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TRANSLATION EQUIVALENCE IN FREE GROUPS

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Abstract. Motivated by the work of Leininger on hyperbolic equivalence of homotopy classes of closed curves on surfaces, we investigate a similar phenomenon for free groups. Namely, we study the situation when two elements g, h in a free group F have the property that for every free isometric action of F on an $\mathbb{R}$-tree X the translation lengths of g and h on X are equal.

1. Introduction

Let S be a closed oriented surface of negative Euler characteristic and let γ be a free homotopy class of essential closed curves on S. If ρ is a hyperbolic metric on S, then γ contains a unique curve c of minimal ρ-length. We denote this length by $\ell_\rho(\gamma)$. The curve c is a closed geodesic on (S, ρ) and $\ell_\rho(\gamma)$ is the translation length of any representative of γ in the action corresponding to ρ of $G = \pi_1(S)$ on $\mathbb{H}^2 = \tilde{S}$. There is an obvious identification between the set of nontrivial conjugacy classes C of G and the set of free homotopy classes of essential closed curves on S and we shall not distinguish between the two. Thus each marked hyperbolic structure on S defines a so-called *marked length spectrum* $l : C \to \mathbb{R}$. It is well known and easy to see that a marked hyperbolic structure on S, considered as a point of the Teichmüller space of S, is uniquely determined by its marked length spectrum.

The dual situation, however, is different. For $\gamma_1, \gamma_2 \in C$ we say that γ_1 is *hyperbolically equivalent to* γ_2, denoted $\gamma_1 \equiv_h \gamma_2$, if for every hyperbolic structure ρ on S we have $\ell_\rho(\gamma_1) = \ell_\rho(\gamma_2)$. In more algebraic terms, for two conjugacy classes $\gamma_1, \gamma_2 \in C$ we have $\gamma_1 \equiv_h \gamma_2$ if for every discrete and cocompact isometric action of G on $\mathbb{H}^2$ the translation lengths of γ_1 and γ_2 are equal. It can happen that $\gamma_1 \neq \gamma_2^{\pm 1}$ and yet $\gamma_1 \equiv_h \gamma_2$. The main source of hyperbolic equivalence comes from "trace identities" in $SL(2, \mathbb{C})$. A number of interesting new results about hyperbolic equivalence were recently obtained by Chris Leininger [12].

We are interested in investigating a similar phenomenon for free groups. In this context the Teichmüller space is replaced by the Culler-Vogtmann outer space [7], so that instead of actions on $\mathbb{H}^2$ we consider free and discrete actions on $\mathbb{R}$-trees. Recall that if G is a group acting by isometries on an $\mathbb{R}$-tree X and $g \in G$, then the *translation length* $\ell_X(g)$ is defined as

$$\ell_X(g) = \inf_{x \in X} d(x, gx).$$

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It is easy to see that $\ell_X(g)$ only depends on the conjugacy class of g and that in the definition above the infimum can be replaced by a minimum. Thus $\ell_X(g) = 0$ if and only if g fixes a point of X. This discussion naturally leads us to the following definition.

Definition 1.1. Let F be a finitely generated free group and let $g, h \in F$ be elements of F. We say that g and h are *translation equivalent in* F, denoted $g \equiv_t h$, if for every free and discrete isometric action of F on an $\mathbb{R}$-tree X we have $\ell_X(g) = \ell_X(h)$.

It is obvious that $\equiv_t$ is an equivalence relation on F. Applying the above definition to the case where X is the Cayley graph of F with respect to some free basis of F implies that every $\equiv_t$-equivalence class is the union of finitely many conjugacy classes in F. Clearly, if $g \equiv_t h$ in F and $\phi : F \to F_1$ is an injective homomorphism to a free group F_1, then $\phi(g) \equiv_t \phi(h)$ in F_1. Indeed, suppose F_1 acts freely and discretely by isometries on an $\mathbb{R}$-tree X. Then, by restriction, we get a free and discrete action of $\phi(F)$ on X, and via a twist by ϕ, we also get a free and discrete action of F itself on X. Namely, $f \cdot p := \phi(f)p$, where $p \in X, f \in F$. Since u and v are translation equivalent in F, it follows that $\ell_X(u) = \ell_X(v)$, that is, $\ell_X(\phi(u)) = \ell_X(\phi(v))$.

A phenomenon related to but different from translation equivalence was studied by Smillie and Vogtmann [17] who, given an arbitrary finite set of conjugacy classes in a free group, constructed multi-parametric families of free discrete actions where the translation length of each conjugacy class from this set remains constant through the family. Results similar in spirit to those of [17] were also obtained by Cohen, Lustig and Steiner [6].

The notion of translation equivalence is also related to the space of geodesic currents on a free group. In [10] the first author studies the properties of an *intersection form*

$$I : FLen(F) \times Curr(F) \to \mathbb{R}.$$

Here $FLen(F)$ is the (non-projectivized) space of hyperbolic length functions corresponding to free and discrete isometric actions of F on $\mathbb{R}$-trees and $Curr(F)$ is the space of *geodesic currents* on F, that is, the space of F-invariant positive Borel measures on the set of all pairs (x, y), where $x, y \in \partial F$ and $x \neq y$. Similarly to Bonahon's notion [4] of the intersection number between geodesic currents on hyperbolic surfaces, it turns out that if η_g is the "counting" current corresponding to a nontrivial $g \in F$ and if $\ell \in FLen(F)$ is a length function, then $I(\ell, \eta_g) = \ell(g)$. Thus $g \equiv_t h$ in F if and only if for every $\ell \in FLen(F)$ we have $I(\ell, \eta_g) = I(\ell, \eta_h)$. Therefore the notion of translation equivalence, in a sense, measures the degeneracy of the intersection form I with respect to its second argument. We refer the reader to [10] for a detailed discussion on this topic.

The first natural problem is to demonstrate that there are nontrivial instances of translation equivalence in free groups. We provide two different sources of translation equivalence: one based on trace identities in $SL(2, \mathbb{C})$ and another based on power redistribution for certain products of translation equivalent elements. Both methods can be iterated and used to produce arbitrarily large finite collections of distinct conjugacy classes in F that are pairwise translation equivalent. Both of these sources can also be used to produce hyperbolic equivalence in the context of $\mathbb{H}^2$-actions, although some distinctions do arise as will be pointed out later. (See Example 7.5.)

Another natural question is to give a more algebraic and combinatorial characterization of translation equivalence.

Recall that an isometric action of a group G on an $\mathbb{R}$-tree X is *very small* [5] if the following conditions hold:

(1) The action is *small*, that is, arc stabilizers do not contain free subgroups of rank two.
(2) Stabilizers of tripods are trivial.
(3) For any $g \in G$ and for each $n \neq 0$ the fixed sets of g and g^n are equal.

In particular, every free action and, more generally, an action with trivial arc stabilizers, is very small. Results of Cohen-Lustig [5] and Bestvina-Feighn [3] imply that an action of F_n on an $\mathbb{R}$-tree is very small if and only if this action represents a point in the standard length functions compactification of the outer space. Thus a very small action can always be approximated in the sense of length functions by a sequence of free simplicial actions.

Notation 1.2. If A is a basis of a free group F, any element $g \in F$ is represented by a unique reduced word w_g over the alphabet $A^{\pm 1}$. The *length of g with respect to the basis A*, denoted $|g|_A$, is the number of letters in w_g. A word is *cyclically reduced* if all its cyclic permutations are reduced. Any reduced word w can be uniquely factored as $w = cuc^{-1}$ where u is cyclically reduced. If $w_g = cuc^{-1}$ is such a factorization, then $|u|_A$ is called *the cyclically reduced length of g with respect to A* is denoted by $||g||_A$. Note that $||g||_A = \ell_X(g)$ where X is the Cayley graph of F with respect to A. A *cyclic word* in $A^{\pm 1}$ is the set of all cyclic permutations of a cyclically reduced word. There is a canonical identification between the set of cyclic words in $A^{\pm 1}$ and the set of conjugacy classes in F.

If w is a cyclic word consisting of all cyclic permutations of a nontrivial word u, and if v is a word in $A^{\pm 1}$, we define the *number of occurrences of v in w* as the number of those i, $0 \le i < |u|$, such that the infinite word $uuu\dots$ begins with $u_i v$, where u_i is the initial segment of u of length i. If w is a cyclic word and $x, y \in A^{\pm 1}$, we use $n_A(w; x, y)$ (or just $n(w; x, y)$ if A is fixed) to denote the total number of occurrences of the subwords xy and $y^{-1}x^{-1}$ in w. Thus $n(w; x, y) = n(w; y^{-1}, x^{-1}) = n(w^{-1}; x, y)$. Similarly, in this case, if w is a cyclic word and $x \in A^{\pm 1}$, we denote by $n(w; x)$ the total number of occurrences of x and x^{-1} in w. Thus again $n(w; x) = n(w; x^{-1}) = n(w^{-1}; x)$. If $[g]$ is a nontrivial conjugacy class in F and w is the unique cyclic word over A representing $[g]$, we denote $n_A([g]; x) := n(w; x)$, where $x \in A^{\pm 1}$.

In studying automorphisms of free groups, Whitehead automorphisms and the Whitehead graph of a cyclically reduced word play a major role. See Lyndon-Schupp [13] for a detailed discussion. Note that in [13] Whitehead graphs are called *star graphs*.

Definition 1.3 (Whitehead graph)**.** Let w be a nontrivial cyclic word in $F(A)$. The *Whitehead graph $\mathcal{W}_A(w)$ of w with respect to A* is the labelled undirected graph defined as follows. The vertex set of $\mathcal{W}_A(w)$ is $A^{\pm 1}$. If $x, y \in A^{\pm 1}$ are such that $x \neq y$, there is an edge in $\mathcal{W}_A(w)$ between x to y with label $n(w; x, y^{-1})$.

If $[g]$ is a nontrivial conjugacy class in F, then $[g]$ is represented by a unique cyclic word w in $F(A)$. Then the *Whitehead graph $\mathcal{W}_A([g])$ of $[g]$ with respect to A* is defined as $\mathcal{W}_A(w)$. Note that $\mathcal{W}_A(w) = \mathcal{W}_A(w^{-1})$ for any nontrivial cyclic word w.

We obtain the following result.

Theorem A. *Let F be a finitely generated free group and let $g, h \in F$ be nontrivial elements.*

Then the following statements are equivalent:

(1) $\ell_X(g) = \ell_X(h)$ *for every very small action of F on an $\mathbb{R}$-tree X.*
(2) $\ell_X(g) = \ell_X(h)$ *for every free action of F on an $\mathbb{R}$-tree X.*
(3) $g \equiv_t h$ *in F.*
(4) $||g||_A = ||h||_A$ *for every free basis A of F.*
(5) $\mathcal{W}_A([g]) = \mathcal{W}_A([h])$ *for every free basis A of F. That is, the conjugacy classes $[g]$ and $[h]$ have the same Whitehead graphs with respect to A.*
(6) $n_A([g]; x) = n_A([h]; x)$ *for every free basis A of F and for every $x \in A$.*

Theorem A immediately yields the following more combinatorial version of translation equivalence.

Corollary 1.4. *Let F be a finitely generated free group with a free basis A and let $g, h \in F$. Then the following conditions are equivalent:*

(1) $g \equiv_t h$ *in F.*
(2) $||\phi(g)||_A = ||\phi(h)||_A$ *for every automorphism ϕ of F.*
(3) $||\phi(g)||_A = ||\phi(h)||_A$ *for every injective endomorphism ϕ of F.*
(4) $||\phi(g)||_B = ||\phi(h)||_B$ *for every free group F_1 with a free basis B and for every injective homomorphism $\phi : F \to F_1$.*

The a priori weakest condition in Theorem A is condition (4) which deserves further comment. Let S be a closed surface as before. If γ and δ are free homotopy classes of closed curves on S, we use $i(\gamma, \delta)$ to denote the *geometric intersection number* of γ and δ. We say that free homotopy classes γ_1, γ_2 of closed curves on S are *simple intersection equivalent* if $i(\gamma_1, [c]) = i(\gamma_2, [c])$ for every essential *simple closed curve* c on S. It is easy to see that hyperbolic equivalence implies simple intersection equivalence. Surprisingly, however, the converse is not true as was recently proved by Chris Leininger [12].

It is therefore natural to consider the analogue of simple intersection equivalence for free groups. If $G = \pi_1(S)$ and c is an essential simple closed curve on S, then c defines a splitting of G as either an amalgamated product or as an HNN-extension over a cyclic subgroup. Let X be the Bass-Serre tree corresponding to that splitting. It is easy to see that for any $g \in G$ we have $i([g], [c]) = \ell_X(g)$. The difficulty is that for a free group F, a single element of F, even if it is a primitive one, does not define a splitting of F. A free basis A of F does, however, define such a splitting. Namely, the splitting of F as the multiple HNN-extension of the trivial group with stable letters corresponding to the elements of A and the Bass-Serre tree X of this splitting is precisely the Cayley graph of F with respect to A. Then for any $g \in F$ we have $\ell_X(g) = ||g||_A$, the cyclically reduced length of g with respect to A. For a free basis A of F and an element $g \in F$ we can therefore define the *intersection number* $i(g, A) := ||g||_A$. By analogy with the surface group case we say that $g_1, g_2 \in F$ are *simple intersection equivalent* in F if for every free basis A of F we have $i(g_1, A) = i(g_2, A)$. Unlike in the surface group case, Theorem A says that in free groups simple intersection equivalence is the same as translation equivalence.

We can now state the main sources of translation equivalence in free groups that we have discovered so far.

If F is a finitely generated free group and $u, v \in F$ are nontrivial elements, we say that u and v are *trace equivalent* or *character equivalent* in F, denoted $u \equiv_c v$, if for every representation $\alpha : F \to SL(2, \mathbb{C})$ we have $tr(\alpha(u)) = tr(\alpha(v))$. Character equivalent words come from the so-called "trace identities" in $SL(2, \mathbb{C})$ and are quite plentiful (see, for example, [9]). A corollary of Theorem A together with a result of Horowitz [9] is:

Theorem B. *Let F be a finitely generated free group and suppose that $u \equiv_c v$ in F. Then $u \equiv_t v$ in F.*

A particularly interesting example of character equivalence comes from two-variable "palindromic reversing". Let $w(x, y) \in F(x, y)$ be a freely reduced word. We denote by $w^R(x, y)$ the word $w(x, y)$ read backwards, but without inverting the letters. Thus $w^R(x, y) = (w(x^{-1}, y^{-1}))^{-1}$. We prove:

Theorem C. *Let F be a free group of rank $k \geq 2$ and let $w(x, y) \in F(x, y)$ be a freely reduced word. Then for any $g, h \in F$ we have*

$$w(g, h) \equiv_t w^R(g, h) \quad \text{in } F.$$

We shall give a direct proof of the above statement as well as a proof via character equivalence.

A very different source of translation equivalence is given by:

Theorem D. *Let F be a free group of rank $k \geq 2$ and let $g, h \in F$ be such that $g \equiv_t h$ but $g \neq h^{-1}$. Then for any positive integers p, q, i, j such that $p + q = i + j$ we have*

$$g^p h^q \equiv_t g^i h^j \quad \text{in } F.$$

Theorem B states that $\equiv_c$ implies $\equiv_t$. However, it turns out that Theorem D does not hold for character equivalence, as demonstrated by Example 7.5 below. This example shows that $\equiv_t$ does not imply $\equiv_c$ and that, although character equivalence and translation equivalence are closely related phenomena, they are not the same and translation equivalence is more general.

The following is an analogue of a result of Randol [20] about hyperbolic surfaces.

Corollary 1.5. *Let F be a free group of rank $k \geq 2$. Then for any integer $M \geq 1$ there exist elements $g_1, \dots, g_M \in F$ such that $g_i \equiv_t g_j$ in F and such that for $i \neq j$ g_i is not conjugate to $g_j^{\pm 1}$.*

Proof. Let $M \geq 1$ be an integer and let $(a, b, \dots)$ be a free basis of F. Consider $g = a$ and $h = ba^{-1}b^{-1}$. Put $g_i = g^i h^{2M-i} = a^i b a^{i-2M} b^{-1}$ for $i = 1, \dots, M$. Then by Theorem D $g_i \equiv_t g_j$ in F. On the other hand, g_i is not conjugate to $g_j^{\pm 1}$ for $i \neq j$. This can be seen, for example, by observing that g_i and $g_j^{\pm 1}$ have distinct images in the abelianization of F. □

Remark 1.6. Theorem C can be iterated to produce other examples with the same properties as in Corollary 1.5. Namely, let $\phi_i : F(x, y) \to F(x, y)$ be injective endomorphisms of $F(x, y)$ for $i = 1, \dots, N$. Let $\phi := \phi_N \circ \dots \circ \phi_1$. For $i = 1, \dots, N-1$ put $\psi_i := \phi_N \circ \dots \circ \phi_{i+1}$ and $\theta_i = \phi_i \circ \dots \circ \phi_1$. Then $\phi = \psi_i \circ \theta_i$. Let $\psi_i(x) = u_i(x, y)$, $\psi_i(y) = v_i(x, y)$, $\theta_i(x) = r_i(x, y)$ and $\theta_i(y) = s_i(x, y)$. Let $w(x, y) = \phi(x)$. Then $w = \psi_i(\theta_i(x)) = \psi_i(r_i(x, y)) = r_i(u_i(x, y), v_i(x, y))$. Let $w_i = r_i^R(u_i(x, y), v_i(x, y))$. Theorem C implies that $w(g, h) \equiv_t w_i(g, h)$ for any $g, h \in F$ for $i = 1, \dots, N-1$. It is possible, for each $N \geq 1$, to choose the endomorphisms ϕ_i and then elements $g, h \in F$ so that $w_1, \dots, w_{N-1}$ are pairwise nonconjugate in F.

Corollary 1.5 also follows from Theorem B and the result of Horowitz [9] establishing (via a more complicated family of words) a similar result for character equivalence. Moreover, as we observe later in Corollary 4.6, it is possible to generalize the proof of Corollary 1.5 to many nonfree groups.

Using SL_2 trace identities, we show that Theorem C and a version of Theorem D also hold for standard hyperbolic equivalence. Via a limiting argument we conclude that these statements also apply to the tree actions that occur in the Thurston boundary of the Teichmüller space of a closed hyperbolic surface, either orientable or nonorientable. Recall that each point μ in the Thurston boundary of the Teichmüller space is a measured lamination. There is an $\mathbb{R}$-tree X_μ, dual to the lift of this lamination, that comes equipped with a small isometric action of the fundamental group of the surface. In the case of a nonorientable surface not all such actions are very small.

Theorem E. *Let S be a possibly nonorientable closed surface of negative Euler characteristic and let $G = \pi_1(S)$. Then the following hold:*

(1) *For any $g, h \in G$ and for any $w(x, y) \in F(x, y)$ and for any tree action μ of G in the Thurston boundary of the Teichmüller space of S we have*

$$\ell_{X_\mu}(w(g, h)) = \ell_{X_\mu}(w^R(g, h)).$$

(2) *For any conjugate elements $g, h \in G$, for any $p, q > 0$ and for any tree action μ of G in the Thurston boundary of the Teichmüller space of S we have*

$$\ell_{X_\mu}(g^p h^q) = \ell_{X_\mu}(g^q h^p).$$

(3) *If S is orientable, then for any conjugate elements $g, h \in G$, for each point $\mu \in \overline{\mathcal{T}(S)} - \mathcal{T}(S)$ and for any positive integers p, q, i, j such that $p+q = i+j$ we have*

$$\ell_{X_\mu}(g^p h^q) = \ell_{X_\mu}(g^i h^j).$$

The paper is organized as follows. In Section 2 we discuss Whitehead graphs and prove Theorem A. In Section 2 we obtain a direct geometric proof of Theorem C about the palindromic sources of translation equivalence. In Section 4 we use the analysis of possible axis configurations for compositions of isometries of $\mathbb{R}$-trees to provide a geometric proof of Theorem D. Section 5 contains a discussion of SL_2 trace identities and a proof of Theorem B. In Section 6 we analyze how our results apply to tree actions occurring in the boundary of the Teichmüller space and prove Theorem E. In Section 7 we discuss various examples and counterexamples, and in Section 8 we list a number of interesting open problems.

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2. Whitehead graphs and a characterization of translation equivalence

Let $F = F(A) = F(a_1, \dots, a_k)$ be a free group with basis $A = \{a_1, \dots, a_k\}$ where $k \geq 2$.

Let $x, y \in \{a_1, \dots, a_k\}^{\pm 1}$ be such that $x \neq y^{\pm 1}$. Denote by $\phi_{x,y}$ the Nielsen automorphism of F that sends x to xy and fixes each generator $a_i \neq x^{\pm 1}$. For simplicity, if $i \neq j, 1 \leq i, j \leq k$, we use $\phi_{i,j}$ to denote ϕ_{a_i, a_j}.

The following lemma is obvious.

Lemma 2.1. *For any cyclic word w and any $x, y \in \{a_1, \dots, a_k\}^{\pm 1}$ such that $x \neq y^{\pm 1}$ we have*

$$||\phi_{x,y}(w)|| - ||w|| = n(w; x) - 2n(w; x, y^{-1}).$$

Lemma 2.2. *Let w be a cyclic word. Then*

$$n(w; a_i) = \frac{1}{k}\big(||w|| + \sum_{j \neq i}(||\phi_{i,j}(w)|| - ||\phi_{j,i}(w)||)\big).$$

Proof. Note that $n(w; x, y^{-1}) = n(w; y, x^{-1})$. Therefore

$$||\phi_{x,y}(w)|| - ||w|| = n(w; x) - 2n(w; x, y^{-1}),$$

$$||\phi_{y,x}(w)|| - ||w|| = n(w; y) - 2n(w; y, x^{-1}) = n(w; y) - 2n(w; x, y^{-1}),$$

and so

$$||\phi_{x,y}(w)|| - ||\phi_{y,x}(w)|| = n(w; x) - n(w; y).$$

Let $x = a_i$ and y vary over $a_j, j \neq i$. Then summing up the instances of the above equality for $x = a_i, y = a_j$ we get

$$(k-1)n(w; a_i) - \sum_{j \neq i} n(w, a_j) = \sum_{j \neq i}(||\phi_{i,j}(w)|| - ||\phi_{j,i}(w)||).$$

On the other hand,

$$n(w; a_i) + \sum_{j \neq i} n(w, a_j) = ||w||.$$

Adding the above formulas we get

$$k\, n(w; a_i) = ||w|| + \sum_{j \neq i}^{n}(||\phi_{i,j}(w)|| - ||\phi_{j,i}(w)||),$$

which yields the statement of the lemma. □

Proposition 2.3. *Let F be a finitely generated free group and let $u, v \in F$ be nontrivial elements such that for every free basis A of F we have $||u||_A = ||v||_A$. Then for every free basis A of F the Whitehead graphs of $[u]$ and $[v]$ with respect to A are equal.*

Proof. Let A be a free basis of F. We consider the conjugacy classes $[u]$, $[v]$ as cyclic words over A. The assumptions of the proposition imply that for every automorphism ϕ of F,

$$||\phi(u)||_A = ||\phi(v)||_A.$$

By Lemma 2.2 it follows that for each $x \in A$ we have $n([u]; x) = n([v], x)$. Therefore by Lemma 2.1 for every $x, y \in A^{\pm 1}$ such that $x \neq y^{\pm 1}$ we have

$$n([u]; x, y) = n([v]; x, y).$$

For a fixed $x \in A$ and an arbitrary cyclic word w,

$$n(w;x) = n(w;x,x) + \sum_{y \neq x^{\pm 1}} n(w;x,y).$$

Since $n([u];x) = n([v];x)$ and $n([u];x,y) = n([v];x,y)$ for any $y \neq x^{\pm 1}$, $y \in A^{\pm 1}$, it follows that $n([u];x,x) = n([v];x,x)$.

Thus we have shown that for any $x, y \in A^{\pm 1}$ such that $x \neq y^{-1}$ we have $n([u];x,y) = n([v];x,y)$. This means that $[u]$ and $[v]$ have equal Whitehead graphs with respect to A, as claimed. □

We can now establish Theorem A from the Introduction:

Theorem A. *Let F be a finitely generated free group and let $g, h \in F$ be nontrivial elements.*

Then the following statements are equivalent:

(1) *$\ell_X(g) = \ell_X(h)$ for every very small action of F on an $\mathbb{R}$-tree X.*
(2) *$\ell_X(g) = \ell_X(h)$ for every free action of F on an $\mathbb{R}$-tree X.*
(3) *$g \equiv_t h$ in F.*
(4) *$||g||_A = ||h||_A$ for every free basis A of F.*
(5) *$\mathcal{W}_A([g]) = \mathcal{W}_A([h])$ for every free basis A of F. That is, the conjugacy classes $[g]$ and $[h]$ have the same Whitehead graphs with respect to A.*
(6) *$n_A([g];x) = n_A([h];x)$ for every free basis A of F and for every $x \in A$.*

Proof. Let k be the rank of F. We may assume that $k \geq 2$ since for $k = 1$ the statement of the theorem is obvious.

The implications (1) ⇒ (2), (2) ⇒ (3), (3) ⇒ (4), (5) ⇒ (6) and (6) ⇒ (4) are obvious. Moreover, (3) implies (1) since by the results of [5] every very small action is the limit (in the sense of length functions) of free discrete actions. Thus (1), (2), (3) are equivalent. The implication (4) ⇒ (6) follows from Lemma 2.2 and the implication (4) ⇒ (5) is Proposition 2.3.

We now show that (5) implies (3). Indeed, suppose that F is acting freely, discretely and isometrically on an $\mathbb{R}$-tree X. Let $Y = X/F$ be the quotient graph of X. Since X is a metric tree, the edges e of Y come equipped with the lengths $l(e)$. Thus every edge-path in Y has a *length* which is the sum of the lengths of the edges of this path. There is an obvious canonical identification between F and $\pi_1(Y, y)$ where y is a vertex of Y. Choose an orientation $EY = E^+Y \sqcup E^-Y$ on Y. Then for every maximal tree in Y there is a canonically associated free basis of $\pi_1(Y, y)$.

Given a conjugacy class $[g]$ of $g \in F$, represent it by an immersed loop γ in Y. Then $\ell_X(g) = \sum_{e \in E^+Y} n(\gamma; e) l(e)$, where $n(\gamma; e)$ is the number of times that γ traverses e (in either direction). We prove that (5) implies (3) by showing that, given $e \in E^+Y$, the number $n(\gamma; e)$ is completely determined by the Whitehead graphs of $[g]$ with respect to free bases of F.

There are two cases.

First, suppose that e does not separate Y. Choose a maximal tree not containing e, and consider the associated free basis A of $F = \pi_1(Y, y)$. Then $n_A(\gamma; e) = n_A([g]; a_e)$, where $a_e \in A$ is the generator corresponding to e.

Suppose next that e separates Y, so that $Y = Y_1 \cup e \cup Y_2$, say with $y \in Y_1$. Choose a basis B of F associated to any maximal tree in Y. This basis is partitioned as

$B = B_1 \sqcup B_2$, with $b \in B_i$ if and only if it corresponds to an edge in Y_i. Then $n(\gamma; e) = \sum_{x \in B_1, y \in B_2} n_B([g]; x, y)$. $\square$

3. Palindromic sources of translation equivalence

Let F be a free group of rank $k \geq 2$ and let $A = \{a_1, \ldots, a_k\}$ be a free basis of F. Let $u = u(a_1, \ldots, a_k) \in F$ be a freely reduced word over A. We define the *palindromic reverse of u with respect to A*, denoted u^R, as

$$u^R := u(a_1^{-1}, \ldots, a_k^{-1})^{-1}.$$

Thus u^R is the word u read backwards without inverting the letters.

Similarly, we define the palindromic reverse w^R of a cyclic word w over A. Thus w^R is again a cyclic word. Namely, if w is represented by a cyclically reduced word u, then w^R is represented by the cyclically reduced word u^R.

Proposition 3.1. *Let $F = F(a, b)$ be free of rank two. Then for any cyclic word w in F over $\{a, b\}^{\pm 1}$ and for any $\phi \in Aut(F)$ we have*

$$(\phi(w))^R = \phi(w^R).$$

Proof. For any $\psi \in Aut(F)$ denote by $\overline{\psi}$ the image of ψ in $Out(F)$. For a free basis $A = (a, b)$ of F denote by τ_A the automorphism of F defined as $\tau_A(a) = a^{-1}, \tau_A(b) = b^{-1}$. It is easy to see that in $Out(F)$ the element $\overline{\tau_A}$ commutes with all the elementary Nielsen automorphisms with respect to A and hence $\overline{\phi_A}$ is central in $Out(F)$. Therefore for any other free basis B of F we have $\overline{\tau_A} = \overline{\tau_B}$.

Suppose now that $w(a, b)$ is a cyclic word in $F(a, b)$ and $\phi \in Aut(F)$. Let $u(a, b) = \phi(w)$. Then by the above observation $\phi(w(a^{-1}, b^{-1})) = u(a^{-1}, b^{-1})$. Since $w^R = (w(a^{-1}, b^{-1}))^{-1}$ and $u^R = (u(a^{-1}, b^{-1}))^{-1}$, this implies that $(\phi(w))^R = \phi(w^R)$. $\square$

Remark 3.2. It is well known that $Out(F_2) \cong GL(2, \mathbb{Z})$ and that the center of $GL(2, \mathbb{Z})$ is cyclic of order two. Thus, in fact, the outer automorphism $\overline{\tau_A}$ is the only nontrivial element of the center of $Out(F_2)$, although we did not need this fact in the above proof.

We can now prove Theorem C from the Introduction:

Theorem C. *Let F be a free group of rank $k \geq 2$ and let $w(x, y) \in F(x, y)$ be a freely reduced word. Then for any $g, h \in F$ we have*

$$w(g, h) \equiv_t w^R(g, h) \quad \textit{in } F.$$

Proof. If g and h commute in F, then the statement is obvious. Suppose now that g and h do not commute and hence $F_1 = \langle g, h \rangle$ is a free group of rank two. Let F act freely and discretely by isometries on an $\mathbb{R}$-tree X. Let X_1 be the minimal F_1-invariant subtree and let $Y = X_1/F_1$ be the quotient graph. Then topologically Y is either a wedge of two circles or is a θ-graph or Y consists of two disjoint circles joined by an edge. In each case Y possesses an involution isometry σ that leaves a maximal subtree T of Y invariant, fixes some point $y \in T$ and takes every edge e outside of a maximal tree to e^{-1}. This is shown in Figure 1.

Thus if c_1, c_2 is the basis of $\pi_1(Y, y)$ corresponding to T and $\sigma_\#$ denotes the isomorphism of $\pi_1(Y, y)$ induced by σ, then for any cyclic word $u(c_1, c_2)$ we have

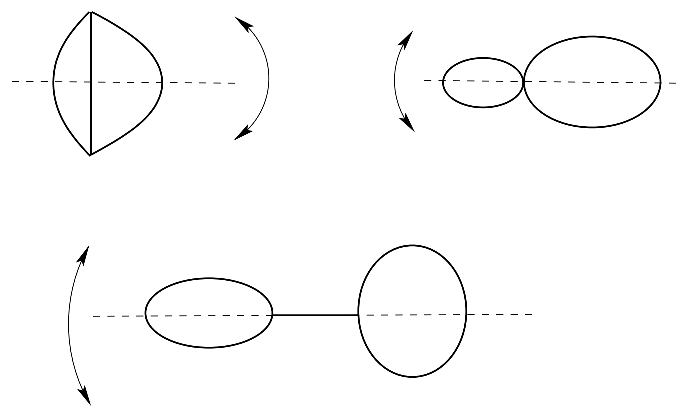

FIGURE 1. "Hyper-elliptic involution"

$\sigma_\# u(c_1, c_2) = u(c_1^{-1}, c_2^{-1})$. Since σ is an isometry of Y, this means that $\ell_X(u(c_1, c_2)) = \ell_X(u(c_1^{-1}, c_2^{-1}))$. On the other hand, $u(c_1^{-1}, c_2^{-1}) = (u^R(c_1, c_2))^{-1}$ and so

$$\ell_X(u(c_1, c_2)) = \ell_X(u^R(c_1, c_2)).$$

The pair (c_1, c_2) is a free basis of $F_1 = F(g, h)$. Write $w(g, h) = u(c_1, c_2)$ and $w^R(g, h) = u'(c_1, c_2)$. By Proposition 3.1 $u' = u^R$. Hence

$$\ell_X(w(g,h)) = \ell_X(u(c_1, c_2)) = \ell_X(u^R(c_1, c_2)) = \ell_X(w^R(g, h)),$$

as required. $\square$

4. Axis diagrams for free actions

Let G be a group acting by isometries on an $\mathbb{R}$-tree X. Recall that $g \in G$ is called *elliptic* if $\ell_X(g) = 0$ and g is called *hyperbolic* if $\ell_X(g) > 0$. Thus g is elliptic if and only if it fixes a point of X.

For a hyperbolic $g \in G$ put

$$L_g = \{x \in X : d(x, gx) = \ell_X(g)\}.$$

Then L_g is the smallest g-invariant subtree of X which is isometric to a line and on which g acts by a translation of magnitude $\ell_X(g)$. The set L_g is called the *axis* of g. In this section if an $\mathbb{R}$-tree X is fixed, we will omit the subscript and denote the translation length of an isometry g of X by $\ell(g)$.

The following simple proposition enumerating all the possibilities for the configuration of L_{gh} with respect to L_g, L_h for two hyperbolic isometries g and h is essentially a restatement of Proposition 1.6 of Paulin [19].

Proposition 4.1. *Let X be an $\mathbb{R}$-tree and let $g, h \in Isom(X)$ be two hyperbolic isometries of X. Then the following hold:*

(1) *Suppose that $|L_g \cap L_h| \leq 1$ and let $D = d(L_g, L_h)$. Then*

$$\ell(gh) = \ell(g) + \ell(h) + 2D.$$

(2) *Suppose that $L_g \cap L_h$ is a nondegenerate segment $[x, y]$.*

 (a) *If the translation directions of g and h on $[x, y]$ coincide, then*

$$\ell(gh) = \ell(g) + \ell(h).$$

(b) *If the translation directions of g and h on $[x,y]$ are opposite, then*

$$\ell(gh) = \begin{cases} \ell(g)+\ell(h)-2d(x,y) & \text{if } \ell(g)\geq d(x,y), \ell(h)\geq d(x,y),\\ |\ell(g)-\ell(h)| & \text{otherwise.}\end{cases}$$

(3) *If L_g and L_h are equal or intersect in a ray, then*

$$\ell(gh) = \begin{cases} \ell(g)+\ell(h) & \text{if } g \text{ and } h \text{ translate in the same direction,}\\ |\ell(g)-\ell(h)| & \text{otherwise.}\end{cases}$$

Remark 4.2. The case where $L_g \cap L_h$ consists of a single point is omitted in Proposition 1.6 of [19]. Only the cases where $L_g \cap L_h$ is empty or contains a nondegenerate segment are explicitly covered there. However, the proofs for the cases where $L_g \cap L_h$ is empty and where $L_g \cap L_h$ is a single point are completely analogous.

We need another simple fact (see Proposition 1.8 of Paulin [19])

Proposition 4.3. *Let g, h be two elliptic isometries of an $\mathbb{R}$-tree X. Then*

$$\ell_X(gh) = 2\min\{d(x,y)|gx = x, hy = y\}.$$

The following statement, together with the definition of translation equivalence, immediately implies Theorem D from the introduction.

Theorem 4.4. *Let G be a group acting isometrically with trivial arc stabilizers on an $\mathbb{R}$-tree X. Let $g, h \in G$ be nontrivial elements of G such that $g \neq h^{-1}$.*

(1) *Suppose that $\ell_X(g) = \ell_X(h) > 0$. Then for any positive integers p, q, i, j such that $p+q = i+j$ we have*

$$\ell_X(g^p h^q) = \ell_X(g^i h^j).$$

(2) *Suppose that $\ell_X(g) = \ell_X(h) = 0$.*

Then for any integers p, q, i, j such that g^p, h^q, g^i, h^j are nontrivial we have

$$\ell_X(g^p h^q) = \ell_X(g^i h^j).$$

Proof. Part (2) follows directly from Proposition 4.3. Indeed, since the action of G has trivial arc stabilizers, there are some $x, y \in X$ such that $Fix(g) = \{x\}$ and $Fix(h) = \{y\}$. Then for any p, q, i, j such that g^p, g^i, h^q, h^j are nontrivial, we have $Fix(g^p) = Fix(g^i) = \{x\}$ and $Fix(h^q) = Fix(h^j) = \{y\}$ and hence by Proposition 4.3, $\ell(g^p h^q) = d(x,y) = \ell(g^i h^j)$.

Suppose now that the assumptions of part (1) of Theorem 4.4 are satisfied.

Denote $a = \ell(g) = \ell(h)$. If $g = h$, then the statement is obvious. Suppose now that $g \neq h$, so that $g \neq h^{\pm 1}$. Let $p, q, i, j \geq 1$ be integers such that $p+q = i+j$.

Observe that $L_g = L_{g^n}$ and $L_h = L_{h^n}$ for any $n \geq 1$. Moreover, in this case $\ell(g^n) = \ell(h^n) = n\ell(h) = n\ell(g) = na$.

Suppose first that $L_g \cap L_h$ consists of at most one point. Put $D = d(L_g, L_h)$. Then by part (1) of Proposition 4.1,

$$\begin{aligned}\ell(g^p h^q) &= \ell(g^p) + \ell(h^q) + 2D = p\ell(g) + q\ell(h) + 2D\\ &= pa + qa + 2D = (p+q)a + 2D.\end{aligned}$$

Thus we see that $\ell(g^p h^q)$ depends only on $p+q$ and hence $\ell(g^p h^q) = \ell(g^i h^j)$, as required.

If the intersection of L_g and L_h contains a ray, then either gh or gh^{-1} fixes a segment of that ray. This is impossible since the action of G on X has trivial arc stabilizers.

Suppose now that $L_g \cap L_h = [x, y]$ and that $d(x, y) > 0$. If the translation directions of g and h on $[x, y]$ coincide, then by part (2a) of Proposition 4.1 we have

$$\ell(g^p h^q) = \ell(g^p) + \ell(h^q) = (p + q)a = (i + j)a = \ell(g^i h^j).$$

Assume now that g and h translate on $[x, y]$ in the opposite directions.

If $a < d(x, y)$, then gh fixes an arc contained in $[x, y]$, yielding a contradiction. Hence $\ell(g) = \ell(h) = a \geq d(x, y)$. Then by part (2b) of Proposition 4.1 we have

$$\begin{aligned}\ell(g^p h^q) &= \ell(g^p) + \ell(h^q) - 2d(x, y) \\ &= (p + q)a - 2d(x, y) = (i + j)a - 2d(x, y) = \ell(g^i h^j).\end{aligned}$$

□

The following lemma is an elementary exercise, but we provide a proof for completeness.

Lemma 4.5. *Let A be a finite abelian group and let $g \in A$ be an element of order bigger than four. Then for any $u \in A$ there is an integer i such that $g^i u$ has order bigger than four.*

Proof. Note that if $a = bc$ in A, then the order of any of these three elements divides the least common multiple of the orders of the other two.

If $u \in \langle g \rangle$, then $u = g^n$ for some n and the conclusion of the lemma holds with $i = -n - 1$. Suppose now that u is not a power of g and that for every i the order of $g^i u$ is at most four. Hence for each integer i the order of $g^i u$ is two, three or four. If i is an integer, then by the observation above the orders of $g^i u$ and $g^{i+1} u$ cannot be both even or both equal to three. Indeed, in that case the order of g would divide either three or four, contrary to the assumption that the order of g is bigger than four. Choose i such that the order of $g^i u$ is three. Then the order of $g^{i+2} u$ is also three and the orders of $g^{i+1} u, g^{i+3} u$ are both even. Hence the order of $g^2 = g^{i+2} u (g^i u)^{-1} = g^{i+3} u (g^{i+1} u)^{-1}$ divides both three and four, yielding a contradiction. □

The following is a generalization of Corollary 1.5 from the Introduction.

Corollary 4.6. *Let G be a group.*

(1) *Suppose that G is nonabelian and that the abelianization of G contains an element of infinite order. Then for every integer $M \geq 1$ there exist nontrivial elements $g_1, \dots, g_M \in G$ such that g_i is not conjugate to $g_j^{\pm 1}$ for $i \neq j$ and such that for every action of G on an $\mathbb{R}$-tree X with trivial arc stabilizers we have $\ell_X(g_i) = \ell_X(g_j)$.*

(2) *Suppose that G is nonabelian and that the abelianization of G contains an element of order bigger than four. Then there exist nontrivial elements $g_1, g_2 \in G$ such that g_1 is not conjugate to $g_2^{\pm 1}$ and such that for every action of G on an $\mathbb{R}$-tree X with trivial arc stabilizers we have $\ell_X(g_1) = \ell_X(g_2)$.*

Proof. Let $\overline{G}$ be the abelianization of G. For an element $x \in G$ we denote by $\overline{x}$ the image of x in $\overline{G}$.

(1) Suppose first that G is nonabelian and that $\overline{G}$ contains an element of infinite order. Then there exists a noncentral element g of G whose image has infinite

order in $\overline{G}$. Indeed, suppose not. Then all noncentral elements of G have finite order images in $\overline{G}$. Take $g_1 \in G$ such that $\overline{g_1}$ has infinite order in $\overline{G}$. Then by assumption g_1 is central in G. Since G is nonabelian, there exists a noncentral element $u \in G$. Again, by assumption, $\overline{u}$ has finite order. But then $g = g_1 u$ is noncentral and has infinite order in $\overline{G}$, yielding a contradiction.

Thus we can choose a noncentral element $g \in G$ such that $\overline{g}$ has infinite order. Then there is some $f \in G$ such that $h := f^{-1}g^{-1}f \neq g^{-1}$. Choose $M \geq 1$ and put $g_i := g^i h^{2M+1-i} = g^i f^{-1} g^{i-2M-1} f$ for $i = 1, \dots, M$. Then $\overline{g}_i = \overline{g}^{2i-2M-1} \neq 1$ for $1 \leq i \leq M$. Since $\overline{g}$ has infinite order in G, for $i \neq j$ we have $\overline{g_i} \neq \overline{g_j}^{\pm 1}$ and therefore g_i is not conjugate to $g_j^{\pm 1}$ in G. Also, since $\overline{g_i} \neq 1$, it follows that $g_i \neq 1$ for $1 \leq i \leq M$. Theorem 4.4 implies that for every action of G on an $\mathbb{R}$-tree X with trivial arc stabilizers we have $\ell_X(g_i) = \ell_X(g_j)$. This establishes part (1) of the corollary.

(2) Suppose now that G is nonabelian and that $\overline{G}$ has an element of order bigger than four. We may assume that $\overline{G}$ is torsion by part (1).

We claim that in this case there exists a noncentral element g of G whose image has order bigger than four in $\overline{G}$. Let $g_0 \in G$ be such that $\overline{g_0}$ has order bigger than four. If g_0 is noncentral, put $g = g_0$. Otherwise, choose a noncentral $u \in G$ and note that $\langle \overline{g_0}, \overline{u} \rangle$ is a finite abelian group. By Lemma 4.5 there is an integer i such that $\overline{g_0^i u}$ has order at least five and hence $g = g_0^i u$ is the desired noncentral element.

Thus let $g \in G$ be a noncentral element such that $\overline{g}$ has order bigger than four. Hence there exists $f \in F$ such that $h := f^{-1}g^{-1}f \neq g^{-1}$.

Put $g_1 = g^4 h = g^4 f^{-1} g^{-1} f$, $g_2 = g^3 h^2 = g^3 f^{-1} g^{-2} f$. Then $\overline{g_1} = \overline{g^3}$ and $\overline{g_2} = \overline{g}$. Since $\overline{g}$ has order bigger than four, the elements $\overline{g}, \overline{g}^{-1}, \overline{g^3}$ are nontrivial and pairwise distinct in $\overline{G}$. Hence $g_2 \neq 1$ and g_1 is not conjugate to $g_2^{\pm 1}$ in G. Theorem 4.4 again implies that for every action of G on an $\mathbb{R}$-tree X with trivial arc stabilizers we have $\ell_X(g_1) = \ell_X(g_2)$. □

5. Trace identities

The following statement is well known and probably goes back to the work of Klein in the late nineteenth century (see, for example, [9, 14] for a proof).

Lemma 5.1. *For any freely reduced word $w(a,b) \in F(a,b)$ there exists a polynomial $f_w \in \mathbb{Z}[x,y,z]$ such that for any field $\mathbb{K}$ and any matrices $A, B \in SL(2,\mathbb{K})$,*

$$tr\, w(A,B) = f_w(tr\, A, tr\, B, tr\, AB).$$

We now obtain Theorem B from the Introduction:

Theorem B. *Let F be a finitely generated free group and suppose that $u \equiv_c v$ in F. Then $u \equiv_t v$ in F.*

Proof. Choose an arbitrary free basis A of F. By Lemma 6.8 of Horowitz [9], established via a careful analysis of traces under an explicit family of representations in $SL(2,\mathbb{C})$, the assumption that $u \equiv_c v$ implies that $||u||_A = ||v||_A$. Since A was an arbitrary free basis of F, Theorem A implies that $u \equiv_t v$ in F. □

Lemma 6.8 of Horowitz [9] was strengthened by Southcott (see Theorem 6.6 of [18]) who proved that character equivalent elements have essentially the same "syllable structure" with respect to every free basis of F.

Corollary 5.2. *Let F be a finitely generated free group and let $u, v \in F$ be trace equivalent elements. Then for any word $w(x,y) \in F(x,y)$ we have $tr(w(u,v)) = tr(w(v,u))$ and hence, by Theorem* B*, $w(u,v)$ is translation equivalent to $w(v,u)$ in F.*

Proof. By Lemma 5.1 there is a polynomial $f(r,s,t) \in \mathbb{Z}[r,s,t]$ such that for every $A, B \in SL(2,\mathbb{C})$ we have $tr(w(A,B)) = f(tr(A), tr(B), tr(AB))$.

Let $\alpha : F \to SL(2,\mathbb{C})$ be an arbitrary representation. Put $A = \alpha(u)$ and $B = \alpha(v)$. Thus $tr(A) = tr(B)$ and, of course, $tr(AB) = tr(BA)$. Therefore $f(tr(A), tr(B), tr(AB)) = f(tr(B), tr(A), tr(BA)$ and hence $tr(w(A,B)) = tr(w(B,A))$. Since α was arbitrary, this implies that $tr(w(u,v)) = tr(w(v,u))$, as required. □

Proposition 5.3. *Let $w(x,y) \in F(x,y)$ be a freely reduced word, let $\mathbb{K}$ be any field and let $A, B \in GL(2,\mathbb{K})$ be arbitrary matrices. Then*

$$tr\, w(A,B) = tr\, w^R(A,B) \tag{†}$$

Proof. We first will prove (†) under the assumption that both $A, B \in SL(2,\mathbb{K})$.

Let $f_w(x,y,z)$ be the polynomial provided by Lemma 5.1. Let $A, B \in SL(2,\mathbb{K})$ be arbitrary matrices. Note that $tr(A) = tr(A^{-1})$, $tr(B) = tr(B^{-1})$, and $tr(AB) = tr(B^{-1}A^{-1}) = tr(A^{-1}B^{-1})$. Then by Lemma 5.1

$$tr(w(A,B)) = f_w(tr(A), tr(B), tr(AB))$$
$$= f_w(tr(A^{-1}), tr(B^{-1}), tr(A^{-1}B^{-1})) = tr(w(A^{-1}, B^{-1})) = tr(w^R(A,B)),$$

where the last equality holds because $w^R(a,b) = [w(a^{-1}, b^{-1})]^{-1}$.

Consider now the general case of $GL(2,\mathbb{K})$. Since every field embeds in an algebraically closed field, it suffices to prove (†) for an algebraically closed field. So let $\mathbb{K}$ be an algebraically closed field, let $w \in F(x,y)$ be a freely reduced word and let $X, Y \in GL(2,\mathbb{K})$ be arbitrary. Since $\mathbb{K}$ is algebraically closed, there exist nonzero $r, q \in \mathbb{K}$ such that $r^2 = \det(X)$ and $q^2 = \det(Y)$. Put $X_1 := X/r$ and $Y_1 := Y/q$. Then $\det(X_1) = \det(Y_1) = 1$, so that $X_1, Y_1 \in SL(2,\mathbb{K})$. Therefore, by the already established fact, we have

$$tr\, w(X_1, Y_1) = tr\, w^R(X_1, Y_1).$$

Let σ be the exponent sum on x in w (and hence in w^R) and let τ be the exponent sum on y in w (and hence in w^R). Then $w(X,Y) = w(rX_1, qY_1) = r^\sigma q^\tau w(X_1, Y_1)$ and, similarly $w^R(X,Y) = w^R(rX_1, qY_1) = r^\sigma q^\tau w^R(X_1, Y_1)$.

Therefore

$$tr\, w(X,Y) = r^\sigma q^\tau tr\, w(X_1,Y_1) = r^\sigma q^\tau tr\, w^R(X_1,Y_1) = tr\, w^R(X,Y),$$

as required. □

We refer the reader to [1] for a detailed discussion on the above proposition.

Remark 5.4. Theorem B and Proposition 5.3 immediately imply Theorem C established earlier by a direct argument.

Proposition 5.5. *For any field $\mathbb{K}$, for any integers p, q and for any conjugate matrices $A, B \in GL(2,\mathbb{K})$ we have*

$$tr\, A^p B^q = tr\, A^q B^p.$$

Proof. Again, we may assume that $\mathbb{K}$ is algebraically closed. For any matrices $A, B \in SL(2,\mathbb{K})$ conjugate in $GL(2,\mathbb{K})$ we have $tr\, A = tr\, B$, and hence $tr\, A^pB^q = tr\, B^pA^p = tr\, A^qB^p$ by Lemma 5.1 applied to $w(x,y) = x^py^q$.

Suppose now $A, B \in GL(2,\mathbb{K})$ are conjugate. Since $\mathbb{K}$ is algebraically closed, there exists $s \in \mathbb{K}$ such that $s^2 = \det(A) = \det(B)$. Then $A_1 = A/s$ and $B_1 = B/s$ have determinant 1 and are still conjugate in $GL(2,\mathbb{K})$. Therefore $tr\, A_1^pB_1^q = tr\, A_1^qB_1^p$. However, $A^pB^q = s^{p+q}A_1^pB_1^q$ and $A^qB^p = s^{p+q}A_1^qB_1^p$ which implies that $tr\, A^pB^q = tr\, A^qB^p$, as required. □

6. Tree actions in the boundary of the Teichmüller space

Let S be a closed surface of negative Euler characteristic, possibly nonorientable, and let $\mathcal{T}(S)$ be the Teichmüller space of S. We think of $\mathcal{T}(S)$ as the set of (isotopy classes of) marked hyperbolic structures on S or, equivalently, as the set of (conjugacy classes of) free discrete and cocompact isometric actions of $G = \pi_1(S)$ on $\mathbb{H}^2$. Let $\overline{\mathcal{T}(S)}$ be the Thurston compactification of $\mathcal{T}(S)$. The points of $\mu \in \overline{\mathcal{T}(S)} - \mathcal{T}(S)$ are measured laminations of S. Each such lamination μ defines a small action of G on an $\mathbb{R}$-tree dual X_μ to the lift of this lamination to $\mathbb{H}^2$ [15, 16]. We refer the reader to [2, 11] for a detailed discussion of this topic.

We next show that our results regarding two-variable palindromes also apply to the elements of $\overline{\mathcal{T}(S)}$.

Theorem 6.1. *Let S be a closed surface (possibly nonorientable) of negative Euler characteristic and let $G = \pi_1(S)$. Then for any $g, h \in G$ and for any $w(x,y) \in F(x,y)$ we have:*

(1) *For each point $p \in \mathcal{T}(S)$, thought of as an action of G on $\mathbb{H}^2$, the elements $w(g,h)$ and $w^R(g,h)$ have equal translation lengths.*
(2) *For each point $\mu \in \overline{\mathcal{T}(S)} - \mathcal{T}(S)$ we have*
$$\ell_{X_\mu}(w(g,h)) = \ell_{X_\mu}(w^R(g,h)).$$

Proof. Let $p \in \mathcal{T}(S)$ and consider the corresponding action $\phi : G \to Isom(\mathbb{H}^2)$ of G by isometries on $\mathbb{H}^2$. Recall that, when $\mathbb{H}^2$ is considered in the upper-half space model, there is a canonical isometric action of $GL(2,\mathbb{R})$ on $\mathbb{H}^2$ whose image is the full isometry group of $\mathbb{H}^2$. Namely, let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2,\mathbb{R})$. If $\det(A) > 0$, then $Az = \frac{az+b}{cz+d}$ for any $z \in \mathbb{H}^2$. If $\det(A) < 0$, then $Az = \frac{a\bar{z}+b}{c\bar{z}+d}$ for any $z \in \mathbb{H}^2$. The first case gives us an orientation-preserving isometry of $\mathbb{H}^2$ and the second case gives an orientation-reversing isometry. It is well known that for $A \in GL(2,\mathbb{R})$ the trace $tr(A)$ and the determinant $\det(A)$ uniquely determine the translation length of A as an isometry of the hyperbolic plane.

Let $w(x,y) \in F(x,y)$ and let $g, h \in G$. Let $A \in GL(2,\mathbb{R})$ represent $\phi(g)$ and let $B \in GL(2,\mathbb{R})$ represent $\phi(h)$. Then by Proposition 5.3, $tr\, w(A,B) = tr\, w^R(A,B)$. Moreover, $\det w(A,B) = \det w^R(A,B)$. Therefore we have
$$\ell_{\mathbb{H}^2}(w(g,h)) = \ell_{\mathbb{H}^2}(w^R(g,h)),$$
as claimed.

Thus part (1) of the theorem is established.

Now, part (1) immediately implies part (2). Indeed, recall that each element $p \in \mathcal{T}(S)$ determines a *marked length spectrum* $\ell_p : G \to \mathbb{R}$ where $\ell_p(g)$ is the translation length of g for the isometric action of G on $\mathbb{H}^2$ corresponding to p. Then it is well known that for any $\mu \in \overline{\mathcal{T}(S)} - \mathcal{T}(S)$ the length-function $\ell_{X_\mu} : G \to \mathbb{R}$ is projectively the limit of marked length spectra ℓ_{p_n} for some sequence of $p_n \in \mathcal{T}(S)$. That is, there exists a sequence of scalars $\lambda_n > 0$ such that for every $f \in G$,

$$\ell_{X_\mu}(f) = \lim_{n\to\infty} \lambda_n \ell_{p_n}(f).$$

Since for $f = w(g,h)$ and $f' = w^R(g,h)$ we know by (1) that $\ell_{p_n}(f) = \ell_{p_n}(f')$ for all n, it follows that $\ell_{X_\mu}(f) = \ell_{X_\mu}(f')$, as required. □

Together with Theorem 6.1 the following result immediately implies Theorem E from the Introduction.

Theorem 6.2. *Let S be a closed surface of negative Euler characteristic, possibly nonorientable, and let $G = \pi_1(S)$. Then for any conjugate $g, h \in G$ and for any integers p, q we have:*

(1) *For each point $p \in \mathcal{T}(S)$, thought of as an action of G on $\mathbb{H}^2$, the elements $g^p h^q$ and $g^q h^p$ have equal translation lengths.*
(2) *For each point $\mu \in \overline{\mathcal{T}(S)} - \mathcal{T}(S)$ we have*

$$\ell_{X_\mu}(g^p h^q) = \ell_{X_\mu}(g^q h^p).$$

(3) *If S is orientable, then for any conjugate elements $g, h \in G$, for each point $\mu \in \overline{\mathcal{T}(S)} - \mathcal{T}(S)$ and for any positive integers p, q, i, j such that $p+q = i+j$ we have*

$$\ell_{X_\mu}(g^p h^q) = \ell_{X_\mu}(g^i h^j).$$

Proof. Parts (1) and (2) are established exactly as Theorem 6.1, but using Proposition 5.5 instead of Proposition 5.3.

To see that part (3) holds observe that, by well-known results, if S is orientable, then every orbit of the action of the mapping class group of S on $\overline{\mathcal{T}(S)} - \mathcal{T}(S)$ is dense in $\overline{\mathcal{T}(S)} - \mathcal{T}(S)$. In particular, this applies to the orbit of a point corresponding to the stable foliation of a pseudo-Anosov homeomorphism of S. Therefore the set of those $\mu \in \overline{\mathcal{T}(S)} - \mathcal{T}(S)$, such that the action of G on the tree X_μ is free, is dense in $\overline{\mathcal{T}(S)} - \mathcal{T}(S)$. Since Theorem 4.4 applies to free actions, part (3) of Theorem 6.2 follows. □

Remark 6.3. The argument used in the proof of part (3) of Theorem 6.2 does not work in the case of nonorientable surfaces, as follows from the results of Danthony and Nogueira [8]. At the moment we do not know how to prove part (3) of Theorem 6.2 in the nonorientable case. The problem is that if S is nonorientable and $\mu \in \overline{\mathcal{T}(S)} - \mathcal{T}(S)$, then the action of $G = \pi_1(S)$ on X_μ need not be very small. Note, however, that G has an index two subgroup whose action on X_μ is very small. Note also that, as Example 7.5 below shows, part (1) of Theorem 4.4 does not have a precise analogue for SL_2 trace identities. Hence it is not possible to argue as in the proof of Theorem 6.1 to establish part (3) of Theorem 6.2 for nonorientable surfaces.

7. Examples

Example 7.1. The palindrome and the $g^p h^q$ phenomena described above no longer hold for the class of small actions (as opposed to free or very small actions).

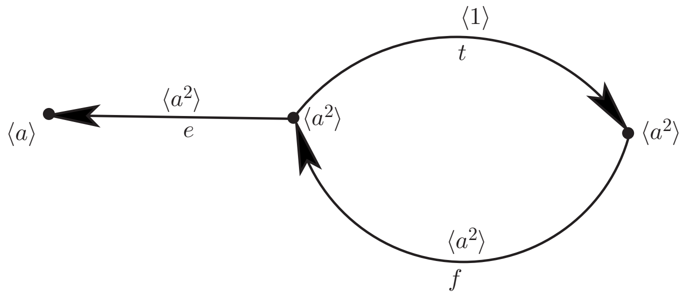

FIGURE 2. Counterexample

Consider the graph of groups $\mathbb{A}$ shown in Figure 2. Let v be the vertex of the graph incident to all three marked up edges e, f, t. Put $G = \pi_1(\mathbb{A}, v)$ and let $X = \widetilde{(\mathbb{A}, v)}$ be the Bass-Serre universal covering tree. It is easy to see that the action of G on X is minimal and that $G = F(x, y)$ is a free group of rank two with free basis $x = eae, y = tf$. Consider now the elements $g = xyx^2y^{-1}$ and $h = y^{-1}x^2yx$. Thus h is the palindromic reverse of g in $F(x, y)$. Moreover, with $a = x$ and $b = yxy^{-1}$ we see that $g = ab^2$ and h is conjugate to a^2b.

In G we have

$$g = xyx^2y^{-1} = eae^{-1}tfea^2e^{-1}f^{-1}t^{-1} = eae^{-1}tfa^2f^{-1}t^{-1} = eaeta^2t^{-1},$$
$$h = y^{-1}x^2yx = f^{-1}t^{-1}ea^2e^{-1}tfeae^{-1} = f^{-1}t^{-1}a^2tfeae^{-1}.$$

Both $eaeta^2t^{-1}$ and $f^{-1}t^{-1}a^2tfeae^{-1}$ are cyclically reduced closed paths in $\mathbb{A}$. Therefore $\ell_X(g) = 4$ while $\ell_X(h) = 6$.

Remark 7.2. We believe that the palindrome phenomenon also holds for small actions of a free group F such that, if $g \in F$ is nontrivial and fixes more than one point, then $Fix(g)$ is a compact segment containing no branch point in its interior.

Example 7.3. We have remarked earlier that translation equivalence is preserved by injective homomorphisms, and hence by passing to larger free groups containing the given free group as a subgroup. However, passing to subgroups, even of finite index, no longer preserves translation equivalence in general.

Let $F = F(a, b)$ and let $H = \langle a, bab^{-1}, b^2\rangle \le F$. It is not hard to see that H is a subgroup of index two in F and that $x = a, y = bab^{-1}, z = b^2$ is a free basis of H, so that $H = F(x, y, z)$. Consider the elements $g = aba^2b^{-1}$ and $h = a^2bab^{-1}$ in F. Then $g \equiv_t h$ in F by, for example, Theorem 4.4. We also have $g, h \in H$ and $g = xy^2, h = x^2y$. Clearly, g and h have different Whitehead graphs in $F(x, y, z)$ and therefore $g \not\equiv_t h$ in H.

On the other hand, it is easy to see that if H is a free factor of F and $g, h \in H$, then $g \equiv_t h$ in F if and only if $g \equiv_t h$ in H.

Example 7.4. Theorem A states, in particular, that two elements of F are translation equivalent in F if and only if they have equal Whitehead graphs with respect to each free basis of F. This statement no longer holds if we consider the obvious "higher rank" analogues of Whitehead graphs where symmetrized numbers of occurrences of subwords of length $m > 2$ are recorded. Again consider $g = aba^2b^{-1}$ and $h = a^2bab^{-1}$ in $F(a, b)$. We already know that $g \equiv_t h$ in F. However, consider the cyclic words w and u defined by g and h accordingly. The number of occurrences of $b^{-1}ab$ in w is equal to 1, while neither $b^{-1}ab$ nor its inverse $b^{-1}a^{-1}b$ occur in u.

Example 7.5. Theorem B shows that in free groups character equivalence implies translation equivalence. However, the converse implication does not hold and these two phenomena are different. For example, we know from Theorem D that if g and h are conjugate in F and $g \neq h^{-1}$, then $g^3h \equiv_t g^2h^2$ in F. On the other hand, let

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 1/2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}, \quad C = BAB^{-1} = \begin{bmatrix} 0 & 1 \\ -1 & 5/2 \end{bmatrix}$$

be matrices in $SL(2,\mathbb{R})$. A direct computation shows that $tr\, A^3C = -79/16$ while $tr\, A^2C^2 = -143/16$.

Thus $a^3bab^{-1} \not\equiv_c a^2ba^2b^{-1}$ while $a^3bab^{-1} \equiv_t a^2ba^2b^{-1}$ in $F(a,b)$ and there is no precise analogue of Theorem D for character equivalence.

8. Open problems

Problem 8.1. Is there an algorithm which, when given two elements in a finitely generated free group, decides whether or not they are translation equivalent?

It can be deduced from results of Leininger [12] that hyperbolic equivalence in surface groups is algorithmically decidable by using standard commutative algebra techniques applied to the representation variety of the surface group. Similarly, one can algorithmically decide whether or not two elements of a free group are character equivalent.

Problem 8.2. Find other sources of translation equivalence in free groups, different from those discussed in this paper.

Problem 8.3. Is it true that whenever $g \equiv_t h$ in F and $w(x,y) \in F(x,y)$ is arbitrary, then $w(g,h) \equiv_t w(h,g)$ in F? It easily follows from Lemma 5.1 that $g \equiv_c h$ (e.g., g is conjugate to h) implies $w(g,h) \equiv_c w(h,g)$ in F and hence $w(g,h) \equiv_t w(h,g)$ in F.

The notion of translation equivalence has several natural generalizations.

Problem 8.4 (Bounded translation equivalence)**.** For nontrivial $g,h \in F$ we say that g is *boundedly translation equivalent* to h, denoted $g \equiv_b h$, if there is $C > 0$ such that for every free and discrete action of F on an $\mathbb{R}$-tree X we have

$$\frac{1}{C} \leq \frac{\ell_X(g)}{\ell_X(h)} \leq C.$$

What are the sources of bounded translation equivalence in free groups and how much more general is it compared to translation equivalence? What are the sources of the failure of bounded translation equivalence? Is bounded translation equivalence, in some natural sense, generic? Is bounded translation equivalence algorithmically decidable?

A similar notion can be defined in the context of hyperbolic metrics on closed surfaces and the above questions make sense there as well.

Problem 8.5 (Volume equivalence of subgroups)**.** If G is a finitely generated group acting discretely isometrically and without a global fixed point on an $\mathbb{R}$-tree X, we denote by $vol_X(G)$ the sum of the lengths of the edges of the metric graph X_G/G, where X_G is the unique minimal G-invariant subtree.

We will say that nontrivial finitely generated subgroups H, K of F are *volume equivalent* in F, denoted $H \equiv_v K$, if for every free and discrete action of F on an $\mathbb{R}$-tree X we have $vol_X(H) = vol_X(K)$.

Thus $g \equiv_t h$ in F iff $\langle g\rangle \equiv_v \langle h\rangle$ in F. Moreover, if H, K have the same finite index n in F, then it is easy to see that $H \equiv_v K$ in F.

Are there any other sources of volume equivalence in free groups? If $H \equiv_v K$ in F, does this imply that H and K are free groups of the same rank? Is volume equivalence algorithmically decidable?

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