

NUMERICAL GODEAUX SURFACES WITH AN INVOLUTION

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ABSTRACT. Minimal algebraic surfaces of general type with the smallest possible invariants have geometric genus zero and $K^2 = 1$ and are usually called *numerical Godeaux surfaces*. Although they have been studied by several authors, their complete classification is not known.

In this paper we classify numerical Godeaux surfaces with an involution, i.e. an automorphism of order 2. We prove that they are birationally equivalent either to double covers of Enriques surfaces or to double planes of two different types: the branch curve either has degree 10 and suitable singularities, originally suggested by Campedelli, or is the union of two lines and a curve of degree 12 with certain singularities. The latter type of double planes are degenerations of examples described by Du Val, and their existence was previously unknown; we show some examples of this new type, also computing their torsion group.

1. INTRODUCTION

In the one-and-a-half century history of classification of algebraic varieties, surfaces having geometric genus $p_g = 0$ and irregularity $q = 0$ have been studied from the very beginning. They were conjectured to be rational by Max Noether (about 1870) until Enriques, in 1894, suggested the existence of surfaces with $p_g = q = 0$ and bi-genus $P_2 = 1$ which now bear his name. After that, Castelnuovo, in 1896, proved his celebrated *rationality criterion*, which says that a surface X is rational if and only if $P_2(X) = q(X) = 0$. Since then, the classification of surfaces with $p_g = 0$ has received particular attention by algebraic geometers, as well as others. Indeed, in the course of the years, it has been discovered that these surfaces are interesting not only for classification purposes, but also for their intriguing relations with other fields of mathematics, e.g. Bloch's conjecture, classification of four-folds, etc. (see [Bl, Fr]; for general information cf. [Ci2]).

In 1931–32, Godeaux and Campedelli gave the first two examples of minimal surfaces of general type with $p_g = 0$. Godeaux considered a quintic surface in \mathbb{P}^3 by a freely acting cyclic group of order 5 of projective transformations. The smooth minimal model of this surface has $K^2 = 1$. Campedelli constructed a double plane, i.e. a double cover of \mathbb{P}^2 , branched along a degree 10 curve with six

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points, not lying on a conic, all of type $[3, 3]$, that is, a triple point with another infinitely near triple point. Here the smooth minimal model has $K^2 = 2$.

Campedelli also proposed the construction of a minimal surface of general type with $p_g = 0$ and $K^2 = 1$ as the minimal model of a double plane branched along a curve of degree 10 with a 4-tuple point and five points of type $[3, 3]$, not lying on a conic. The actual existence of such a curve was proved only 50 years later by Kulikov, Oort and Peters in [OP]. We will say that a double plane with branch curve having the singularities suggested by Campedelli is of *Campedelli type*.

Minimal surfaces of general type with $p_g = 0$ and $K^2 = 1$ are nowadays called *numerical Godeaux surfaces*. They have been studied classically, but also by several authors in the last 30 years: it would be too long to recall here all the relevant contributions, anyway most of them appear in our list of references.

As its construction shows, the original Godeaux's example has non-trivial torsion, more precisely its torsion group is cyclic of order 5. It is actually a result of Miyaoka in [Mi] that the torsion group of a numerical Godeaux surface is cyclic of order at most 5 and surfaces S with $\text{Tors}(S) = \mathbb{Z}/5\mathbb{Z}$ fill up an irreducible component of the moduli space of the expected dimension $8 = \chi(\mathcal{T}_S)$, consisting of quotients of quintics in \mathbb{P}^3 by a $\mathbb{Z}/5\mathbb{Z}$ -action, as in the original Godeaux's construction.

It has to be expected that the larger the torsion is, the easier the study and the classification of these surfaces. This viewpoint has been pursued by Miles Reid in [Re1], who proved that also surfaces with $\text{Tors}(S) = \mathbb{Z}/3\mathbb{Z}$ or $\mathbb{Z}/4\mathbb{Z}$ fill up an 8-dimensional irreducible component of the moduli space, giving an explicit description of their canonical rings.

Although a few examples of numerical Godeaux surfaces S with no torsion (cf. [CG, DW]) or even simply connected (see [Bar2]), or with $\text{Tors}(S) = \mathbb{Z}/2\mathbb{Z}$ (see [Bar1, We1, We2]) are known, neither a classification result, nor a description of the moduli space are known in these cases. Furthermore, all the known examples with no torsion or $\text{Tors}(S) = \mathbb{Z}/2\mathbb{Z}$ turn out to possess an involution, i.e. a birational automorphism of order 2.

This motivates the study of numerical Godeaux surfaces with an involution. A first investigation of this subject has been done by J. Keum and Y. Lee in [KL], under the assumption that the bicanonical system has no fixed components. In this paper, we make no assumption of this sort, and we prove the following.

Classification Theorem. *A numerical Godeaux surface S with an involution σ is birationally equivalent to one of the following:*

- (1) *a double plane of Campedelli type;*
- (2) *a double plane branched along a reduced curve which is the union of two distinct lines r_1, r_2 and a curve of degree 12 with the following singularities:*
 - *the point $q_0 = r_1 \cap r_2$ of multiplicity 4;*
 - *a point $q_i \in r_i$, $i = 1, 2$, of type $[4, 4]$, where the tangent line is r_i ;*
 - *further, three points q_3, q_4, q_5 of multiplicity 4 and a point q_6 of type $[3, 3]$, such that there is no conic through q_1, \dots, q_6 ;*
- (3) *a double cover of an Enriques surface branched along a curve of arithmetic genus 2.*

In case (3), the torsion group of S is $\text{Tors}(S) = \mathbb{Z}/4\mathbb{Z}$, whilst in case (2) $\text{Tors}(S)$ is either $\mathbb{Z}/2\mathbb{Z}$ or $\mathbb{Z}/4\mathbb{Z}$.

As we said, examples of surfaces of type (1) are known in the literature. Surfaces of type (3) will be called of *Enriques type*. Examples of such surfaces have been produced by Keum and Naie (cf. [Ke, Na]). Double planes as in case (2) are, to the best of our knowledge, new in the literature. They appear to be degenerations of double planes with $p_g = 4$ and $K^2 = 8$, introduced by Du Val in 1952 (see [Du]), when classifying surfaces with non-birational bicanonical map (cf. [Ci1] for a modern reference). For this reason, we call numerical Godeaux surfaces as in (2) of *Du Val type*. We give explicit examples of these surfaces, both with torsion group $\mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}/4\mathbb{Z}$, in section 9.

In order to prove our classification theorem, we proceed as follows. First of all, in section 3, we prove some relevant properties of the fixed locus of an involution acting on a surface of general type with geometric genus zero. In order to do so, we follow ideas contained in joint work of the third author and Rita Pardini (e.g. [MP1]), namely we combine the topological and holomorphic fixed point formulas with Kawamata-Viehweg’s vanishing theorem.

Applying these results to numerical Godeaux surfaces S with an involution, in section 4 we give a rather precise description of the fixed locus of the involution and we prove that the bicanonical system is invariant under the involution. This is actually the key ingredient for the proof of the classification and explains why degenerations of Du Val double planes, which have non-birational bicanonical map, come into play (cf. also recent results of Borrelli in [Bo] on the classification of surfaces with non-birational bicanonical map and low invariants).

Since the pencil $|2K_S|$ is invariant, we can in fact consider its image on the quotient surface under the involution, which is a pencil \mathcal{D} of curves of arithmetic genus 2. Using this, one sees that the quotient surface is either rational or birational to an Enriques surface. The latter case, i.e. the Enriques type, is worked out in section 5, where in particular we prove that numerical Godeaux surfaces of this type have torsion group of order 4 and are birational to the double cover of an Enriques surface branched along a curve of arithmetic genus 2, with at most irrelevant singularities, which moves in a linear system with no fixed component.

In case the quotient of S by the involution is rational, one studies the pencil \mathcal{D} using adjunction. This leads to two different cases: one in which the adjoint to this system is a base-point-free pencil of rational curves, the other in which the adjoint is a pencil of curves of genus 1. These two cases are analysed separately in sections 6 and 7, respectively. The former case leads to double planes of Campedelli type: it suffices to suitably use the pencil of rational curves to map the quotient surface to \mathbb{F}_1 , and then to the plane. The latter case leads to double planes of Du Val type. Here the quotient surface is mapped to a weak Del Pezzo surface X , i.e. $-K_X$ is big and nef, having $K_X^2 = 1$ and four disjoint (-2) -curves, whose sum is an even divisor in X . Rational surfaces with an even set of (-2) -curves have been studied by Dolgachev, Mendes Lopes and Pardini in [DMP] and, more recently, by us in [CCM]. Indeed we apply the main result in [CCM] to find a suitable birational morphism of the weak Del Pezzo surface X to \mathbb{P}^2 , which realizes the original numerical Godeaux surface as a Du Val double plane.

Then we give more information about the previously unknown case of numerical Godeaux surfaces of Du Val type. In particular, in section 8, we examine the interplay between reducibility of the branch curve and torsion. Our main result in this direction is Theorem 8.5, which, under some assumptions on the branch curve,

gives a useful criterion to decide whether the torsion is $\mathbb{Z}/4\mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z}$, based on the existence or not of plane curves of degree 8 with suitable singularities.

We remark that our results concern the birational classification of *pairs* (S, σ) , where S is a numerical Godeaux surface and σ an involution of S . We do not treat here, in general, the interesting problem of determining how many involutions can occur on a given numerical Godeaux surface and of which type, according to our classification theorem. However, our results do give some partial information. For instance, if the ramification curve R on S has an irreducible component of genus 2, then we are in case (3) (see Proposition 7.10). In any case, using the 2-torsion, we give a criterion, i.e. Corollary 8.9, based on the irreducibility of a certain plane cubic, which allows us to distinguish between the Du Val and the Campedelli types.

As we said, in section 9, we prove the existence of numerical Godeaux double planes of Du Val type and we are able to compute the torsion of these examples. We do this using MapleTM [Ma]¹ in two different ways. One way is to find branch curves, with irreducible degree 12 component, which are invariant under a projective automorphism of order 2 of the plane, which is an idea originally due to Stagnaro (cf. [St]). The resulting examples turn out to have torsion group $\mathbb{Z}/4\mathbb{Z}$.

Another way is to try and find the degree 12 component of the branch curve suitably reducible in a line and an irreducible component of degree 11. We find examples of this type with torsion $\mathbb{Z}/2\mathbb{Z}$ and with torsion $\mathbb{Z}/4\mathbb{Z}$, and we prove that the former example does not have a different involution, which makes it a double plane of Campedelli type, therefore it is certainly new in the literature. Although we do not treat moduli problems here, we prove that both our examples vary in families whose images in the moduli space have dimension 5, and the general member of each family is a Du Val double plane with an irreducible degree 12 component of the branch curve (cf. Corollaries 9.6 and 9.7).

2. NOTATION AND CONVENTIONS

In this section we fix the notation which will be used in this paper.

Let S be a complex projective surface. We set:

- $\kappa(S)$: the Kodaira dimension of S ;
- $\chi(\mathcal{F})$: the Euler characteristic of a sheaf \mathcal{F} on S ;
- $\text{Pic}(S)$: the Picard group of S ;
- $\text{Tors}(S)$: the subgroup of $\text{Pic}(S)$ composed of torsion elements;
- $\text{Tors}_n(S)$: the subgroup of $\text{Pic}(S)$ composed of elements of torsion n ;
- $\rho(S)$: the rank of the Néron–Severi group of S ;
- K_S : a canonical divisor of S ;
- $p_g(S)$: the geometric genus of S , that is, $h^0(S, \mathcal{O}_S(K_S))$;
- $q(S)$: the irregularity of S , that is, $h^1(S, \mathcal{O}_S)$;
- $P_m(S)$: the m -th pluri-genus of S , that is, $h^0(S, \mathcal{O}_S(mK_S))$, $m \geq 1$.

If S is clear from the context, sometimes we will write p_g , q , K^2 , etc., instead of $p_g(S)$, $q(S)$, K_S^2 , etc.

Let X be a complex projective variety. We denote by $e(X)$ the topological Euler characteristic of X and by $p_a(X)$ the arithmetic genus of X . Recall that if D is a curve on a surface S , then $p_a(D) = D(D + K_S)/2 + 1$.

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We denote by \equiv the linear equivalence of divisors on a surface and by \sim the numerical equivalence. We usually omit the sign \cdot of the intersection product of two divisors on a surface.

Recall that a (-1) -curve is a smooth irreducible rational curve C with $C^2 = -1$. More generally, one says that a smooth irreducible rational curve C with $C^2 = -n < 0$ is a $(-n)$ -curve.

We say that a divisor D on a surface is *nef and big* if $D^2 > 0$ and $DE \geq 0$ for every irreducible curve E .

If x is a real number, we denote by $[x]$ its integer part, i.e. the largest integer number less than or equal to x .

A singular point of type $[m, m]$ on a curve is a point of multiplicity m with an infinitely near point again of multiplicity m .

3. INVOLUTIONS ON SURFACES

Let S be a smooth, irreducible, projective surface over the field \mathbb{C} of complex numbers. An *involution* of S is an automorphism σ of S of order 2. Remark that if S is a minimal surface of general type, then any birational automorphism is an isomorphism, therefore any birational automorphism of order 2 is an involution. If X is any variety and $\psi : S \dashrightarrow X$ is a rational map, one says that ψ is *composed* with the involution σ if $\psi \circ \sigma = \psi$.

Given an involution σ on S , its fixed locus is the union of a smooth, possibly reducible, curve R and of k isolated points p_1, \dots, p_k .

Let $\pi : S \rightarrow \Sigma := S/\sigma$ be the quotient map and set $B := \pi(R)$. The surface Σ is normal and $\pi(p_1), \dots, \pi(p_k)$ are ordinary double points, which are the only singularities of Σ . In particular, the singularities of Σ are canonical and the adjunction formula gives $K_S \equiv \pi^*K_\Sigma + R$.

Let $\epsilon : V \rightarrow S$ be the blowing-up of S at p_1, \dots, p_k and let E_i be the exceptional curve over p_i , $i = 1, \dots, k$. Then σ induces an involution $\tilde{\sigma}$ of V whose fixed locus is the union of $R_0 := \epsilon^*(R)$ and of E_1, \dots, E_k . Denote by $\tilde{\pi} : V \rightarrow W := V/\tilde{\sigma}$ the projection onto the quotient and set $B_0 := \tilde{\pi}(R_0)$, $C_i := \tilde{\pi}(E_i)$, $i = 1, \dots, k$. The surface W is smooth and the C_i are disjoint (-2) -curves. Denote by $\eta : W \rightarrow \Sigma$ the map induced by ϵ . The map η is the minimal resolution of the singularities of Σ , and there is a commutative diagram:

$$(1) \quad \begin{array}{ccc} V & \xrightarrow{\epsilon} & S \\ \tilde{\pi} \downarrow & & \downarrow \pi \\ W & \xrightarrow{\eta} & \Sigma \end{array}$$

The map $\tilde{\pi}$ is a flat double cover branched on $\tilde{B} = B_0 + \sum_{i=1}^k C_i$, hence there exists a divisor L on W such that $2L \equiv \tilde{B}$ and

$$(2) \quad \tilde{\pi}_* \mathcal{O}_V = \mathcal{O}_W \oplus \mathcal{O}_W(-L).$$

Also $K_V \equiv \tilde{\pi}^*(K_W + L)$. With this notation:

Proposition 3.1. *Let S be a minimal surface of general type and let σ be an involution of S . Then:*

- (i) $2K_W + B_0$ is nef and big;
- (ii) $(2K_W + B_0)^2 = 2K_S^2$;
- (iii) $H^i(W, \mathcal{O}_W(2K_W + L)) = 0$, $i = 1, 2$.

Proof. By the adjunction formula and commutativity of diagram (1), we have

$$(3) \quad \tilde{\pi}^*(2K_W + B_0) \equiv 2K_V - 2 \sum_{i=1}^k E_i \equiv \epsilon^*(2K_S).$$

Then $2K_W + B_0$ is nef and big because so is $2K_S$, proving part (i). Statement (ii) also follows by formula (3).

Finally for part (iii), we have the equivalence of \mathbb{Q} -divisors:

$$K_W + L \equiv \frac{1}{2}(2K_W + B_0) + \frac{1}{2} \sum_{i=1}^k C_i.$$

The divisor $(2K_W + B_0)/2 = \eta^*(2K_\Sigma + B)/2$ is nef and big, because so is $2K_W + B_0$, whereas $\frac{1}{2} \sum_{i=1}^k C_i$ is effective, with zero integral part, and its support has normal crossings. Thus $h^i(W, \mathcal{O}_W(2K_W + L)) = 0$ for $i > 0$ by the Kawamata-Viehweg vanishing theorem (see, e.g., Corollary 5.12, c), of [EV]). \square

For surfaces of general type with $p_g = 0$ having an involution, one can be more specific. As shown in [KL], Lemma 1.3 or [DMP], the holomorphic and topological fixed point formulas (see p. 566 in [AS] and formula (30.9) in [Gr]) yield:

Lemma 3.2. *Let S be a surface with $p_g = q = 0$ and σ an involution of S . Then the number of isolated fixed points of σ is $k = K_S R + 4$.* \square

Also the following properties, which will be very useful in the sequel, hold.

Proposition 3.3. *Let S be a minimal surface of general type with $p_g = 0$ and let σ be an involution of S . Then:*

- (i) $k \geq 4$;
- (ii) $K_W L + L^2 = -2$;
- (iii) $h^0(W, \mathcal{O}_W(2K_W + L)) = K_W^2 + K_W L$;
- (iv) $K_W^2 + K_W L \geq 0$;
- (v) $k = K_S^2 + 4 - 2h^0(W, \mathcal{O}_W(2K_W + L))$.

Proof. Since S is minimal of general type, K_S is nef, and so statement (i) follows from Lemma 3.2.

Since S is of general type with $p_g(S) = 0$, also $q(S) = 0$. Therefore

$$(4) \quad p_g(\Sigma) = p_g(W) = 0, \quad q(\Sigma) = q(W) = 0.$$

By standard double cover formulas, we have $\chi(\mathcal{O}_V) = 2\chi(\mathcal{O}_W) + (L^2 + K_W L)/2$, thus statement (ii) follows from $p_g(W) = q(W) = 0$.

By Proposition 3.1(iii), and the Riemann-Roch Theorem, one has

$$\begin{aligned} h^0(W, \mathcal{O}_W(2K_W + L)) &= \chi(\mathcal{O}_W(2K_W + L)) = (2K_W + L)(K_W + L)/2 + 1 \\ &= K_W(K_W + L) + L(K_W + L)/2 + 1. \end{aligned}$$

Thus (iii) follows by statement (ii), and (iv) is a trivial consequence of (iii).

Finally, as for statement (v), it suffices to remember that $k = K_S^2 - K_V^2$, $K_V^2 = 2(K_W + L)^2$ and use statements (ii), (iii). \square

Corollary 3.4. *Let S be a minimal surface of general type with $p_g = 0$ and let σ be an involution of S . Then, with W as above,*

- (i) $|2K_W + B_0| \neq \emptyset$, $|2K_W + B_0 + \sum_{i=1}^k C_i| = |2K_W + B_0| + \sum_{i=1}^k C_i$ and $h^0(W, \mathcal{O}_W(2K_W + B_0)) = K_S^2 + 1 - h^0(W, \mathcal{O}_W(2K_W + L))$;

- (ii) if $D \in |2K_W + B_0|$, then D is nef and big, 1-connected and thus $h^0(D, \mathcal{O}_D) = 1$, $h^1(D, \mathcal{O}_D) = p_a(D)$;
- (iii) $p_a(D) > 0$.

Proof. The Hurwitz formula gives $2K_V \equiv \tilde{\pi}^*(2K_W + B_0 + \sum_{i=1}^k C_i)$. By the projection formula (2), one has

$$(5) \quad H^0(V, \mathcal{O}_V(2K_V)) = H^0(W, \mathcal{O}_W(2K_W + L)) \oplus H^0(W, \mathcal{O}_W(2K_W + B_0 + \sum_{i=1}^k C_i)).$$

By Proposition 3.3(v) and (i), $K_S^2 - 2h^0(W, \mathcal{O}_W(2K_W + L)) = k - 4 \geq 0$.

Since $h^0(V, \mathcal{O}_V(2K_V)) = h^0(S, \mathcal{O}_S(2K_S)) = K_S^2 + 1$, we conclude that the second summand in (5) is not 0. For every $i = 1, \dots, k$, one has that

$$C_i(2K_W + B_0 + \sum_i C_i) = -2,$$

and so $|2K_W + B_0 + \sum_i C_i| = |2K_W + B_0| + \sum_i C_i$. This proves (i).

In Proposition 3.1(i), we already proved that D is nef and big. Therefore D is 1-connected (see e.g. Lemma 2.6 in [Me]), hence $h^0(D, \mathcal{O}_D) = 1$, and the final assertion of (ii) follows by the Riemann–Roch Theorem.

Finally for assertion (iii), it suffices to note that, by Propositions 3.1 and 3.3, $K_W D \geq 0$ and $D^2 > 0$. □

Corollary 3.5. *Let S be a minimal surface of general type with $p_g = 0$ and σ an involution of S . Then, with W as above, $K_W^2 \geq K_V^2$.*

Proof. By Noether’s formula, because $\chi(\mathcal{O}_W) = \chi(\mathcal{O}_V)$ and $e(W) \leq e(V)$, since the map $\tilde{\pi}$ determines an injection of $H^2(W, \mathbb{C})$ into $H^2(V, \mathbb{C})$. □

Corollary 3.6. *Let S be a minimal surface of general type with $p_g = 0$, let $\varphi : S \rightarrow \mathbb{P}^{K_S^2}$ be the bicanonical map of S and let σ be an involution of S . Then the following conditions are equivalent:*

- (i) φ is composed with σ ;
- (ii) $h^0(W, \mathcal{O}_W(2K_W + L)) = 0$;
- (iii) $K_W(K_W + L) = 0$;
- (iv) the number of isolated fixed points of σ is $k = K_S^2 + 4$.

Proof. The bicanonical φ is composed with σ if and only if one of the summands in formula (5) above vanishes. By Corollary 3.4(i), the second summand is never 0, and so φ is composed with σ if and only if $h^0(W, \mathcal{O}_W(2K_W + L)) = 0$, which in turn, by Proposition 3.3(iii) and (v), is equivalent to $K_W(K_W + L) = 0$ and $k = K_S^2 + 4$. □

Corollary 3.7. *Let S be a minimal surface of general type with $p_g = 0$ and σ an involution of S . If the bicanonical map φ is composed with σ , then:*

- (i) $h^0(W, \mathcal{O}_W(2K_W + B_0)) = P_2(S) = 1 + K_S^2$;
- (ii) for $D \in |2K_W + B_0|$, $h^0(D, \mathcal{O}_D(D)) = K_S^2$;
- (iii) $K_W D = 0$, $D^2 = 2K_S^2$ and $p_a(D) = K_S^2 + 1$;
- (iv) $-4 \leq K_W^2 \leq 0$, and $K_W^2 = 0$ if and only if $K_W \sim 0$;
- (v) either W is rational or the minimal model of W is an Enriques surface.

Proof. Part (i) is immediate from Corollaries 3.4(i) and 3.6(ii). Part (ii) follows from part (i) by considering the long exact sequence obtained from

$$0 \rightarrow \mathcal{O}_W \rightarrow \mathcal{O}_W(D) \rightarrow \mathcal{O}_D(D) \rightarrow 0$$

and using that $h^1(W, \mathcal{O}_W) = 0$. Now note that $DK_W = 2(K_W + L)K_W$, so $K_W D = 0$ follows from Corollary 3.6(iii). Note also that $D^2 = 2K_S^2$ by Proposition 3.1(ii), therefore the adjunction formula implies that $p_a(D) = K_S^2 + 1$. Since $k = K_S^2 + 4$ one has that $K_V^2 = -4$, and so the first inequality in (iv) follows from Corollary 3.5. The second inequality and the remainder of statement (iv) follow from the Index Theorem because D is nef and big by Proposition 3.1(i).

Finally, again because D is nef and big, $K_W D = 0$ implies that $\kappa(W) \leq 0$. So statement (v) follows by the classification of surfaces, since $p_g(W) = q(W) = 0$. \square

The following lemma will be used later. We keep the notation introduced above. Namely D is a general member of the non-empty linear system $|2K_W + B_0|$.

Lemma 3.8. *Let S be a minimal surface of general type with $p_g = 0$ and an involution σ and let W be as above. If $E \subset W$ is a curve such that $ED = 0$, then $E^2 < 0$ and the intersection form on the components of E is negative definite. In particular if E is a curve such that $E^2 = -1$ and $ED = 0$, then $E(\sum_{i=1}^k C_i) \leq 0$.*

Proof. The first part of the lemma is obvious by the Index Theorem, because D is nef and big by Proposition 3.1(i). For the second part, note that, since $E(B_0 + \sum_{i=1}^k C_i)$ is even and $ED = 0$, $E(\sum_{i=1}^k C_i)$ is also even. By the first part applied to the curve $E + C_i$, $i = 1, \dots, k$, one has $EC_i \leq 1$. Suppose that $E(\sum_{i=1}^k C_i) > 0$. Then there would be at least two curves, say C_1 and C_2 , such that $EC_1 = EC_2 = 1$, therefore $A = 2E + C_1 + C_2$ would satisfy $A^2 = 0$ and $AD = 0$, a contradiction. \square

Proposition 3.9. *Let S be a minimal surface of general type with $p_g = 0$ and an involution σ and let W be as above. Then there exists a birational morphism $f : W \rightarrow W'$ and an effective divisor D' on W' with the following properties:*

- (i) *there are k (-2) -curves C'_i on W' such that $f^*(C'_i) = C_i$, $i = 1, \dots, k$;*
- (ii) *D' is nef and such that $f^*(D') = D$, $D'^2 = D^2$ and $p_a(D') = p_a(D)$;*
- (iii) *$K_{W'} + D'$ is nef.*

Proof. If $K_W + D$ is nef, there is nothing to prove. Otherwise, since $K_W + D$ is effective (see Corollary 3.4), there is an irreducible curve E_1 such that $E_1^2 < 0$ and $E_1(K_W + D) < 0$. Then $K_W E_1 < 0$, thus E_1 is a (-1) -curve and $E_1 D = 0$. Hence $B_0 E_1 = 2$, and furthermore $E_1 \cap \bigcup_{i=1}^k C_i = \emptyset$ by Lemma 3.8.

Let $f_1 : W \rightarrow W_1$ be the contraction of E_1 to a point p_1 . On W_1 we have k (-2) -curves $C_i^{(1)}$ such that $f_1^*(C_i^{(1)}) = C_i$, $i = 1, \dots, k$, and a nef divisor D_1 on it such that $D_1^2 = D^2$ and $f_1^*(D_1) = D$. We set $B_0^{(1)} = f_1(B_0)$. Note that $B_0^{(1)}$ has a double point at p_1 , and it does not meet $\bigcup_{i=1}^k C_i^{(1)}$. Note also that $D_1 \equiv 2K_{W_1} + B_0^{(1)}$.

If $K_{W_1} + D_1$ is nef, we have finished. Otherwise, reasoning as above, there is a (-1) -curve E_2 on W_1 such that $E_2(K_{W_1} + D_1) < 0$, $D_1 E_2 = 0$ and $E_2 B_0^{(1)} = 2$. The curve $E' = f_1^*(E_2)$ satisfies $E' D = 0$, $E'^2 = -1$ and $K_W E' = -1$ and, since E_1 is disjoint from the curve $\sum_{i=1}^k C_i^{(1)}$, none of the curves C_i is a component of E' . Hence, by Lemma 3.8, $E' \cap \sum_{i=1}^k C_i^{(1)} = \emptyset$, implying that E_2 is also disjoint from the curves $C_i^{(1)}$. So we can contract E_2 and proceed.

Finally, the existence of the morphism $f : W \rightarrow W'$ is shown by iterating the above procedure. \square

Remark 3.10. We note that, by the proof of Proposition 3.9, the divisor $D' + C'_1 + \dots + C'_k$ is divisible by 2 in $\text{Pic}(W')$.

Later, we will need the following lemma, due to Beauville (Lemme 2 in [Be]).

Lemma 3.11. *Let $\tilde{\pi} : V \rightarrow W$ be a flat double cover between two smooth surfaces V and W , branched over the smooth curve $\tilde{B} = B_1 + \dots + B_n$, where B_1, \dots, B_n are the irreducible components of \tilde{B} . Suppose that $\text{Pic}(W)$ has no 2-torsion element. Define a group homomorphism $\psi : \mathbb{Z}_2^{\oplus n} \rightarrow \text{Pic}(W) \otimes \mathbb{Z}_2$ by $\psi(\varepsilon_1, \dots, \varepsilon_n) = \sum_{i=1}^n \varepsilon_i B_i$. Then $\text{Tors}_2(V) \cong \ker(\psi) / \langle (1, \dots, 1) \rangle$. \square*

4. NUMERICAL GODEAUX SURFACES WITH AN INVOLUTION

In the remainder of this paper, we focus on the study of a *numerical Godeaux* surface S , i.e. a minimal surface of general type with $p_g = 0$ and $K_S^2 = 1$, having an involution σ . We will freely use the notation introduced so far.

We start by recalling the following.

Theorem 4.1 (Miyaoka, [Mi]). *Let S be a numerical Godeaux surface. Then the torsion group $\text{Tors}(S)$ of S is cyclic of order $n \leq 5$. Moreover the linear system $|3K_S|$ has no fixed part and has at most 2 base points. If there is no base point, then $|\text{Tors}(S)| \leq 2$. If there is one base point, then $3 \leq |\text{Tors}(S)| \leq 4$. If there are two base points, then $|\text{Tors}(S)| = 5$. \square*

Lemma 4.2 (Lemma 5 in [Mi]). *Let S be a numerical Godeaux surface. If D is an effective divisor with $h^0(S, D) \geq 2$, then $DK_S \geq 2$. \square*

Lemma 4.3 (Reid, see p. 158 in [Do]). *Let S be a numerical Godeaux surface. If η is a non-trivial element of $\text{Tors}(S)$, then there is a unique element P_η in $|K_S + \eta|$. Furthermore, if η and η' , $\eta \neq \eta'$, are non-trivial elements of $\text{Tors}(S)$, then P_η and $P_{\eta'}$ have no common components. \square*

As an immediate consequence of Lemma 4.3, we have the following.

Corollary 4.4. *Let S be a numerical Godeaux surface. If $|\text{Tors}(S)| = 4$ or 5, then $|2K_S|$ has no fixed components. \square*

Our results in section 3 imply that S enjoys the following properties (cf. also [KL]).

Proposition 4.5. *If S is a numerical Godeaux surface with an involution σ , then:*

- (i) *the number of isolated fixed points of σ is $k = 5$;*
 - (ii) *the bicanonical map φ is composed with σ ;*
 - (iii) *$K_S R = 1$;*
 - (iv) *R^2 is odd and $-7 \leq R^2 \leq 1$.*
- Furthermore $R = \Gamma + Z_1 + \dots + Z_h$ where:*
- (v) *Γ is a smooth curve with $K_S \Gamma = 1$, $0 \leq p_a(\Gamma) \leq 2$ and $\Gamma^2 = 2p_a(\Gamma) - 3$;*
 - (vi) *if $p_a(\Gamma) = 2$, then $\Gamma \sim K_S$ and S has non-trivial torsion;*
 - (vii) *Z_1, \dots, Z_h are disjoint (-2) -curves, which are disjoint also from Γ , and*
- (6)
$$h = p_a(\Gamma) - K_W^2 - 2 \geq 0.$$

Proof. By Proposition 3.3(i) and (v), the number k is odd and $4 \leq k \leq 5$. Hence $k = 5$, that is, part (i). Then, part (ii) follows by Corollary 3.6, whilst part (iii) follows from (i) and Lemma 3.2.

Let us prove part (iv). Since $2R^2 = B^2$, one has $4L^2 = 2R^2 - 10$ and thus $L^2 = (R^2 - 5)/2$. By Proposition 3.3(ii) and Corollary 3.6(iii), $L^2 = -2 - K_W L = -2 + K_W^2$, hence $R^2 = 1 + 2K_W^2$. Then by Corollary 3.7(iv), $-7 \leq R^2 \leq 1$.

Part (iii) implies that R has a unique irreducible component Γ such that $K_S \Gamma = 1$. Since R is smooth, so is Γ . We can write $R = \Gamma + Z$, where Z is effective. Then $K_S Z = 0$, thus the irreducible components of Z are (-2) -curves, which are pairwise disjoint and disjoint from Γ because R is smooth.

By the Index Theorem, $\Gamma^2 \leq 1$. Since $K_S \Gamma = 1$, by adjunction one has $\Gamma^2 = 2p_a(\Gamma) - 3$, which ends the proof of statement (v).

If $\Gamma^2 = 1$, then Γ is homologous to K_S , but not linearly equivalent to K_S , because $p_g = 0$. Therefore $\Gamma - K_S$ is a non-trivial torsion element of the Néron-Severi group of S , which shows (vi).

Finally (vii) follows from $R^2 = 1 + 2K_W^2 = \Gamma^2 - 2h$ and $\Gamma^2 = 2p_a(\Gamma) - 3$. □

Remark 4.6. The statements about R on S in Proposition 4.5 can be read as well as about the curve B_0 on the surface W ; cf. diagram (1) and Corollary 3.7. For instance, $\tilde{\pi}(\epsilon^{-1}(Z_i))$ is a smooth rational curve on W with self-intersection -4 and is an irreducible component of B_0 .

Now Beauville’s Lemma 3.11 and Theorem 4.1 imply the following.

Corollary 4.7. *Let h be the number of (-2) -curves of R as in Proposition 4.5. If W is a rational surface, then*

$$(7) \quad h \leq 1 + \left\lfloor \frac{-K_W^2}{2} \right\rfloor,$$

and if equality holds, then S has non-trivial 2-torsion.

Proof. Consider the map ψ of Lemma 3.11. The domain of ψ is $\mathbb{Z}_2^{\oplus(h+6)}$, because the branch locus of $\tilde{\pi} : V \rightarrow W$ has $h + 6$ irreducible components. Moreover the image of ψ is a totally isotropic subspace of $\text{Pic}(W) \otimes \mathbb{Z}_2 = H^2(W, \mathbb{Z}_2)$.

Since W is simply connected, $h^2(W, \mathbb{Z}_2) = h^2(W, \mathbb{Z})$. Noether’s formula implies that $b_2(W) = 10 - K_W^2$, hence a totally isotropic subspace of $H^2(W, \mathbb{Z}_2)$ has dimension at most $5 + \lfloor -K_W^2/2 \rfloor$. By Theorem 4.1, $\dim \ker(\psi) \leq 2$, therefore $h + 6 \leq 2 + 5 + \lfloor -K_W^2/2 \rfloor$, which proves (7). If equality holds in (7), then $\dim \ker(\psi) = 2$, so $\text{Tors}_2(V)$ and $\text{Tors}_2(S)$ are not trivial by Lemma 3.11. □

In order to prove the Classification Theorem, stated in the Introduction, we need to understand the surface W and the divisor $D \equiv 2K_W + B_0$ on W . By Proposition 4.5 and Corollaries 3.4 and 3.7, one has that D is nef, $D^2 = 2$, $K_W D = 0$, $p_a(D) = 2$, and $h^0(W, \mathcal{O}_W(D)) = 2$.

If R has a component Γ with $p_a(\Gamma) = 2$, we have further information on W, D .

Corollary 4.8. *Let S be a numerical Godeaux surface with an involution σ . Suppose that R has an irreducible component Γ of genus 2. Then either one of the following two cases occurs:*

- (i) W is a minimal Enriques surface, $R = \Gamma$ and $|D| = |B_0|$;
- (ii) W is a rational surface, $-2 \leq K_W^2 \leq -1$ and $\Gamma - K_S \in \text{Tors}_2(S)$.

Proof. On W , let $\Gamma_0 = \eta^*(\pi(\Gamma)) \leq B_0$, thus $\Gamma_0^2 = 2$ and $K_W\Gamma_0 = 0$. Write

$$(8) \quad D \equiv 2K_W + B_0 \equiv 2K_W + \Gamma_0 + (B_0 - \Gamma_0).$$

Now $\Gamma_0 D = \Gamma_0^2 = D^2$, so, by the Index Theorem, $\Gamma_0 \sim D$. Hence (8) implies that $2K_W + B_0 - \Gamma_0 \sim 0$. By Corollary 3.7(v), either the minimal model of W is an Enriques surface or W is rational. In the former case $2K_W$ is an effective divisor. Since $B_0 - \Gamma_0$ is also effective, $2K_W + B_0 - \Gamma_0 \sim 0$ implies that $0 \equiv 2K_W \equiv B_0 - \Gamma_0$, that is, case (i) of the statement.

If W is rational, numerical equivalence is the same as linear equivalence and so $\Gamma_0 \equiv D$ and $B_0 - \Gamma_0 \equiv -2K_W$, therefore $2\Gamma \equiv 2K_S$. By formula (6), B_0 has $1 - K_W^2$ irreducible components, i.e. $h = -K_W^2$, thus formula (7) implies $K_W^2 \geq -2$. \square

Remark 4.9. We will see later, in Proposition 7.10, that case (ii) in Corollary 4.8 does not actually occur.

We now study W and the pencil $|D|$ on W by using adjunction.

Lemma 4.10. *With the above notation, write $|K_W + D| = F + |M|$, where F denotes the fixed part and $|M|$ the movable part. Then:*

- (i) $h^0(W, \mathcal{O}_W(M)) = 2$;
- (ii) *the general curve of the pencil $|M|$ is irreducible;*
- (iii) $MD = 2$;
- (iv) *if $F \neq 0$, then every component E of F is such that $DE = 0$ and $E^2 < 0$.*

Proof. Assertion (i) follows since $p_a(D) = 2$ and W is regular. Assertion (ii) follows by Bertini's Theorem, since every pencil on the regular surface W is rational.

Let us prove part (iii). Since D is nef and $D(M + F) = 2$, one has $MD \leq 2$. Suppose by contradiction that $MD \leq 1$. Consider the pull-back $|\tilde{M}|$ of $|M|$ to V , which is also the pull-back of a pencil $|N|$ on S . Since $\tilde{\pi}^*(D) = \epsilon^*(2K_S)$, one would have that $NK_S \leq 1$, which is impossible by Lemma 4.2. This proves (iii).

Assertion (iv) now follows by the Index Theorem. \square

We have already seen that $-4 \leq K_W^2 \leq 0$. Now we want to consider the surface W' as in Proposition 3.9.

Lemma 4.11. *One has $-2 \leq K_{W'}^2 \leq 0$. Furthermore $K_{W'}^2 = 0$ if and only if W' is a minimal Enriques surface.*

Proof. Since $D'(K_{W'} + D') = 2$, the Index Theorem implies that $(K_{W'} + D')^2 \leq 2$, or equivalently $K_{W'}^2 \leq 0$, because $K_{W'}D' = 0$ and $D'^2 = 2$. On the other hand, $(K_{W'} + D')^2 \geq 0$ and therefore $K_{W'}^2 \geq -2$.

If $K_{W'}^2 = 0$, then $K_{W'} \sim 0$, and therefore W' is an Enriques surface. The converse is trivial. \square

Lemma 4.12. *If $K_{W'}^2 < 0$, then $|K_{W'} + D'| = |M'|$ has no fixed part.*

Proof. Write, as usual, $|K_{W'} + D'| = F' + |M'|$, where F' is the fixed part and $|M'|$ is the movable part. By Lemma 4.10(iii), and the construction of the morphism $f : W \rightarrow W'$, which contracts only curves in F , we see that $M'D' = 2$.

Note that $F'K_{W'} = F'M' + F'^2$, so $M'F'$ is even. Since $M'F' \leq M'F' + M'^2 \leq (K_{W'} + D')^2 = K_{W'}^2 + 2 < 2$, it follows that $M'F' = 0$. This forces F' to be 0. Otherwise, since $D'F' = 0$, then $F'^2 < 0$, implying that $F'(K_{W'} + D') = F'^2 + M'F' < 0$, a contradiction because $(K_{W'} + D')$ is nef. \square

Remark 4.13. Since $K_W^2 \geq -4$ by Corollary 3.7(iv), the birational map $f : W \rightarrow W'$ of Proposition 3.9 is the contraction of at most 3 (resp., at most 4) exceptional curves if W' is rational (resp., an Enriques surface).

Note that an unessential singularity of B'_0 corresponding to a triple point requires at least 4 blowing-ups to be resolved, and exactly 4 unless there is a double point infinitely near to the triple point. Thus B'_0 can have a triple point only if W' is a minimal Enriques surface, $K_{W'}^2 = -4$ and B'_0 has only one triple point with no infinitely near double point.

If $K_W^2 = -4$, then $\tilde{\pi}$ induces an isomorphism between $H^2(W, \mathbb{Z})$ and $H^2(V, \mathbb{Z})$. This implies that there is no rational curve on W that does not meet the branch locus \tilde{B} . As a consequence, an unessential singularity of B'_0 can only be an ordinary triple point, a node or a cusp.

5. ON A NUMERICAL GODEAUX SURFACE OF ENRIQUES TYPE

In this section we keep the notation we introduced in the above sections.

Let S be a numerical Godeaux surface with an involution. Here we analyze the case in which W is birational to an Enriques surface (see Corollary 3.7(v)). This situation corresponds to the case in which W' is a minimal Enriques surface and $B'_0 = D'$ (see Lemma 4.11). The main information is given by the following result.

Proposition 5.1. *In the above setting, the general curve in the linear system $|D'|$ is smooth and irreducible.*

Proof. Since $D'^2 = 2$ and $h^0(W', \mathcal{O}_{W'}(D')) = 2$ by Lemma 4.10 and Proposition 3.9, it suffices to prove that the linear system $|D'|$ has no fixed component. Assume the contrary and write $|D'| = \Phi + |\Psi|$, where Φ is the fixed part. Since D' is nef, then $D'\Psi \leq 2$. On the other hand, by Lemma 4.2 we have $D'\Psi = 2$. This implies $\Psi^2 = 0$, $\Psi\Phi = 2$, and $\Phi^2 = -2$. Moreover by Proposition 3.9, one has $B'_0 \cap (C'_1 + \dots + C'_5) = \emptyset$, which implies $\Psi(C'_1 + \dots + C'_5) = 0$. Now we remark that $|\Psi|$ is a pencil of elliptic curves on the Enriques surface W' , thus Ψ is divisible by 2 in $\text{Pic}(W')$. Since $\Psi + \Phi + C'_1 + \dots + C'_5$ is also divisible by 2, we have that $\Phi + C'_1 + \dots + C'_5$ is divisible by 2. Since $\Psi(\Phi + C'_1 + \dots + C'_5) = \Psi\Phi = 2$, we find a contradiction. □

Remark 5.2. The above proposition also tells us that every numerical Godeaux surface with an involution of Enriques type is deformation equivalent to the double cover of a minimal Enriques surface W with five nodes C_1, \dots, C_5 , branched along a smooth curve $B_0 + C_1 + \dots + C_5$, where B_0 is irreducible of genus 2. There are examples of surfaces of this type (see Example 4.3 of [KL]; cf. Corollary 4.8 and Proposition 7.10 below).

Next we describe the torsion group of these surfaces.

Proposition 5.3. *The torsion group of a numerical Godeaux surface S of Enriques type is \mathbb{Z}_4 .*

Proof. By Remark 5.2, we may assume that W is a minimal Enriques surface and that B_0 is a smooth, irreducible curve of genus 2. One has a cartesian diagram

$$(9) \quad \begin{array}{ccc} Y & \longrightarrow & V \\ p \downarrow & & \downarrow \tilde{\pi} \\ T & \xrightarrow{g} & W \end{array}$$

where $g : T \rightarrow W$ is the K3-double cover of W , so $Y \rightarrow V$ is an étale double cover, where the minimal model of Y has $K^2 = 2$ and $p_g = 1$. Thus V , and hence S , has 2-torsion. Moreover $\text{Tors}(V) = \mathbb{Z}_4$ if and only if $\text{Tors}(Y) = \mathbb{Z}_2$.

Let us now look at the double cover $p : Y \rightarrow T$ which is branched along the curve $H := g^*(B_0)$ and the ten (-2) -curves which are the pull-back via g of the five (-2) -curves C_1, \dots, C_5 . Standard double cover considerations show that the bicanonical map of Y factors through p , in particular it is not birational. By Theorem 6.1 of [CaD], this implies that Y has 2-torsion, which implies the assertion. \square

Remark 5.4. By Proposition 5.3 and Corollary 4.4, it follows that, if S is a numerical Godeaux surface of Enriques type, then $|2K_S|$ has no fixed component.

6. ON NUMERICAL GODEAUX SURFACES OF CAMPEDELLI TYPE

In this section we follow the notation introduced above and we study the case $K_{W'}^2 = -2$ (cf. Lemma 4.11). As we will see, this case corresponds to numerical Godeaux surfaces which are birationally equivalent to double planes branched along a curve of degree 10 with a point of multiplicity 4 and five points of type $[3, 3]$. As we said in the Introduction, we call these surfaces of *Campedelli type*.

Lemma 6.1. *Let S be a numerical Godeaux surface with an involution and let W' be as in sections 3, 4. Suppose that $K_{W'}^2 = -2$. Then $|M'|$ is a base point free pencil of rational curves. Furthermore, if A is an irreducible component of a reducible fibre of $|M'|$, then one of the following occurs:*

- (i) *either A is a (-1) -curve such that $D'A = 1$ and $B'_0A = 3$;*
- (ii) *or A is a (-2) -curve such that $D'A = B'_0A = 0$.*

Proof. The hypothesis $K_{W'}^2 = -2$ implies that $M'^2 = 0$ and $K_{W'}M' = -2$, so that $|M'|$ is a base point free pencil of rational curves.

An irreducible component of a reducible fibre of $|M'|$ is a rational curve A with $A^2 < 0$. Statements (i) and (ii) follow from $0 = M'A = (K_{W'} + D')A \geq K_{W'}A = -2 - A^2$, where the inequality holds because D' is nef. \square

Lemma 6.1 immediately implies the following corollary (cf. also [Xi], [Ho]).

Corollary 6.2. *In the above setting, if $K_{W'}^2 = -2$, a reducible fibre of $|M'|$ is of one of the following types:*

- (i) *either it contains two irreducible (-1) -curves, in which case the dual graph of the fibre is a chain as in Figure 1(i);*
- (ii) *or it contains only one (-1) -curve with multiplicity 2, in which case the dual graph of the fibre is shown in Figure 1(ii) and (ii').*

In Figure 1 the black (resp. white) vertices correspond to curves contained (resp. not contained) in the branch locus. \square

Remark 6.3. Note that the number of the vertices in the graphs in Figure 1 is always odd. For every $i = 1, \dots, 5$, one has $C'_iM' = C'_i(K_{W'} + D') = 0$, thus the (-2) -curve C'_i is contained in a curve of the pencil $|M'|$. It is easy to verify (cf. also [Ho]) that the black vertices in Figure 1 correspond to the curves C'_1, \dots, C'_5 , except one of the (-2) -curves in (ii'), which is an irreducible component of B'_0 .

Since $M'D' = 2$, the pencil $(f \circ \tilde{\pi})^*(|M'|)$ is a genus 2 pencil without base points on V which descends via ϵ to a genus 2 pencil without base points on S .

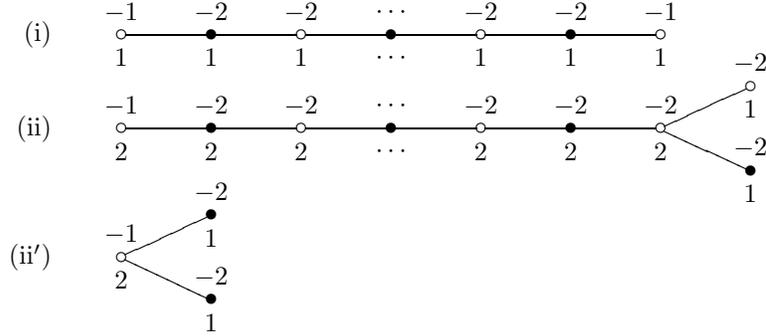


FIGURE 1. Reducible fibres of M' with self-intersections and multiplicities.

Proposition 6.4. *In the above setting, if $K_{W'}^2 = -2$, then there exists a birational morphism $g : W' \rightarrow \mathbb{F}_a$ with either $a = 1$ or $a = 3$. Let Δ be the exceptional divisor of g , let E be the $(-a)$ -curve on \mathbb{F}_a and let $\Theta = g^*(E)$ be its total transform on W' . Then*

$$(10) \quad D' \equiv 2\Theta + (a + 3)M' - \Delta.$$

Furthermore, if $a = 3$, then Θ is also the proper transform of E on W' .

Proof. As seen in Remark 6.3, each (-2) -curve C'_i is contained in a (reducible) curve of the pencil $|M'|$.

We claim that there exists a birational morphism $g : W' \rightarrow \mathbb{F}_a$, for some $a \geq 0$, which contracts each of the curves C'_i to points. Suppose that the fibre to which C'_i belongs is of type (i) of Corollary 6.2. In a birational morphism of W' to a \mathbb{F}_a , every component of such a fibre is contracted to a point, except one of the components corresponding to an end-point. This proves our claim in this case.

Otherwise, the fibre is of type either (ii) or (ii') of Corollary 6.2. In a birational morphism of W' to a \mathbb{F}_a , every component of such a fibre is contracted to a point, except either one of the two (-2) -components of multiplicity 1, corresponding to one of the two right end-points of the graphs in Figure 1. In both cases, there is only one of the (-2) -components which is one of the C'_i 's (cf. Remark 6.3), thus we may and will choose to contract it. This concludes the proof of our claim.

Recall that, since $K_{\mathbb{F}_a} \equiv -2E - (a + 2)F$, where F is the ruling of \mathbb{F}_a , $K_{W'} \equiv -2\Theta - (a + 2)M' + \Delta$. Then formula (10) follows from $K_{W'} + D' \equiv M'$.

Since D' is nef, one has that $0 \leq D'\Theta = -2a + a + 3$ by (10), hence $a \leq 3$.

Let $\bar{\Theta}$ be the proper transform of E on W' . Again, since D' is nef, one has

$$(11) \quad 0 \leq D'\bar{\Theta} = -2a + a + 3 - \Delta\bar{\Theta} = -a + 3 - \Delta\bar{\Theta}.$$

In particular, if $a = 3$, it follows that $\Delta\bar{\Theta} = 0$, which means that $\bar{\Theta} = \Theta$.

It remains to prove that a is odd. Note that g induces a double cover $\bar{\pi} : \bar{V} \rightarrow \mathbb{F}_a$ branched along the curve $\bar{B}_0 = g(B'_0)$, which has to be an even divisor in $\text{Pic}(\mathbb{F}_a)$. Since

$$(12) \quad B'_0 \equiv D' - 2K_{W'} \equiv 6\Theta + (3a + 7)M' - 3\Delta,$$

then $\bar{B}_0 \equiv 6E + (3a + 7)F$, which shows that a has to be odd. □

Remark 6.5. Passing from W' to \mathbb{F}_a creates only essential singularities of the branch curve \bar{B}_0 of the double cover $\bar{\pi} : \bar{V} \rightarrow \mathbb{F}_a$. More precisely, a fibre of type (i), as in Figure 1, with $2l + 1$ vertices produces $2l + 1$ a point x of type $[3^{2l}]$, whose tangent direction is different from the tangent direction of the fibre F_x of \mathbb{F}_a through x . Here we denote by $[3^n]$ the sequence $[3, \dots, 3]$ of length n , so a point of type $[3^n]$ is a triple point with an infinitely near point of type $[3^{n-1}]$.

Instead, as in Figure 1(ii) and (ii'), a fibre with $2l + 1$ vertices again produces a point x of type $[3^{2l}]$, but the tangent direction coincides with the one of F_x . In case (ii') (resp. case (ii)), the fibre F_x is (resp. is not) a component of \bar{B}_0 .

Now we show that the case $a = 3$ can be reduced to the case $a = 1$.

Lemma 6.6. *If $a = 3$, the branch curve \bar{B}_0 of the double cover $\bar{\pi} : \bar{V} \rightarrow \mathbb{F}_3$ is of the form $\bar{B}_0 = E + \bar{B}$, where E is the (-3) -curve in \mathbb{F}_3 and $E\bar{B} = 1$. Furthermore, there is a birational map $h : \mathbb{F}_3 \dashrightarrow \mathbb{F}_1$ such that $h \circ g : W' \rightarrow \mathbb{F}_1$ is a birational morphism contracting all the (-2) -curves C'_i , $i = 1, \dots, 5$, to points as in Proposition 6.4.*

Proof. Formula (10) says that $\bar{B}_0 \equiv 6E + 16F$, which implies the first assertion.

By Remark 6.5, \bar{B} has five points of type $[3, 3]$, which are off E , because $E\bar{B} = 1$, and can be proper or infinitely near. Let x be a $[3, 3]$ -point of \bar{B}_0 and let F_x be the fibre of the ruling of \mathbb{F}_3 through x . Since $\bar{B}F_x = 5$, Remark 6.5 again implies that the fibre F_x is not of type (ii) as in Figure 1. If F_x is of type (ii'), then F_x is a component of \bar{B} and $E\bar{B} = F_xE = 1$, thus there is at most one such fibre. Therefore there are fibres of type (i), and we may choose a point $x \in \mathbb{F}_3$ such that \bar{B} has a $[3, 3]$ -point at x , whose tangent direction is different from the one of the fibre F_x through x (and F_x is not a component of \bar{B}).

Now we define the birational map $h : \mathbb{F}_3 \dashrightarrow \mathbb{F}_1$ as the composition of two elementary transformations:

- the first one is based at the point x . This leads to \mathbb{F}_2 , and the proper transform of \bar{B} shows the triple point y which was infinitely near to x ;
- the second elementary transformation is based at y .

The birational map $h \circ g$ is a morphism because the exceptional curves created by the two elementary transformations are contracted by g , by construction. □

Remark 6.7. In the proof of the above lemma, we saw that, if $a = 3$, then there is no fibre of type (ii), and at most one fibre of type (ii').

Since \bar{B}_0 has a double point where \bar{B} meets E , then \bar{B} has at most one unessential singularity, which is resolved by one blowing-up (cf. Remark 4.13).

Finally we deal with the case $a = 1$ of Proposition 6.4.

Lemma 6.8. *In the above setting, if $a = 1$ and E is a component of the branch locus \bar{B}_0 of $\bar{\pi} : \bar{V} \rightarrow \mathbb{F}_1$, then $\bar{B}_0 = E + \bar{B}$, where \bar{B} has a tacnode at a point $x \in E$ where the tangent direction is also tangent to E at x . Moreover \bar{B} meets E transversally at another point $y \neq x$.*

Proof. By formula (12), one has $E\bar{B} = 5$. Since \bar{B}_0 has at most two unessential singularities (cf. Remark 4.13), then a $[3, 3]$ -point of \bar{B}_0 must lie on E . □

Now let $g' : \mathbb{F}_1 \rightarrow \mathbb{P}^2$ be the contraction of the (-1) -curve E of \mathbb{F}_1 to a point $p \in \mathbb{P}^2$. This induces a double plane $\pi^\bullet : V^\bullet \rightarrow \mathbb{P}^2$ branched along $B_0^\bullet = g'(\bar{B}_0)$.

The following corollary proves the Classification Theorem in the case $K_{W'}^2 = -2$.

Corollary 6.9. *Let S be a numerical Godeaux surface with an involution and suppose that $K_{W'}^2 = -2$. Then the double plane $\pi^\bullet : V^\bullet \rightarrow \mathbb{P}^2$, constructed as above, is branched along a reduced curve B_0^\bullet of degree 10 with a point q of multiplicity at least 4 and five points of type $[3, 3]$, which can be distinct or infinitely near, not lying on, a conic, and possibly further unessential singularities resolved by at most two blowing-ups.*

Proof. The curve B_0^\bullet has degree 10 by (12) and multiplicity either 5 or 4 at q , depending on whether E is or is not a component of \bar{B}'_0 . By Remark 6.5, B_0^\bullet has five $[3, 3]$ -points, which can be proper or infinitely near. The unessential singularities of B_0^\bullet are at most two, because $K_{W'}^2 - K_W^2 = -2 - K_W^2 \leq 2$ (cf. Remark 4.13).

The fact that there is no conic through q and the five $[3, 3]$ -points follows by standard double plane considerations, since $p_g(S) = 0$. □

Remark 6.10. Lemma 6.8 completely describes the nature of the singularity at q in the case where B^\bullet has a point of multiplicity 5 at q (cf. also Example 13.3 of [CF]).

Remark 6.11. Note that, if B^\bullet is irreducible, then its geometric genus is 0. Examples of this type can be found in [Re2], [St], [CG] (as shown by [DW]).

7. ON NUMERICAL GODEAUX SURFACES OF DU VAL TYPE

In this section we go on using the notation introduced above, and we study the remaining case $K_{W'}^2 = -1$ (cf. Lemma 4.11). We will show that it gives rise to numerical Godeaux surfaces which are birationally equivalent to double planes branched along a curve of degree 14 as in case (2) of the Classification Theorem, stated in the Introduction.

According to the terminology introduced in Example 3.7(c) of [Ci1], this double plane is a degeneration of a Du Val *ancestor* with invariants $p_g = 4$ and $K^2 = 8$. As we said in the Introduction, we will call these surfaces of *Du Val type*.

We start by studying the linear systems $|M'|$ and $|K_{W'} + M'|$. Recall that by definition $|M'| = |K_{W'} + D'|$ and $|M'|$ has no fixed part by Lemma 4.12.

Lemma 7.1. *Let S be a numerical Godeaux surface with an involution and let W' be as in sections 3, 4. Suppose that $K_{W'}^2 = -1$. Then:*

- (i) $|M'|$ is a pencil of elliptic curves with a simple base point;
- (ii) $h^0(W', \mathcal{O}_{W'}(K_{W'} + M')) = 1$;
- (iii) if G is the unique curve in $|K_{W'} + M'|$, then $GM' = 0$ and $G = 2E + C'_5$, where E is a (-1) -curve and $EC'_5 = ED' = 1$;
- (iv) the divisor $C'_1 + \dots + C'_4$ is even in $\text{Pic}(W')$.

Proof. Since $K_{W'}^2 = -1$, one has $M'^2 = (K_{W'} + D')^2 = 1$. Moreover $K_{W'}M' = -1$. This proves (i). The long exact sequence obtained from

$$0 \rightarrow \mathcal{O}_{W'}(K_{W'}) \rightarrow \mathcal{O}_{W'}(K_{W'} + M') \rightarrow \mathcal{O}_{M'} \rightarrow 0$$

implies part (ii), because W' is rational.

Let us prove part (iii). Since $p_a(M') = 0$, one has that $M'G = 0$, and every component θ of G is such that $M'\theta = 0$, because M' is nef. Furthermore $M'G = 0$ implies that the intersection form on the components of G is negative definite. Since $G^2 = -2$, there is an irreducible curve E in G such that $E^2 < 0$ and $E(K_{W'} + M') = EG < 0$. From $M'E = 0$, we conclude that $EK_{W'} < 0$, thus E is a (-1) -curve and $ED' = 1, EG = -1$. Furthermore $E(G - E) = 0$.

We claim that E intersects one of the (-2) -curves C'_i , say C'_5 . Indeed, since $D' + C'_1 + \dots + C'_5$ is divisible by 2 by Remark 3.10, the claim follows, because $ED' = 1$.

Note that $C'_5G = C'_5(2K_{W'} + D') = 0$. Since $EC'_5 > 0$, we have $C'_5 \leq G - E$. Since $E(G - E) = 0$, then E is necessarily a component of $G - E$. Note that $(2E + C'_5)^2 = 4E^2 + 4EC'_5 + C'^2_5 = 4EC'_5 - 6$ and so, because the intersection form on the components of G is negative definite, we conclude that $(2E + C'_5)^2 = -2$. Since $(2E + C'_5)G = -2$, we have $(G - (2E + C'_5))^2 = 0$ and, so again by negative definiteness, we see that $G = 2E + C'_5$.

So the proof of part (iii) is concluded.

Finally we prove part (iv). Since $D' + C'_1 + \dots + C'_5$ is divisible by 2, then $2K_{W'} + D' + C'_1 + \dots + C'_5 \equiv C'_5 + 2E + C'_1 + \dots + C'_5$ is divisible by 2, and therefore $C'_1 + \dots + C'_4$ is also divisible by 2. \square

Remark 7.2. Note that $C_1 + \dots + C_4$ is an even divisor in W by the previous lemma, hence $B_0 + C_5$ is even, too. So, by Beauville’s Lemma 3.11, $\text{Tors}_2(S)$ is not trivial and thus, by Theorem 4.1, $\text{Tors}(S)$ is either \mathbb{Z}_2 or \mathbb{Z}_4 .

We will later need the following.

Corollary 7.3. *In the above setting, the curve B'_0 is 1-connected.*

Proof. Suppose that B'_0 is not 1-connected and write $B'_0 = G + H$ with $GH = 0$. Then say $D'G = 0$ and $D'H = 2$. Since D' is nef, $G^2 < 0$, and therefore $0 = D'G = 2K_{W'}G + GH + G^2 = 2K_{W'}G + G^2$ implies that $K_{W'}G > 0$. Also we note that $H^2 \leq 1$. In fact the Index Theorem applied to D' and $D' - H$ implies that $H^2 \leq 2$ and that $H^2 = 2$ if and only if $D' \equiv H$ (because W' is rational). But if $D' \equiv H$, then $G \equiv -2K_{W'}$ and, since also $C_1 + \dots + C_4$ is even, this would imply by Beauville’s Lemma 3.11 that $|\text{Tors}_2(S)| \geq 2$, contradicting Theorem 4.1. Then $2 = D'H = 2K_{W'}H + H^2$ also implies that $K_{W'}H > 0$. Since $K_{W'}B'_0 = 2$, it follows that $K_{W'}H = K_{W'}G = 1$. From $0 = D'G = 2K_{W'}G + G^2$, we conclude that $G^2 = -2$, which contradicts the adjunction formula. \square

By Lemma 7.1, one has that

$$(13) \quad K_{W'} \equiv -M' + 2E + C'_5.$$

Furthermore, since $C'_1 + \dots + C'_4$ is an even divisor, then also $D' - C'_5 \equiv 2\Delta$, where $\Delta^2 = 0$ and $K_{W'}\Delta = 0$. By the Riemann-Roch Theorem, $|\Delta| \neq \emptyset$. The following identities are easy to check:

$$(14) \quad M' \equiv \Delta + E + C'_5,$$

$$(15) \quad B'_0 \equiv 4\Delta - 2E + C'_5 \equiv 4M' - 6E - 3C'_5,$$

$$(16) \quad K_{W'} + \Delta \equiv E.$$

Remark 7.4. Going back to our original surface W , the above formulas mean that $D \equiv 2\hat{\Delta} + C_5$, where $\hat{\Delta}$ is the pull-back of Δ to W .

Note that there is a birational morphism $g : W' \rightarrow X$ which contracts $G = 2E + C'_5$ to a smooth point q . The rational surface X is such that $K_X^2 = 1$ and $g^*(K_X) \equiv -M'$. Therefore $-K_X$ is nef and big.

We denote by \mathcal{D} the pencil $g_*(|D'|)$ and by $D'' \in \mathcal{D}$ its general element. Then $D''^2 = 4$, $K_X D'' = -2$ and \mathcal{D} has at q a base point with a fixed tangent. Furthermore $D'' \equiv -2K_X$.

We note that X still contains an even set of four disjoint (-2) -curves. Now we can apply the following result from [CCM], regarding rational surfaces with an even set of nodes (cf. also §4 in Chapter 0 of [CoD]).

Theorem 7.5. *Let X be a weak Del Pezzo surface, i.e. $-K_X$ is big and nef. Assume that $K_X^2 = 1$ and that X has four disjoint (-2) -curves C''_1, \dots, C''_4 such that $C''_1 + \dots + C''_4$ is even in $\text{Pic}(X)$. Then there exists a birational morphism $h : X \rightarrow X'$, where X' is obtained from \mathbb{F}_a , with $a = 0, 1$, or 2 , by blowing up:*

- two points q'_1, q'_2 in distinct fibres F_1, F_2 of the same ruling of \mathbb{F}_a ;
- the point which is the intersection of the strict transform of F_i with the exceptional curve corresponding to $q_i, i = 1, 2$;

in case $a = 2$, none of the blown-up points lies on the (-2) -curve of \mathbb{F}_2 . The morphism h maps isomorphically the (-2) -curves C''_1, \dots, C''_4 onto the proper transforms in X' of F_1, F_2 and of the exceptional curves corresponding to q'_1 and q'_2 . \square

Next corollary proves the Classification Theorem in the last case (2).

Corollary 7.6. *Let S be a numerical Godeaux surface with an involution and suppose that $K_W^2 = -1$. Then there exists a birational morphism $g' : W' \rightarrow \mathbb{P}^2$ such that the induced double cover $\pi^\bullet : V^\bullet \rightarrow \mathbb{P}^2$ is a double plane of Du Val type, branched along a reduced curve B^\bullet , which possibly has irrelevant singularities, resolved with at most three blowing-ups.*

Proof. Suppose that $a = 1$ in the statement of Theorem 7.5, i.e. there exists a birational morphism $X \rightarrow \mathbb{F}_1$. By blowing-down the (-1) -curve of \mathbb{F}_1 , it determines a birational morphism $h' : X \rightarrow \mathbb{P}^2$ and hence a birational morphism $g' = h' \circ g \circ f : W \rightarrow \mathbb{P}^2$. Up to reordering the indices, the (-2) -curves C_1 and C_2 are mapped via g' to two distinct lines, say r_1 and r_2 , whilst C_3 and C_4 are contracted to two distinct points $q_1 \in r_1$ and $q_2 \in r_2$. The anti-canonical pencil $| -K_X |$ on X is mapped via h' to a pencil of plane cubics with the following eight base points:

- $q_0 = r_1 \cap r_2$;
- $q_i, i = 1, 2$, where the cubics are tangent to r_i at q_i ;
- three further simple base points q_3, q_4, q_5 .

Therefore the pencil $|D|$ on W is mapped via g' to a pencil of plane sextics with the following base points:

- double points at q_0, q_3, q_4, q_5 ;
- a tacnode at $q_i, i = 1, 2$, where the tacnodal tangent is r_i ;
- a further simple base point q_6 with fixed tangent.

Standard double plane considerations imply the assertion in this case $a = 1$.

Suppose that $a = 2$ in the statement of Theorem 7.5, i.e. there exist birational morphisms $X \rightarrow X' \rightarrow \mathbb{F}_2$. Note that on X' there is a (-2) -curve N , which is the proper transform of the (-2) -curve on \mathbb{F}_2 , and two reducible fibres of type $A_1 + A_3 + 2N_1$ and $A_2 + A_4 + N_2$, where A_1, \dots, A_4 are (-2) -curves and N_1, N_2 are (-1) -curves with $NA_1 = NA_2 = 1, NA_3 = NA_4 = NN_1 = NN_2 = 0$. By first blowing-down N_1, N_2 and then the image of A_1 and of A_4 , one arrives at \mathbb{F}_1 , and the image of N is the (-1) -curve. At this point, one proceeds as in the previous case. Note that, in the present situation, either q_1 or q_2 is infinitely near to q_0 .

Suppose finally that $a = 0$. On X' there are two reducible fibres of the same ruling of the type $A_1 + A_3 + 2N_1$ and $A_2 + A_4 + 2N_2$, where A_1, \dots, A_4 are (-2) -curves and N_1, N_2 are (-1) -curves such that, if N_0 is a general fibre of the other

ruling, $N_0A_1 = N_0A_2 = 1$, $N_0A_3 = N_0A_4 = N_0N_1 = N_0N_2 = 0$. By first blowing-down N_1, N_2 and then the image of A_1 and of A_4 , one arrives at \mathbb{F}_1 , because the image of N_0 is a section with self-intersection 1, and one concludes as before. \square

Remark 7.7. Note that, if the degree 12 component of B^\bullet is irreducible, then its geometric genus is 1.

Remark also that all irrelevant singularities are double points, because four blowing-ups are needed to resolve an irrelevant triple point (cf. Remark 4.13).

Remark 7.8. The morphism $g' : W \rightarrow \mathbb{P}^2$ defined in the proof of Corollary 7.6 maps the curve $\tilde{\Delta}$ of Remark 7.4 to a plane cubic $\tilde{\Delta}$. Indeed the curve Δ on W' (cf. formulas (13)–(16)) is mapped to $\tilde{\Delta}$ via $h' \circ g : W' \rightarrow \mathbb{P}^2$.

Remark 7.9. Consider X' as in Theorem 7.5. Let $\tilde{C}_1, \dots, \tilde{C}_4$ be the (-2) -curves of X' such that $\tilde{C}_1 + \dots + \tilde{C}_4$ is even. Let $\tilde{\Upsilon}$ be the proper transform of a general conic through q_1 and q_2 and suppose that $q_i, i = 1, 2$, is not infinitely near to q_0 . Clearly $-2K_{X'} \equiv 2\tilde{\Upsilon} + \tilde{C}_1 + \dots + \tilde{C}_4$. Going back to W , setting H and Υ the pull-back of $-K_{X'}$ and $\tilde{\Upsilon}$, respectively, then $C_1 + \dots + C_4 \equiv 2(H - \Upsilon)$.

In Corollary 4.8 we saw that if R has an irreducible component Γ of genus 2, then W is either an Enriques surface or a rational surface with $K_W^2 = -2$ or -1 . Now we conclude this section by ruling out the case that W is rational, as announced.

Proposition 7.10. *Let S be a numerical Godeaux surface with an involution (cf. notation in section 4). Suppose that $-2 \leq K_W^2 \leq -1$ and $\Gamma^2 = 1$. Then W is not rational.*

Proof. Suppose by contradiction that W is rational. Let $\Gamma_0 = \eta^*(\pi\Gamma)$. By the proof of Corollary 4.8, one has that $B_0 - \Gamma_0 \equiv -2K_W$. Beauville’s Lemma 3.11 and Theorem 4.1 imply that in this case $\sum_{i=1}^4 C_i$ cannot be divisible by 2.

By the above Lemma 7.1, it follows that $W' = W$ and $K_W^2 = -2$. In this case $B_0 = \Gamma_0 + N_1 + N_2$, where N_1, N_2 are smooth rational curves with self-intersection -4 by formula (6). Then, because $B_0 - \Gamma_0 \equiv -2K_W$, $N_1 + N_2 \equiv -2K_W$. Note that $D - N_1 - N_2 \equiv D + 2K_W$ cannot be effective because $|M| = |D + K_W|$ is a base point free pencil of rational curves by Lemma 6.1.

Since $\Gamma_0 \in |D|$ is irreducible, the pencil $|D|$ has no fixed part. In particular any two distinct curves of D have no common component. Since $DN_1 = DN_2 = 0$, we can write $D \equiv G + N_1 \equiv H + N_2$. Since, as we saw, $D - N_1 - N_2$ is not effective, the two curves $G + N_1$ and $H + N_2$ in $|D|$ are distinct, hence G, H are effective divisors without common components. Note that $G^2 = H^2 = -2$ and $K_W G = K_W H = -2$. Since $2D \equiv G + H + N_1 + N_2 \equiv G + H - 2K_W$, one has $G + H \equiv 2(D + K_W) \equiv 2M$.

By Lemma 6.1, every component A of G is either a (-1) -curve or a (-2) -curve. In the latter case, $0 = AD = AH + AN_2$ implies that $AH = AN_2 = 0$, because G has no common components with H . Since $N_2 \equiv -2K_W - N_1$, also $AN_1 = 0$, implying that $AG = 0$. Moreover, since $K_W G = -2$, one has that G contains either two (-1) -curves A_1, A_2 or one (-1) -curve A with multiplicity 2. In the first case, $A_i G \geq A_i^2 = -1$, hence, since $G^2 = -2$, the equality holds and $(G - A_1 - A_2)^2 = 0$, implying by Zariski’s Lemma that $G = A_1 + A_2$. In the second case, one sees that $AG = -1$, implying that A meets $G - 2A$ transversally at one point of a (-2) -curve θ . As above we conclude that $G = 2A + \theta$. Similarly one shows that either $H = A'_1 + A'_2$ or $H = 2A' + \theta'$, where A', A'_i are (-1) -curves and θ' is a (-2) -curve.

Now recall that $2M \equiv G + H$, G and H have no common components and $|D - M| = \emptyset$. So $G + H = M_1 + M_2$, where M_1, M_2 are distinct fibres of $|M|$ and this excludes the possibility that $G = 2A + \theta$ or $H = 2A' + \theta'$.

Therefore the only remaining possibility is that $G = A_1 + A_2$ and $H = A'_1 + A'_2$. This implies that $M_1 = A_1 + A'_1$ and $M_2 = A_2 + A'_2$, after possibly reordering the indices, which is impossible by Corollary 6.2. \square

8. ON THE TORSION OF NUMERICAL GODEAUX SURFACES OF DU VAL TYPE

In this section we will study the torsion of a numerical Godeaux surface S of Du Val type. We will freely use the notation of the previous sections.

Proposition 8.1. *Let S be a numerical Godeaux surface of Du Val type. Then there is a non-trivial 2-torsion element in $\text{Tors}(S)$, and accordingly, there is an étale 2-to-1 cover $\tilde{S} \rightarrow S$, where \tilde{S} is a regular surface with $p_g(\tilde{S}) = 1$ and $K_{\tilde{S}}^2 = 2$. Therefore $\text{Tors}(S)$ is either $\mathbb{Z}/2\mathbb{Z}$ or $\mathbb{Z}/4\mathbb{Z}$, depending on whether \tilde{S} has either no torsion or 2-torsion.*

Proof. Recall that, by Remark 7.2, the (-2) -curves C_1, \dots, C_4 in W are such that $C_1 + \dots + C_4$ is even and so S has non-trivial 2-torsion, which defines an étale cover $\tilde{S} \rightarrow S$. The rest of the assertion follows from Theorem 4.1. \square

We can now consider the following commutative diagram:

$$\begin{array}{ccc} \tilde{V} & \longrightarrow & V \\ \downarrow \Pi & & \downarrow \tilde{\pi} \\ \tilde{W} & \xrightarrow{t} & W \end{array}$$

where $t : \tilde{W} \rightarrow W$ is the double cover branched over $C_1 + \dots + C_4$, $\Pi : \tilde{V} \rightarrow \tilde{W}$ is the double cover branched over $t^*(B_0 + C_5)$ and $\tilde{V} \rightarrow V$ is the étale double cover associated to the 2-torsion of V .

Note that the minimal model of \tilde{V} is obviously isomorphic to the surface \tilde{S} of Proposition 8.1. Also remark that any smooth rational curve θ on W disjoint from the curves C_1, \dots, C_4 pulls back on \tilde{W} to two disjoint rational curves with the same self-intersection number as θ .

Note also that $t^*(B_0 + C_5) = C_{5,1} + C_{5,2} + \tilde{B}_0$, where \tilde{B}_0 is an étale double cover of B_0 , which is disjoint from the two (-2) -curves $C_{5,1}$ and $C_{5,2}$. Furthermore, since $B_0 + C_5$ is divisible by 2 in $\text{Pic}(W)$, then also $C_{5,1} + C_{5,2} + \tilde{B}_0 \equiv 2\Phi$ with Φ in $\text{Pic}(\tilde{W})$.

Lemma 8.2. *Let S be a numerical Godeaux surface of Du Val type, and suppose that $\text{Tors}(S) = \mathbb{Z}/4\mathbb{Z}$. Then the double cover t splits over $B_0 + C_5$.*

Proof. By Proposition 8.1, \tilde{V} has 2-torsion if and only if $\text{Tors}(S) = \mathbb{Z}/4\mathbb{Z}$. On the other hand, \tilde{W} is rational and thus, by Beauville’s Lemma 3.11, \tilde{W} has 2-torsion if and only if there exists a curve $G \not\lesssim \tilde{B}_0 + C_{5,1} + C_{5,2}$ divisible by 2 in $\text{Pic}(\tilde{W})$. Let ι be the involution on $\text{Pic}(\tilde{W})$ (and $H^2(\tilde{W}, \mathbb{Z})$) induced by the involution on \tilde{W} corresponding to the double cover t . We note that Φ , where $C_{5,1} + C_{5,2} + \tilde{B}_0 \equiv 2\Phi$, is invariant under ι .

Suppose that there exists a curve $G \not\lesssim \tilde{B}_0 + C_{5,1} + C_{5,2}$ divisible by 2 in $\text{Pic}(\tilde{W})$. Then also $H := \tilde{B}_0 + C_{5,1} + C_{5,2} - G$ is divisible by 2, and we can write $G \equiv 2L_1$,

$H \equiv 2L_2$, where $L_1 + L_2 \equiv \Phi$. Note that neither L_1 nor L_2 can be invariant under ι , because otherwise there would be too many divisibility relations in the branch locus in W , implying again by Beauville’s Lemma 3.11 that $|\text{Tors}_2(S)| \geq 2$, which is impossible by Theorem 4.1.

Then write $G = G' + J_1$, and $H = H' + J_2$, where all divisors appearing are effective, and we assume that $\iota(G') = G'$, $\iota(H') = H'$, $\iota(J_i) \neq J_i$, $i = 1, 2$ (and $\iota(\theta) \neq \theta$ for any component θ of J_i). Since $\iota(G+H) = G+H$, necessarily $\iota(J_1) = J_2$. Note also that $J_i \neq 0$, $i = 1, 2$, otherwise $\iota(G) = G$ and so $\iota(2L_1) = 2\iota(L_1) = 2L_1$, implying, because \tilde{W} is rational, that $\iota(L_1) = L_1$. This would mean that $\iota(G) = G$ and the divisibility would be coming already from W , which is impossible.

Now we have $G + \iota(G) = G' + J_1 + G' + J_2 = 2G' + J_1 + J_2$. On the other hand, $G + \iota(G) \equiv 2L_1 + 2\iota(L_1)$, and so we conclude that $J_1 + J_2 \equiv 2L_1 + 2\iota(L_1) - 2G'$ is divisible by 2. Again by Beauville’s Lemma 3.11 and Theorem 4.1, there cannot further exist divisibility relations in $\tilde{B}_0 + C_{5,1} + C_{5,2}$, hence we conclude that $G' = H' = 0$, $\iota(L_1) = L_2$, $\iota(J_1) = J_2$ and therefore t splits over $B_0 + C_5$. \square

Lemma 8.3. *Let S be a numerical Godeaux surface of Du Val type and suppose that the double cover t splits over $f^*(B'_0 + C'_5)$, where $f : W \rightarrow W'$ is the birational morphism of Proposition 3.9. Then $\text{Tors}(S) = \mathbb{Z}/4\mathbb{Z}$.*

Proof. Note that the components of $f^*(B'_0)$, which are not components of B_0 , form trees of rational curves; each one of them is double in $f^*(B'_0)$ and does not meet C_1, \dots, C_4 , so t splits over it.

Write $t^*(f^*(B'_0 + C'_5)) = A_1 + C_{5,1} + A_2 + C_{5,2}$ where $\iota(A_1) = A_2$ and A_i , $i = 1, 2$, is isomorphic to $f^*(B'_0)$, so it is connected by Corollary 7.3, and each component of A_i , which is not a component of $t^*(B_0)$, is double in A_i . Recall that $B'_0 + C'_5 \equiv 4\Delta - 2E + 2C'_5$ by formula (15). Denote the two connected components of $t^*(f^*(E))$ by E_1 and E_2 , where $E_1C_{5,1} = 1$ and $E_1C_{5,2} = 0$. Since $EB'_0 = 3$ and $(E+B'_0)^2 > 0$, $t^*(f^*(E + B'_0))$ is connected and therefore, say, $E_2A_1 = 2$ and $E_2A_2 = 1$.

Then we claim that $A_1 \equiv t^*(f^*(2\Delta)) - 2E_2 + C_{5,1}$. In fact we have $t^*(D)A_1 = 2$ and $t^*(D)(t^*(f^*(2\Delta)) - 2E_2 + C_{5,1}) = 2$. Since $(A_1 - t^*(f^*(2\Delta)) + 2E_2 - C_{5,1})^2 = 0$, by the Index Theorem we obtain the claim, because \tilde{W} is a rational surface.

Hence $A_1 + C_{5,1}$ is divisible by 2 in $\text{Pic}(\tilde{W})$, and therefore also $t^*(B_0 + C_5)$ strictly contains an effective divisor divisible by 2, implying by Beauville’s Lemma 3.11 that \tilde{V} (and \tilde{S}) has 2-torsion. \square

Remark 8.4. Note that if t splits over $f^*(B'_0 + C'_5)$, then t splits also over $B_0 + C_5$, but the converse is not necessarily true.

For the examples of the next section, we will need the following criterion.

Theorem 8.5. *Let S be a numerical Godeaux surface S of Du Val type (cf., e.g., Corollary 7.6 and the notation therein). Suppose that B_0 on W is a (smooth) irreducible curve with $p_a(B_0) = 1$, and that q_i , $i = 1, 2$, is not infinitely near to q_0 .*

Then $\text{Tors}(S) = \mathbb{Z}/4\mathbb{Z}$ if and only if there exists a (unique) plane curve of degree 8 with the following singularities:

- a point of multiplicity 2 at q_0 ;
- a point of type $[3, 2]$ at q_i , $i = 1, 2$, where the infinitely near double point is in the direction of the line r_i ;
- triple points at q_3, q_4, q_5 ;
- a tacnode at q_6 , where the tacnodal tangent is also tangent to B^\bullet .

Proof. The hypotheses imply that $W' = W$. By Lemmas 8.2-8.3, $\text{Tors}(S) = \mathbb{Z}/4\mathbb{Z}$ if and only if the double cover t splits on B_0 , which happens by Remark 7.9 if and only if the restriction of $-(C_1 + \dots + C_4)/2 \equiv -H + \Upsilon$ to B_0 is trivial, or, equivalently, if and only if $\mathcal{O}_{B_0}(K_W + B_0 - H + \Upsilon) \cong \mathcal{O}_{B_0}$. Note that, since $B_0 \cap C_i = \emptyset$, $i = 1, \dots, 4$, $\mathcal{O}_{B_0}(K_W + B_0 - H + \Upsilon)$ has degree 0. The sequence

$$0 \rightarrow \mathcal{O}_W(K_W - H + \Upsilon) \rightarrow \mathcal{O}_W(K_W + B_0 - H + \Upsilon) \rightarrow \mathcal{O}_{B_0}(K_W + B_0 - H + \Upsilon) \rightarrow 0$$

is exact. Note that $H^0(W, \mathcal{O}_W(K_W - H + \Upsilon)) = 0$, because

$$H^0(W, \mathcal{O}_W(2K_W - C_1 - \dots - C_4)) = 0,$$

and, since $H^2(W, \mathcal{O}_W(K_W - H + \Upsilon)) = 0$, the Riemann-Roch Theorem implies that $H^1(W, \mathcal{O}_W(K_W - H + \Upsilon)) = 0$.

Thus $\mathcal{O}_{B_0}(K_W + B_0 - H + \Upsilon) \cong \mathcal{O}_{B_0}$ if and only if $h^0(W, \mathcal{O}_W(K_W + B_0 - H + \Upsilon)) = 1$. The morphism $g' : W \rightarrow \mathbb{P}^2$ (cf. the proof of Theorem 7.6) maps the unique curve in $|K_W + B_0 - H + \Upsilon|$ to a plane curve of degree 8 as in the statement. \square

Theorem 8.5 can be extended, under suitable assumptions, e.g. as follows.

Corollary 8.6. *Let S be a numerical Godeaux surface S of Du Val type. Suppose that q_i , $i = 1, 2$, is not infinitely near to q_0 and the curve B_0 on W has two irreducible components, Γ_0 of genus 1 and Z of genus 0, such that B'_0 on W' is the union of two curves meeting transversally at a point. Then $\text{Tors}(S) = \mathbb{Z}/4\mathbb{Z}$ if and only if there exists a plane curve of degree 8, as in the statement of Theorem 8.5, containing the image of Z in the plane.*

Proof. Lemmas 8.2 and 8.3 again imply that the torsion group of S is $\mathbb{Z}/4\mathbb{Z}$ if and only if the double cover $t : \tilde{W} \rightarrow W$ splits over Γ_0 . The same argument of the proof of Theorem 8.5 shows that there is the plane curve of degree 8 as above and it contains the rational curve which is the image of Z in the plane. \square

Remark 8.7. By Propositions 4.5 and 7.10 (cf. Remarks 6.11 and 7.7), it follows that numerical Godeaux surfaces described in Example 4.2 in [KL], which have $\text{Tors} = \mathbb{Z}/4\mathbb{Z}$, are birational to double planes of Du Val type. In that example, indeed, the ramification curve R is irreducible of genus 1, and this may happen, by our classification results, only if W is rational and S is of Du Val type.

A natural question is whether a numerical Godeaux surface S of Du Val type can also be of Campedelli type, of course for different involutions on S .

In order to give a result about this problem, we first prove the following.

Proposition 8.8. *Let S be a numerical Godeaux surface of Campedelli type. If there is a non-trivial element η_2 in $\text{Tors}_2(S)$, then the unique curve Δ_2 in $|K_S + \eta_2|$ is reducible and has a common component A with the ramification curve R . Furthermore, A is rational.*

Proof. Note that Δ_2 is fixed by the involution, and it is not contained in R , because Δ_2 has arithmetic genus 2 (cf. Corollary 4.8 and Proposition 7.10). Suppose that Δ_2 and R have no common component. Looking at the Campedelli double plane representation (cf. Corollary 6.9), the image of Δ_2 is a plane curve Δ' , which has no common component with the branch curve B^\bullet . Since $2\Delta_2 \in |2K_S|$, the curve $2\Delta'$ is contained in the linear system of quartics with a double point at p and passing simply through the five points of type $[3, 3]$ with tangent lines also tangent to the $[3, 3]$ -point. Then Δ' should be a conic passing through all the given

points, a contradiction. We conclude that Δ_2 and R have a common component A . Reasoning as above, one sees that the image of A in the plane is a conic, hence A is rational. \square

Corollary 8.9. *Let S be a numerical Godeaux surface of Du Val type and $\pi^\bullet : V^\bullet \rightarrow \mathbb{P}^2$ the corresponding double plane, branched along the curve B^\bullet as in Corollary 7.6. Let $\bar{\Delta}$ be the unique plane cubic curve passing through $q_0, q_i, i = 1, 2$, where it is tangent to r_i, q_3, q_4, q_5 and q_6 . Suppose that $\bar{\Delta}$ is irreducible and does not pass through any irrelevant singularity of B^\bullet . Then S has no involution which realizes it as a double plane of Campedelli type.*

Proof. By Lemma 7.1 and Remark 7.2, S has an element η_2 of 2-torsion. According to Remark 7.8, the unique curve Δ_2 in $|K_S + \eta_2|$ is mapped to the plane cubic $\bar{\Delta}$. The hypotheses on $\bar{\Delta}$ imply that Δ_2 is irreducible, hence there is no involution on S which realizes it as a double plane of Campedelli type by Proposition 8.8. \square

9. EXAMPLES OF NUMERICAL GODEAUX SURFACES OF DU VAL TYPE

As we already remarked, there was no previously known construction of numerical Godeaux surfaces as double planes of Du Val type.

In this section we produce such a construction. In order to do that, one has to find a reduced curve B^\bullet of degree 12 with singularities at the points q_0, \dots, q_6 as described in the Classification Theorem stated in the Introduction.

If one chooses the points in general position, then one expects no curve like B^\bullet , because the virtual dimension of the linear system of curve of degree 12 with those singularities is -2 . However it is possible to find such irreducible curves. One way is to look for a curve which is invariant under a linear transformation of the plane of order 2. This is an idea originally used by Stagnaro in order to construct numerical Godeaux surfaces as double planes of Campedelli type; cf. [St] and [We3].

Example 9.1. Let $[x, y, z]$ be homogeneous coordinates in \mathbb{P}^2 . Let r_1 be the line $x = y$ and choose the following points:

$$q_0 = [0, 0, 1], \quad q_1 = [1, 1, 1], \quad q_3 = [1, 0, 0], \quad q_4 = [0, 1, 1], \quad q_6 = [-2, 0, 1],$$

and let r_3 be the line $x + 2z = 0$, which passes through q_6 . The linear involution

$$\phi : [x, y, z] \in \mathbb{P}^2 \rightarrow [x, -y, z] \in \mathbb{P}^2$$

fixes $[0, 1, 0]$ and all the points of the line $y = 0$. Consider the line $r_2 = \phi(r_1)$ and the points $q_2 = \phi(q_1) = [1, -1, 1]$ and $q_5 = \phi(q_4) = [0, -1, 1]$.

Note that there is no conic passing through q_1, q_2, \dots, q_6 .

One sees that there are 49 monomials of degree 12 in x, y, z whose equations are invariant under ϕ , i.e. their degree in y is even. Hence the curves of degree 12, which are invariant under ϕ , form a linear system of dimension 48. Now we impose:

- quadruple points at q_0, q_3 and q_4 ;
- a point of type $[4, 4]$ at q_1 , where the tangent direction is the line r_1 ;
- a point of type $[3, 3]$ at q_6 , where the tangent direction is the line r_3 .

Note that, in this way, we are also imposing another quadruple point at $q_5 = \phi(q_4)$ and another point of type $[4, 4]$ at $q_2 = \phi(q_1)$, where the tangent direction is the line $r_2 = \phi(r_1)$, so the resulting curve should have the required singularities.

The interesting observation is that q_0 and q_3 , instead of imposing 10 independent conditions each, impose only 6 conditions each, because of the symmetry of

the configuration. For the same reason, the singularities at q_6 do not impose 12 independent conditions as expected, but only 6. In conclusion, the number of independent conditions we are imposing is no more than 48, so that there is surely a curve with at least the required singularities.

Using a computer algebra program (we used Maple for this), it turns out that there is one curve of degree 12 satisfying the above conditions and having exactly the required singularities, and not worse than those. Its equation is:

$$\begin{aligned} & 15625 x^8 y^4 - 117650 x^8 y^2 z^2 + 81289 x^8 z^4 + 604400 x^7 y^4 z - 908128 x^7 y^2 z^3 + 386672 x^7 z^5 + \\ & - 867475 x^6 y^6 + 1931246 x^6 y^4 z^2 - 1557283 x^6 y^2 z^4 + 369096 x^6 z^6 - 561632 x^5 y^6 z + \\ & + 429504 x^5 y^4 z^3 + 777504 x^5 y^2 z^5 - 562432 x^5 z^7 + 857979 x^4 y^8 - 520134 x^4 y^6 z^2 + \\ & - 2103701 x^4 y^4 z^4 + 2553616 x^4 y^2 z^6 - 808496 x^4 z^8 - 572400 x^3 y^8 z + 2171200 x^3 y^6 z^3 + \\ & - 2711600 x^3 y^4 z^5 + 1112800 x^3 y^2 z^7 + 802575 x^2 y^{10} - 4569750 x^2 y^8 z^2 + 8348975 x^2 y^6 z^4 + \\ & - 6199000 x^2 y^4 z^6 + 1617200 x^2 y^2 z^8 + 540000 x y^{10} z - 1620000 x y^8 z^3 + 1620000 x y^6 z^5 + \\ & - 540000 x y^4 z^7 - 810000 y^{12} + 3240000 y^{10} z^2 - 4860000 y^8 z^4 + 3240000 y^6 z^6 - 810000 y^4 z^8 = 0. \end{aligned}$$

According to Maple, this curve is irreducible over the algebraic closure of \mathbb{Q} .

Proposition 9.2. *The above curve of degree 12, together with the lines $x = y$ and $x = -y$, is the branch curve of a double plane whose smooth minimal model is a numerical Godeaux surface S of Du Val type with $\text{Tors}(S) = \mathbb{Z}/4\mathbb{Z}$.*

Proof. By Theorem 8.5, it remains only to check that there exists a plane curve of degree 8 with the singularities described in that statement. Again using Maple, we found that such a curve actually exists and is the union of the sextic

$$\begin{aligned} & 900 y^6 - 300 y^4 x z - 1800 y^4 z^2 - 519 y^4 x^2 + 900 y^2 z^4 + 106 y^2 x^3 z + 1597 y^2 x^2 z^2 + \\ & - 365 y^2 x^4 + 300 y^2 x z^3 + 130 x^5 z - 884 x^2 z^4 + 299 x^4 z^2 - 364 x^3 z^3 = 0. \end{aligned}$$

and the lines $y = z$, $y = -z$. \square

In [We3], Caryn Werner shows that all numerical Godeaux surfaces of Campedelli type, whose branch curve is invariant under a linear involution of the plane, have $\text{Tors}(S) = \mathbb{Z}/4\mathbb{Z}$. It would be interesting to see whether a similar statement is still true for numerical Godeaux surfaces of Du Val type.

The following example is computationally more complicated, but it also gives instances of numerical Godeaux surfaces S of Du Val type with $\text{Tors}(S) = \mathbb{Z}/2\mathbb{Z}$.

Example 9.3. Let $[x, y, z]$ be homogeneous coordinates in \mathbb{P}^2 . Consider the lines $r_1 : x = 0$, $r_2 : x = y$, and the points

$$q_0 = [0, 0, 1], q_1 = [0, 1, 0], q_2 = [1, 1, 0], q_3 = [1, 0, 0], q_4 = [-1, 1, 1], q_5 = [1, -2, 1].$$

We want to find a curve B^\bullet which is reducible in the line $r_3 : y = 0$, which passes through q_0 and q_3 , and in a curve B_1^\bullet of degree 11 with the following singularities:

- two points of type $[4, 4]$ in q_1, q_2 , with tangent lines r_1, r_2 ;
- two quadruple points at q_4 and q_5 ;
- two triple points at q_0, q_3 ;
- a tacnode at a point $q_6 = [t, 0, 1]$ on r_3 , with tacnodal tangent r_3 .

Clearly, we want q_6 to be different from q_0 , i.e. $t \neq 0$.

Note that the virtual dimension of the linear system of curves of degree 11 with these singularities is -1 . Thus it is reasonable to expect that for finitely many points on r_3 there is a curve with the required singularities. Using Maple, we found a polynomial $p(t)$ of degree 15 in t such that, if t_0 is a root of $p(t)$, then there is a curve B_1^\bullet with at least the above singularities, with $q_6 = [t_0, 0, 1] \neq q_0$.

Luckily, the polynomial $p(t)$ factors over \mathbb{Q} , and there is an irreducible factor of degree 5 and another of degree 10. Maple is able to work out the computations on the corresponding algebraic extensions of \mathbb{Q} and to verify that the corresponding curves B_1^\bullet have the required, and not worse, singularities and are irreducible over the algebraic closure of \mathbb{Q} .

Unfortunately, as we said, the computations are complicated and the results are very cumbersome, and it is not the case to exhibit the explicit equations here. They can be found at the following Web address:

<http://www.mat.uniroma2.it/~calabri/duValeqs.pdf>.

It is interesting to remark that the solutions relative to the degree 5 factor have torsion $\mathbb{Z}/4\mathbb{Z}$, whereas the ones of the factor of degree 10 have torsion $\mathbb{Z}/2\mathbb{Z}$. This is verified by applying Corollary 8.6, i.e. by checking the existence or not of the curve of degree 8 with the appropriate singularities, which in this case is the union of the line r_3 and a curve of degree 7.

Finally, one can check, again by using Maple, that the unique cubic plane curve $\bar{\Delta}$ as in Corollary 8.9 is irreducible and does not pass through the irrelevant singularities of B^\bullet , which is just a point in $B_1^\bullet \cap r_3$ different from q_1, q_3, q_6 . Therefore these surfaces cannot be of Campedelli type by Corollary 8.9.

Similar computations can also be tried in order to find irreducible curves B^\bullet , giving $\text{Tors}(S) = \mathbb{Z}/2\mathbb{Z}$. Unfortunately, Maple is not able to carry out all the computations in a reasonable time. However we will see in a moment, in Corollary 9.7, that there are equisingular deformations of our curve which are irreducible.

Remark 9.4. Using Maple, we verified that, in both of the above examples, imposing all the required singularities but q_4 , then the quadruple point q_4 can be chosen only in finitely many ways. The verification is performed by checking that there is no solution to the problem of finding the quadruple point q_4 on a given line, say $y - z = 0$ in Example 9.1, and $x + y = 0$ in Example 9.3.

This agrees with the following fact. The linear system of curves of degree 12 with the given singularities at all the points but q_4 has virtual dimension 8, which coincides with the actual dimension. This can be verified using Maple. Imposing a further fixed quadruple point q_4 is 10 more conditions. However, by moving q_4 with two parameters in the plane, we expect only eight more conditions. The above computation shows that this naive expectation is actually right.

We want to finish by giving information about the number of parameters on which our constructions of numerical Godeaux surfaces of Du Val type depend.

Proposition 9.5. *Let B^\bullet be the plane curve of degree 12 as in one of the two examples above. Let \mathcal{V} be a complete family of plane curves of degree 12 which is maximal under the property that its general element is an equisingular deformation of B^\bullet . Then $\dim \mathcal{V} = 13$.*

Proof. It suffices to prove that $\dim(\mathcal{V}/PGL(3, \mathbb{C})) = 5$. Up to projective transformations, we can fix distinct points q_0, q_1, q_2 and q_3 , so that $r_1 = \overline{q_0q_1}$ and $r_2 = \overline{q_0q_2}$ are also fixed. Let \mathcal{L} be the linear system of curves of degree 12 with quadruple points at q_0, q_3 and points of type $[4, 4]$ at q_i , where the tangent line is r_i , $i = 1, 2$. Consider the following quadruplets $(B, p, q, \xi) \in \mathcal{L} \times \mathbb{P}^2 \times \mathbb{P}^2 \times \Delta$, where:

- $\Delta \subset \mathbb{P}^2 \times \mathbb{P}^{2*}$ is the incidence correspondence so that $\xi \in \Delta$ is a pair (x, r) , with x a point of the line r ;

- $B \in \mathcal{L}$ is irreducible and reduced;
- q_0, \dots, q_3, p, q, x are all distinct;
- B has 4-uple points at p, q and a $[3, 3]$ -point at x with tangent direction r .

Let $\bar{\mathcal{V}}$ be an irreducible component of the closure of the set of such quadruplets, containing the point $\beta = (B^\bullet, q_4, q_5, (q_6, r_3))$. Note that $\dim \bar{\mathcal{V}} \geq 5$. In fact, $\bar{\mathcal{V}}$ is defined in a neighbourhood of β inside the variety $\mathcal{L} \times \mathbb{P}^2 \times \mathbb{P}^2 \times \Delta$, of dimension at least 37, by 32 equations. Since the projection on the first factor from $\bar{\mathcal{V}}$ to \mathcal{L} is generically finite to its image, it suffices to prove that $\dim \bar{\mathcal{V}} = 5$.

Then consider the projection $\rho : \bar{\mathcal{V}} \rightarrow \mathbb{P}^2 \times \Delta$ to the last two factors. Remark 9.4 shows that ρ is generically finite. This concludes the proof. \square

Corollary 9.6. *The components of the moduli space of numerical Godeaux surfaces with an involution of Du Val type, containing our two Examples 9.1 and 9.3, are 5-dimensional.* \square

Another interesting consequence is the following.

Corollary 9.7. *In the case of Example 9.3, the general element in $\bar{\mathcal{V}}$ corresponds to an irreducible curve.*

Proof. Since, as we saw in the proof of Proposition 9.5, the map ρ is generically finite, then it is also surjective. This means that the point q_6 can be moved out of the line passing through q_0 and q_3 , hence, for the general member of $\bar{\mathcal{V}}$, the line cannot be a component of B^\bullet . \square

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