

ALGEBRAIC INDEPENDENCE IN THE GROTHENDIECK RING OF VARIETIES

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ABSTRACT. We give sufficient cohomological criteria for the classes of given varieties over a field k to be algebraically independent in the Grothendieck ring of varieties over k and construct some examples.

1. INTRODUCTION

Let k be a field. The Grothendieck ring of varieties over k , denoted $K_0(\text{Var}_k)$, was defined by A. Grothendieck in [G].

The abelian group $K_0(\text{Var}_k)$ is generated by the isomorphism classes $[X]$ of separated k -schemes of finite type X/k subject to the relations $[X] = [Y] + [X - Y]$ for $Y \subset X$ a closed subscheme. Multiplication is given by $[X_1] \cdot [X_2] := [X_1 \times_k X_2]$ and makes $K_0(\text{Var}_k)$ a commutative ring with unit $[\text{Spec}(k)]$. By its very definition $K_0(\text{Var}_k)$ is the value group of the universal Euler-Poincaré characteristic with compact support for varieties over k and is thus a fundamental invariant of the algebraic geometry over k . The most outstanding result on the structure of $K_0(\text{Var}_k)$ for k of characteristic zero is a presentation of the ring $K_0(\text{Var}_k)$ in terms of generators and relations anticipated by E. Looijenga and proved by his student F. Bittner [Bi]. B. Poonen showed that for k of characteristic zero a linear combination of the classes of suitable abelian varieties is a zero divisor in $K_0(\text{Var}_k)$ [P], and J. Kollár has computed the subring of $K_0(\text{Var}_k)$ generated by the classes of conics for suitable base fields k [Ko]. A deep problem pertaining to the structure of $K_0(\text{Var}_k)$ is the rationality of motivic zeta-functions posed by M. Kapranov [K] on which there has been recent progress due to M. Larsen and V. Lunts [LL1]. This problem is intimately related to the finite-dimensionality of motives; cf. [A] for an exposition. Furthermore, the ring $K_0(\text{Var}_k)$ plays a central rôle in the theory of motivic integration; cf. [Lo]. The present note is dedicated to the following problem about the structure of $K_0(\text{Var}_k)$.

Given varieties X_i/k , when are their classes $[X_i] \in K_0(\text{Var}_k)$ algebraically independent (in the sense made precise at the end of this Introduction)?

We give a number of sufficient cohomological conditions for this to be the case and construct some examples. Our main results are also valid in positive characteristic where there is no known Theorem about the structure of $K_0(\text{Var}_k)$. We now review the content of the individual sections.

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In section 2 we construct various ring homomorphisms $\mu_k : K_0(\text{Var}_k) \longrightarrow R$ (so called motivic measures) all based on étale cohomology. The weight filtration is used to show that the classes of proper smooth varieties over a finitely generated field k are algebraically independent in $K_0(\text{Var}_k)$ as soon as their first étale cohomology groups are algebraically independent in the ring of virtual l -adic Galois representations (Corollary 3). We then reformulate known properties of rationality and l -independence for varieties over \mathbb{F}_q (Proposition 7).

In Subsection 3.1 we reduce the problem of algebraic independence of virtual l -adic Galois representations to a problem about representations of (possibly non-connected) reductive groups, give a result illustrating the subtleties arising from this non-connectedness (Theorem 11) and construct an infinite sequence of curves over a given finite field k the classes of which are algebraically independent in $K_0(\text{Var}_k)$ (Theorems 12 and 17). This shows in particular that $K_0(\text{Var}_k)$ contains a polynomial ring in infinitely many variables and supports the intuition that $K_0(\text{Var}_k)$ should be a large ring. We obtain a similar result for number fields using only elliptic curves (Theorem 13).

In Subsection 3.2 we use a lemma of Skolem together with the aforementioned rationality properties to give another approach to the algebraic independence of virtual l -adic Galois representations and show that a generic pair of curves over a finite field has algebraically independent classes in the Grothendieck ring of varieties (Theorem 26 in Subsection 3.5). We also determine the structure of the subring of $K_0(\text{Var}_{\mathbb{F}_q})$ generated by zero-dimensional varieties showing in particular that $K_0(\text{Var}_{\mathbb{F}_q})$ contains infinitely many zero divisors (Theorem 25).

The following definition will be central throughout the rest of this note.

Definition 1. Let R be a commutative unitary ring and let $n\mathbb{Z}$ be the kernel of $\mathbb{Z} \longrightarrow R$. We call a set $\{x_i : i \in I\}$ of elements $x_i \in R$ algebraically independent if the morphism

$$(\mathbb{Z}/n)[T_i : i \in I] \longrightarrow R, T_i \mapsto x_i$$

is injective.

If R is a field, then this means algebraically independent over the prime field of R in the usual sense. Note that $1/2 \in \mathbb{Q}$ is algebraically dependent though not integral over \mathbb{Z} ; cf. Theorem 24. All the rings appearing below will have $n = 0$, i.e. $\mathbb{Z} \subset R$. In this case, elements $x_i \in R$ are algebraically independent if and only if the subring of R they generate is a polynomial ring over \mathbb{Z} in the variables x_i .

2. A MOTIVIC MEASURE

2.1. General construction. Let k be a field. We fix a separable closure \bar{k} of k and a rational prime l different from the characteristic of k . For X/k separated and of finite type we denote by

$$H_c^i(\bar{X}) := H_c^i(X \times_k \bar{k}, \mathbb{Q}_l) \text{ for } i \geq 0$$

the étale cohomology with compact support and constant coefficients \mathbb{Q}_l of the base change $X \times_k \bar{k}$. Then $H_c^i(\bar{X})$ is naturally a $G_k := \text{Aut}(\bar{k}/k)$ -module, and denoting by $\text{Rep}_{G_k} \mathbb{Q}_l$ the category of finite-dimensional continuous representations of G_k over \mathbb{Q}_l (i.e. l -adic Galois representations) we have a motivic measure

$$(1) \quad \mu_k : K_0(\text{Var}_k) \longrightarrow K_0(\text{Rep}_{G_k} \mathbb{Q}_l), [X] \mapsto \sum_i (-1)^i [H_c^i(\bar{X})].$$

In fact, (1) is (well defined and) a homomorphism of abelian groups by excision, multiplicative by the Künneth formula and clearly preserves the unit.

For a finite extension $k \subset L \subset \bar{k}$ we have a ring homomorphism (base change)

$$-\times_k L : K_0(\text{Var}_k) \longrightarrow K_0(\text{Var}_L), [X] \mapsto [X \times_k L]$$

and a homomorphism of abelian groups (restriction of scalars)

$$/k : K_0(\text{Var}_L) \longrightarrow K_0(\text{Var}_k), [X \longrightarrow \text{Spec}(L)] \mapsto [X \longrightarrow \text{Spec}(L) \longrightarrow \text{Spec}(k)].$$

From the inclusion $G_L \subset G_k$ we have an exact tensor functor (restriction)

$$(2) \quad \text{Res}_{G_L}^{G_k} : \text{Rep}_{G_k} \mathbb{Q}_l \longrightarrow \text{Rep}_{G_L} \mathbb{Q}_l$$

and an exact functor (induction)

$$(3) \quad \text{Ind}_{G_L}^{G_k} : \text{Rep}_{G_L} \mathbb{Q}_l \longrightarrow \text{Rep}_{G_k} \mathbb{Q}_l, V \mapsto \mathbb{Q}_l[G_k] \otimes_{\mathbb{Q}_l[G_L]} V$$

inducing a ring homomorphism and a homomorphism of abelian groups on the level of K_0 to be denoted by the same symbols. The following diagram is commutative:

$$\begin{array}{ccc} K_0(\text{Var}_k) & \xrightarrow{\mu_k} & K_0(\text{Rep}_{G_k} \mathbb{Q}_l) \\ -\times_k L \updownarrow /k & & \text{Res}_{G_L}^{G_k} \updownarrow \text{Ind}_{G_L}^{G_k} \\ K_0(\text{Var}_L) & \xrightarrow{\mu_L} & K_0(\text{Rep}_{G_L} \mathbb{Q}_l) \end{array}$$

2.2. Weight filtration. Notation being as in Subsection 2.1 we now assume in addition that k is finitely generated. In this subsection we will incorporate the weight filtration on $H_c^*(\bar{X})$ into the motivic measure (1). We need to assume that k is finitely generated in order to have a theory of weights. Vaguely speaking, this is a substitute (valid in any characteristic) of Hodge theory and will allow us to isolate H^1 from the Euler-Poincaré characteristic in Theorem 2 below.

We refer the reader to [J], §6, for the definition of the full subcategory

$$\text{WRep}_{G_k} \mathbb{Q}_l \subset \text{Rep}_{G_k} \mathbb{Q}_l$$

of l -adic Galois representations having a weight filtration. From [J], Lemma 6.8.2 and (6.8.3), we know that $H_c^i(\bar{X}) \in \text{WRep}_{G_k} \mathbb{Q}_l$ holds for all X/k separated and of finite type and hence (1) factors as

$$(4) \quad \mu_k : K_0(\text{Var}_k) \longrightarrow K_0(\text{WRep}_{G_k} \mathbb{Q}_l).$$

For $V \in \text{WRep}_{G_k} \mathbb{Q}_l$ we denote by $W_\bullet V$ the weight filtration of V and by $\text{Gr}_i^W(V) := W_i V / W_{i-1} V$ ($i \in \mathbb{Z}$) the associated graded of weight i . As any morphism in $\text{WRep}_{G_k} \mathbb{Q}_l$ is strictly compatible with the weight filtration ([J], Lemma 6.8.1), the functor

$$\text{Gr}_i^W : \text{WRep}_{G_k} \mathbb{Q}_l \longrightarrow \text{Rep}_{G_k} \mathbb{Q}_l, V \mapsto \text{Gr}_i^W(V)$$

is exact ([D1], Proposition 1.1.11, ii)) and thus induces a homomorphism

$$\text{Gr}_i^W : K_0(\text{WRep}_{G_k} \mathbb{Q}_l) \longrightarrow K_0(\text{Rep}_{G_k} \mathbb{Q}_l).$$

So we have a homomorphism of abelian groups

$$(5) \quad \Phi : K_0(\text{WRep}_{G_k} \mathbb{Q}_l) \longrightarrow K_0(\text{Rep}_{G_k} \mathbb{Q}_l)[T, T^{-1}], x \mapsto \sum_{i \in \mathbb{Z}} \text{Gr}_i^W(x) T^i$$

which is a ring homomorphism as follows from

$$\mathrm{Gr}_i^W(V \otimes W) \simeq \bigoplus_{a+b=i} \mathrm{Gr}_a^W(V) \otimes \mathrm{Gr}_b^W(W) \text{ for } i \in \mathbb{Z} \text{ and } V, W \in \mathrm{WRep}_{G_k} \mathbb{Q}_l.$$

As $\mathrm{Gr}_i^W(H_c^*(\overline{X})) = 0$ for $i < 0$ (see [Ka]) we obtain, composing (4) and (5), a motivic measure

$$(6) \quad \mu_k : K_0(\mathrm{Var}_k) \longrightarrow K_0(\mathrm{Rep}_{G_k} \mathbb{Q}_l)[T]$$

given explicitly by

$$(7) \quad \mu_k([X]) = \sum_{i \geq 0} \left(\sum_j (-1)^j [\mathrm{Gr}_i^W(H_c^j(\overline{X}))] \right) T^i.$$

The slight abuse of notation in denoting (1) and (6) by the same symbol μ_k will cause no confusion.

For a finite extension $k \subset L \subset \bar{k}$ we extend the morphisms induced on K_0 by (2) (resp. (3)) to $K_0(\mathrm{Rep}_{G_k} \mathbb{Q}_l)[T]$ (resp. $K_0(\mathrm{Rep}_{G_L} \mathbb{Q}_l)[T]$) by demanding that $T \mapsto T$. With μ_k as in (6) we then have a commutative diagram

$$(8) \quad \begin{array}{ccc} K_0(\mathrm{Var}_k) & \xrightarrow{\mu_k} & K_0(\mathrm{Rep}_{G_k} \mathbb{Q}_l)[T] \\ -\times_k L \downarrow \uparrow /k & & \mathrm{Res}_{G_L}^{G_k} \downarrow \uparrow \mathrm{Ind}_{G_L}^{G_k} \\ K_0(\mathrm{Var}_L) & \xrightarrow{\mu_L} & K_0(\mathrm{Rep}_{G_L} \mathbb{Q}_l)[T] \end{array}$$

The commutativity of (8) follows from the compatibility of the weight filtration with restriction and induction, which we leave to the reader to verify using the uniqueness of the weight filtration ([J], Lemma 6.8.1, a)). We now use the weight filtration to establish the following criterion for algebraic independence in $K_0(\mathrm{Var}_k)$.

For X/k separated and of finite type and $i \geq 0$ we write

$$\mathrm{Gr}_i^W(X) := \sum_j (-1)^j [\mathrm{Gr}_i^W(H_c^j(\overline{X}))] \in K_0(\mathrm{Rep}_{G_k} \mathbb{Q}_l).$$

Then $\mu_k([X]) = \sum_i \mathrm{Gr}_i^W(X) T^i \in K_0(\mathrm{Rep}_{G_k} \mathbb{Q}_l)[T]$; see (7).

Theorem 2. *Let k be finitely generated and let $X_1, \dots, X_n/k$ be separated and of finite type and assume that $\mathrm{Gr}_0^W(X_i) \in \mathbb{Z} \subseteq K_0(\mathrm{Rep}_{G_k} \mathbb{Q}_l)$ for all i and that the $\mathrm{Gr}_1^W(X_i)$ are algebraically independent in $K_0(\mathrm{Rep}_{G_k} \mathbb{Q}_l)$. Then the $\mu_k([X_i])$ are algebraically independent in $K_0(\mathrm{Rep}_{G_k} \mathbb{Q}_l)[T]$ and hence the $[X_i]$ are algebraically independent in $K_0(\mathrm{Var}_k)$.*

Note that $K_0(\mathrm{Rep}_{G_k} \mathbb{Q}_l)$ is augmented over \mathbb{Z} by the degree map, hence $\mathbb{Z} \subseteq K_0(\mathrm{Rep}_{G_k} \mathbb{Q}_l)$. The last assertion of the Theorem follows from the existence of the motivic measure (6).

Corollary 3. *Let k be finitely generated and let $X_1, \dots, X_n/k$ be proper and smooth. Then, if the $[H_c^1(\overline{X}_i)]$ are algebraically independent in $K_0(\mathrm{Rep}_{G_k} \mathbb{Q}_l)$, so are the $\mu_k([X_i])$ in $K_0(\mathrm{Rep}_{G_k} \mathbb{Q}_l)[T]$ and hence also the $[X_i]$ in $K_0(\mathrm{Var}_k)$.*

Proof. As the X_i/k are proper and smooth we have $\mathrm{Gr}_0^W(X_i) = [H_c^0(\overline{X}_i)] = [\mathbb{Q}_l^{\oplus |\pi_0(\overline{X}_i)|}] = |\pi_0(\overline{X}_i)| \in \mathbb{Z}$ and $\mathrm{Gr}_1^W(X_i) = -[H_c^1(\overline{X}_i)]$. Since the $[H_c^1(\overline{X}_i)]$ are algebraically independent by assumption, so are the $\mathrm{Gr}_1^W(X_i)$, and Theorem 2 applies. □

For the proof of Theorem 2 we need some elementary preparation. Let R be a commutative unitary ring. On $R[T_1, \dots, T_n]$ we have the usual derivations $\frac{\partial}{\partial T_i}$, and for $\underline{\alpha} = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$ we write $\frac{\partial^{|\underline{\alpha}|}}{\partial T_1^{\alpha_1} \dots \partial T_n^{\alpha_n}}$ for their iterations; here $|\underline{\alpha}| := \sum \alpha_i$.

Now let $f_1, \dots, f_n \in R[T], G \in \mathbb{Z}[T_1, \dots, T_n]$ be given and put $\tilde{G}(T) := G(f_1, \dots, f_n) \in R[T]$. For $N \geq 1$ let $I_N \subseteq R[T]$ be the ideal generated by $\frac{\partial^{|\underline{\alpha}|} G}{\partial T_1^{\alpha_1} \dots \partial T_n^{\alpha_n}}(f_1, \dots, f_n)$ for $|\underline{\alpha}| \leq N$ and put $I_0 := 0$. The $(I_N)_{N \geq 0}$ form an ascending chain of ideals of $R[T]$ and $\frac{d}{dT}(I_N) \subset I_{N+1}$ for $N \geq 0$.

Proposition 4. *For $N \geq 1$ we have*

$$\frac{d^N \tilde{G}}{dT^N} \equiv \sum_{1 \leq i_1, \dots, i_N \leq n} \frac{\partial^N G}{\partial T_{i_1} \dots \partial T_{i_N}}(f_1, \dots, f_n) \frac{df_{i_1}}{dT} \dots \frac{df_{i_N}}{dT} \pmod{I_{N-1}}.$$

Proof. We use induction on $N \geq 1$. We have

$$\frac{d\tilde{G}}{dT} = \frac{d}{dT}G(f_1, \dots, f_n) = \sum_{1 \leq i_1 \leq n} \frac{\partial G}{\partial T_{i_1}}(f_1, \dots, f_n) \frac{df_{i_1}}{dT},$$

i.e. the result for $N = 1$. We assume $N \geq 2$ and compute, using the induction hypothesis:

$$\frac{d^N \tilde{G}}{dT^N} = \frac{d}{dT} \left(\sum_{1 \leq i_1, \dots, i_{N-1} \leq n} \frac{\partial^{N-1} G}{\partial T_{i_1} \dots \partial T_{i_{N-1}}} (f_1, \dots, f_n) \frac{df_{i_1}}{dT} \dots \frac{df_{i_{N-1}}}{dT} + \alpha \right),$$

some $\alpha \in I_{N-2}$. As $\frac{d}{dT}(I_{N-2}) \subseteq I_{N-1}$ this equals, modulo I_{N-1} :

$$\begin{aligned} & \sum_{1 \leq i_1, \dots, i_{N-1} \leq n} \left[\left(\sum_{j=1}^n \frac{\partial^N G}{\partial T_{i_1} \dots \partial T_{i_{N-1}} \partial T_j} (f_1, \dots, f_n) \frac{df_j}{dT} \right) \frac{df_{i_1}}{dT} \dots \frac{df_{i_{N-1}}}{dT} \right. \\ & \quad \left. + \frac{\partial^{N-1} G}{\partial T_{i_1} \dots \partial T_{i_{N-1}}} (f_1, \dots, f_n) \frac{d}{dT}(\dots) \right] \\ & \stackrel{i_N := j}{\equiv} \sum_{1 \leq i_1, \dots, i_N \leq n} \frac{\partial^N G}{\partial T_{i_1} \dots \partial T_{i_N}} (f_1, \dots, f_n) \frac{df_{i_1}}{dT} \dots \frac{df_{i_N}}{dT}, \end{aligned}$$

as claimed. □

Proof of Theorem 2. Put

$$R := K_0(\text{Rep}_{G_k} \mathbb{Q}_l), f_i(T) := \mu_k([X_i]) = \sum_{\nu \geq 0} \text{Gr}_\nu^W(X_i) T^\nu \in R[T],$$

$a_i := \text{Gr}_0^W(X_i) \in \mathbb{Z} \subset R$ by assumption. Assume that we have $G \in \mathbb{Z}[T_1, \dots, T_n]$ with $G(\mu_k([X_1]), \dots, \mu_k([X_n])) = 0$, i.e.

$$(9) \quad \tilde{G}(T) := G(f_1(T), \dots, f_n(T)) = 0.$$

We have to show that $G = 0$ which follows from the

Claim. For $N \geq 0$ and any $\underline{\alpha} \in \mathbb{N}_0^n$ with $|\underline{\alpha}| \leq N$ we have

$$\frac{\partial^{|\underline{\alpha}|} G}{\partial T_1^{\alpha_1} \dots \partial T_n^{\alpha_n}}(a_1, \dots, a_n) = 0,$$

which in turn will be established by induction on N . Note that $f_i(0) = a_i$, so from (9) putting $T = 0$ we get $G(a_1, \dots, a_n) = 0$, i.e. the claim for $N = 0$. Now assume that $N \geq 1$. By applying Proposition 4 to (9) and putting $T = 0$ we obtain the following relation in R :

$$0 = \frac{d^N \tilde{G}}{dT^N}(T = 0) = \sum_{1 \leq i_1, \dots, i_N \leq n} \frac{\partial^N G}{\partial T_{i_1} \dots \partial T_{i_N}}(a_1, \dots, a_n) \frac{df_{i_1}}{dT}(0) \dots \frac{df_{i_N}}{dT}(0) + \alpha(0).$$

Here $\alpha \in I_{N-1}$, hence $\alpha(0) = 0$ by induction hypothesis. As $\frac{df_i}{dT}(0) = \text{Gr}_1^W(X_i)$ we have the following relation in R :

$$(10) \quad 0 = \sum_{1 \leq i_1, \dots, i_N \leq n} \frac{\partial^N G}{\partial T_{i_1} \dots \partial T_{i_N}}(a_1, \dots, a_n) \text{Gr}_1^W(X_{i_1}) \dots \text{Gr}_1^W(X_{i_N}).$$

We collect terms in this expression: Consider

$$\begin{aligned} \pi : I := \{(i_1, \dots, i_N) : 1 \leq i_j \leq n\} &\longrightarrow \{\underline{\alpha} = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n : |\underline{\alpha}| = N\} \\ (i_1, \dots, i_N) &\longmapsto (\alpha_k := |\{1 \leq j \leq N : i_j = k\}|)_{k=1, \dots, n}. \end{aligned}$$

Given $(i_1, \dots, i_N) \in I$ with $\pi((i_1, \dots, i_N)) = (\alpha_1, \dots, \alpha_n)$ we have

$$\frac{\partial^N G}{\partial T_{i_1} \dots \partial T_{i_N}} = \frac{\partial^N G}{\partial T_1^{\alpha_1} \dots \partial T_n^{\alpha_n}}$$

and

$$\text{Gr}_1^W(X_{i_1}) \dots \text{Gr}_1^W(X_{i_N}) = \text{Gr}_1^W(X_1)^{\alpha_1} \dots \text{Gr}_1^W(X_n)^{\alpha_n}.$$

So (10) may be written as

$$(11) \quad \sum_{\underline{\alpha} = (\alpha_1, \dots, \alpha_n) \in \frac{\mathbb{N}_0^n}{|\underline{\alpha}| = N}} |\pi^{-1}(\underline{\alpha})| \frac{\partial^N G}{\partial T_1^{\alpha_1} \dots \partial T_n^{\alpha_n}}(a_1, \dots, a_n) \cdot \text{Gr}_1^W(X_1)^{\alpha_1} \dots \text{Gr}_1^W(X_n)^{\alpha_n} = 0.$$

The $\text{Gr}_1^W(X_i) \in R$ are algebraically independent by assumption and π is surjective, hence $|\pi^{-1}(\underline{\alpha})| \neq 0$. So (11) implies

$$\frac{\partial^N G}{\partial T_1^{\alpha_1} \dots \partial T_n^{\alpha_n}}(a_1, \dots, a_n) = 0$$

for all $\underline{\alpha} \in \mathbb{N}_0^n$ with $|\underline{\alpha}| = N$, concluding the proof of the Claim and of Theorem 2. □

2.3. Finite base field. In the situation of Section 2.1 we now assume in addition that the base field $k = \mathbb{F}_q$ is a finite field. We will use the fact that $G_{\mathbb{F}_q}$ is topologically cyclic, generated by the geometric Frobenius $F_q \in G_{\mathbb{F}_q}$, to rewrite the results of Sections 2.1 and 2.2 in terms of characteristic polynomials of Frobenius. We need to recall some facts about the “universal ring” from [DG], V, §5, no. 2.

For any commutative unitary ring R the set $\Lambda(R) := 1 + tR[[t]]$ of formal power series with coefficients in R and constant coefficient 1 is an abelian group under multiplication. The resulting functor $R \mapsto \Lambda(R)$ can be endowed with the structure of a functor in commutative unitary rings such that for all R and $a, b \in R$

$$(12) \quad (1 - at)(1 - bt) = 1 - (ab)t \text{ in } \Lambda(R) = 1 + tR[[t]];$$

see *loc. cit.* Section 2.2. The multiplicative unit of $\Lambda(R)$ is $1 - t$. The ring structure of $\Lambda(R)$ may cause confusion: The addition is the usual multiplication of power series, but the multiplication in $\Lambda(R)$ is a rather unusual operation on power series. We will denote addition and multiplication in $\Lambda(R)$ by the usual symbols but make clear that the composition is to be understood in $\Lambda(R)$. In this sense, the additive analogue of (12) is $(1 - at) + (1 - bt) = 1 - (a + b)t + abt^2$ in $\Lambda(R)$.

Let R and $f \in \Lambda(R)$ be given. One defines $\partial_\nu(f) \in R$ for all $\nu \geq 1$ by the expansion $\text{dlog}(f) =: -\sum_{\nu \geq 1} \partial_\nu(f)t^{\nu-1}$. Here, $\text{dlog}(f) = \frac{df/dt}{f}$ (usual quotient of power series) is the logarithmic derivative of f . If we denote by \mathcal{O} the identity functor on the category of commutative unitary rings, then each $\partial_\nu : \Lambda \rightarrow \mathcal{O}$ is a natural transformation (*loc. cit.* 2.4) and if R is a \mathbb{Q} -algebra

$$(13) \quad \Lambda(R) \xrightarrow{\sim} R^{\mathbb{N}}, f \mapsto (\partial_\nu(f))_{\nu \geq 1}$$

is an isomorphism of rings in which $R^{\mathbb{N}}$ is the usual product of countably many copies of the ring R . For any $\nu \geq 1$ there is a natural transformation of ring-valued functors (“Frobenius”) $F_\nu : \Lambda \rightarrow \Lambda$ characterized by

$$(14) \quad F_\nu(1 - at) = 1 - a^\nu t \text{ in } \Lambda(R) = 1 + tR[[t]],$$

for all R and $a \in R$ and a natural transformation of functors in abelian groups (“Verschiebung”) $V_\nu : \Lambda \rightarrow \Lambda$ characterized by $V_\nu(1 - at) = 1 - at^\nu$ in $\Lambda(R)$. In fact, we have

$$(15) \quad V_\nu(f(t)) = f(t^\nu) \text{ for all } f(t) \in \Lambda(R) = 1 + tR[[t]];$$

see *loc. cit.* 2.5.

The rings $\Lambda(R)$ are convenient tools to keep track of characteristic polynomials.

Proposition 5. *Let k be a field and V, W finite-dimensional k -vector spaces.*

- i) *For $\phi \in \text{End}_k(V)$ and $\psi \in \text{End}_k(W)$ we have*
 $\det(1 - (\phi \oplus \psi)t|V \oplus W) = \det(1 - \phi t|V) + \det(1 - \psi t|W)$ and
 $\det(1 - (\phi \otimes \psi)t|V \otimes W) = \det(1 - \phi t|V)\det(1 - \psi t|W)$ in $\Lambda(k) = 1 + tk[[t]]$.
For any $\nu \geq 1$:
 $\det(1 - \phi^\nu t|V) = F_\nu(\det(1 - \phi t|V))$ in $\Lambda(k)$ and
 $\partial_\nu(\det(1 - \phi t|V)) = \text{tr}(\phi^\nu|V)$ in k .
- ii) *Let G be a group and $N \subset G$ a normal subgroup such that G/N is finite cyclic of order ν generated by the class of $\pi \in G$. If V is a $k[N]$ -module, finite-dimensional as a k -vector space, then we have*

$$(16) \quad \det(1 - \pi t|\text{Ind}_N^G(V)) = V_\nu(\det(1 - \pi^\nu t|V)) \text{ in } \Lambda(k) = 1 + tk[[t]].$$

In ii) we denoted by $\text{Ind}_N^G(V) := k[G] \otimes_{k[N]} V$ the $k[G]$ -module induced by V . Note that $\pi^\nu \in N$, hence the right-hand side of (16) is meaningful.

Proof. Part i) follows from equations (12) and (14) and [H], Appendix C, Lemma 4.1. Using (15), part ii) is equivalent to $\det(1 - \pi t|\text{Ind}_N^G(V)) = \det(1 - \pi^\nu t|V)$. This has to be checked by an explicit computation which we omit. \square

For a ring R we define a subset $\mathcal{R}(R) \subset \Lambda(R)$ by

$$\mathcal{R}(R) := \{f - g \mid f, g \in 1 + tR[[t]]\} \subset \Lambda(R) = 1 + tR[[t]].$$

It is clear that $\mathcal{R} \subset \Lambda$ is a subfunctor in abelian groups. One should think of the elements of $\mathcal{R}(R)$ as the the power series expansions of certain rational functions, namely the quotients of polynomials with constant coefficient 1.

Proposition 6. $\mathcal{R} \subset \Lambda$ is a subfunctor in rings.

Proof. First, $1_{\Lambda(R)} = 1 - t \in \mathcal{R}(R) \subset \Lambda(R) = 1 + tR[[t]]$. Now it suffices to check that for $f, g \in 1 + tR[[t]]$ we have $fg \in \mathcal{R}(R) \subset \Lambda(R)$. Indeed, fg is again a polynomial according to [DG], p. 632, formula (**). \square

Using Proposition 5 one sees that there is a unique ring homomorphism

$$\chi_q : K_0(\text{Rep}_{G_{\mathbb{F}_q}} \mathbb{Q}_l) \hookrightarrow \Lambda(\mathbb{Q}_l)$$

such that

$$\chi_q([V]) = \det(1 - F_q t|V) \in \Lambda(\mathbb{Q}_l) = 1 + t\mathbb{Q}_l[[t]] \text{ for } V \in \text{Rep}_{G_{\mathbb{F}_q}} \mathbb{Q}_l$$

and χ_q is injective as follows from [Algèbre], §12,1, Proposition 3.

Composing (6) with χ_q we obtain a motivic measure

$$(17) \quad \mu_q : K_0(\text{Var}_{\mathbb{F}_q}) \longrightarrow \Lambda(\mathbb{Q}_l)[T]$$

given explicitly by

$$(18) \quad \mu_q([X]) = \sum_{i \geq 0} \left(\sum_j (-1)^j \det(1 - F_q t | \text{Gr}_i^W(H_c^j(\overline{X}))) \right) T^i.$$

Proposition 7. The image of (17) is contained in $\mathcal{R}(\mathbb{Q})[T] \subset \Lambda(\mathbb{Q}_l)[T]$.

Proof. Let X/\mathbb{F}_q be separated and of finite type. The cohomological formula for the zeta function $Z_{X/\mathbb{F}_q}(t)$ of X/\mathbb{F}_q ([Mi2], Theorem 13.1) gives in $\Lambda(\mathbb{Q}_l) = 1 + t\mathbb{Q}_l[[t]]$:

$$\begin{aligned} Z_{X/\mathbb{F}_q}(t) &= \sum_{j \geq 0} (-1)^{j+1} \det(1 - F_q t | H_c^j(\overline{X})) \\ &= \sum_{j \geq 0} (-1)^{j+1} \left(\sum_i \det(1 - F_q t | \text{Gr}_i^W(H_c^j(\overline{X}))) \right) \\ &= \sum_i \left(\sum_{j \geq 0} (-1)^{j+1} \det(1 - F_q t | \text{Gr}_i^W(H_c^j(\overline{X}))) \right). \end{aligned}$$

In this expression, every summand is in $\mathbb{Q}_l(t)$, and we have that $Z_{X/\mathbb{F}_q}(t) \in \mathbb{Q}[[t]]$ from the very definition of $Z_{X/\mathbb{F}_q}(t)$. For weight reasons there can be no cancellation among different summands (recall that the sums are actually products of power series), hence every summand lies in $\mathbb{Q}_l(t) \cap \mathbb{Q}[[t]] \subset \mathbb{Q}(t)$. As every summand is in addition a quotient of polynomials with constant coefficient 1, every summand lies in fact in $\mathcal{R}(\mathbb{Q})$ and comparison with (18) concludes the proof. \square

The relevance of having a motivic measure taking values in $\mathcal{R}(\mathbb{Q})[T]$ rather than in $\Lambda(\mathbb{Q}_l)[T]$ will become clear in Section 3.2. Finally we record some formulae relating the various operations on Λ with the motivic measure μ_q .

Proposition 8.

- i) If X/\mathbb{F}_q is separated and of finite type, then, for all $\nu \geq 1$,

$$\partial_\nu(\mu_q([X])) = \sum_{i \geq 0} \left(\sum_j (-1)^j \text{tr}(F_q^\nu | \text{Gr}_i^W(H_c^j(\overline{X}))) \right) T^i \text{ in } \mathbb{Q}[T].$$

ii) For $\mathbb{F}_q \subset \mathbb{F}_{q^\nu} \subset \overline{\mathbb{F}_q}$ the finite extension of degree $\nu \geq 1$ we have a commutative diagram

$$(19) \quad \begin{array}{ccc} K_0(\text{Var}_{\mathbb{F}_q}) & \xrightarrow{\mu_q} & \mathcal{R}(\mathbb{Q})[T] \\ \begin{array}{c} \uparrow \\ -\times_{\mathbb{F}_q} \mathbb{F}_{q^\nu} \\ \downarrow \end{array} / \mathbb{F}_q & & \begin{array}{c} \uparrow \\ F_\nu \\ \downarrow \end{array} V_\nu \\ K_0(\text{Var}_{\mathbb{F}_{q^\nu}}) & \xrightarrow{\mu_{q^\nu}} & \mathcal{R}(\mathbb{Q})[T] \end{array}$$

In i) ∂_ν denotes the restriction of $\partial_\nu : \Lambda(\mathbb{Q}) \rightarrow \mathbb{Q}$ to $\mathcal{R}(\mathbb{Q}) \subset \Lambda(\mathbb{Q})$ extended to $\partial_\nu : \mathcal{R}(\mathbb{Q})[T] \rightarrow \mathbb{Q}[T]$ by demanding that $\partial_\nu(T) = T$, and in ii) F_ν and V_ν denote the maps which on $\mathcal{R}(\mathbb{Q})$ are the restrictions of $F_\nu, V_\nu : \Lambda(\mathbb{Q}) \rightarrow \Lambda(\mathbb{Q})$ and with $F_\nu(T) = T$ and $V_\nu(T) = T$.

Proof. Part i) follows from (18) and Proposition 5i). Part ii) follows from (8) and Proposition 5i) and ii). □

3. ALGEBRAIC INDEPENDENCE

3.1. Virtual continuous representations. In view of Corollary 3 we are led to consider the following problem:

Given a topological group T and a finite extension K/\mathbb{Q}_l denote by $\text{Rep}_T K$ the category of finite-dimensional continuous representations of T over K . Given finitely many $V_i \in \text{Rep}_T K$, when are their classes $[V_i] \in K_0(\text{Rep}_T K)$ algebraically independent?

In this subsection we will use a Tannakian argument to reduce this question to a problem about algebraic independence of representations of a (possibly non-connected) reductive algebraic group and in a special case of a diagonalizable group. For this latter case we establish a rather explicit Jacobi criterion (Lemma 10). The subtleties arising from the possible non-connectedness of the reductive group above will be illustrated by an example (Theorem 11) and motivate the following definition.

Definition 9. Representations $V_i \in \text{Rep}_T K$ are *geometrically* algebraically independent if for all open subgroups of finite index $T' \subset T$ the classes of the restrictions $[\text{Res}_{T'}^T(V_i)] \in K_0(\text{Rep}_{T'} K)$ are algebraically independent.

We now give a series of reduction steps to obtain workable criteria for the $[V_i] \in K_0(\text{Rep}_T K)$ to be algebraically (in)dependent. The reader is invited to glance at a simple application, Theorem 11, first.

Denoting by V_i^{ss} the semi-simplification of V_i we have

$$(20) \quad [V_i] = [V_i^{ss}] \text{ in } K_0(\text{Rep}_T K),$$

and we can assume the V_i to be semi-simple themselves. In the sequel we will repeatedly use the obvious fact that elements $x_i \in R \subset S$ of a subring R of S are algebraically independent in R if and only if they are so in S . We now use a Tannakian argument to find an accessible subring of $K_0(\text{Rep}_T K)$ containing all the $[V_i]$. The reference for Tannakian categories is [D2]. With the obvious notion of tensor product, dual and using the forgetful functor $\text{Rep}_T K \rightarrow \text{Vect}_K$ to the category of finite-dimensional K -vector spaces as a fibre functor, the category $\text{Rep}_T K$ is a neutral Tannakian category over K . We put $V := \bigoplus_{i=1}^n V_i$ and consider

$$i : \mathcal{C} := \langle V \rangle^\otimes \hookrightarrow \text{Rep}_T K,$$

the Tannakian subcategory generated by V , i.e. the smallest strictly full subcategory of $\text{Rep}_T K$ containing V and stable under the formation of finite direct sums, tensor products, duals, subobjects and quotients. As i is an exact tensor functor it induces a ring homomorphism

$$(21) \quad K_0(\mathcal{C}) \hookrightarrow K_0(\text{Rep}_T K)$$

which is injective: one checks that i preserves simple objects and uses the fact that the K_0 of an abelian category with all objects of finite length is \mathbb{Z} -free on the isomorphism classes of its simple objects [Q], §5, Corollary 1. By construction we have $[V_i] \in K_0(\mathcal{C})$, and our initial problem is reduced to considering the algebraic (in)dependence of the $[V_i]$ as elements of $K_0(\mathcal{C})$. Let

$$(22) \quad \phi : T \longrightarrow \text{Gl}(V)(K)$$

be the continuous homomorphism giving the action of T on V . Here $\text{Gl}(V)$ denotes the algebraic group and $\text{Gl}(V)(K)$ its group of K -points endowed with the topology inherited from the one of K . The Zariski closure of the image of ϕ , say

$$(23) \quad G := \overline{\text{im}(\phi)} \subset \text{Gl}(V),$$

is an affine algebraic group over K ([Bo], Proposition 1.3, b)) and can be identified with the Tannaka group of \mathcal{C} as follows ([S4], 1.3). From ϕ we have a continuous homomorphism $\tilde{\phi} : T \longrightarrow G(K)$, and the restriction of the fibre functor $\mathcal{C} \longrightarrow \text{Vect}_K$ induces an equivalence of Tannakian categories

$$(24) \quad \mathcal{C} \xrightarrow{\sim} \underline{\text{Rep}}_K G,$$

where $\underline{\text{Rep}}_K G$ denotes the category of finite-dimensional representations of the algebraic group G over K . A quasi-inverse of (24) is given by

$$\underline{\text{Rep}}_K G \xrightarrow{\sim} \mathcal{C}, (G \xrightarrow{\psi} \text{Gl}(W)) \mapsto \psi(K) \circ \tilde{\phi}.$$

As \mathcal{C} is semi-simple by (20), then [DMOS], Remark 2.28, implies that G is (not necessarily connected) reductive. The isomorphism $K_0(\mathcal{C}) \simeq K_0(\underline{\text{Rep}}_K G)$ induced by (24) reduces our initial problem to studying the algebraic (in)dependence of given representations of a reductive group. We next show that we may allow finite extensions of the coefficients K of these representations. For $K \subset L$ a finite extension, the exact tensor functor

$$\underline{\text{Rep}}_K G \longrightarrow \underline{\text{Rep}}_L G \otimes L, V \mapsto V \otimes_K L$$

induces a ring homomorphism

$$(25) \quad K_0(\underline{\text{Rep}}_K G) \hookrightarrow K_0(\underline{\text{Rep}}_L G \otimes L)$$

which is injective; cf. [Mi1], Lemma 3.14, for a similar argument. This allows us to replace G by $G \otimes L$ and to assume that the connected component of identity $G^0 \subset G$ is a connected split reductive group and that the finite étale group G/G^0 is in fact constant. Extending K further to contain a splitting field of G/G^0 , the ring $K_0(\underline{\text{Rep}}_K G/G^0)$ is a familiar object from the representation theory of finite (abstract) groups [S3], Chapter II. Also $K_0(\underline{\text{Rep}}_K G^0)$ can be described quite precisely (cf. below) but without imposing further, rather restrictive, conditions (e.g. $G \simeq G^0 \times G/G^0$) I cannot say much about the structure of $K_0(\underline{\text{Rep}}_K G)$. We will describe two special situations in which the above line of thought may be continued.

So K/\mathbb{Q}_l is a finite extension, G is a split reductive group over K and we need to study algebraic (in)dependence in the ring $K_0(\underline{\text{Rep}}_K G)$. We assume in addition that G is connected and fix a K -split maximal torus $S \subset G$. Then restriction induces an injective ring homomorphism

$$K_0(\underline{\text{Rep}}_K G) \hookrightarrow K_0(\underline{\text{Rep}}_K S)$$

[S1], 3.4, and the structure of $K_0(\underline{\text{Rep}}_K S)$ is particularly simple; cf. (28) for the more general case of a diagonalizable group. It then becomes important to determine the character group of S which in the present case (the topological group T being arbitrary) we do not know how to do. The most satisfactory solution of our initial problem can be obtained in the case that $T = \langle F \rangle$ is topologically cyclic. Though very special, this case will be of interest for us because it covers the case of the Galois group of a finite field. So let

$$\phi : T = \langle F \rangle \longrightarrow \text{Gl}(V)(K) \simeq \text{Gl}_m(K)$$

be as in (22) (so $m = \sum_{i=1}^n \dim_K V_i$). By our preliminary reductions (20) and (25) and conjugating $\phi(F)$ suitably we can assume that

$$(26) \quad \phi(F) = \text{diag}(\alpha_1, \dots, \alpha_m) \in \mathbb{D}_m(K)$$

for some $\alpha_i \in K^*$, where $\mathbb{D}_m \subset \text{Gl}_m$ is the torus of diagonal matrices. Then G as in (23) is the smallest algebraic subgroup $G \subset \text{Gl}_m$ defined over K such that $\phi(F) \in G(K)$, and $D := G \cap \mathbb{D}_m$ is a closed subgroup, hence diagonalizable by [Bo], Proposition 8.4. The character group $X(D)$ of D can be identified with the subgroup of K^* generated by the eigenvalues of $\phi(F)$ as follows from [Bo], 8.2, Corollary:

$$(27) \quad X(D) \simeq \langle \alpha_1, \dots, \alpha_m \rangle \subset K^*.$$

We recall the structure of $K_0(\underline{\text{Rep}}_K D)$ from [S1], 3.4. Giving $V \in \underline{\text{Rep}}_K D$ is equivalent to giving a “grading of type $X(D)$ ” of the finite-dimensional K -vector space V , i.e. $V = \bigoplus_{\chi \in X(D)} V_\chi$, where $V_\chi \subset V$ is the χ -eigenspace and

$$(28) \quad \text{ch} : K_0(\underline{\text{Rep}}_K D) \xrightarrow{\sim} \mathbb{Z}[X(D)], [V] \mapsto \sum_{\chi} \dim_K(V_\chi) e^\chi$$

is a ring isomorphism. Here $\mathbb{Z}[X(D)]$ is the group algebra over $X(D)$, i.e. $\mathbb{Z}[X(D)]$ is \mathbb{Z} -free on the set $\{e^\chi : \chi \in X(D)\}$ and $e^\chi e^{\chi'} = e^{\chi+\chi'}$. Decomposing the finitely generated abelian group $X(D)$ as

$$X(D) = X(D)^{\text{tor}} \bigoplus_{i=1}^d \mathbb{Z}e_i,$$

where $X(D)^{\text{tor}} \subset X(D)$ is the torsion subgroup and the $e_i \in X(D)$ are suitably chosen, the natural map

$$\mathbb{Z}[X(D)^{\text{tor}}] \otimes_{\mathbb{Z}} \mathbb{Z} \left[\bigoplus_{i=1}^d \mathbb{Z}e_i \right] \xrightarrow{\sim} \mathbb{Z}[X(D)], e^t \otimes e^f \mapsto e^{t+f}$$

is an isomorphism and

$$\mathbb{Z} \left[\bigoplus_{i=1}^d \mathbb{Z}e_i \right] \xrightarrow{\sim} \mathbb{Z}[T_1, \dots, T_d, T_1^{-1}, \dots, T_d^{-1}], e^{\sum a_i e_i} \mapsto T_1^{a_1} \dots T_d^{a_d}$$

is an isomorphism to the ring of Laurant polynomials over \mathbb{Z} in

$$d = \text{rk}(X(D)) = \dim(D)$$

variables. Since $X(D)^{\text{tor}}$ is finite, $A := \mathbb{Z}[X(D)^{\text{tor}}]$ is a finite flat \mathbb{Z} -algebra and summing up we have

$$(29) \quad K_0(\underline{\text{Rep}}_K D) \simeq A[T_1, \dots, T_d, T_1^{-1}, \dots, T_d^{-1}].$$

Note that, if $X(D)$ is as in (27), then $X(D)^{\text{tor}} \subset K^*$ is a finite subgroup, hence cyclic of order N , say. Then we have

$$(30) \quad A = \mathbb{Z}[\mathbb{Z}/N] \simeq \mathbb{Z}[S]/(S^N - 1).$$

So our initial problem is finally reduced to deciding whether the

$$f_i := \text{ch}([V_i]) \in A[T_1, \dots, T_d, T_1^{-1}, \dots, T_d^{-1}],$$

i.e. the characters of the given representations V_i , are algebraically independent. For this problem one can prove a Jacobian criterion as follows.

Let A be a finite flat \mathbb{Z} -algebra. We will prove a criterion for given

$$f_1, \dots, f_n \in A[T_1, \dots, T_m, T_1^{-1}, \dots, T_m^{-1}]$$

to be algebraically independent in terms of the Jacobian

$$J := \left(\frac{\partial f_i}{\partial T_j} \right)_{1 \leq i \leq n, 1 \leq j \leq m}$$

which is an n by m matrix with entries from $A[T_1, \dots, T_m, T_1^{-1}, \dots, T_m^{-1}]$.

For a homomorphism $\phi : A \rightarrow \overline{\mathbb{Q}}$ we denote by

$$\begin{aligned} \phi \otimes 1 : A[\underline{T}, \underline{T}^{-1}] &:= A[T_1, \dots, T_m, T_1^{-1}, \dots, T_m^{-1}] = A \otimes \mathbb{Z}[\underline{T}, \underline{T}^{-1}] \\ &\rightarrow \overline{\mathbb{Q}}[\underline{T}, \underline{T}^{-1}] := \overline{\mathbb{Q}}[T_1, \dots, T_m, T_1^{-1}, \dots, T_m^{-1}] = \overline{\mathbb{Q}} \otimes \mathbb{Z}[\underline{T}, \underline{T}^{-1}] \end{aligned}$$

the base change of ϕ . Then $(\phi \otimes 1)(J)$ is a matrix with coefficients in

$$\overline{\mathbb{Q}}[\underline{T}, \underline{T}^{-1}] \subset Q := \overline{\mathbb{Q}}(T_1, \dots, T_m),$$

where Q is the field of rational functions in m variables over $\overline{\mathbb{Q}}$. For a homomorphism $\psi : \overline{\mathbb{Q}}[\underline{T}, \underline{T}^{-1}] \rightarrow K$ with K some field, $(\psi \circ (\phi \otimes 1))(J)$ is a matrix with coefficients in K . Finally, by $\text{rk}_L(M)$ we will denote the rank of a matrix M having coefficients in the field L . The Jacobi criterion for algebraic independence mentioned above is the following.

Lemma 10. *Keeping the above notations we have:*

- 1) *The following are equivalent:*
 - i) *The $f_i \in A[\underline{T}, \underline{T}^{-1}]$ are algebraically dependent.*
 - ii) *For all $\phi : A \rightarrow \overline{\mathbb{Q}}$ we have*

$$\text{rk}_Q((\phi \otimes 1)(J)) < n.$$

- 2) *The following are equivalent:*
 - i) *The $f_i \in A[\underline{T}, \underline{T}^{-1}]$ are algebraically independent.*
 - ii) *There are homomorphisms $\phi : A \rightarrow \overline{\mathbb{Q}}$ and $\psi : \overline{\mathbb{Q}}[\underline{T}, \underline{T}^{-1}] \rightarrow K$ (K some field) such that*

$$\text{rk}_K(\psi \circ (\phi \otimes 1)(J)) = n.$$

The proof of Lemma 10 consists in a reduction, which we will omit, to the case when A is a field of characteristic zero. In this case the Lemma is [Bo], Chapter AG, Theorem 17.3. We conclude this subsection by giving two applications.

Theorem 11. *Let k be a finite field and let $E_1, E_2/k$ be non-isogeneous ordinary elliptic curves. Let E'_2/k be the quadratic twist of E_2 , and consider the abelian surfaces $A_1 := E_1 \times E_2$ and $A_2 := E_1 \times E'_2$ over k and the associated Galois representations $V_i := H_c^1(\overline{A_i}) \in \text{Rep}_{G_k} \mathbb{Q}_l$. Let $k \subset L$ be the unique quadratic extension of k inside \overline{k} . Then:*

- i) $\text{Res}_{G_L}^{G_k}(V_1) \simeq \text{Res}_{G_L}^{G_k}(V_2)$.
- ii) *The classes $[V_1]$ and $[V_2]$ are algebraically independent in $K_0(\text{Rep}_{G_k} \mathbb{Q}_l)$. In particular, the classes $[A_1]$ and $[A_2]$ are algebraically independent in $K_0(\text{Var}_k)$.*

Remark. The representations V_1 and V_2 are algebraically independent but not geometrically algebraically independent. In fact, after restriction to the subgroup of index two of G_k their classes become algebraically dependent and in fact equal. We also see that $x := [A_1] - [A_2] \in K_0(\text{Var}_k)$ generates a polynomial ring over \mathbb{Z} inside $K_0(\text{Var}_k)$ even though x lies in the kernel of the base change homomorphism $K_0(\text{Var}_k) \rightarrow K_0(\text{Var}_L)$. This shows that $K_0(\text{Var}_k)$ encodes fine arithmetic invariants of varieties over k .

Proof. Part i) is clear because $A_1 \times_k L \simeq A_2 \times_k L$, and the last assertion follows from part ii) and Corollary 3. We prove ii): Fix $\pi_i \in \overline{\mathbb{Q}}^*$ a Weil number attached to E_i/k and denote by $\overline{\pi}_i$ the conjugate of π_i . Note that $\mathbb{Q}(\pi_1)$ and $\mathbb{Q}(\pi_2)$ are distinct imaginary quadratic fields, hence

$$(31) \quad \mathbb{Q}(\pi_1) \cap \mathbb{Q}(\pi_2) = \mathbb{Q}.$$

The eigenvalues of the geometric Frobenius F_k acting on the H_c^1 of E_1, E_2 and E'_2 are (respectively) $\{\pi_1, \overline{\pi}_1\}$, $\{\pi_2, \overline{\pi}_2\}$ and $\{-\pi_2, -\overline{\pi}_2\}$. Fixing a finite extension $\mathbb{Q}_l \subset K$ containing π_1 and π_2 we get in the notation of (26):

$$\phi(F_k) = \text{diag}(\pi_1, \overline{\pi}_1, \pi_2, \overline{\pi}_2, \pi_1, \overline{\pi}_1, -\pi_2, -\overline{\pi}_2) \in \mathbb{D}_8(K)$$

and

$$X(D) \simeq \langle \pi_1, \overline{\pi}_1, \pi_2, \overline{\pi}_2, -1 \rangle \subset K^*.$$

Using (31) and the fact that E_1 and E_2 are ordinary, one checks that the structure of $X(D)$ is as follows:

$$(32) \quad X(D) = \{\pm 1\} \oplus \pi_1^{\mathbb{Z}} \oplus (\overline{\pi}_1)^{\mathbb{Z}} \oplus \pi_2^{\mathbb{Z}}.$$

From (32) we get as in (29) together with (30)

$$(33) \quad K_0(\text{Rep}_K D) \simeq (\mathbb{Z}[S]/(S^2 - 1))[T_1, T_1^{-1}, \dots, T_3, T_3^{-1}]$$

and compute the characters $f_i := \text{ch}([V_i])$ as $f_1 = e^{\pi_1} + e^{\overline{\pi}_1} + e^{\pi_2} + e^{\overline{\pi}_2}$ and $f_2 = e^{\pi_1} + e^{\overline{\pi}_1} + e^{-\pi_2} + e^{-\overline{\pi}_2}$. We have $\overline{\pi}_2 = \pi_1 \overline{\pi}_1 \pi_2^{-1}$ and using this, (32) and (33) we get (denoting abusively the images of $[V_i]$ under (33) by f_i again)

$$\begin{aligned} f_1 &= T_1 + T_2 + T_3 + T_1 T_2 T_3^{-1} \text{ and} \\ f_2 &= T_1 + T_2 + S T_3 + S T_1 T_2 T_3^{-1}. \end{aligned}$$

To show that f_1 and f_2 are algebraically independent we compute the Jacobian of f_1 and f_2

$$J = \begin{pmatrix} 1 + T_2T_3^{-1} & 1 + T_1T_3^{-1} & 1 - T_1T_2T_3^{-2} \\ 1 + ST_2T_3^{-1} & 1 + ST_1T_3^{-1} & S - ST_1T_2T_3^{-2} \end{pmatrix}$$

which we specialize by putting $S = -1, T_1 = -1, T_2 = T_3 = 1$ to obtain

$$\begin{pmatrix} 2 & 0 & 2 \\ 0 & 2 & -2 \end{pmatrix}$$

which has rank two, hence f_1 and f_2 are algebraically independent by Lemma 10, as was to be shown. □

Our next application of the methods developed so far is the following.

Theorem 12. *Let k be a finite field. Then there is a sequence of proper, smooth and geometrically connected curves X_i/k ($i \geq 1$) such that the classes $[X_i] \in K_0(\text{Var}_k)$ are algebraically independent.*

Theorem 13. *Let k be a number field and $\{E_i\}_{i \in I}$ a set of elliptic curves over k such that the E_i are pairwise non-isogeneous and satisfy $\text{End}_{\bar{k}}(E_i) = \mathbb{Z}$. Then the classes $[E_i] \in K_0(\text{Var}_k)$ are algebraically independent.*

Note that over any number field there are infinite sets of elliptic curves satisfying the assumptions of Theorem 13.

To prove these results we will need the following sufficient criterion for geometric algebraic independence in terms of Frobenius eigenvalues.

Lemma 14. *Let $\hat{\mathbb{Z}}$ denote the pro-finite completion of \mathbb{Z} and let K/\mathbb{Q}_l be a finite extension and let $V_1, \dots, V_n \in \text{Rep}_{\hat{\mathbb{Z}}} K$ be given. For $1 \leq i \leq n$ put $m_i := \dim_K V_i$ and let*

$$\{\lambda_{i,j}\}_{1 \leq j \leq m_i} \subset \overline{K}^*$$

be the eigenvalues of $F := 1$ acting on V_i . For any $1 \leq k \leq n$ consider the finite-dimensional \mathbb{Q} -vector space

$$(34) \quad A_k := \langle \lambda_{i,j} \rangle_{1 \leq i \leq k, 1 \leq j \leq m_i} \otimes_{\mathbb{Z}} \mathbb{Q}$$

and set $A_0 := 0$. If

$$(35) \quad \dim_{\mathbb{Q}} A_k > \dim_{\mathbb{Q}} A_{k-1} \text{ for } k = 1, \dots, n,$$

then the $V_1, \dots, V_n \in \text{Rep}_{\hat{\mathbb{Z}}} K$ are geometrically algebraically independent.

For the application of this lemma in the proof of Theorem 13 we make its assumptions more explicit in the case when the V_i are (dual to) the Tate-modules of elliptic curves.

Lemma 15. *Let \mathbb{F}_q be a finite field and identify $G_{\mathbb{F}_q} = \hat{\mathbb{Z}}$ using the geometric Frobenius. Let $E_1, \dots, E_n/\mathbb{F}_q$ be elliptic curves, assume that E_1 is ordinary, put $V_i := H_c^1(\overline{E_i}) \in \text{Rep}_{G_{\mathbb{F}_q}}(\mathbb{Q}_l)$ and define the A_i as in Lemma 14. Then the assumptions of this Lemma are satisfied if and only if $\dim_{\mathbb{Q}} A_n = n + 1$.*

Proof. Since E_1 is ordinary we have $\dim_{\mathbb{Q}} A_1 = 2$. For $k = 2, \dots, n$ we have $\dim_{\mathbb{Q}} A_k \leq \dim_{\mathbb{Q}} A_{k-1} + 1$ because $\lambda_{k,1}\lambda_{k,2} = q \in A_{k-1}$. The assertion is obvious now. □

Proof of Lemma 14. In the notation of (26) we have

$$\phi(F) = \text{diag}(\lambda_{1,1}, \dots, \lambda_{1,m_1}, \dots, \lambda_{n,1}, \dots, \lambda_{n,m_n}) \in \mathbb{D}_{\sum_{i=1}^n m_i}(K).$$

As a first step we will prove that the characters $f_i := \text{ch}([V_i]) \in K_0(\underline{\text{Rep}}_K D)$ are algebraically independent. Here $D \subset \mathbb{D}_{\sum_{i=1}^n m_i}$ is the smallest algebraic subgroup with $\phi(F) \in D(K)$. After suitably enlarging K (legitimate by (25)) we identify (as in (27)) the character group $X(D)$ of D ,

$$(36) \quad X(D) \simeq \langle \lambda_{i,j} \rangle_{1 \leq i \leq n, 1 \leq j \leq m_i} \subset K^*,$$

and choose a decomposition

$$(37) \quad X(D) = \langle \zeta \rangle \bigoplus_{i=1}^N \pi_i^{\mathbb{Z}}$$

with $\zeta \in K^*$ a root of unity of order M , say, and $\pi_1, \dots, \pi_N \in K^*$ a \mathbb{Z} -basis of the free part of $X(D)$. We order the π_i 's such that their images $\pi_i \otimes 1$ in $X(D) \otimes \mathbb{Q} = A_n$ satisfy the following: For all $1 \leq k \leq n$, $\pi_1 \otimes 1, \dots, \pi_{d_k} \otimes 1$ is a basis of A_k . Here $d_k := \dim_{\mathbb{Q}} A_k$ and (35) gives

$$(38) \quad d_0 = 0 < d_1 < \dots < d_n = N.$$

Note that

$$(39) \quad \begin{aligned} N &= \dim(D) = \dim_{\mathbb{Q}}(X(D) \otimes \mathbb{Q}) \\ &\stackrel{(36)}{=} \dim_{\mathbb{Q}}(\langle \lambda_{i,j} \rangle_{1 \leq i \leq n, 1 \leq j \leq m_i} \otimes \mathbb{Q}) \\ &\stackrel{(34)}{=} \dim_{\mathbb{Q}}(A_n) \stackrel{(35)}{\geq} n. \end{aligned}$$

For any $1 \leq i \leq n$ and $1 \leq j \leq m_i$ we decompose, according to (37),

$$(40) \quad \lambda_{i,j} = \zeta^{a_{i,j}} \pi_1^{a_{1,i,j}} \dots \pi_N^{a_{N,i,j}}$$

for suitable $a_{i,j}, a_{k,i,j} \in \mathbb{Z}$. Using (29)

$$K_0(\underline{\text{Rep}}_K D) \simeq A[T_1, T_1^{-1}, \dots, T_N, T_N^{-1}]$$

where, according to (30), we have $A \simeq \mathbb{Z}[S]/(S^M - 1)$. The characters under consideration are given as follows (taking (33) as an identification):

$$(41) \quad \begin{aligned} f_i &= \text{ch}([V_i]) = \sum_{j=1}^{m_i} \lambda_{i,j} \stackrel{(40)}{=} \sum_{j=1}^{m_i} \zeta^{a_{i,j}} \pi_1^{a_{1,i,j}} \dots \pi_N^{a_{N,i,j}} \\ &= \sum_{j=1}^{m_i} S^{a_{i,j}} T_1^{a_{1,i,j}} \dots T_N^{a_{N,i,j}}, \quad 1 \leq i \leq n. \end{aligned}$$

We define $\phi : A \rightarrow \overline{\mathbb{Q}}$ by $S \mapsto 1$ and

$$\psi : \overline{\mathbb{Q}}[T_1, T_1^{-1}, \dots, T_N, T_N^{-1}] \hookrightarrow Q := \overline{\mathbb{Q}}(T_1, \dots, T_N)$$

to be the natural inclusion. Now we show that the $[V_i] \in K_0(\underline{\text{Rep}}_{\overline{\mathbb{Q}}} K)$ are algebraically independent. Using Lemma 10 it suffices to show that $\psi \circ (\phi \otimes 1)(J)$ has rank n over Q . Here J is the Jacobi matrix $J = \left(\frac{\partial f_i}{\partial T_j} \right)_{1 \leq i \leq n, 1 \leq j \leq N}$. Since $n \leq N$, by (39) this is equivalent to the vectors

$$v_i := \left(\psi \circ (\phi \otimes 1) \left(\frac{\partial f_i}{\partial T_j} \right) \right)_{1 \leq j \leq N}, \quad 1 \leq i \leq n,$$

being linearly independent over Q . Let us abbreviate $x_{i,j} := \psi \circ (\phi \otimes 1) \left(\frac{\partial f_i}{\partial T_j} \right) \in Q$, hence $v_i = (x_{i,j})_{1 \leq j \leq N}$. We first prove the following claim concerning the shape of the matrix $(x_{i,j})$.

Claim. For every $1 \leq i \leq n$ there is some $1 \leq j_i \leq N$ such that

$$(42) \quad \begin{aligned} x_{i',j_i} &= 0 \text{ for } i' < i \text{ and} \\ x_{i,j_i} &\neq 0. \end{aligned}$$

Indeed, given $1 \leq i \leq n$ let j_i be any integer satisfying $d_{i-1} < j_i \leq d_i$ which exists by (38). This choice guarantees that, in the notation of (41), we have

$$\begin{aligned} a_{j_i,i,l} &\geq 1 \text{ for some } 1 \leq l \leq m_i \text{ and} \\ a_{j_i,i',l} &= 0 \text{ for } i' < i \text{ and all } l, \end{aligned}$$

and the claim follows. Put a little less formally: In (41) there is, for every i , some variable T_{j_i} which occurs in f_i but does not occur in any $f_{i'}$ with $i' < i$. This is where (35) is used. Now the linear independence of v_1, \dots, v_n over Q follows. Assume by contradiction that we are given $\alpha_1, \dots, \alpha_k \in Q$, $\alpha_k \neq 0$, such that $\sum_{i=1}^k \alpha_i v_i = 0$. Taking the j_k -component of this relation we get $\sum_{i=1}^k \alpha_i x_{i,j_k} = 0$ in Q which by (42) simplifies to $\alpha_k x_{k,j_k} = 0$ with $x_{k,j_k} \neq 0$, hence $\alpha_k = 0$, a contradiction. At this point the proof of the fact that the classes $[V_i] \in K_0(\text{Rep}_{\mathbb{Z}} K)$ are algebraically independent is complete.

To show that they are in fact geometrically algebraically independent, let an open subgroup of finite index $T' \subset \hat{\mathbb{Z}}$ be given. There is an integer $L \geq 1$ such that $T' = L\hat{\mathbb{Z}}$ and the set of eigenvalues of the topological generator $L \in T'$ acting on $\text{Res}_{T'}^{\hat{\mathbb{Z}}}(V_i)$ is $\{\lambda'_{i,j}\}_{i,j} = \{\lambda^L_{i,j}\}_{i,j}$. So the homomorphism

$$\langle \lambda_{i,j} \rangle_{1 \leq i \leq k, 1 \leq j \leq m_i} \longrightarrow \langle \lambda'_{i,j} \rangle_{1 \leq i \leq k, 1 \leq j \leq m_i}, \quad x \mapsto x^L,$$

is surjective with finite kernel and induces an isomorphism

$$A_k \xrightarrow{\sim} \langle \lambda'_{i,j} \rangle_{1 \leq i \leq k, 1 \leq j \leq m_i} \otimes_{\mathbb{Z}} \mathbb{Q}$$

for all $1 \leq k \leq n$. Thus the tuple $(T', \text{Res}_{T'}^{\hat{\mathbb{Z}}}(V_i))$ satisfies the analogue of condition (35), and by what has already been proved we see that the $[\text{Res}_{T'}^{\hat{\mathbb{Z}}}(V_i)] \in K_0(\text{Rep}_{T'} K)$ are algebraically independent. As T' was arbitrary this shows that the V_i are in fact geometrically algebraically independent and concludes the proof. \square

Before engaging into the slightly involved proof of Theorem 12 we remark that if in this result one replaces curves by abelian varieties one can give a quick proof as follows: The subgroup $\Gamma \subset \overline{\mathbb{Q}}^*$ generated by q -Weil numbers of weight one has infinite rank. Choosing a suitable sequence of such Weil numbers and corresponding abelian varieties (which exist by T. Honda), one obtains the result using Lemma 14 and Corollary 3. Even though $\text{rk}(\Gamma) = \infty$ is well known we could not find it stated explicitly in the literature, so we record this result (which will be clear from our proof of Theorem 12) for reference.

Corollary 16. *Let q be a prime power. Then the subgroup of $\overline{\mathbb{Q}}^*$ generated by q -Weil numbers of weight one has infinite rank.*

We now start the proof of Theorem 12. In fact, we will give a family of curves satisfying the conclusion of Theorem 12 explicitly.

Let $\mathbb{F}_p \subset k$ be the prime field of k and for any $a \geq 1$ let Y_a/\mathbb{F}_p be the proper smooth curve which is birational to the affine plane curve given by

$$y^p - y = x^{p^a - 1}.$$

The Y_a are geometrically connected of genus $\frac{1}{2}(p^a - 2)(p - 1)$. The arithmetic of these curves has been studied by H. Davenport and H. Hasse [DH]. Unfortunately, we cannot assert that the sequence $\{Y_a\}_{a \geq 1}$ satisfies the conclusion of Theorem 12 but a suitable subsequence will do. Let $T = T(p)$ be the integer determined by Lemma 19 below and let $\{n_i\}_{i \geq 1}$ be the sequence of prime numbers greater than T ordered increasingly. Then the

$$X_i := Y_{n_i} \otimes_{\mathbb{F}_p} k, \quad i \geq 1,$$

are proper, smooth and geometrically connected curves over k , and the following result makes the existence statement of Theorem 12 explicit.

Theorem 17. *The classes $[X_i] \in K_0(\text{Var}_k)$ ($i \geq 1$) are algebraically independent.*

For any $i \geq 1$ let us write $V_i := H_c^1(\overline{Y}_{n_i}) \in \text{Rep}_{G_{\mathbb{F}_p}} \mathbb{Q}_l$. Using Corollary 3, Theorem 17 results from the following.

Theorem 18. *The classes $V_i \in \text{Rep}_{G_{\mathbb{F}_p}} \mathbb{Q}_l$ ($i \geq 1$) are geometrically algebraically independent.*

This will be proved using Lemma 14 for which we need to review some results concerning the Weil numbers attached to the Jacobians of the curves Y_a/\mathbb{F}_p .

We fix a valuation v of $\overline{\mathbb{Q}}$ lying over p and normalized by $v(p) = 1$. We will denote by the same letter v the restriction of v to any subfield $K \subset \overline{\mathbb{Q}}$.

Let F_p be the geometric Frobenius of \mathbb{F}_p . Then the eigenvalues of F_p^a acting on $H_c^1(\overline{Y}_a)$ are given by generalized Jacobi sums

$$(43) \quad \tau_j(\chi^t)^{(a)} := - \sum_{u \in \mathbb{F}_{p^a}^*} \chi^t(u) \exp\left(\frac{2\pi i}{p} \cdot j \cdot \text{tr}(u)\right), \quad 1 \leq j \leq p - 1, \quad 1 \leq t \leq p^a - 2,$$

where $\chi : \mathbb{F}_{p^a}^* \hookrightarrow \mathbb{C}^*$ is a certain faithful character of $\mathbb{F}_{p^a}^*$ (a Teichmüller lift depending on the choice of v ; see [Ma], §4) and $\text{tr} : \mathbb{F}_{p^a} \rightarrow \mathbb{F}_p$ denotes the trace, [Y], (2).

We will abbreviate $\tau(\chi^t)^{(a)} := \tau_1(\chi^t)^{(a)}$ in the following. Let $1 \leq t \leq p^a - 2$ be given and let

$$t = \sum_{i=0}^{a-1} j_i p^i, \quad 0 \leq j_i \leq p - 1,$$

be the p -adic expansion of t and put $\sigma(t) := \sum_{i=0}^{a-1} j_i$. Then [Ma], §4, gives the valuations of the generalized Jacobi sums as

$$(44) \quad v(\tau(\chi^t)^{(a)}) = \frac{\sigma(t)}{p - 1}.$$

We will need the following lower bound on Euler's phi-function, $\phi(n) := |(\mathbb{Z}/n)^*|$ for $n \geq 1$.

Lemma 19. *Let p be a prime number. Then there is an integer $T = T(p)$ such that*

$$\frac{\phi(p^n - 1)}{n} > \phi(p - 1) \text{ for all } n \geq T.$$

Proof. This follows from the inequality, valid for any $m \geq 1$,

$$m - 1 \geq \phi(m) \geq \frac{m}{C \log \log m},$$

with a suitable positive constant C . The first inequality is trivial and the second is [RS], Theorem 15. \square

Proof of Theorem 18. It suffices to show that for any given $M \geq 1$ the $V_1, \dots, V_M \in \text{Rep}_{G_{\mathbb{F}_p}} \mathbb{Q}_l$ are geometrically algebraically independent. So we fix some $M \geq 1$. For any $1 \leq i \leq M$ let $\{\lambda_{i,j}\}_{1 \leq j \leq (p^{n_i}-2)(p-1)} \subset \overline{\mathbb{Q}}^*$ be the eigenvalues of F_p acting on V_i . Note that

$$(45) \quad \{\lambda_{i,j}^{n_i}\}_{1 \leq j \leq (p^{n_i}-2)(p-1)} = \{\tau_j(\chi^t)^{(n_i)}\}_{1 \leq j \leq p-1, 1 \leq t \leq p^{n_i}-2}$$

by (43). We define $B_0 := \{1\} \subset \overline{\mathbb{Q}}^*$ and for $1 \leq k \leq M$ we define $B_k := \langle \lambda_{i,j} \rangle_{1 \leq i \leq k, \text{all } j} \subset \overline{\mathbb{Q}}^*$ to be the subgroup of $\overline{\mathbb{Q}}^*$ generated by the F_p -eigenvalues occurring among the V_1, \dots, V_k and $A_k := B_k \otimes_{\mathbb{Z}} \mathbb{Q}$, $d_k := \dim_{\mathbb{Q}} A_k$. Using Lemma 14 the proof will be complete if we establish the following:

$$(46) \quad \text{We have } d_k > d_{k-1} \text{ for } k = 1, \dots, M.$$

As $B_{k-1} \subset B_k$ we always have $d_k \geq d_{k-1}$, and we need to show that this inequality is strict. For any $n \geq 1$ we denote by $K_n \subset \overline{\mathbb{Q}}$ the field of n -th roots of unity and we fix isomorphisms $\text{Gal}(K_n/\mathbb{Q}) \simeq (\mathbb{Z}/n)^*$ using a compatible system of roots of unity in $\overline{\mathbb{Q}}^*$. Recall that $K_n \cap K_m = K_{\text{gcd}(n,m)}$ and $K_n K_m = K_{\text{lcm}(n,m)}$, where gcd and lcm mean greatest common divisor and least common multiple, respectively. From (45) and (43) we see that

$$(47) \quad \lambda_{i,j}^{n_i} \in K_{p^{n_i-1}} K_p = K_{p(p^{n_i-1})},$$

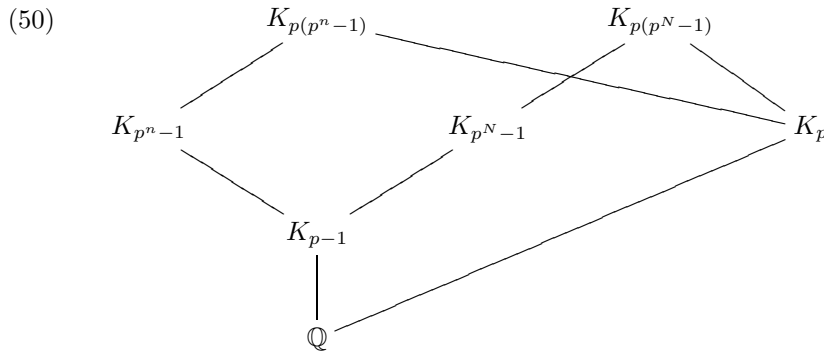
hence, introducing $N_k := \prod_{i=1}^k n_i$ for $0 \leq k \leq M$, we have that

$$(48) \quad B_k^{N_k} := \{x^{N_k} : x \in B_k\} \subset K_{p(p^{N_k-1})} \text{ for } k = 0, \dots, M.$$

We now prove (46) by contradiction, assuming that we are given some $1 \leq k \leq M$ such that

$$(49) \quad d_k = d_{k-1}.$$

As k will be fixed from now on, we put $n := n_k$ and $N := N_{k-1}$ to ease the reading. The cyclotomic fields to play a role are organized in the following diagram:



Observe that by construction $\text{gcd}(n, N) = 1$ (N is a product of primes different from the prime n) and hence $\text{gcd}(p^n - 1, p^N - 1) = p - 1$ which implies as recalled

above that $K_{p^n-1} \cap K_{p^{n-1}} = K_{p-1}$ as asserted by (50). The remaining assertions implicit in (50) are checked similarly. According to (43) there are λ_1 and λ_2 which are eigenvalues of F_p acting on V_k (hence $\lambda_i \in B_k$ by the definition of B_k) and such that

$$(51) \quad \begin{aligned} \lambda_1^n &= \tau(\chi^1)^{(n)}, \\ \lambda_2^n &= \tau(\chi^{p^n-2})^{(n)}; \end{aligned}$$

hence $\lambda_i^n \in K_{p(p^n-1)}$ by (43) and by [Y], (14) we have

$$(52) \quad \tau_i := \lambda_i^{n(p-1)} \in K_{p^n-1}, \quad i = 1, 2.$$

The idea now is the following. We will exhibit two places v_1 and v_2 of K_{p^n-1} lying above the same place of K_{p-1} such that a suitable linear combination x of τ_1 and τ_2 has different valuations at v_1 and v_2 , and using (49) we will see that we may choose x to lie in K_{p-1} which will give the desired contradiction. Here are the details.

The inclusion $K_{p-1} \subset K_{p^n-1}$ corresponds to a surjection of Galois groups

$$\pi : (\mathbb{Z}/p^n - 1)^* \twoheadrightarrow (\mathbb{Z}/p - 1)^*.$$

The decomposition group of the place $v_1 := v$ of K_{p^n-1} is generated by the residue of p (since p is unramified in K_{p^n-1}) and π factors over a surjection

$$(53) \quad \bar{\pi} : (\mathbb{Z}/p^n - 1)^* / \langle \bar{p} \rangle \twoheadrightarrow (\mathbb{Z}/p - 1)^*.$$

The left-hand side of (53) acts simply transitively by conjugation on the set of places of K_{p^n-1} lying above p , and this will be used to construct a second such place v_2 . The order of the left-hand side of (53) is $\phi(p^n - 1)/n > \phi(p - 1) = |(\mathbb{Z}/p - 1)^*|$ by Lemma 19 and our choice that $n \geq T(p)$ so we may choose

$$(54) \quad 1 \neq \gamma \in \ker(\bar{\pi}).$$

We denote by v_2 the place of K_{p^n-1} conjugate by γ^{-1} to v_1 , i.e. $v_2(x) = v_1(\gamma.x) = v(\gamma.x)$ for all $x \in K_{p^n-1}$. Now we compute some valuations:

$$\begin{aligned} v_1(\tau_1) &\stackrel{(52)}{=} (p-1)v(\lambda_1^n) \stackrel{(51),(44)}{=} (p-1)\frac{\sigma(1)}{p-1} = 1 \text{ and} \\ v_1(\tau_2) &\stackrel{(52)}{=} (p-1)v(\lambda_2^n) \stackrel{(51),(44)}{=} (p-1)\frac{\sigma(p^n-2)}{p-1} = n(p-1) - 1, \end{aligned}$$

using for the computation of $\sigma(p^n - 2)$ that

$$p^n - 2 = p^n - 1 - 1 = \sum_{i=0}^{n-1} (p-1)p^i - 1 = (p-2) + \sum_{i=1}^{n-1} (p-1)p^i.$$

Let the p -adic expansion of γ be given by

$$(55) \quad \gamma = \sum_{i=0}^{n-1} \alpha_i p^i, \quad 0 \leq \alpha_i \leq p-1.$$

Observe that

$$\sigma(\gamma) = \sum_{i=0}^{n-1} \alpha_i \equiv \bar{\pi}(\gamma) \stackrel{(54)}{\equiv} 1 \pmod{p-1}.$$

If we had $\sigma(\gamma) = 1$, then [Y], Lemma 3, ii), would imply that $\gamma \in p^{\mathbb{Z}}$, hence $\gamma = 1$ in $(\mathbb{Z}/p^n - 1)^*/\langle \bar{p} \rangle$, contrary to our choice (54). So we have

$$(56) \quad \sigma(\gamma) = 1 + a(p - 1) \text{ for some } a \geq 1.$$

Now we compute the valuations at v_2 :

$$\begin{aligned} v_2(\tau_1) &\stackrel{(52),(51)}{=} (p - 1)v_2(\tau(\chi^1)^{(n)}) = (p - 1)v(\gamma \cdot \tau(\chi^1)^{(n)}) \\ &= (p - 1)v(\tau(\chi^\gamma)^{(n)}) \stackrel{(44)}{=} (p - 1)\frac{\sigma(\gamma)}{p - 1} \stackrel{(56)}{=} 1 + a(p - 1). \end{aligned}$$

In the third term we consider γ as an element of $\text{Gal}(K_{p^n-1}/\mathbb{Q})$, and the second equality follows from our definition of v_2 . The third equality uses $\gamma \cdot \tau(\chi^t)^{(n)} = \tau(\chi^{\gamma t})^{(n)}$ which follows from (43). Similarly we have

$$\begin{aligned} v_2(\tau_2) &\stackrel{(52),(51)}{=} (p - 1)v_2(\tau(\chi^{p^n-2})^{(n)}) = (p - 1)v(\tau(\chi^{\gamma(p^n-2)})^{(n)}) \\ &= (p - 1)v(\tau(\chi^{-\gamma})^{(n)}) \stackrel{(44)}{=} (p - 1)\frac{\sigma(-\gamma)}{p - 1} = \sigma(-\gamma) \stackrel{(*)}{=} n(p - 1) - \sigma(\gamma) \\ &\stackrel{(56)}{=} n(p - 1) - (1 + a(p - 1)) = (n - a)(p - 1) - 1. \end{aligned}$$

In this sequence of equalities, we would like to justify (*), which is an elementary property of the function σ : From (64) we get the p -adic expansion of $-\gamma$:

$$-\gamma = -\sum_{i=0}^{n-1} \alpha_i p^i \stackrel{(\text{mod } p^n-1)}{=} p^n - 1 - \sum_{i=0}^{n-1} \alpha_i p^i = \sum_{i=0}^{n-1} (p - 1 - \alpha_i) p^i,$$

hence

$$\sigma(\gamma) + \sigma(-\gamma) = \sum_{i=0}^{n-1} \alpha_i + \sum_{i=0}^{n-1} (p - 1 - \alpha_i) = n(p - 1),$$

as used in (*). To sum up, we have computed the matrix

$$X := \begin{pmatrix} v_1(\tau_1) & v_1(\tau_2) \\ v_2(\tau_1) & v_2(\tau_2) \end{pmatrix} = \begin{pmatrix} 1 & n(p - 1) - 1 \\ 1 + a(p - 1) & (n - a)(p - 1) - 1 \end{pmatrix}$$

and compute its determinant

$$\begin{aligned} (57) \quad \det(X) &= (n - a)(p - 1) - 1 - (1 + a(p - 1))(n(p - 1) - 1) \\ &= (n - a)(p - 1) - 1 - (n(p - 1) - 1 + an(p - 1)^2 - a(p - 1)) \\ &= -an(p - 1)^2 \neq 0, \end{aligned}$$

because $a \neq 0$ by (56). If $\mu, \nu \in \mathbb{Z}$ are given and we put $x := \tau_1^\mu \tau_2^\nu \in K_{p^n-1}$ we have

$$\begin{pmatrix} v_1(x) \\ v_2(x) \end{pmatrix} = X \begin{pmatrix} \mu \\ \nu \end{pmatrix}.$$

By (57) there is some integer $R \geq 1$ and $\mu, \nu \in \mathbb{Z}$ such that $X \begin{pmatrix} \mu \\ \nu \end{pmatrix} = \begin{pmatrix} R \\ 0 \end{pmatrix}$, so if we put $x := \tau_1^\mu \tau_2^\nu$ for these particular values of μ and ν we get $x \in K_{p^n-1}$ with

$$(58) \quad v_1(x) = R \geq 1 \text{ and } v_2(x) = 0.$$

Note that for the restrictions of v_1 and v_2 to K_{p-1} we have

$$(59) \quad v_1|_{K_{p-1}} = v_2|_{K_{p-1}}$$

because the restriction of $\gamma \in \text{Gal}(K_{p^n-1}/\mathbb{Q})$ to K_{p-1} is trivial, i.e. $\overline{\pi}(\gamma) = 1$; cf. (54). Now we can finally derive a contradiction. By (49) we know that B_k/B_{k-1} is torsion. As $\lambda_1, \lambda_2 \in B_k$ we find an integer $S \geq 1$ with

$$(60) \quad \lambda_i^S \in B_{k-1}^N \stackrel{(48)}{\subset} K_{p(p^N-1)}, \quad i = 1, 2$$

(recall that $N = N_{k-1}$), and furthermore $x^S = (\tau_1^\mu \tau_2^\nu)^S \stackrel{(52)}{=} (\lambda_1^\mu \lambda_2^\nu)^{n(p-1)S}$, so $x^S \in K_{p^n-1}$ by (52) and $x^S \in K_{p(p^N-1)}$ by (60), from which we conclude that

$$(61) \quad x^S \in K_{p^n-1} \cap K_{p(p^N-1)} \stackrel{(50)}{=} K_{p-1}.$$

This implies $SR \stackrel{(58)}{=} v_1(x^S) \stackrel{(61),(59)}{=} v_2(x^S) \stackrel{(58)}{=} 0$, a contradiction which concludes the proof. \square

Proof of Theorem 13. We can assume that the given set of elliptic curves $\{E_i\}_{i \in I} = \{E_1, \dots, E_n\}$ is finite. We will show that the classes of $V_i := H_c^1(\overline{E}_i)$ in $K_0(\text{Rep}_{G_k} \mathbb{Q}_l)$ are algebraically independent which is sufficient by Corollary 3. We consider the subring $R := \mathbb{Z}[[V_i]] \subset K_0(\text{Rep}_{G_k} \mathbb{Q}_l)$ generated by the $[V_i]$. For any finite place v of k with residue field $k(v)$ at which all the E_i have good reduction, we have a canonical homomorphism $\phi_v : R \rightarrow K_0(\text{Rep}_{G_{k(v)}} \mathbb{Q}_l)$ because all V_i are unramified at v . We are going to construct such a place v so that the $\phi_v([V_i])$ are algebraically independent. For $A := E_1 \times \dots \times E_n$ we have $H_c^1(\overline{A}) \simeq \bigoplus_{i=1}^n V_i$ and denote by $G_l \subset \text{Gl}(H_c^1(\overline{A}))(\mathbb{Q}_l)$ the closure of the image of Galois. Since $\text{End}_{\overline{k}}(E_i) = \mathbb{Z}$ we know that the Hodge group $\text{Hdg}(H^1(E_i))$ is isogeneous to Sl_2 . As the E_i are pairwise non-isogeneous we have that $\text{Hgd}(H^1(A))$ is isogeneous to Sl_2^n , and hence the Mumford-Tate group $\text{MT}(H^1(A))$ has rank $n + 1$. Now observe that the Mumford-Tate conjecture is known to be true for A/k : For $n = 1$ it is a celebrated result of J-P. Serre, and one reduces to this case using [R], Lemma on page 790. We conclude that G_l has rank $n + 1$. By [S5] we find a finite place v of k of good reduction for all E_i and such that the corresponding Frobenius torus $T_v \subset G_l^0$ is maximal, i.e. T_v has rank $n + 1$. But this rank is nothing other than $\dim_{\mathbb{Q}} A_n$ in the notation of Lemma 15 (applied to the reductions at v of the E_i), and using this Lemma we conclude that the $\phi_v([V_i])$ are indeed algebraically independent as was to be shown. \square

3.2. Using a result of Skolem. In the previous subsection we used a Tannakian argument to establish the algebraic independence of l -adic Galois representations given by the cohomology of suitable varieties. In this subsection we approach the same goal using a lemma of Skolem on the shape of the power series expansion of rational functions together with the rationality properties given in Section 2.3. The possibility of this was suggested by reading a note of J-P. Serre [S2].

Let k be a finite field. We will use the motivic measure from Proposition 7,

$$\mu_k : K_0(\text{Var}_k) \rightarrow \mathcal{R}(\mathbb{Q})[T],$$

to give sufficient conditions for the classes of given varieties over k to be algebraically independent in $K_0(\text{Var}_k)$.

The main observation is that, if there is a non-trivial algebraic relation among the motivic measures of some varieties, then, at least after a finite extension of the base field, there will be an *irreducible* relation; see Section 3.3.

For $\nu \geq 1$ we denote by k_ν the unique extension of degree ν of k inside some fixed algebraic closure of k .

3.3. Irreducible equations.

Theorem 20. *Let k be a finite field and let $X_1, \dots, X_n/k$ ($n \geq 1$) be separated and of finite type and assume that $\mu_k([X_1]), \dots, \mu_k([X_n])$ are algebraically dependent in $\mathcal{R}(\mathbb{Q})[T]$. Then there are $M \geq 1$ and an irreducible polynomial $G \in \mathbb{Z}[T_1, \dots, T_n]$ such that*

$$G(\mu_{k_M}([X_1 \times_k k_M]), \dots, \mu_{k_M}([X_n \times_k k_M])) = 0.$$

Remark. This result is non-trivial even for $n = 1$ because $\mathcal{R}(\mathbb{Q})[T]$ contains zero divisors. In the remark following the proof of Theorem 25 we give an example showing the necessity of allowing the finite extension $k \subset k_M$ in Theorem 20.

The starting point of the proof of Theorem 20 is a result due to Skolem which we now explain. For brevity, we will call a subset $X \subset \mathbb{N}$ *good* if it is the union of a finite set and finitely many arithmetic progressions for a single modulus, i.e. X is good if there are $\Sigma \subset \mathbb{N}$ finite, $M \geq 1$ and $I \subset \{0, \dots, M - 1\}$ such that

$$X = \Sigma \bigcup_{i \in I} (i + M\mathbb{N}).$$

The collection of good subsets is stable under finite union and finite intersection. Skolem’s result is the following [S2].

Proposition 21. *Let $f \in \mathbb{Q}(t)$ be a rational function with power series expansion $f = \sum_{\nu \in \mathbb{Z}} a_\nu t^\nu$. Then the set $\{\nu \geq 0 : a_\nu = 0\}$ is good.*

We will use the following consequence of Skolem’s result.

Corollary 22. *For $f \in \mathcal{R}(\mathbb{Q})$ the set $\{\nu \geq 1 : \partial_\nu(f) = 0\}$ is good.*

Indeed, for $f \in \mathcal{R}(\mathbb{Q})$ we have that $\text{dlog}(f)$ is a rational function to which we apply Proposition 21. As the collection of good subsets is stable under finite intersection, we also have the following.

Corollary 23. *For $F \in \mathcal{R}(\mathbb{Q})[T]$ the set $\{\nu \geq 1 : \partial_\nu(F) = 0 \text{ in } \mathbb{Q}[T]\}$ is good.*

Corollary 23 follows from Corollary 22 applied to the coefficients of F . By abuse of notation, in Corollary 23, we have written $\partial_\nu : \mathcal{R}(\mathbb{Q})[T] \rightarrow \mathbb{Q}[T]$ for the map derived from $\partial_\nu : \mathcal{R}(\mathbb{Q}) \subset \Lambda(\mathbb{Q}) \rightarrow \mathbb{Q}$ by sending $T \mapsto T$.

Proof of Theorem 20. By assumption we have some $0 \neq H \in \mathbb{Z}[T_1, \dots, T_n]$ such that

$$(62) \quad H(\mu_k([X_1]), \dots, \mu_k([X_n])) = 0 \text{ in } \mathcal{R}(\mathbb{Q})[T].$$

Decompose $H = \prod_{i=1}^N H_i$ into a product of irreducible $H_i \in \mathbb{Z}[T_1, \dots, T_n]$ and consider

$$(63) \quad X_i := \{\nu \geq 1 | \partial_\nu(H_i(\mu_k([X_1]), \dots, \mu_k([X_n]))) = 0\}, \quad 1 \leq i \leq N.$$

By Corollary 23 the $X_i \subset \mathbb{N}$ are good and for fixed $\nu \geq 1$ we have

$$0 \stackrel{(62)}{=} \partial_\nu(H(\mu_k([X_1]), \dots, \mu_k([X_n]))) = \prod_{i=1}^N \partial_\nu(H_i(\mu_k([X_1]), \dots, \mu_k([X_n]))) \text{ in } \mathbb{Q}[T].$$

As $\mathbb{Q}[T]$ is an integral domain this implies that $\bigcup_{i=1}^N X_i = \mathbb{N}$.

We claim that this implies the existence of $1 \leq i_0 \leq N$ and $M \geq 1$ such that $M\mathbb{N} \subset X_{i_0}$. Indeed, assume that this is not the case and write

$$X_i = \Sigma_i \bigcup_{j \in I_i} (j + M_i\mathbb{N}), \quad i = 1, \dots, N,$$

for suitable finite sets Σ_i , integers $M_i \geq 1$ and subsets $I_i \subset \{0, \dots, M_i - 1\}$. Then $0 \notin I_i$ for $i = 1, \dots, N$. Consider a positive multiple x of the product $M_1 \dots M_N$ so large that $x \notin \bigcup \Sigma_i$. As $\mathbb{N} = \bigcup X_i$ we find i_0 and $j \in I_{i_0}$ such that $x \in j + M_{i_0}\mathbb{N}$ hence $x \equiv j \pmod{M_{i_0}}$ and $x \equiv 0 \pmod{M_{i_0}}$, a contradiction because $j \in I_{i_0}$, hence $j \not\equiv 0 \pmod{M_{i_0}}$, so there do indeed exist $M \geq 1$ and i_0 such that $M\mathbb{N} \subset X_{i_0}$. We will see that $G := H_{i_0}$ and M satisfy the requirements of Theorem 20. By construction, G is irreducible, and it remains to be checked that

$$G(\mu_{k_M}([X_1 \times_k k_M]), \dots, \mu_{k_M}([X_n \times_k k_M])) = 0.$$

Equivalently, as $\bigcap_{\nu \geq 1} \ker(\partial_\nu : \mathcal{R}(\mathbb{Q})[T] \rightarrow \mathbb{Q}[T]) = 0$ we need to show that for all $\nu \geq 1$

$$(64) \quad \partial_\nu(G(\mu_{k_M}([X_1 \times_k k_M]), \dots, \mu_{k_M}([X_n \times_k k_M]))) = 0.$$

Using (19) and $\partial_\nu F_M = \partial_{\nu M} [\text{DG}]$, V, §5, 2.6, we obtain

$$\begin{aligned} &= G(\partial_\nu(\mu_{k_M}([X_1 \times_k k_M]), \dots, \partial_\nu(\mu_{k_M}([X_n \times_k k_M]))) \\ &= G(\partial_\nu(F_M(\mu_k([X_1])), \dots, \partial_\nu(F_M(\mu_k([X_n]))) \\ &= \partial_{\nu M}(G(\mu_k([X_1]), \dots, \mu_k([X_n])). \end{aligned}$$

However, $\nu M \in M\mathbb{N} \subset X_{i_0}$, so this vanishes by construction as $G = H_{i_0}$; see (63). □

3.4. Zero-dimensional varieties. Let k be a finite field. Before applying the results of Section 3.3 we include in this subsection a digression on the subring of $K_0(\text{Var}_k)$ generated by zero-dimensional varieties.

Generally speaking, the difficulty of the problem of deciding whether the classes of given varieties $[X_1], \dots, [X_n]$ are algebraically independent in $K_0(\text{Var}_k)$ grows quite rapidly with n . For $n = 1$ a complete answer can be given.

Theorem 24. *Let k be a finite field and let X/k be separated and of finite type. Then the following are equivalent:*

- 1) $[X] \in K_0(\text{Var}_k)$ is algebraically dependent.
- 2) $[X]$ is integral over \mathbb{Z} .
- 3) $\dim(X) = 0$.

Condition 1) means that $[X]$ satisfies some polynomial with coefficients in \mathbb{Z} but this polynomial does not have to be monic. So, a priori, condition 2) is stronger than condition 1). Let $\tilde{\mathbb{Z}} \subset K_0(\text{Var}_k)$ denote the integral closure of \mathbb{Z} inside $K_0(\text{Var}_k)$ and let $S \subset K_0(\text{Var}_k)$ be the subring generated by the classes of zero-dimensional varieties. By Theorem 24 we have $S \subset \tilde{\mathbb{Z}}$ and even more, for all X/k , if $[X] \in \tilde{\mathbb{Z}}$, then $[X] \in S$. Note that this does not imply $\tilde{\mathbb{Z}} \subset S$ because an element of $K_0(\text{Var}_k)$ is in general only a formal difference of varieties.

In fact, I do not know whether $S \subset \tilde{\mathbb{Z}}$ is an equality.

One can give a simple presentation of the ring S ; see Theorem 25 below.

Proof of Theorem 24. We start by showing that 3) implies 2). Let X/k be zero dimensional, i.e. $X = \text{Spec}(A)$ for some Artinian k -algebra A . As $[X] = [X^{red}]$ we can assume that A is reduced, i.e. $A = \bigoplus L_i$ for $L_i|k$ finite field extensions. Then $[X] = \sum [\text{Spec}(L_i)]$, and we are reduced to showing that for a finite extension field $k \subset L$ we have $[\text{Spec}(L)] \in \tilde{\mathbb{Z}}$. But, if $d = [L : k]$, then

$$\begin{aligned} [\text{Spec}(L)]^2 - d [\text{Spec}(L)] &= [\text{Spec}(L \otimes_k L)] - d [\text{Spec}(L)] \\ &= [\text{Spec}(L^{\oplus d})] - d [\text{Spec}(L)] = 0. \end{aligned}$$

We used that $L|k$ is Galois to have $L \otimes_k L \simeq L^{\oplus d}$ as k -algebras.

That 2) implies 1) is trivial, and we now show that 1) implies 3). Let X/k be separated and of finite type and such that $[X]$ is algebraically dependent. We need to show that $\dim(X)=0$. As $[X] = [X^{red}]$ and $\dim(X)=\dim(X^{red})$ we assume that X is reduced. As k is perfect this implies that the smooth locus of X/k is (open and) dense. This will be used later, but first we exploit the assumption that $[X]$ is algebraically dependent. There is $0 \neq F \in \mathbb{Z}[T_1]$ with $F([X]) = 0$. Let us consider $f := \mu_k([X]) \in \mathcal{R}(\mathbb{Q})[T]$. For any $\nu \geq 1$ we have

$$F(\partial_\nu(f)) = F(\partial_\nu(\mu_k([X]))) = (\partial_\nu \circ \mu_k)(F([X])) = 0.$$

As $\partial_\nu(f)$ lies in $\mathbb{Q}[T]$ and $F \neq 0$, this implies that $\partial_\nu(f)$ is a constant, i.e. $\frac{d}{dT}(\partial_\nu(f)) = 0$. Then also

$$\partial_\nu\left(\frac{df}{dT}\right) = \frac{d}{dT}(\partial_\nu(f)) = 0 \text{ for all } \nu \geq 1$$

and using $\bigcap_{\nu \geq 1} (\ker(\partial_\nu : \mathcal{R}(\mathbb{Q})[T] \rightarrow \mathbb{Q}[T])) = 0$ we get $\frac{df}{dT} = 0$, i.e. f itself is a constant, $f \in \mathcal{R}(\mathbb{Q})$. Now assume by contradiction that $d := \dim(X) \geq 1$. The coefficient of T^{2d} in f is, by definition of μ_k , the image under $\chi_q : K_0(\text{Rep}_{G_k} \mathbb{Q}_l) \hookrightarrow \Lambda(\mathbb{Q}_l)$ of the virtual Galois representation $\sum_j (-1)^j [\text{Gr}_{2d}^W H_c^j(\bar{X})]$, and this virtual representation is zero because f is a constant and $2d > 0$, i.e.

$$(65) \quad \sum_j (-1)^j [\text{Gr}_{2d}^W H_c^j(\bar{X})] = 0.$$

From [D3], 3.3.4, we have

$$(66) \quad \text{Gr}_{2d}^W H_c^j(\bar{X}) = 0 \text{ for } j \neq 2d \text{ and}$$

$$(67) \quad \text{Gr}_\nu^W H_c^{2d}(\bar{X}) = 0 \text{ for } \nu \neq 2d.$$

From (65) and (66) we get $\text{Gr}_{2d}^W H_c^{2d}(\bar{X}) = 0$ which by (67) implies

$$(68) \quad H_c^{2d}(\bar{X}) = 0.$$

By the introductory remarks concerning the smooth locus of X/k we can choose $U \subset X$ open, smooth and purely d -dimensional such that the dimension of $X - U$ is strictly less than d (U need not be dense in X ; it may be chosen as the union of the smooth loci of the irreducible components of X of dimension d). The last condition assures that the natural map $H_c^{2d}(\bar{U}, \mathbb{Q}_l) \rightarrow H_c^{2d}(\bar{X}, \mathbb{Q}_l)$ is an isomorphism, hence (68) gives $H_c^{2d}(\bar{U}, \mathbb{Q}_l) = 0$. Poincaré duality gives $H^0(\bar{U}, \mathbb{Q}_l) = 0$ which is a contradiction because $\dim_{\mathbb{Q}_l} H^0(\bar{U}, \mathbb{Q}_l)$ is the number of connected components of \bar{U} and $U \neq \emptyset$. □

We now give a presentation of the subring $S \subset K_0(\text{Var}_k)$ generated by zero-dimensional varieties. From the proof of Theorem 24, 3) \Rightarrow 2), we know that S is

generated by the classes of finite field extensions of k . Putting $x_\nu := [\text{Spec}(k_\nu)] \in K_0(\text{Var}_k)$ ($\nu \geq 1$) we have

$$S = \mathbb{Z}[x_\nu : \nu \geq 1] \subset K_0(\text{Var}_k).$$

For $\mu, \nu \geq 1$ we write $d(\mu, \nu)$ (resp. $m(\mu, \nu)$) for the greatest common divisor (resp. the least common multiple) of μ and ν . Computing tensor products of finite extensions of finite fields, one checks that $x_\mu x_\nu = d(\mu, \nu)x_{m(\mu, \nu)}$, and we get a surjective ring homomorphism

$$(69) \quad \phi : \mathbb{Z}[T_\nu : \nu \geq 1] / (T_\nu T_\mu - d(\mu, \nu)T_{m(\mu, \nu)} : \mu, \nu \geq 1) \longrightarrow S, \quad T_\nu \mapsto x_\nu.$$

Theorem 25. *The above ϕ is an isomorphism. In particular, the x_ν ($\nu \geq 2$) are pairwise distinct zero divisors in $K_0(\text{Var}_k)$.*

Remark. The existence of zero divisors in $K_0(\text{Var}_k)$ was first addressed by B. Poonen in [P] where he showed their existence for k of characteristic zero. One can show that for any finitely generated field k the images of the classes of the non-trivial finite Galois extensions of k in $K_0(\text{Var}_k)[[\mathbb{A}_k^1]^{-1}]$ are an infinite set of zero divisors.

Proof of Theorem 25. The left-hand side of (69) is a free \mathbb{Z} -module on the set $\{T_\nu : \nu \geq 1\}$. To show that ϕ is injective we use that, for all $n \geq 1$, sending $[X] \mapsto |X(k_n)|$ defines a motivic measure $\psi_n : K_0(\text{Var}_k) \rightarrow \mathbb{Z}$ and that, writing $X_\nu := \text{Spec}(k_\nu)$, we have

$$\begin{aligned} |X_\nu(k_n)| &= \nu, \text{ if } \nu \text{ divides } n, \\ |X_\nu(k_n)| &= 0, \text{ otherwise,} \end{aligned}$$

for all $\nu, n \geq 1$. Now assume that $\phi(\sum_{\nu \geq 1} a_\nu T_\nu) = \sum_{\nu \geq 1} a_\nu [X_\nu] = 0$ for some $a_\nu \in \mathbb{Z}$, almost all zero. We need to show that $a_\nu = 0$ for all ν . For any $n \geq 1$ we have

$$(*_n) \quad 0 = \psi_n(\sum_{\nu \geq 1} a_\nu [X_\nu]) = \sum_{\nu \geq 1} a_\nu |X_\nu(k_n)| = \sum_{\nu|n} a_\nu \nu.$$

To show that all a_ν are zero we proceed by induction on the number of prime divisors, counting multiplicity, of ν .

For $\nu = 1$ we have $(*_1) : a_1 = 0$.

For $\nu \neq 1$ we have from $(*_\nu)$ that $\sum_{d|\nu} a_d d = 0$. By induction hypothesis we have $a_d = 0$ for all proper divisors d of ν , hence $a_\nu \nu = 0$ and $a_\nu = 0$.

This shows that ϕ is injective, hence an isomorphism. To prove the last claim of Theorem 25 we use that, since ϕ is an isomorphism, S is \mathbb{Z} -free on the set $\{x_\nu : \nu \geq 1\}$ and in particular that the x_ν are all distinct. Finally, for $\nu \geq 2$, $0 = x_\nu^2 - \nu x_\nu = x_\nu(x_\nu - \nu)$. As $x_\nu - \nu = x_\nu - \nu x_1 \neq 0$ (note that $x_1 = 1$) we see that x_ν is a zero divisor. \square

Remark. The above considerations suggest a simple example illustrating Theorem 20. We take $n := 1$, $X_1 := \text{Spec}(k_2)$ in this result and compute $\mu_k([X_1]) \in \mathcal{R}(\mathbb{Q})[T]$. We have $H_c^i(\overline{X}_1) = 0$ for $i \neq 0$ and $H_c^0(\overline{X}_1)$ is two dimensional with geometric Frobenius F_k acting by exchanging the vectors of a (fixed) basis. This gives for $\nu \geq 1$

$$(70) \quad c_\nu := \text{tr}(F_k^\nu | H_c^0(\overline{X}_1)) = 2 \text{ for } \nu \text{ even}$$

and $c_\nu = 0$ for ν odd. For arbitrary $G \in \mathbb{Z}[T_1]$ we have $G(\mu_k([X_1])) = 0$ if and only if $G(\partial_\nu(\mu_k([X_1]))) = 0$ for all $\nu \geq 1$, and from the above computation of the cohomology of X_1 and Proposition 8i) we get that $\partial_\nu(\mu_k([X_1])) \in \mathbb{Q}[T]$ is in fact a constant equal to c_ν . So $G(\mu_k([X_1])) = 0$ is equivalent to $G(c_\nu) = 0$ for all $\nu \geq 1$. From (70) we see that the assertion of Theorem 20 holds with $M = 2$ and $G = T_1 - 2$.

However, there is no *irreducible* polynomial $G \in \mathbb{Z}[T_1]$ satisfying $G(\mu_k([X_1])) = 0$ because such a G must, by (70), satisfy $G(0) = G(2) = 0$ and hence must be divisible by $T(T - 2)$.

This shows that in Theorem 20 one has to allow for a finite extension of the base field in order to find an irreducible relation.

3.5. Two curves. Let k be a finite field. In this subsection by a curve over k we shall always mean a proper, smooth and geometrically connected curve. We are going to treat in detail the following special case of the problem of algebraic independence in $K_0(\text{Var}_k)$. Let X and Y be curves over k . When are $[X]$ and $[Y]$ algebraically independent in $K_0(\text{Var}_k)$, i.e. when is the subring $\mathbb{Z}[[X], [Y]] \subset K_0(\text{Var}_k)$ a polynomial ring in the variables $[X]$ and $[Y]$? We will see that this is the case for a generic pair (X, Y) .

First we define special curves which will turn out to be those for which our methods can shed no light on the above question. A curve X/k is *special* if i) the degree of k over its prime field is even: $k = \mathbb{F}_q$, $q = p^{2n}$ ($n \geq 1$) and ii) all eigenvalues of the geometric Frobenius attached to k acting on $H_c^1(\overline{X})$ are equal to $+q^{1/2}$. Note that by i) $+q^{1/2}$ is a rational integer so ii) is meaningful. Being special is not a property of the curve alone but depends on the base field, too. The projective line is special over any k satisfying condition i). An elliptic curve X/k becomes special after a finite extension of the base field if and only if it is super-singular. If X/k is special, then the Jacobian of X is k -isogeneous to the power of a super-singular elliptic curve, and we see that most curves of genus at least one are non-special. The main result of this subsection is the following.

Theorem 26. *Let X_1 and X_2 be proper, smooth and geometrically connected curves over the finite field k such that $[X_1]$ and $[X_2]$ are algebraically dependent in $K_0(\text{Var}_k)$. Then there is a finite extension $k \subset K$ such that at least one of the following holds:*

- 1) *Both $X_1 \times_k K/K$ and $X_2 \times_k K/K$ are special.*
- 2) *The Jacobians of X_1 and X_2 are K -isogeneous.*

Corollary 27. *If E/k is an ordinary elliptic curve, then $[\mathbb{P}^1]$ and $[E]$ are algebraically independent in $K_0(\text{Var}_k)$.*

This is clear using Theorem 26 and the discussion preceding it. Theorem 26 can be applied to the following problem: It is difficult to exhibit relations in $K_0(\text{Var}_k)$. One such relation, which follows directly from the definition, is that if $X \rightarrow Y$ is a *Zariski*-locally trivial fibre bundle with fibre G , then $[X] = [G][Y]$ in $K_0(\text{Var}_k)$. One might hope for a similar relation for more general $X \rightarrow Y$, e.g. étale covers. However:

Corollary 28. *Let X/k be a curve of genus at least two which does not become special after any finite extension of k . If $Y \rightarrow X$ is a non-trivial finite étale cover, then $[X]$ and $[Y]$ are algebraically independent in $K_0(\text{Var}_k)$.*

Proof. By the assumption made on X , condition 1) of Theorem 26 is excluded. The Hurwitz formula implies that the genus of Y is strictly larger than the genus of X , excluding condition 2) of Theorem 26. Hence Theorem 26 implies the algebraic independence of $[X]$ and $[Y]$. \square

Remark. Rephrasing Theorem 26 we have: If for two given curves $X_1, X_2/k$ none of 1) or 2) from Theorem 26 holds after any finite extension of k , then $[X_1]$ and $[X_2]$ are algebraically independent. We will in fact show that in this situation $\mu_k([X_1])$ and $\mu_k([X_2])$ are algebraically independent in $\mathcal{R}(\mathbb{Q})[T]$. We have seen in Corollary 3 that for the algebraic independence of the $\mu_k([X_i])$ it is sufficient that $[H_c^1(\overline{X}_1)]$ and $[H_c^1(\overline{X}_2)]$ are algebraically independent in $K_0(\text{Rep}_{G_k}\mathbb{Q}_l)$. The following example shows that Theorem 26 is sharper than just the algebraic independence of H^1 's. Let X_2/k be a curve such that its Jacobian is k -isogeneous to the square of an ordinary elliptic curve X_1/k (such a curve exists for suitable k by [BDS], Theorem 2). Then $H_c^1(\overline{X}_2) \simeq H_c^1(\overline{X}_1)^{\oplus 2}$ as G_k -modules and so $[H_c^1(\overline{X}_1)]$ and $[H_c^1(\overline{X}_2)]$ are algebraically dependent in $K_0(\text{Rep}_{G_k}\mathbb{Q}_l)$. But Theorem 26 still shows that the $\mu_k([X_i])$ ($i = 1, 2$) (and hence the $[X_i]$ themselves) are algebraically independent. The curve $X_1 \times_k K/K$ is not special for any finite extension $k \subset K$ as explained before Theorem 26 and the Jacobians of X_1 and X_2 are not isogeneous because they are of different dimensions.

Another consequence of Theorem 26 is that two curves of different genera have algebraically independent classes unless they both become special after some finite extension of the base field. As the referee points out, over $k = \mathbb{C}$ one can easily show that two curves of different genera have algebraically independent classes using a motivic measure based on the Hodge polynomial; see [LL2], 3.4.

Proof of Theorem 26. Since we are allowed to make a finite extension of k we can assume by Theorem 20 that there is an irreducible $G \in \mathbb{Z}[T_1, T_2]$ with $G(\mu_k([X_1]), \mu_k([X_2])) = 0$. Writing

$$f_{i,\nu} := \partial_\nu(\mu_k([X_i])) \in \mathbb{Q}[T], \quad i = 1, 2, \nu \geq 1,$$

we have

$$(71) \quad G(f_{1,\nu}, f_{2,\nu}) = 0 \text{ for all } \nu \geq 1.$$

Introducing, for $\nu \geq 1$,

$$\begin{aligned} a_\nu &:= -\text{tr}(F_k^\nu | H_c^1(\overline{X}_1)), \\ b_\nu &:= -\text{tr}(F_k^\nu | H_c^1(\overline{X}_2)), \end{aligned}$$

where F_k is the geometric Frobenius of $k = \mathbb{F}_q$, say, we have

$$\begin{aligned} f_{1,\nu} &= 1 + a_\nu T + q^\nu T^2, \\ f_{2,\nu} &= 1 + b_\nu T + q^\nu T^2 \end{aligned}$$

because $H_c^0(\overline{X}_i) = \mathbb{Q}_l$ and $H_c^2(\overline{X}_i) = \mathbb{Q}_l(-1)$ and using Proposition 8i). We consider

$$(72) \quad F(T_1, T_2) := q(T_2 - T_1)^2 - (a_1 - b_1)(a_1 T_2 - b_1 T_1) + (a_1 - b_1)^2.$$

The discriminant of F , considered as a polynomial in T_2 , equals

$$(73) \quad 4q(a_1 - b_1)^2 T_1 + (a_1 - b_1)^2 (a_1^2 - 4q).$$

The choice of F is made such that we have

$$\begin{aligned}
 (74) \quad F(f_{1,1}, f_{2,1}) &= F(1 + a_1T + qT^2, 1 + b_1T + qT^2) \\
 &= q(b_1 - a_1)^2T^2 - (a_1 - b_1)(a_1 + a_1b_1T + a_1qT^2 \\
 &\quad - b_1 - b_1a_1T - b_1qT^2) + (a_1 - b_1)^2 \\
 &= q(b_1 - a_1)^2T^2 - (a_1 - b_1)^2(1 + qT^2) + (a_1 - b_1)^2 \\
 &= (a_1 - b_1)^2(qT^2 - 1 - qT^2) + (a_1 - b_1)^2 = 0.
 \end{aligned}$$

To continue, we make the following additional assumption:

1) $a_1 \neq b_1$: The prime ideal

$$P_1 := \ker(\mathbb{Q}[T_1, T_2] \longrightarrow \mathbb{Q}[T], T_i \mapsto f_{i,1})$$

is proper and non-zero, hence of height 1 in $\mathbb{Q}[T_1, T_2]$. By (71) $0 \neq G \in P_1$ and G is irreducible. Since $\mathbb{Q}[T_1, T_2]$ is factorial we have $P_1 = (G)$, the principal ideal generated by G . As $a_1 \neq b_1$ the discriminant (73) of F is not a square in $\mathbb{Q}[T_1]$, so F is irreducible in $\mathbb{Q}[T_1, T_2]$. By (74) we have $F \in P_1$, and the same reasoning as for G gives $P_1 = (F) = (G)$. So there is some $\alpha \in \mathbb{Q}^*$ with $F = \alpha G$ from which we get $F(f_{1,\nu}, f_{2,\nu}) = \alpha G(f_{1,\nu}, f_{2,\nu}) \stackrel{(71)}{=} 0$ for all $\nu \geq 1$ which we make more explicit:

$$\begin{aligned}
 0 = F(f_{1,\nu}, f_{2,\nu}) &= F(1 + a_\nu T + q^\nu T^2, 1 + b_\nu T + q^\nu T^2) \\
 &\stackrel{(72)}{=} q((b_\nu - a_\nu)T)^2 - (a_1 - b_1) \\
 &\quad (a_1 + a_1b_\nu T + a_1q^\nu T^2 - b_1 - b_1a_\nu T - b_1q^\nu T^2) + (a_1 - b_1)^2 \\
 &= T^2(q(b_\nu - a_\nu)^2 - (a_1 - b_1)(a_1q^\nu - b_1q^\nu)) + T(-(a_1 - b_1) \\
 &\quad (a_1b_\nu - b_1a_\nu)) - (a_1 - b_1)^2 + (a_1 - b_1)^2 \\
 &= T^2(q(b_\nu - a_\nu)^2 - q^\nu(a_1 - b_1)^2) + T((a_1 - b_1)(b_1a_\nu - a_1b_\nu)).
 \end{aligned}$$

So for any $\nu \geq 1$ (using $a_1 - b_1 \neq 0$):

$$(75) \quad q(b_\nu - a_\nu)^2 = q^\nu(a_1 - b_1)^2,$$

$$(76) \quad b_1a_\nu = a_1b_\nu.$$

Now we assume in addition to $a_1 \neq b_1$ that we also have

1.1) $a_1 \neq 0$ and $b_1 \neq 0$: From (75) we have

$$b_\nu^2 - 2a_\nu b_\nu + a_\nu^2 = q^{\nu-1}(a_1 - b_1)^2$$

into which we substitute (76) to get

$$a_\nu^2 \frac{b_1^2}{a_1^2} - 2a_\nu^2 \frac{b_1}{a_1} + a_\nu^2 = q^{\nu-1}(a_1 - b_1)^2.$$

Multiplying this by a_1^2 gives

$$a_\nu^2(b_1^2 - 2a_1b_1 + a_1^2) = q^{\nu-1}(a_1 - b_1)^2 a_1^2,$$

and cancelling $(a_1 - b_1)^2 \neq 0$ we obtain

$$(77) \quad a_\nu^2 = q^{\nu-1} a_1^2 \text{ for } \nu \geq 1.$$

We have $a_\nu^2 = \text{tr}(F_k^\nu | H_c^1(\overline{X}_1)^{\otimes 2})$, hence

$$\begin{aligned}
 (78) \quad \det(1 - F_k t | H_c^1(\overline{X}_1)^{\otimes 2}) &= \exp\left(-\sum_{\nu \geq 1} a_\nu^2 \frac{t^\nu}{\nu}\right) \\
 &\stackrel{(77)}{=} \exp\left(-\sum_{\nu \geq 1} q^{\nu-1} a_1^2 \frac{t^\nu}{\nu}\right) = \exp\left(q^{-1} a_1^2 \left(\sum_{\nu \geq 1} -\frac{(qt)^\nu}{\nu}\right)\right) \\
 &= (1 - qt)^{a_1^2/q}.
 \end{aligned}$$

The first equality in (78) is [H], Appendix C, Lemma 4.1. We denote by $g(X_i)$ the genus of X_i . As $\dim_{\mathbb{Q}_l} H_c^1(\overline{X}_i) = 2g(X_i)$ we obtain from (78) by comparing degrees:

$$(79) \quad a_1^2 = 4g(X_1)^2 q.$$

Denoting by $\alpha_1, \dots, \alpha_{2g(X_1)}$ the eigenvalues of F_k acting on $H_c^1(\overline{X}_1)$ considered as complex numbers via some isomorphism $\overline{\mathbb{Q}_l} \xrightarrow{\sim} \mathbb{C}$, we have $|\alpha_i| = q^{1/2}$ and thus

$$|a_1^2| = \left| -\sum_{i=1}^{2g(X_1)} \alpha_i \right|^2 \leq (2g(X_1)q^{1/2})^2 = 4g(X_1)^2 q.$$

By (79) this upper bound is in fact an equality which forces all α_i to be equal to either $+q^{1/2}$ or $-q^{1/2}$. After possibly a quadratic extension of k we will have all α_i equal to $+q^{1/2}$ and then X_1/k will be special by definition. As the additional assumptions we have made so far, namely that $a_1 \neq b_1$ and $a_1, b_1 \neq 0$, are symmetric in X_1 and X_2 the same conclusion holds for X_2/k , and we are in case 1) of Theorem 26. Now we assume:

1.2) $a_1 = 0$: We then have, as we still assume that $a_1 \neq b_1$, that $b_1 \neq 0$ and (76) implies that $a_\nu = 0$ for all $\nu \geq 1$, hence $H_c^1(\overline{X}_1) = 0$ and $X_1 \simeq \mathbb{P}^1$. Note that there are no non-trivial \overline{k}/k -forms of \mathbb{P}^1 because the Brauer group of a finite field is trivial. From (75) we then get $b_\nu^2 = q^{\nu-1} b_1^2$, and the same argument as in case 1.1) (cf. (77)) shows that X_2 becomes special after at most a quadratic extension. So we are again in case 1) of Theorem 26.

1.3) $b_1 = 0$: this case is symmetric to the case 1.2) above.

There is one case left to be considered.

2) $a_1 = b_1$: We now have $f_{1,1} = f_{2,1}$ and $F := T_1 - T_2$ satisfies $F(f_{1,1}, f_{2,1}) = 0$. By the same arguments as in case 1) we get $F(f_{1,\nu}, f_{2,\nu}) = 0$ for all $\nu \geq 1$, i.e.

$$0 = F(1 + a_\nu T + q^\nu T^2, 1 + b_\nu T + q^\nu T^2) = (a_\nu - b_\nu)T,$$

so $a_\nu = b_\nu$ for all $\nu \geq 1$ from which we get

$$\det(1 - F_k t | H_c^1(\overline{X}_1)) = \det(1 - F_k t | H_c^1(\overline{X}_2))$$

which is equivalent to the Jacobians of X_1 and X_2 being k -isogeneous [T], Theorem 1, c1) \Leftrightarrow c2), and hence we are in case 2) of Theorem 26. \square

Remark. We conserve the notations of the above proof. The simplest case in which Theorem 26 does not guarantee the algebraic independence of the classes of two curves in $K_0(\text{Var}_k)$ is if $X_1 = \mathbb{P}^1$ and $X_2 = E$ is a super-singular elliptic curve. Let k be so large that X_2/k is special and put $q := |k|$. Then $F := q(T_2 - T_1)^2 - 4qT_1 + 4q$ satisfies

$$(80) \quad F(\mu_k([X_1]), \mu_k([X_2])) = 0.$$

In fact, we have

$$\begin{aligned} f_{1,\nu} &= 1 + q^\nu T^2 &= \partial_\nu(\mu_k([X_1])), \\ f_{2,\nu} &= 1 - 2q^{\nu/2}T + q^\nu T^2 &= \partial_\nu(\mu_k([X_2])) \end{aligned}$$

and compute

$$\begin{aligned} F(\partial_\nu(\mu_k([X_1])), \partial_\nu(\mu_k([X_2]))) &= q(-2q^{\nu/2}T)^2 - 4q(1 + q^\nu T^2) + 4q \\ &= 4q^{\nu+1}T^2 - 4q^{\nu+1}T^2 - 4q + 4q = 0, \end{aligned}$$

for all $\nu \geq 1$, hence (80) follows. On the other hand

$$F([X_1], [X_2]) = [Z^+] - [Z^-] \text{ in } K_0(\text{Var}_k),$$

where Z^+ and Z^- are the (non-connected) proper, smooth varieties

$$\begin{aligned} Z^+ &= (E^2)^{\sqcup q} \sqcup ((\mathbb{P}^1)^2)^{\sqcup q} \sqcup \text{Spec}(k)^{\sqcup 4q}, \\ Z^- &= (E \times \mathbb{P}^1)^{\sqcup 2q} \sqcup (\mathbb{P}^1)^{\sqcup 4q}. \end{aligned}$$

The sign \sqcup means disjoint union of schemes. I cannot decide whether $[Z^+]$ and $[Z^-]$ are equal (i.e. whether $F([X_1], [X_2]) = 0$), but I suspect not and even more, that $[\mathbb{P}^1]$ and $[E]$ should be algebraically independent. Note at least that $Z^+ \not\cong Z^-$, e.g. Z^+ has isolated points but Z^- does not.

This illustrates that our methods can only give *sufficient* conditions for varieties to be algebraically independent in the Grothendieck ring of varieties, but we have no idea how to produce interesting relations in $K_0(\text{Var}_k)$.

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