LATTICE-ORDERED ABELIAN GROUPS
AND SCHAUDER BASES OF UNIMODULAR FANS

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ABSTRACT. Baker-Beynon duality theory yields a concrete representation of any finitely generated projective Abelian lattice-ordered group $G$ in terms of piecewise linear homogeneous functions with integer coefficients, defined over the support $|\Sigma|$ of a fan $\Sigma$. A unimodular fan $\Delta$ over $|\Sigma|$ determines a Schauder basis of $G$: its elements are the minimal positive free generators of the pointwise ordered group of $\Delta$-linear support functions. Conversely, a Schauder basis $H$ of $G$ determines a unimodular fan over $|\Sigma|$; its maximal cones are the domains of linearity of the elements of $H$. The main purpose of this paper is to give various representation-free characterisations of Schauder bases. The latter, jointly with the De Concini-Procesi starring technique, will be used to give novel characterisations of finitely generated projective Abelian lattice ordered groups. For instance, $G$ is finitely generated projective iff it can be presented by a purely lattice-theoretical word.

1. Background: $\ell$-groups and fans

We assume familiarity with lattice-ordered Abelian groups (for short, $\ell$-groups [4, 7]) and fans. By a fan we shall always understand a finite rational polyhedral fan, as defined in [6, 15].

Throughout the paper, $\mathbb{N} = \{1, 2, \ldots\}$, $\mathbb{Z}$ is the set of integers, and $\mathbb{R}$ is the set of reals. If $G$ is an $\ell$-group, we let $G^+ = \{g \in G \mid g \geq 0\}$. By $\ell$-homomorphisms we mean homomorphisms of $\ell$-groups; the symbol $\cong_\ell$ denotes $\ell$-isomorphism. Kernels of $\ell$-homomorphisms are precisely $\ell$-ideals, always denoted by Gothic letters $a, m, p, \ldots$. An $\ell$-ideal is principal iff it is finitely generated (which for $\ell$-groups is equivalent to being singly generated). Maximal $\ell$-ideals are defined in the obvious manner. A finitely generated $\ell$-group $G$ is Archimedean iff it has no "infinitesimal elements": thus, whenever $0 < x \leq y$ holds, there is $n \in \mathbb{N}$ such that $nx \not\leq y$; equivalently, the intersection of all maximal $\ell$-ideals in $G$ is $\{0\}$; see [7, 4.1] and [4] 10.2 for details. An $\ell$-ideal $m$ is maximal in $G$ iff $G/m$ is Archimedean totally ordered. As explained in [4] 13.2.6, Archimedean $\ell$-groups with a strong order unit (that is, an element $u \in G$ such that for every $x \in G$ there is $n \in \mathbb{N}$ with $nu \geq x$) are precisely those representable as $\ell$-groups of real-valued continuous functions over some compact Hausdorff space (operations being defined pointwise); finitely generated $\ell$-groups may always be endowed with a strong order unit.
Suppose $G$ is a finitely generated $\ell$-group. Equipped with the spectral topology, the set $\text{MaxSpec}(G)$ of maximal $\ell$-ideals of $G$ is a (nonempty) compact Hausdorff space. The closed sets in $\text{MaxSpec}(G)$ are given by the zero sets $\mathfrak{Z}(S)$ of arbitrary subsets $S$ of $G$, where $\mathfrak{Z}(S) = \bigcap_{g \in S} \{ m \in \text{MaxSpec}(G) \mid g \in m \}$.

We let $|\Sigma| \subseteq \mathbb{R}^n$ denote the (closed) support of the fan $\Sigma$. A continuous function $f: |\Sigma| \to \mathbb{R}$ is said to be an (integral) $\ell$-function over $|\Sigma|$ iff it is (always finitely) piecewise linear homogeneous, each piece having the form $c_1 x_1 + \cdots + c_n x_n$, for suitable integers $c_1, \ldots, c_n$.

By $A_n$ we denote the $\ell$-group of all $\ell$-functions over $\mathbb{R}^n$, with pointwise operations. As is well known, $A_n$ is free in the equational class of $\ell$-groups. The (coordinate) projection functions $\pi_i: \mathbb{R}^n \to \mathbb{R}$ are free generators of $A_n$. For $\Sigma$ a fan in $\mathbb{R}^n$, we let

$$A_n|_{|\Sigma|}$$

denote the $\ell$-group of all $\ell$-functions over $|\Sigma|$. This notation is in agreement with the fact that $\ell$-functions over $|\Sigma|$ are the same as restrictions to $|\Sigma|$ of $\ell$-functions over $\mathbb{R}^n$.

Elements of fans are called cones; thus, a cone is always a rational polyhedral cone. A fan $\Theta$ is simplicial iff every one of its cones is simplicial (i.e., taking an affine section in general position yields a simplex). A simplicial cone $\sigma \in \Theta$ is unimodular iff its 1-dimensional faces are spanned (over the positive reals) by primitive vectors that are columns of some unimodular matrix (an integral square matrix with determinant $\pm 1$). A fan is unimodular iff all of its cones are unimodular.$^3$

Given two fans $\Sigma$ and $\Delta$, if $|\Delta| = |\Sigma|$ and all the cones of $\Sigma$ are unions of cones of $\Delta$, then we say that $\Delta$ refines $\Sigma$.

For $\Delta$ a unimodular fan in $\mathbb{R}^n$, let $\sigma \in \Delta$ be a 1-dimensional cone. Following [12, 13] (also see [17]), the Schauder hat of $\Delta$ at $\sigma$ is the unique $\ell$-function $h_{\sigma}$ that is linear over each cone of $\Delta$, has value 1 at the primitive vector along $\sigma$, and value 0 at every point of every other 1-dimensional cone of $\Delta$. The Schauder basis over $\Delta$, denoted Hats$_{\Delta}$, is the collection of all Schauder hats of $\Delta$. Integrality of the linear pieces is guaranteed by unimodularity of the fan.

Every Schauder hat of $\Delta$ is a particular case of a $\Delta$-linear support function [13, page 66]. Specifically, the Schauder basis over $\Delta$ is a free generating set in the free group $G$ of $\Delta$-linear support functions. If, in addition, $F$ is equipped with the pointwise order, then the Schauder hats over $\Delta$ are just the minimal positive free generators; see [13].

2. Schauder bases and De Concini-Procesi refinements

A nontrivial exercise shows that there are $\ell$-automorphisms of, say, $A_3$, carrying a Schauder basis to a set of $\ell$-functions which is not a Schauder basis. Thus, the property of being a Schauder basis is not invariant under $\ell$-automorphisms.

A useful representation-independent generalisation is the following.

**Definition 2.1.** Let $G$ be an $\ell$-group and $B = \{b_1, \ldots, b_t\} \subseteq G$, where $t \in \mathbb{N}$. We say that $B$ is an abstract Schauder basis of $G$ iff for some $n \in \mathbb{N}$ there exist an unimodular fan $\Delta$ in $\mathbb{R}^n$, and an $\ell$-isomorphism $\phi: A_n|_{|\Delta|} \cong \ell G$, such that,

1. All fans are denoted by capital Greek letters.
2. An integral vector is primitive iff the greatest common divisor of its coordinates is 1.
3. Unimodular fans are also known as nonsingular fans, as they correspond precisely to smooth toric varieties under the fan-toric dictionary. In [6], they are called regular fans.
letting $\text{Hats}_\Delta = \{h_1, \ldots, h_t\}$ be the Schauder basis over $\Delta$, we have $\phi(h_i) = b_i$ for every $i \in \{1, \ldots, t\}$.

When we wish to emphasise that $B$ is a Schauder basis, we sometimes write that it is a \textit{concrete} Schauder basis. Trivially, any concrete Schauder basis is an abstract Schauder basis.

\textbf{Proposition 2.2.} If the $\ell$-group $G$ has an abstract Schauder basis $B$, then

\begin{enumerate}
\item $B \subseteq G^+ \setminus \{0\}$,
\item $B$ is linearly independent in the $\mathbb{Z}$-module $G$, and
\item $G$ is Archimedean.
\end{enumerate}

\textit{Proof.} In view of the existence of the above $\ell$-isomorphism $\phi: A_n \to |\Delta| \cong G$, it is enough to prove the proposition for $A_n \to |\Delta|$ equipped with the concrete Schauder basis $\text{Hats}_\Delta = \{h_1, \ldots, h_t\}$ corresponding to $B$ via $\phi$. Then (1) and (2) are immediate. Property (3) follows upon observing that $A_n \to |\Delta|$ is an $\ell$-group of real-valued functions. \hfill $\square$

In Proposition 2.5 below we shall prove that $B$ generates $G$. The proof requires some background material on stellar subdivisions \cite{DeConcini, Procesi}. Barycentric stellar subdivisions (\textit{starrings} for short) can be used to subdivide a unimodular fan, preserving unimodularity. If $\Delta$ is obtained from $\Sigma$ via a finite number (possibly zero) of starings, we write $\Delta \preceq \Sigma$; and if all such subdivisions are along 2-dimensional cones only (\textit{binary} starings for short), we write $\Delta \preceq_2 \Sigma$.

\textbf{Lemma 2.3} (The De Concini-Procesi Lemma). \textit{Let $\Sigma_1$ and $\Sigma_2$ be unimodular fans with the same support. There exists a unimodular fan $\Delta$ such that $\Delta \preceq_2 \Sigma_1$ and $\Delta$ refines $\Sigma_2$.}

\textit{Proof.} See \cite{DeConcini}; for a short elementary proof, see \cite{Procesi} or \cite{Sturmfels}. \hfill $\square$

For our purposes, it is crucial to realise that starings of unimodular fans are definable in the language of $\ell$-groups. Although this is really a remark, we record it for easier reference.

\textbf{Lemma 2.4.} \textit{Let $\Sigma$ be a unimodular fan and $H = \text{Hats}_\Sigma$ its associated Schauder basis. Let $h_1, h_2 \in H$ be hats at the 1-cones $\sigma_1$ and $\sigma_2$, respectively. Suppose $\sigma_1$ and $\sigma_2$ span a 2-cone $\tau$ of $\Sigma$ (equivalently, $h_1 \wedge h_2 \neq 0$). Let $\Delta \preceq_2 \Sigma$ be the unimodular fan obtained from $\Sigma$ by starring along $\tau$. Then $K = \text{Hats}_\Delta$ is the Schauder basis obtained from $H$ by removing $h_1$ and $h_2$, and adjoining the three hats $h_1 - (h_1 \wedge h_2), h_2 - (h_1 \wedge h_2), h_1 \wedge h_2$.}

\textit{Conversely, if $K$ is obtained from $H$ by such substitutions, then there is a unique unimodular fan $\Delta$, obtained from $\Sigma$ by starring along $\tau$, such that $K = \text{Hats}_\Delta$.}

\textit{Proof.} A straightforward computation. \hfill $\square$

\textbf{Proposition 2.5.} \textit{If an $\ell$-group $G$ has an abstract Schauder basis $B$, then $B$ generates $G$.}

\footnote{In Proposition 2.5 below we shall prove that if $G$ has a Schauder basis, then $G$ is finitely generated.}

\footnote{Starings correspond to (equivariant) blow-ups of toric varieties via the fan-toric dictionary \cite{DeConcini, Sturmfels}.}
Proof: Using the $\ell$-isomorphism $\phi : A_n \upharpoonright |\Delta| \cong_\ell G$, without loss of generality we can assume $G = A_n \upharpoonright |\Delta|$ equipped with the concrete Schauder basis $B = \text{Hats}_\Delta = \{b_1, \ldots, b_t\}$. By [6, Theorem 8.5], for every $\ell$-function $f$ over $|\Delta|$, there exists a unimodular refinement $\Delta_f$ of $\Delta$ such that $f$ is linear over every cone of $\Delta_f$. By Lemma 2.3, an appropriate sequence of binary starrings transforms $\Delta$ into a refinement $\Delta^*$ of $\Delta_f$. By Lemma 2.4, all hats of $\Delta^*$ are in the $\ell$-group $H$ of real-valued functions over $|\Delta|$ generated by $B$. Since, by the construction of $\Delta^*$, $f$ must take an integer value at each primitive generating vector of $\Delta^*$, then $f$ is a linear combination of the hats of $\Delta^*$ with integer coefficients. This shows that $H = A_n \upharpoonright |\Delta|$.

The concept of concrete Schauder bases only makes sense for $\ell$-groups of the form $G = A_n \upharpoonright |\Delta|$, where $|\Delta|$ is a unimodular fan. The main question we shall address is whether one can characterise abstract Schauder bases without reference to their geometric realisations.

3. Topological characterisation of abstract Schauder bases

Let $G$ be an $\ell$-group, and $D$ be a finite subset of $G$. Then by an abstract $k$-simplex of $D$ we mean a set $Y \subseteq D$ of cardinality $k + 1$ such that $\bigwedge Y \neq 0$.

Theorem 3.1. Let $G$ be an $\ell$-group, and let $B = \{b_1, \ldots, b_t\} \subseteq G^+ \setminus \{0\}$ be a finite subset. Then the following conditions are equivalent:

1. $B$ is an abstract Schauder basis of $G$.
2. $G$ is Archimedean, $B$ generates $G$, and for every abstract $k$-simplex $Y$ of $B$, the zero set $\mathfrak{z}(B \setminus Y)$ is homeomorphic to the closed $k$-dimensional ball.

Proof. (2 $\to$ 1) Let the $\ell$-homomorphism $\eta : A_t \to G$ be the unique extension of the map $\pi_i \mapsto b_i$ ($i = 1, \ldots, t$). Then $\eta$ is surjective, because $B$ generates $G$. Since $G$ is Archimedean, $G$ is $\ell$-isomorphic to the $\ell$-group $A_t \upharpoonright W$ of restrictions of the $\ell$-functions of $A_t$ to a suitable homogeneous closed subset $W$ of $\mathbb{R}^t$. We shall show that $W$ is the support of a unimodular fan $\Theta$, and $B$ is the $\eta$-image of Hats$\Theta$.

The assumption that each $b_i$ is positive implies that $W$ is contained in the positive orthant $O$ of $\mathbb{R}^t$. Let $L$ be the affine hyperplane determined by the unit basic vectors $e_i$,

$$L = \left\{(\xi_1, \ldots, \xi_t) \in \mathbb{R}^t \mid \sum \xi_i = 1\right\}. $$

Let $W' = W \cap L$. Via the canonical $\ell$-isomorphism of $A_t \upharpoonright W$ onto $A_t \upharpoonright W'$, we shall identify $G$ with the $\ell$-group of restrictions to $W'$ of the functions of $A_t$; in symbols,

$$(2) \quad G = A_t \upharpoonright W'. $$

Under this identification, $\eta$ is just the restriction map, and every element $b_i \in B$ coincides with the restriction to $W'$ of the $i$th coordinate function $\pi_i$ of $A_t$. Since the rays of $W$ are in one-one correspondence with the maximal $\ell$-ideals of $G$, there also is a natural one-one correspondence

$m \in \text{MaxSpec} (G) \mapsto z_m \in W'$

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6 The techniques for the proof of Theorem 3.1 were developed, in a different but related context, in [13]. For partitions of unity, a similar result is stated in [8].

7 Meaning that whenever $W$ contains a point $z$, then $W$ also contains the ray through 0 and $z$.  

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between maximal $\ell$-ideals $m$ and points in $W'$. This correspondence is also a homeomorphism of $\text{MaxSpec}(G)$ (equipped with the spectral topology; see [4, Chapter 13]) onto $W'$ (with the natural topology of $\mathbb{R}^\ell$). Accordingly, we shall identify $\text{MaxSpec}(G)$ and $W'$ as topological spaces. The quotient map $f \in G \mapsto f/m \in G/m$ coincides with the evaluation map of every $f \in G$ at the point $z_m = (\xi_1, \ldots, \xi_\ell)$ corresponding to the maximal $\ell$-ideal $m$; in symbols,

$$f/m = f(z_m), \text{ for all } m \in \text{MaxSpec}(G).$$

Claim 1. Let $Y = \{b_1, \ldots, b_k\}$ be an abstract simplex of $B$. Then the zero set $\mathfrak{z}(B \setminus Y)$ coincides with the set $\{z \in W' \mid b_1(z) + \cdots + b_k(z) = 1\}$.

As a matter of fact, by (3), $\mathfrak{z}(B \setminus Y)$ is the set of points of $W'$ where all functions of $B \setminus Y$ vanish. On the other hand, the sum of all functions $b_j$ of $B$ (equivalently, the sum of the corresponding $\pi_j$) constantly equals 1 over $W'$.

Claim 2. For each $i = 1, \ldots, t$, the singleton $\{b_i\}$ forms an abstract 0-simplex of $B$, and the zero set $\mathfrak{z}(B \setminus \{b_i\})$ is the (singleton) standard basis vector $e_i$. Thus $e_i \in W'$.

By assumption $\{b_i\}$ is nonzero. By definition it forms an abstract 0-simplex of $B$.

By our topological assumption, $\mathfrak{z}(B \setminus \{b_i\})$ is a singleton in the compact Hausdorff space $\text{MaxSpec}(G) = W'$. The only possible point $z$ in $\mathcal{O} \cap L$ where $b_i(z) = 1$ is the basis vector $e_i$. Thus, by Claim 1, $e_i \in W'$.

Claim 3. For every abstract $(k - 1)$-simplex $Y = \{b_1, \ldots, b_k\}$ of $B$, the zero set $\mathfrak{z}(B \setminus Y)$ is the convex hull $[e_{i_1}, \ldots, e_{i_k}]$ of the basis vectors $e_{i_1}, \ldots, e_{i_k}$. Thus, in particular, $[e_{i_1}, \ldots, e_{i_k}] \subseteq W'$.

The proof is by induction on $k$.

\textbf{Basis.} Suppose $\{b_i, b_j\}$ forms an abstract 1-simplex $Y$ of $B$. By Claim 1, $\mathfrak{z}(B \setminus Y)$ coincides with the set $V = \{z \in W' \mid b_i(z) + b_j(z) = 1\}$. By definition of $W'$, together with (3), $V$ is a subset of the closed segment $[e_i, e_j]$. By Claim 2, both $e_i$ and $e_j$ belong to $V$, because both $b_i$ and $b_j$ form an abstract 0-simplex of $B$. If $V$ were a proper subset of $[e_i, e_j]$, then it would not be connected, against our topological assumption. Thus, $\mathfrak{z}(B \setminus Y) = [e_i, e_j]$.

\textbf{Induction step.} Let $D = \{b_1, \ldots, b_{k+1}\}$ be an abstract $k$-simplex of $B$. A fortiori, every subset $D' = \{b_{j_1}, \ldots, b_{j_k}\}$ of $D$ is an abstract $(k - 1)$-simplex of $B$. By induction hypothesis, the zero set $\mathfrak{z}(B \setminus D')$ coincides with the (topological) $(k - 1)$-simplex $[e_{j_1}, \ldots, e_{j_k}]$. Thus, by (3), $\mathfrak{z}(B \setminus D)$ is a certain subset $V \subseteq [e_{i_1}, \ldots, e_{i_{k+1}}]$, containing the union of $S^{k-1}$ all $(k - 1)$-dimensional faces of $[e_{i_1}, \ldots, e_{i_{k+1}}]$. Suppose $V$ were a proper subset of $[e_{i_1}, \ldots, e_{i_{k+1}}]$ (absurdum hypothesis). Write

$$V = [e_{i_1}, \ldots, e_{i_{k+1}}] \setminus U$$

for a suitable nonempty subset $U$ of the relative interior of $[e_{i_1}, \ldots, e_{i_{k+1}}]$. As in [13], the intuitively obvious fact that $V$ cannot be homeomorphic to the $k$-ball is confirmed by the verification that the singular homology groups $\tilde{H}$ of $V$ and of $[e_{i_1}, \ldots, e_{i_{k+1}}]$ are not isomorphic.

\footnote{One passes from $m$ to $z_m$ by taking the intersection of all zero sets of functions in $m$. Conversely, one passes from any point $z \in W'$ to its corresponding maximal $\ell$-ideal $m$ by taking all functions of $G$ that vanish at $z$.}
On the one hand, \( H_{k-1}(\{e_1, \ldots, e_{k+1}\}) \) is zero, because \( \{e_1, \ldots, e_{k+1}\} \) is contractible. On the other hand, letting \( u \) be any point in set \( U \), the inclusion maps
\[
S^{k-1} \to V \to \mathbb{R}^n \setminus \{u\}
\]
induce homomorphisms of homology groups
\[
Z = H_{k-1}(S^{k-1}) \to H_{k-1}(V) \to H_{k-1}(\mathbb{R}^n \setminus \{u\}) = Z.
\]
Moreover, the composition of these two homomorphisms is the homomorphism induced by the inclusion
\[
S^{k-1} \to \mathbb{R}^n \setminus \{u\}.
\]
Because the latter map is a homotopy equivalence, it induces an isomorphism on homology groups. We have shown that \( H_{k-1}(V) \) contains the group of integers as a summand. This contradicts our topological assumption, and also settles our claim.

To conclude the proof of \((2 \to 1)\), for each abstract \((k-1)\)-simplex \( Y \) of \( B \) let \( \sigma_Y \) be an abbreviation of the zero set \( \mathfrak{Z}(B \setminus Y) \). By Claim 3, \( \sigma_Y \) is a \((k-1)\)-dimensional simplex. For any two abstract simplices \( Y', Y'' \) of \( B \), their corresponding simplices \( \sigma_{Y'}, \sigma_{Y''} \) intersect in a common face, namely the simplex \( \sigma_{Y' \cap Y''} \). Here it is convenient to write \( \sigma_0 = \emptyset \).

The abstract simplices of \( B \) then determine a (concrete, topological) simplicial complex \( S \) in \( \mathbb{R}^t \), and \( W' \) is the support of \( S \).\( ^9 \) The \( t \) vertices of \( S \) are the standard basis vectors \( e_i \). The Schauder hats of \( \Theta \) are precisely the restrictions \( \tau_i \upharpoonright W \), and hence they correspond to the elements of \( B \), via our standing \( \ell \)-isomorphism \( b_i \cong \tau_i \upharpoonright W \).

This completes the proof of \((2 \to 1)\).

\((1 \to 2)\) For some \( n \in \mathbb{N} \) and unimodular fan \( \Delta \) there is an \( \ell \)-isomorphism
\[
\phi: A_n \upharpoonright |\Delta| \cong_{\ell} G
\]
that geometrically realises \( B \) by the Schauder basis over \( \Delta \). By Propositions \( 2.2 \) and \( 2.5 \), \( G \) is Archimedean and \( B \) generates \( G \). Without loss of generality we may assume \( |\Delta| \) to lie entirely in the positive orthant of \( \mathbb{R}^n \).\(^{10}\) Let \( L \) be the hyperplane
\[
\{(\xi_1, \ldots, \xi_n) \in \mathbb{R}^n \mid \xi_1 + \cdots + \xi_n = 1\}.
\]
Let \( L_\Delta = |\Delta| \cap L \) equipped with the natural topology. Then \( \text{MaxSpec}(A_n \upharpoonright |\Delta|) \) is canonically homeomorphic to \( L_\Delta \); let us identify these topological spaces. Since \( \Delta \) is a complex of simplicial cones, its affine section \( L_\Delta \) is a simplicial complex. For every abstract simplex \( Y = \{f_1, \ldots, f_k\} \) of \( B \), let \( v_i \in \mathbb{R}^n \) be the point of \( L_\Delta \) where the Schauder hat \( f_i \) attains its maximum value. Then direct inspection shows that the zero set \( T = \mathfrak{Z}(B \setminus Y) \) is the convex hull in \( \mathbb{R}^n \) of the points \( v_1, \ldots, v_k \).

Therefore, \( T \) is homeomorphic to the \((k-1)\)-dimensional ball.

The proof is complete. \( \square \)

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\(^9\)That is, \( W' \) is the set-theoretical union of all simplices of \( S \).

\(^{10}\)Indeed, we can further assume that the primitive generating vectors of the rays of \( \Delta \) coincide with the standard basis vectors \( e_1, \ldots, e_n \) of \( \mathbb{R}^n \).
4. Characterising finitely generated projective $\ell$-groups

As we have seen, abstract Schauder bases are in one-one correspondence with unimodular fans. In this section they are used to characterise projective $\ell$-groups.

By definition, an $\ell$-group $G$ is projective if, whenever $\varphi: H \rightarrow K$ is a surjective $\ell$-homomorphism and $\alpha: G \rightarrow K$ is an $\ell$-homomorphism, there is an $\ell$-homomorphism $\theta: G \rightarrow H$ such that $\varphi \circ \theta = \alpha$.

It follows that an $\ell$-group $G$ is finitely generated projective iff it is a retract of some $A_n$. In other words, there are maps $\iota: G \rightarrow A_n$ and $\sigma: A_n \rightarrow G$ such that $\sigma \circ \iota$ is the identity over $G$.

The following well-known characterisation is due to Baker and Beynon [1, 2, 3].

Theorem 4.1 ([7, Corollary 5.2.2], [3, Theorem 3.1]). For any $\ell$-group $G$ the following conditions are equivalent:

(1) $G$ is finitely generated projective;
(2) for some $n \in \mathbb{N}$, $G$ is $\ell$-isomorphic to an $\ell$-group of $\ell$-functions over the support of some fan $\Sigma$ in $\mathbb{R}^n$, in symbols, $G \cong_\ell A_n|\Sigma$; and
(3) for some $n \in \mathbb{N}$, $G$ is $\ell$-isomorphic to the quotient $A_n/p$ for some principal $\ell$-ideal $p$ of $A_n$.

Hence:

Proposition 4.2. For any $\ell$-group $G$ the following are equivalent:

(1) $G$ is finitely generated projective.
(2) $G$ has an abstract Schauder basis.

Proof. (1 $\rightarrow$ 2) By Theorem 4.1 we can write $\psi: G \cong_\ell A_n|\Sigma$ for some fan $\Sigma$ in $\mathbb{R}^n$. Now, every fan can be unimodularised (see, e.g., [6, Theorem 8.5]), hence its support is also the support of a unimodular fan. Accordingly, we shall assume that $\Sigma$ is unimodular. Let Hats$_\Sigma$ be the concrete Schauder basis over $\Sigma$. The $\ell$-isomorphism $\psi^{-1}$ induces a one-one correspondence between Hats$_\Sigma$ and an abstract Schauder basis of $G$.

(2 $\rightarrow$ 1) If $G$ has an abstract Schauder basis, then $G \cong_\ell A_n|\Delta$ for some unimodular fan $\Delta$. By Theorem 4.1, $A_n|\Delta$ is finitely generated projective, hence so is $G$. \qed

Now recalling Theorem 3.1 we immediately obtain from the proposition above:

Corollary 4.3. For any $\ell$-group $G$ the following are equivalent:

(1) $G$ is finitely generated projective.
(2) $G$ is Archimedean and is generated by a finite set $B \subseteq G^+ \setminus \{0\}$ such that for every abstract $k$-simplex $Y$ of $B$, the zero set $\mathfrak{Z}(B \setminus Y)$ is homeomorphic to the $k$-ball.

The proof of Theorem 3.1 in conjunction with Proposition 4.2 actually yields the following interesting variant of (1 $\leftrightarrow$ 2) in Theorem 4.1.

Corollary 4.4. For any $\ell$-group $G$ the following are equivalent:

(1) $G$ is finitely generated projective.
(2) For some positive integer $n$, $G$ is $\ell$-isomorphic to $A_n|\Theta$, where $\Theta$ is a (necessarily unimodular) fan in $\mathbb{R}^n$ whose 1-dimensional cones are generated by the standard basis vectors $e_1, \ldots, e_n$. 

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5. Refinements of Abstract Schauder Bases

As shown in [6, 15], refinements of unimodular fans are fundamental to the fan-theoretic treatment of birational toric geometry. In this section we derive abstract versions of two basic refinement procedures for unimodular fans.

Let \( \Sigma \) and \( \Delta \) be unimodular fans in \( \mathbb{R}^n \) with the same support. Let \([\text{Hats}_\Sigma]\) and \([\text{Hats}_\Delta]\) be the submonoids of \( A_n \setminus |\Sigma| \) generated by the Schauder bases over \( \Sigma \) and \( \Delta \), respectively. Direct inspection shows that \( \Sigma \) refines \( \Delta \), henceforth written \( \Sigma \preceq \Delta \). There exists a unimodular fan \( \Xi \) with \( \delta \in \Sigma \setminus \Xi \) such that \( \Xi \) refines \( \delta \), and \( \Xi \) is unimodular. Now \( \Xi \) is inscribed in \( \Theta \), meaning that \( \Xi \subseteq \Theta \). Let \( \phi \colon G \cong \ell A_n \setminus |\Sigma| \) be an \( \ell \)-isomorphism carrying \( B_1 \) to \( \text{Hats}_\Sigma \), for some unimodular fan \( \Sigma \) in \( \mathbb{R}^n \). Then \( \phi \) induces \( \Xi \) to \( \text{Hats}_\Sigma \), whereby \( \Xi \) is linear over each cone of \( \Delta \).

To prove this, let \( L_1 \) be the collection of all linear pieces of all \( \ell \)-functions in \( S \cup \text{Hats}_\Sigma \), regarded by linear extension \([1] \) as linear \( \ell \)-functions from \( \mathbb{R}^n \) into itself. Further, let \( L_2 \) be a collection of rational homogeneous hyperplanes in \( \mathbb{R}^n \) defining the rational homogeneous polyhedral set \( |\Sigma| \), again regarded as linear \( \ell \)-functions from \( \mathbb{R}^n \) into itself. Set \( L = L_1 \cup L_2 \), say \( |L| = m \); display the elements of \( L \) as \( L = \{ l_1, \ldots, l_m \} \). We construct a complete unimodular fan \( \Xi \) in \( \mathbb{R}^n \) as follows. For every permutation \( \sigma \in \{ 1, \ldots, m \} \), let

\[
E_\sigma = \{ x \in \mathbb{R}^n \mid l_{\sigma(1)}(x) \geq l_{\sigma(2)}(x) \geq \cdots \geq l_{\sigma(m)}(x) \}
\]

Then \( E_\sigma \subseteq \mathbb{R}^n \) is a rational homogeneous polyhedral set; we do not exclude the trivial case \( E_\sigma = \{ 0 \} \), in which case there is nothing to prove. In all remaining cases, the dimension of \( E_\sigma \) is \( n \). It is an exercise to check that there exists a unique refinement-maximal (not necessarily unimodular, nor even simplicial) fan \( \Xi \) in \( \mathbb{R}^n \) whose \( n \)-dimensional cones are precisely the nontrivial \( E_\sigma \), as \( \sigma \) ranges over all permutations of \( \{ 1, \ldots, m \} \). Furthermore, it is clear that \( \bigcup_\sigma E_\sigma = \mathbb{R}^n \), whence \( \Xi \) is complete. By e.g. [6, Theorem 8.5], every fan can be unimodularised; let \( \Theta \) be a unimodular refinement of \( \Xi \). Now \( |\Sigma| \) is inscribed in \( \Theta \), meaning that \( |\Sigma| \) is a union of cones of \( \Theta \), because \( L \) includes \( L_2 \), a collection of hyperplanes defining \( \Sigma \). Let \( \Delta \) be the fan obtained from \( \Xi \) by selecting all those cones \( \delta \in \Theta \) that intersect \( |\Sigma| \) nontrivially, i.e. \( \delta \cap |\Sigma| \neq \emptyset \). Then clearly \( |\Delta| = |\Sigma| \), \( \Delta \) is unimodular, and every \( \ell \)-function in \( S \cup \text{Hats}_\Sigma \) is linear over each cone of \( \Delta \); the latter condition holds because \( L \) includes \( L_2 \), the collection of all linear pieces of all \( \ell \)-functions in \( S \cup \text{Hats}_\Sigma \). This settles our first claim.

\footnote{Such an extension is generally not unique, but the choice of the extension is immaterial.}
Claim. \( \text{Hats}_\Delta = \{h_1, \ldots, h_t\} \) spans \( S \cup \text{Hats}_\Sigma \) positively.

Indeed, let \( f : \Sigma \to \mathbb{R} \) be an \( \ell \)-function that is linear over each cone of \( \Delta \). For each 1-cone \( \delta_i \in \Delta, i \in \{1, \ldots, t\} \), \( f \) attains an integral value \( z_i \in \mathbb{Z} \) at the primitive vector along \( \delta_i \). The \( \ell \)-function \( f = z_1h_1 + \cdots + z_th_t \) coincides with \( f \); it indeed does by construction over 1-cones, hence it does over any cone because both \( f \) and \( f \) are linear over each cone of \( \Delta \). This settles our second claim.

We thus have \( S, \text{Hats}_\Sigma \subseteq [\text{Hats}_\Delta] \). Set \( C = \phi^{-1}(\text{Hats}_\Delta) \). Since \( \phi^{-1} \) is an \( \ell \)-isomorphism, \( B_1, B_2 \subseteq [C] \) holds, and the proof is complete. \( \square \)

If \( \Sigma \) refines \( \Delta \), there is generally no way of obtaining \( \Sigma \) from \( \Delta \) via a sequence of starring operations; thus, \( \Sigma \preceq_2 \Delta \) implies \( \Sigma \preceq \Delta \), but not conversely. The stronger \( \Sigma \preceq_2 \Delta \) condition that \( \Sigma \) be a starring refinement of \( \Delta \) also has an algebraic counterpart via Lemma 2.4. Specifically, let us write \( [\text{Hats}_\Sigma] \preceq_2 [\text{Hats}_\Delta] \) iff \( \text{Hats}_\Sigma \) can be obtained from \( \text{Hats}_\Delta \) via a finite number of transformations of the type

\[
\{h_1, h_2\} \mapsto \{h_1 - (h_1 \land h_2), h_2 - (h_1 \land h_2), h_1 \land h_2\},
\]

with the proviso that \( h_1 \land h_2 \neq 0 \). We can then prove the following purely algebraic version of the De Concini-Procesi Lemma.

**Proposition 5.2** (The abstract De Concini-Procesi Lemma). Let \( B_1, B_2 \) be abstract Schauder bases of an \( \ell \)-group \( G \). There exists an abstract Schauder basis \( C \) of \( G \) such that:

1. \( C \preceq B_2 \), in the sense that \( B_2 \subseteq [C] \).
2. \( C \preceq_2 B_1 \), in the sense that \( C \) is obtainable from \( B_1 \) via a finite number of transformations of the type (4), where \( h_1 \land h_2 \neq 0 \).

**Proof.** Let \( \phi : G \cong \ell A_n \mid \Sigma \) be an \( \ell \)-isomorphism carrying \( B_1 \) to \( \text{Hats}_\Sigma \), for some unimodular fan \( \Sigma \) in \( \mathbb{R}^n \), and let \( S = \phi(B_2) \). By the same argument as the one in Proposition 5.1 there is a unimodular fan \( \Delta \) with \( |\Delta| = |\Sigma| \) such that \( S \cup \text{Hats}_\Sigma \subseteq [\text{Hats}_\Delta] \). By the De Concini-Procesi Lemma, there is a unimodular fan \( \Theta \) such that \( \Theta \preceq_2 \Sigma \), and \( \Theta \) refines \( \Delta \). It follows that \( \text{Hats}_\Sigma, S \subseteq [\text{Hats}_\Theta] \), because every \( \ell \)-function in \( \text{Hats}_\Sigma \cup S \) is linear over each cone of \( \Delta \), hence over each cone of its refinement \( \Theta \). Thus, if \( C = \phi^{-1}(\text{Hats}_\Theta) \), we have \( B_1, B_2 \subseteq [C] \). Since \( \Theta \preceq_2 \Sigma \), Lemma 2.4 shows \( \text{Hats}_\Theta \) is obtainable from \( \text{Hats}_\Sigma \) via a finite number of transformations of type (4), with \( h_1 \land h_2 \neq 0 \). Since these transformations are preserved by \( \ell \)-isomorphism, we obtain \( C \preceq_2 B_1 \), which completes the proof. \( \square \)

The above result should be compared with Lemma 2.3 to which it reduces when \( B_1 \) and \( B_2 \) are concrete Schauder bases.

6. Finitely generated projective = presentable by a lattice word

We refer to [7, Section 5.2], [13, Section 4] and [8] for background on finite presentations of \( \ell \)-groups. As an immediate consequence of Corollary 4.4 we have:

**Proposition 6.1.** Finitely presented \( \ell \)-groups \( G \cong \langle \pi_1, \ldots, \pi_n : v = 0 \rangle \) are exactly the same as finitely generated projective \( \ell \)-groups.
We are in a position to prove\footnote{Compare with [10] page 47.}

**Theorem 6.2.** For any \(\ell\)-group \(G\) the following are equivalent:

1. \(G\) is finitely generated projective.
2. \(G\) is finitely presented by \(\langle \pi_1, \ldots, \pi_n : l = 0 \rangle\), where the \(\ell\)-function \(l \in A_n\) is obtainable from \(\pi_1, \ldots, \pi_n, 0\) by applying the lattice operations only.

**Proof.** (2 \(\rightarrow\) 1) Follows immediately from Proposition 6.1.

(1 \(\rightarrow\) 2) In light of Corollary \footnote{Also, if needed, such orthographic symbols as parentheses and commas.} it is sufficient to prove the theorem for any \(\ell\)-group \(G\) of the form \(G = A_n \upharpoonright |\Theta|\), where \(\Theta\) is a (necessarily) unimodular fan in \(\mathbb{R}^n\) whose 1-dimensional cones are generated by the standard basis vectors \(e_1, \ldots, e_n\). In order to get the desired presentation of such a \(G\) it is enough to exhibit an \(\ell\)-function \(l \in A_n\) having the following two properties:

- the vanishing locus \(l^{-1}(0) \subseteq \mathbb{R}^n\) of \(l\) coincides with \(|\Theta|\), and
- \(l\) is obtainable from \(\pi_1, \ldots, \pi_n, 0\) by applying the lattice operations only.

To this purpose one first notes that \(|\Theta|\) is contained in the positive orthant \(O\) of \(\mathbb{R}^n\), and in addition, \(\text{Hats}_O = \{\pi_1 \upharpoonright |\Theta|, \ldots, \pi_n \upharpoonright |\Theta|\}\).

Let the \(\ell\)-function \(w \in A_n\) be defined by

\[
w = \bigvee_{i=1}^n \pi_i \land 0.
\]

The vanishing locus of \(w\) coincides with the intersection \(O\) of the vanishing loci of the \(\ell\)-functions \(\pi_i \land 0\). If \(|\Theta| = O\) we are done. Otherwise, for every cone \(\theta\) of \(\Theta\), let \(e_{\theta,1}, \ldots, e_{\theta,r}\) be the primitive generating vectors of \(\theta\), and let \(e_{\theta,r+1}, \ldots, e_{\theta,n}\) be the remaining basis vectors of \(\mathbb{R}^n\). Note that this latter set is nonempty. Let the \(\ell\)-function \(l_\theta\) be defined by

\[
l_\theta = w \lor \pi_{\theta,r+1} \lor \cdots \lor \pi_{\theta,n}.
\]

Then the vanishing locus of \(l_\theta\) coincides with \(\theta\), as is immediately seen. Finally, let the \(\ell\)-function \(l\) be defined by

\[
l = \bigwedge_{\theta \in \Theta} l_\theta.
\]

The vanishing locus of \(l\) coincides with the union \(|\Theta|\) of the cones of \(\Theta\). This completes the proof. \(\square\)

**Final remark.** Suppose the finitely generated projective \(\ell\)-group \(G\) is presented by \(\langle \pi_1, \ldots, \pi_m : v = 0 \rangle\). Suppose the \(\ell\)-function \(v\) is explicitly written as an \(\ell\)-group word \(\omega\), i.e., as a string of symbols over the alphabet \(\{0, +, -, \land, \lor\}\) of \(\ell\)-groups, plus symbols \(X_1, \ldots, X_m\) for the variables\footnote{\textit{Here, by abuse of notation,} \(\pi_j\) may denote a free generator both in \(A_m\) and in \(A_n\).} using the familiar composition rules. Then the proof of the theorem above yields an alternative presentation \(G \cong \langle \pi_1, \ldots, \pi_n : l = 0 \rangle\), where the \(\ell\)-function \(l\) can be explicitly written as a \(\{0, \land, \lor\}\)-word \(\lambda\) in the variables \(Y_1, \ldots, Y_n\), and the map \(\omega \mapsto \lambda\) is effective\footnote{\textit{See [10] Proposition 4.3} for details.}

As a matter of fact, from \(\omega(X_1, \ldots, X_m)\) one can effectively\footnote{\textit{See [10] Proposition 4.3} for details.} construct a unimodular fan \(\Delta\) whose support \(|\Delta| \subseteq \mathbb{R}^m\) coincides with the vanishing locus of \(v\).
Let $p_1, \ldots, p_n$ display the primitive generating vectors of the rays in $\Delta$. By sending each $p_i$ to the standard basis vector $e_i$ of $\mathbb{R}^n$ one effectively obtains from $\Delta$ a unimodular fan $\Theta$ in $\mathbb{R}^n$: the cones in $\Delta$ precisely correspond to the cones in $\Theta$ via the map $p_i \mapsto e_i$. The desired lattice-word $\lambda$ is now effectively obtainable from $\Theta$ as in the proof of Theorem 6.2. The $\ell$-isomorphisms

$$A_m \mid |\Delta| \cong A_n \mid |\Theta| \cong \langle \pi_1, \ldots, \pi_m : v = 0 \rangle \cong \langle \pi_1, \ldots, \pi_n : l = 0 \rangle$$

follow from the trivial piecewise homogeneous linear homeomorphism existing between $|\Delta|$ and $|\Theta|$ in the light of Baker-Beynon duality (see [7, 5.2]).

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