is interesting to point out that $\mathrm{B}SL(2,q)$ cannot be the $\mathrm{B}\mathbb{Z}/p$-cellularization of $\mathrm{B}PSL(2,q)$, not even in the case of p odd, in which we know $SL(2,q)$ to be the $\mathbb{Z}/p$-cellularization of $PSL(2,q)$.

5.3. **Symmetric and alternating groups.** In this section we will be concerned with the permutation group S_n and the alternating group A_n, whose orders are $n!$ and $n!/2$, respectively.

a) *Nullification.*

In the sequel we will study the nullification and cellularization of the classifying spaces of symmetric and alternating groups. The symmetric group of n letters ($n \geq 2$) admits the following presentation:

$$S_n = \{X_1, \ldots, X_{n-1}; X_i^2 = 1, X_iX_{i+1}X_i = X_{i+1}X_iX_{i+1}, X_iX_{i+j}X_i = X_{i+j} \text{ for } j \geq 2\}.$$

This shows that S_n is generated by order 2 elements, and in particular is equal to its $\mathbb{Z}/2$-radical. Hence, its $\mathrm{B}\mathbb{Z}/2$-nullification is homotopy equivalent to its $\mathbb{Z}[1/2]$-completion. If p is odd, we can consider the fibration defined by the inclusion of the alternating group:

$$\mathrm{B}A_n \longrightarrow \mathrm{B}S_n \longrightarrow \mathrm{B}\mathbb{Z}/2.$$

As the base space is $\mathrm{B}\mathbb{Z}/p$-null for p odd, the fibration is preserved under $\mathrm{B}\mathbb{Z}/p$-nullification. If $n = 3$, $A_3 = \mathbb{Z}/3$, and then $\mathbf{P}_{\mathrm{B}\mathbb{Z}/3}\mathrm{B}S_3 \simeq \mathrm{B}\mathbb{Z}/2$. In the case $n \geq 4$, A_n is always generated by order p elements: if $n = 4$, it is known that $A_4 = PSL(2,3)$, and hence is generated by the 3-transvections, as we have said before; if $n > 4$, A_n is simple, and then the $\mathbb{Z}/p$-radical, which is normal, is the whole group. So, these considerations prove that for $n \geq 4$ and p odd, $\mathbf{P}_{\mathrm{B}\mathbb{Z}/p}\mathrm{B}A_n$ is homotopy equivalent to $\mathbb{Z}[1/p]_\infty \mathrm{B}A_n$. In particular, the fibration which defines the $\mathrm{B}\mathbb{Z}/p$-nullification of $\mathrm{B}S_n$ takes the form

$$\mathbb{Z}[1/p]_\infty \mathrm{B}A_n \longrightarrow \mathbf{P}_{\mathrm{B}\mathbb{Z}/p}\mathrm{B}S_n \longrightarrow \mathrm{B}\mathbb{Z}/2$$

which turns out to be a covering fibration, because $\mathbb{Z}[1/p]_\infty \mathrm{B}A_n$ is simply connected.

The previous arguments also prove that $\mathbf{P}_{\mathrm{B}\mathbb{Z}/2}\mathrm{B}A_n \simeq \mathbb{Z}[1/2]_\infty \mathrm{B}A_n$ if $n \geq 5$. If $n = 4$ the alternating group is isomorphic to the tetrahedral group and this case has been studied in the previous section.

b) *Cellularization.*

In [RS98, 7.5], Rodríguez-Scherer effectively built S_n as a colimit of copies of $\mathbb{Z}/2$, so S_n is $\mathbb{Z}/2$-cellular. If p is odd, every permutation of order p is even, and this means that the $\mathbb{Z}/p$-socle of S_n is a subgroup of A_n, which is normal because the socle is always a characteristic subgroup; this already proves that $\mathrm{S}_{\mathbb{Z}/p}S_n = A_n$, and in particular $\mathrm{CW}_{\mathbb{Z}/3}S_4 = \mathrm{CW}_{\mathbb{Z}/3}A_4 = SL(2,3)$. So, we fix our attention again on the alternating groups A_n with $n \geq 5$. It is known ([Asc00, 33.15]) that the Schur multiplier of A_n is $\mathbb{Z}/2$ for every $n \neq 6, 7$, and $H_2(A_6) = H_2(A_7) = \mathbb{Z}/6$. So, if $p \geq 5$, Corollary 4.10 implies that the $\mathbb{Z}/p$-cellularization of A_n is in this case isomorphic to the universal central extension of A_n. If $p = 3$, according to Proposition 4.11 the $\mathbb{Z}/3$-cellularization is given by the unique nontrivial extension of A_n by $\mathbb{Z}/2$, and finally, if $p = 2$, A_n is $\mathbb{Z}/2$-cellular for every n different from 6 or 7; in the pathological case, Proposition 4.9 tells us that the $\mathbb{Z}/2$-cellularization of A_n for $n = 6, 7$ is the extension of A_n by $\mathbb{Z}/3$ induced by the corresponding universal central extension.

As the order of S_n and A_n is respectively $n!$ and $n!/2$, these groups are never p-groups for $n \geq 2$, and hence their classifying spaces cannot be $\mathrm{B}\mathbb{Z}/p$-cellular.

5.4. **p-groups.** We finish by describing the effect of the $\mathrm{B}\mathbb{Z}/p$-cellularization functor on the classifying spaces of three families of p-groups, namely the quaternionic groups, semidihedral groups and $M_m(p)$-groups.

Consider the quaternion group $Q_{m+1} = \{H, K;\ H^{2^{m-1}} = K^2 = 1,\ HK = KH^{-1}\}$, with $m \geq 3$. This group has order 2^{m+1}, and the $\mathbb{Z}/2$-socle is the center, which is the subgroup generated by $H^{2^{m-1}}$. This subgroup is isomorphic to $\mathbb{Z}/2$, and hence $\mathbf{CW}_{\mathrm{B}\mathbb{Z}/2}\mathrm{B}Q_m \simeq \mathrm{B}\mathbb{Z}/2$ for every m.

We now turn our attention to the semidihedral groups of order 2^m, that admit a presentation $SD_m = \{X, Y;\ X^{2^{m-1}} = Y^2 = 1,\ YXY^{-1} = X^{-1+2^{m-2}}\}$. The $\mathbb{Z}/2$-socle of SD_m is generated by $X^{2^{m-2}}$ and Y, and it is isomorphic to $D_{2^{m-2}}$. So, $\mathbf{CW}_{\mathrm{B}\mathbb{Z}/2}\mathrm{B}SD_m$ is homotopy equivalent to $\mathrm{B}D_{2^{m-2}}$.

Finally, if $p = 2$ and $m > 3$, or if p is odd and $m > 2$, we define the group $M_m(p) = \{X, Y;\ X^{p^{m-1}} = Y^p = 1,\ YXY^{-1} = X^{1+p^{m-2}}\}$. The $\mathbb{Z}/p$-socle of $M_m(p)$ is generated by $X^{p^{m-2}}$ and Y, and it is isomorphic to $\mathbb{Z}/p \times \mathbb{Z}/p$. Thus, the $\mathrm{B}\mathbb{Z}/p$-cellularization of $\mathrm{B}M_m(p)$ is $\mathrm{B}\mathbb{Z}/p \times \mathrm{B}\mathbb{Z}/p$.

Remark 5.1. The classification results that can be found in [Gor80, 5.4] establish that the last computations give an explicit formula for the $\mathrm{B}\mathbb{Z}/p$-cellularization of the classifying spaces of a wide family of p-groups, such as for example the p-groups of order p^m which contain a cyclic subgroup of order p^{m-1}, the p-groups with no noncyclic abelian normal subgroup, or the groups of order p^3.

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