TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 359, Number 4, April 2007, Pages 1485–1498
S 0002-9947(06)04205-X
Article electronically published on November 3, 2006

A GENERALIZATION OF HALF-PLANE MAPPINGS TO THE BALL IN $\mathbb{C}^n$

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Abstract. Let F be a normalized ($F(0) = 0$, $DF(0) = I$) biholomorphic mapping of the unit ball $B \subseteq \mathbb{C}^n$ onto a convex domain $\Omega \subseteq \mathbb{C}^n$ that is the union of lines parallel to some unit vector $u \in \mathbb{C}^n$. We consider the situation in which there is one infinite singularity of F on ∂B. In one case with a simple change-of-variables, we classify all convex mappings of B that are half-plane mappings in the first coordinate. In the more complicated case, when u is not in the span of the infinite singularity, we derive a form of the mappings in dimension $n = 2$.

1. Introduction

In this article, the authors continue a classification started in [3] of biholomorphic mappings F of the unit ball $B = B_n = \{z \in \mathbb{C}^n : \|z\| < 1\}$ of $\mathbb{C}^n$ onto unbounded convex domains in $\mathbb{C}^n$ that can be written as the union of parallel lines. We assume, as is common, that F is normalized to satisfy $F(0) = 0$ and $DF(0) = I$, where I is the identity operator.

We pause to standardize some notation. We write $\|\cdot\|$ for the Euclidean norm in $\mathbb{C}^n$ and $\langle\cdot,\cdot\rangle$ for the standard Hermitian inner product. The standard basis vectors of $\mathbb{C}^n$ are written e_k, $k = 1, \ldots, n$. Given $z \in \mathbb{C}^n$, with $n \geq 2$, it will be convenient to write $z = (z_1, \hat{z})$, where $\hat{z} \in \mathbb{C}^{n-1}$ contains the last $n-1$ components of z. Let $\mathbb{N}_0$ stand for the set of nonnegative integers. Then $\mathbb{N}_0^n$ is the set of multi-indices, and in common shorthand for $z \in \mathbb{C}^n$ and $\alpha \in \mathbb{N}_0^n$, we write

$$z^\alpha = \prod_{k=1}^{n} z_k^{\alpha_k}, \qquad |\alpha| = \sum_{k=1}^{n} \alpha_k.$$

The group of biholomorphic automorphisms of B is denoted $\operatorname{Aut} B$. Furthermore, given a locally biholomorphic function $G : B \to \mathbb{C}^n$ and $\varphi \in \operatorname{Aut} B$, the Koebe transform of G with respect to φ is denoted $\Lambda_\varphi(G)$. More explicitly,

$$\Lambda_\varphi(G)(z) = D\varphi(0)^{-1} DG(\varphi(0))^{-1}[G(\varphi(z)) - G(\varphi(0))], \qquad z \in B.$$

If $\mathcal{K}$ is the family of all normalized biholomorphic convex mappings of B into $\mathbb{C}^n$, then $\mathcal{K}$ is known [4] to be compact (in the topology of uniform convergence on compact subsets of B) and linearly invariant. (The latter meaning $\Lambda_\varphi(G) \in \mathcal{K}$ for

Received by the editors January 4, 2005.
2000 *Mathematics Subject Classification.* Primary 32H02; Secondary 30C55.
Key words and phrases. Biholomorphic, convex mapping, holomorphic automorphism.

all $G \in \mathcal{K}$ and $\varphi \in \operatorname{Aut} B$.) Furthermore, each $G \in \mathcal{K}$ satisfies the useful growth condition

$$\|G(z)\| \leq \frac{\|z\|}{1 - \|z\|}, \qquad z \in B. \tag{1.1}$$

It was noted in [3] that if $\Omega = F(B)$ is unbounded, then the set $A = \{u \in \partial B : ru \in \Omega,\ r \geq 0\}$ is nonempty. We are particularly interested in the case where $u, -u \in A$ for some $u \in \partial B$. (Incidentally, $A = \{u, -u\}$ is the only case that is possible if A is disconnected.) Convexity implies that Ω is the union of lines parallel to the vector u. In [3], it is revealed that the family $\{\psi_t : t \in \mathbb{R}\} \subseteq \operatorname{Aut} B$ given by

$$\psi_t(z) = F^{-1}(F(z) + tu), \qquad z \in B,\ t \in \mathbb{R}, \tag{1.2}$$

is a one-parameter subgroup of $\operatorname{Aut} B$. (The mapping $t \mapsto \psi_t$ is a continuous homomorphism of $\mathbb{R}$ into $\operatorname{Aut} B$ with the topology of uniform convergence on compact subsets of B.) There exist $a, b \in \partial B$ for which $\lim_{t\to\infty} \psi_t \equiv a$ and $\lim_{t\to-\infty} \psi_t \equiv b$ uniformly on compact subsets of B. It is evident that a and b are infinite singularities of F. If $a \neq b$, it was shown that, up to a Koebe transform, F has the form

$$F(z) = \left(\frac{1}{2}\log\frac{1+z_1}{1-z_1}\right)e_1 + K\left(\frac{\hat{z}}{\sqrt{1-z_1^2}}\right), \tag{1.3}$$

where $K : B_{n-1} \to \mathbb{C}^n$ is holomorphic. This is clearly a generalization of a strip mapping of the unit disk.

We will now address what occurs when $a = b$, so that F has only one singularity on ∂B. Assume that the singularity is e_1. If u lies in the span of e_1, we will show that

$$F(z) = \frac{z}{1-z_1} + K\left(\frac{\hat{z}}{1-z_1}\right),$$

where $K : \mathbb{C}^{n-1} \to \mathbb{C}^n$ is a homogeneous polynomial of degree 2. If u lies outside of the span of e_1, then a more complicated situation results. In dimension $n = 2$, we will prove that

$$F(z) = \frac{z}{1-z_1} - \frac{1}{2}\left(\frac{z_2}{1-z_1}\right)^2 e_1 + K\left(\frac{z_1}{1-z_1} - \frac{1}{2}\left(\frac{z_2}{1-z_1}\right)^2 - \frac{u_1}{u_2}\frac{z_2}{1-z_1}\right),$$

where K is holomorphic and $\mathbb{C}^2$-valued in a neighborhood of 0 in $\mathbb{C}$ with terms only of degree ≥ 2.

These clearly generalize the half-plane mappings of the unit disk, and so it comes as no surprise that the familiar mappings

$$z \mapsto \frac{z}{1 - \langle z, v\rangle}, \qquad z \in B \tag{1.4}$$

(where $v \in \partial B$), are part of the classification. As a surprising corollary to our analysis, we find that these mappings are *not* extreme points of $\mathcal{K}$ when $n \geq 2$. (They are exactly the extreme points when $n = 1$.)

The classification provided in this article addresses a portion of part (b) of the following conjecture, proposed by the authors in [3], and of which that paper addresses part (c).

Conjecture 1.1. *If $F : B \to \mathbb{C}^n$ is a normalized univalent holomorphic mapping of the ball onto a convex domain, then*

(a) *$F(B)$ is bounded and F extends continuously to ∂B, or*
(b) *F extends continuously to ∂B except for one point that is an infinite discontinuity, or*
(c) *F is given by* (1.3)*, and K extends continuously to ∂B_{n-1}.*

2. A remark on the extreme points of $\mathcal{K}$

A family of examples of unbounded convex mappings is furnished by the following lemma and leads to an interesting result.

Lemma 2.1. *If $Q : \mathbb{C}^{n-1} \to \mathbb{C}$ is a homogeneous polynomial of degree* 2 *such that $\|Q\| = \sup_{\|w\|\le 1} |Q(w)| \le 1/2$, then the function $H : B \to \mathbb{C}^n$ given by*

$$H(z) = \frac{z}{1-z_1} + \left(\frac{Q(\hat{z})}{(1-z_1)^2}\right) e_1 \tag{2.1}$$

lies in $\mathcal{K}$.

Proof. It follows from Lemma 2.1 of [4] that $G : B \to \mathbb{C}^n$ defined by

$$G(z) = z + Q(\hat{z})e_1$$

lies in $\mathcal{K}$. For $r \in (0,1)$, let $\varphi_r \in \operatorname{Aut} B$ be given by

$$\varphi_r(z) = \frac{(z_1 - r, \sqrt{1-r^2}\,\hat{z})}{1 - rz_1}, \qquad z \in B.$$

Since $\mathcal{K}$ is a linearly invariant family, $\Lambda_{\varphi_r}(G) \in \mathcal{K}$ for each r. Now

$$H = \lim_{r\to 1^-} \Lambda_{\varphi_r}(G),$$

and therefore $H \in \mathcal{K}$ by compactness. □

Corollary 2.2. *The mappings* (1.4) *are not extreme points of $\mathcal{K}$ when $n \ge 2$.*

Proof. Let $Q : \mathbb{C}^{n-1} \to \mathbb{C}$ be a homogeneous polynomial of degree 2 such that $\|Q\| = 1/2$. The mapping $z \mapsto z/(1-z_1)$ can be written as the mean of the two mappings of the type (2.1) formed using Q and $-Q$. Any other mapping of the form (1.4) is obtained by composition on the left and right by U and U^*, respectively, for some unitary operator U on $\mathbb{C}^n$. □

Corollary 2.2 is somewhat unexpected, and not just because the mappings (1.4) are precisely the extreme points of $\mathcal{K}$ when $n = 1$. If $G \in \mathcal{K}$ is written $G = I + \sum_{k=2}^{\infty} P_k$, where each P_k is a homogeneous polynomial of degree k, then $\|P_k\| \le 1$ for all $k = 2, 3, \ldots$ (see [5]). Equality is achieved in these inequalities for every k for any mapping of the form (1.4).

Example 2.3. The function

$$H(z) = \frac{z}{1-z_1} - \frac{1}{2}\left(\frac{z_2}{1-z_1}\right)^2 e_1, \qquad z \in B,$$

is of the form (2.1), and hence lies in $\mathcal{K}$. One can easily verify that $H(B)$ is the union of parallel lines in the direction ie_1 as well as the union of parallel lines in the direction e_2. It follows that $H(B)$ contains the real 2-dimensional space spanned by these vectors. This illustrates that there is not necessarily uniqueness in the choice of the direction vector.

3. The automorphisms

As described in the introduction, we consider an appropriate function F such that the one-parameter group of automorphisms described in (1.2) tend to the same point on ∂B as $t \to \pm\infty$. Without loss of generality, we may assume that e_1 is the fixed boundary point. (Indeed, replace F by $U^* \circ F \circ U$ for an appropriately chosen unitary operator U on $\mathbb{C}^n$.)

Recall [6] that the transformation

$$T(z) = \frac{e_1 + z}{1 - z_1}, \qquad z \in \mathbb{C}^n,\ z_1 \neq 1, \tag{3.1}$$

maps B biholomorphically onto the Siegel right half-space $R = \{z \in \mathbb{C}^n : \operatorname{Re} z_1 > \|\hat{z}\|^2\}$, while sending e_1 to "∞". If $\varphi \in \operatorname{Aut} B$ is such that e_1 is its only fixed point in $\overline{B}$, then $T \circ \varphi \circ T^{-1} \in \operatorname{Aut} R$ has ∞ as its only fixed point in $\overline{R}$. (By $\overline{R}$, we mean the closure of R in the one-point compactification of $\mathbb{C}^n$.) It follows from Proposition 2.2.10 of [1] that

$$T \circ \varphi \circ T^{-1}(z) = (z_1 + a_1 + 2\langle V\hat{z}, \hat{a}\rangle, V\hat{z} + \hat{a}), \qquad z \in R, \tag{3.2}$$

for some choice of $a \in \partial R$ and unitary operator V on $\mathbb{C}^{n-1}$ such that if $(I - V)w = \hat{a}$ for some $w \in \mathbb{C}^{n-1}$, then $2\langle w, \hat{a}\rangle \neq \overline{a}_1$.

For all $t \in \mathbb{R}$, the automorphism ψ_t must satisfy (3.2) for some appropriate a and V, and therefore we solve to find

$$\psi_t(z) = \frac{(2z_1 + 2\langle V\hat{z}, \hat{a}\rangle + (1 - z_1)a_1, 2V\hat{z} + 2(1 - z_1)\hat{a})}{2 + 2\langle V\hat{z}, \hat{a}\rangle + (1 - z_1)a_1}, \qquad z \in B,\ t \in \mathbb{R}, \tag{3.3}$$

where a and V are functions of t satisfying $a(0) = 0$ and $V(0) = I$. It is useful to observe that, in fact, a and V are holomorphic in the real variable t. (That is, they are locally represented by a convergent power series in t.)

Lemma 3.1. *There is an Hermitian operator A on $\mathbb{C}^{n-1}$ such that $V(t) = e^{-itA}$ for all $t \in \mathbb{R}$.*

Proof. Let $\{\psi_t^k\}_{k=1}^{\infty}$ be the iterates of ψ_t for any $t \in \mathbb{R}$. Inductively, this means that $\psi_t^1 = \psi_t$ and $\psi_t^{k+1} = \psi_t \circ \psi_t^k$ for all positive integers k. Clearly, $\psi_t^k = \psi_{kt}$. Therefore $V(t)^k = V(kt)$ as a direct result of (3.3). Equivalently, $V(t) = V(t/k)^k$. Differentiation of this expression in t yields

$$V'(t) = V\left(\frac{t}{k}\right)^{k-1} V'\left(\frac{t}{k}\right) = V(t)V\left(\frac{t}{k}\right)^{-1} V'\left(\frac{t}{k}\right). \tag{3.4}$$

Taking the limit of (3.4) as $k \to \infty$ gives the operator differential equation $V'(t) = V(t)V'(0)$. Hence $V(t) = e^{tM}$ for some operator M on $\mathbb{C}^{n-1}$. Clearly, $V(t)^{-1} = V(-t)$ and $V(t)^* = e^{tM^*}$. Since $V(t)$ is unitary for all t, M must be skew-Hermitian ($M^* = -M$). The lemma follows by letting $A = iM$. □

We will now perform some calculations to analyze the relationship between a and u.

Differentiate the expression

$$F(z) + tu = F(\psi_t(z)), \qquad z \in B,\ t \in \mathbb{R}, \tag{3.5}$$

with respect to t and choose $z = \psi_t^{-1}(0)$. (Then z satisfies the equations $V\hat{z} = -(1-z_1)\hat{a}$ and $z_1 = (\|\hat{a}\|^2 - a_1/2)(1-z_1)$.) The result is

$$u \equiv (i\langle A\hat{a}, \hat{a}\rangle - \langle \hat{a}, \hat{a}'\rangle + a_1'/2, iA\hat{a} + \hat{a}') \tag{3.6}$$

because $DF(0) = I$.

Setting the last $n-1$ components of (3.6) equal gives the first-order equation $\hat{a}' + iA\hat{a} \equiv \hat{u}$. It follows that

$$\hat{a}(t) = \sum_{k=1}^{\infty} \frac{(-i)^{k-1} t^k A^{k-1} \hat{u}}{k!}.$$

Equating the first components of (3.6) reveals that $a_1'(0) = 2u_1$. Using the facts that $\hat{a}'(t) = e^{-itA}\hat{u}$, $A\hat{a}(t) = i(e^{-itA} - I)\hat{u}$, and $\hat{a}''(t) = -iAe^{-itA}\hat{u}$, we differentiate to see that

$$a_1''(t) = 2\langle e^{-itA}\hat{u}, \hat{u}\rangle.$$

With initial values $a_1(0) = 0$ and $a_1'(0) = 2u_1$,

$$\frac{a_1(t)}{2} = \left\langle \sum_{k=2}^{\infty} \frac{(-i)^{k-2} t^k A^{k-2} \hat{u}}{k!}, \hat{u} \right\rangle + tu_1.$$

Recall that $\operatorname{Re} a_1(t) = \|\hat{a}(t)\|^2$ and therefore contains terms in t only of degree ≥ 2. This gives the following result.

Lemma 3.2. *The component u_1 is purely imaginary.*

As is evident by Example 2.3, we cannot make any strong conclusions regarding $\hat{u}$.

4. Mappings such that u lies in the span of the singularity

With the assumption that the boundary singularity is e_1, we assume that $\hat{u} = 0$. Since u_1 is purely imaginary, $u = ie_1$. (The geometry of the situation renders the possibility $u = -ie_1$ redundant.) Furthermore, the work in the previous section leading to Lemma 3.2 implies $a_1 = 2it$ and $\hat{a} = 0$. Accordingly, the form of the automorphisms in (3.3) is simplified to become

$$\psi_t(z) = \frac{(z_1 + it(1-z_1), V\hat{z})}{1 + it(1-z_1)}, \qquad z \in B,\ t \in \mathbb{R}.$$

We now prove the following.

Lemma 4.1. *The function F has the form*

$$F(z) = \left(\frac{z_1}{1-z_1}, \exp\left(\frac{z_1}{1-z_1}A\right)\frac{\hat{z}}{1-z_1}\right) + K\left(\exp\left(\frac{z_1}{1-z_1}A\right)\frac{\hat{z}}{1-z_1}\right),$$

where K is a holomorphic function of $n-1$ complex variables containing terms only of degree ≥ 2 in its expansion about 0.

Proof. Define the function $H : B \to \mathbb{C}^n$ by

$$H(z) = \left(\frac{z_1}{1-z_1}, \exp\left(\frac{z_1}{1-z_1}A\right)\frac{\hat{z}}{1-z_1}\right).$$

Due to our above observations concerning ψ_t, we can calculate for all $z \in B$ and $t \in \mathbb{R}$,

$$\begin{aligned} H(\psi_t(z)) &= \left(\frac{z_1 + it(1-z_1)}{1-z_1}, \exp\left(\frac{z_1 + it(1-z_1)}{1-z_1} A \right) \frac{V\hat{z}}{1-z_1} \right) \\ &= \left(\frac{z_1}{1-z_1} + it, \exp\left(\frac{z_1}{1-z_1} A \right) \frac{\hat{z}}{1-z_1} \right) \\ &= H(z) + ite_1. \end{aligned}$$

Let $G = F \circ H^{-1}$. (Clearly, H is univalent.) For $w \in H(B)$, write $z = H^{-1}(w)$. Then

$$G(w + ite_1) = F(\psi_t(z)) = F(z) + ite_1 = G(w) + ite_1.$$

Define $K_0 : H(B) \to \mathbb{C}^n$ to be given by $K_0(w) = G(w) - w$. Then in its expansion about 0, K_0 contains terms only of degree ≥ 2 in w. Furthermore,

$$K_0(w + ite_1) = K_0(w), \qquad w \in H(B).$$

This indicates that K_0 is independent of the first coordinate of its argument. We may then define $K(\hat{w}) = K_0(w)$ for $w \in H(B)$. It follows that $F(z) = w + K(\hat{w})$, as desired. □

It is interesting to note that functions of the form given in Lemma 4.1 have the property that the image of the ball is the union of lines parallel to the vector ie_1. However, most will fail to be convex mappings. As we will show in the following lemmas, convexity implies that $A = 0$ and K is a homogeneous polynomial of degree 2.

In the following, let $\Delta \subseteq \mathbb{C}$ be the open unit disk.

Lemma 4.2. *If $(w_1, \zeta\hat{w}) + o(\zeta) \in F(B)$ for some $w \in \mathbb{C}^n$ and all $\zeta \in \overline{\Delta}$, then*

$$\left\| \frac{e^{-w_1 A}\hat{w}}{1 + w_1} \right\| \leq 1.$$

Proof. By comparing series terms, define $f : \overline{\Delta} \to B$ by

$$f(\zeta) = F^{-1}((w_1, \zeta\hat{w}) + o(\zeta))$$

to calculate

$$\begin{aligned} f(\zeta) &= F^{-1}(w_1 e_1) + DF^{-1}(w_1 e_1)(0, \hat{w})\zeta + o(\zeta) \\ &= F^{-1}((w_1, \zeta\hat{w}) + K(\zeta\hat{w})) + o(\zeta) \\ &= \left(\frac{w_1}{1+w_1}, e^{-w_1 A} \frac{\zeta\hat{w}}{1+w_1} \right) + o(\zeta). \end{aligned}$$

By Cauchy's estimates,

$$\left\| \left(0, e^{-w_1 A} \frac{\hat{w}}{1+w_1} \right) \right\| = \|f'(0)\| \leq 1.$$

□

We now find that $V(t) = I$ for all $t \in \mathbb{R}$, which greatly simplifies the form of F.

Theorem 4.3. $A = 0$.

Proof. Let $\hat{v}$ be an eigenvector of A with (real) eigenvalue λ, and assume $\|\hat{v}\| = 1$. The result will follow once it is shown that λ must be 0.

Suppose that $\lambda < 0$, and choose $c \in (0,1)$. We may choose $r \in (0,1)$ such that for any $x \in (r,1)$,

$$c\sqrt{\frac{1+x}{1-x}}\exp\left(\frac{\lambda x}{1-x}\right) < 1.$$

For $x \in (r,1)$ and $\zeta \in \overline{\Delta}$, define

$$p_1 = (x, c\zeta\sqrt{1-x^2}\,\hat{v}). \tag{4.1}$$

Now $p_1 \in B$ and

$$F(p_1) = \left(\frac{x}{1-x}, c\zeta\sqrt{\frac{1+x}{1-x}}\exp\left(\frac{\lambda x}{1-x}\right)\hat{v}\right) + o(\zeta).$$

Define

$$d = 1 - c\sqrt{\frac{1+x}{1-x}}\exp\frac{\lambda x}{1-x}.$$

Then $d \in (0,1)$ and hence

$$p_2 = (0, d\zeta\hat{v})$$

lies in B. Now $F(p_2) = (0, d\zeta\hat{v}) + o(\zeta)$, and the convexity of $F(B)$ implies

$$\frac{F(p_1)+F(p_2)}{2} = \left(\frac{x}{2(1-x)}, \frac{\zeta\hat{v}}{2}\right) + o(\zeta) \in F(B).$$

From Lemma 4.2, we see that

$$\begin{aligned} 1 &\geq \left\|\zeta\exp\left(\frac{-x}{2(1-x)}A\right)\left(\frac{2(1-x)}{2-x}\right)\frac{\hat{v}}{2}\right\| \\ &= \left\|\zeta\exp\left(\frac{-\lambda x}{2(1-x)}\right)\frac{1-x}{2-x}\hat{v}\right\|. \end{aligned} \tag{4.2}$$

But (4.2) clearly tends to ∞ as $x \to 1^-$ for nonzero $\zeta \in \overline{\Delta}$. This is a contradiction.

We now have that $\lambda \geq 0$. Suppose that $\lambda > 0$. Let p_1 remain as defined in (4.1) with $c = \sqrt{1-x^2}$. The convexity of $F(B)$ implies that

$$\frac{F(p_1)}{2} = \left(\frac{x}{2(1-x)}, \zeta(1+x)\exp\left(\frac{\lambda x}{1-x}\right)\frac{\hat{v}}{2}\right) + o(\zeta) \in F(B).$$

But Lemma 4.2 implies that

$$\left\|\exp\left(\frac{\lambda x}{2(1-x)}\right)\frac{\zeta(1-x^2)}{2-x}\hat{v}\right\| \leq 1. \tag{4.3}$$

But as $x \to 1^-$, the left-hand side of (4.3) tends to ∞, a contradiction. Hence $\lambda = 0$. □

The next theorem completes our analysis of this case, and provides a nice final form for F.

Theorem 4.4. *The function K is a homogeneous polynomial of degree 2.*

Of course, this allows for the possibility that $K \equiv 0$.

Proof. It remains to show that K cannot contain terms of degree ≥ 3. In a neighborhood of 0, we may write

$$K(\hat{w}) = \sum_{\alpha \in \mathbb{N}_0^{n-1}} w^\alpha a_\alpha,$$

where the multi-indices have the form $\alpha = (\alpha_2, \ldots, \alpha_n)$, and $a_\alpha \in \mathbb{C}^n$ for each $\alpha \in \mathbb{N}_0^{n-1}$.

Let $z_1 = r \in (0,1)$. Pick $\rho_2, \ldots, \rho_n \in (0,1)$ such that $\sum_{k=2}^n \rho_k^2 < 1/4$. Choose $z_2, \ldots, z_n$ so that $|z_k| = \rho_k \sqrt{1-r^2}$. This forces $\|z\| < (1+r)/2$. Writing $w_k = z_k/(1-r)$ for $k = 2, \ldots, n$, we have

$$F(z_1, z_2 e^{i\theta_2}, \ldots, z_n e^{i\theta_n}) = \left(\frac{r}{1-r}, w_2 e^{i\theta_2}, \ldots, w_n e^{i\theta_n}\right) + K(w_2 e^{i\theta_2}, \ldots, w_n e^{i\theta_n})$$

for all $\theta_2, \ldots, \theta_n \in \mathbb{R}$.

Let m be Lebesgue measure in $\mathbb{R}^{n-1}$ with the normalization $m([0,2\pi]^{n-1}) = 1$, and write $\theta = (\theta_2, \ldots, \theta_n) \in \mathbb{R}^{n-1}$. If $|\alpha| \geq 3$, then

$$\int_{[0,2\pi]^{n-1}} e^{-i\alpha\cdot\theta} \left(\frac{r}{1-r}, w_2 e^{i\theta_2}, \ldots, w_n e^{i\theta_n}\right) dm(\theta) = 0.$$

We can therefore calculate the Cauchy integrals

$$\begin{aligned} w^\alpha a_\alpha &= \int_{[0,2\pi]^{n-1}} e^{-i\alpha\cdot\theta} K(w_2 e^{i\theta_2}, \ldots, w_n e^{i\theta_n})\, dm(\theta) \\ &= \int_{[0,2\pi]^{n-1}} e^{-i\alpha\cdot\theta} F(z_1, z_2 e^{i\theta_2}, \ldots, z_n e^{i\theta_n})\, dm(\theta). \end{aligned}$$

Using the growth bound (1.1), we have

$$\|w^\alpha a_\alpha\| \leq \int_{[0,2\pi]^{n-1}} \|F(z_1, z_2 e^{i\theta_2}, \ldots, z_n e^{i\theta_n})\|\, dm(\theta) < \frac{1+r}{1-r}. \tag{4.4}$$

We now calculate

$$|w^\alpha| = \prod_{k=2}^n |w_k|^{\alpha_k} = \left(\frac{1+r}{1-r}\right)^{|\alpha|/2} \prod_{k=2}^n \rho_k^{\alpha_k}.$$

Now (4.4) becomes

$$\|a_\alpha\| \left(\frac{1+r}{1-r}\right)^{|\alpha|/2} \prod_{k=2}^n \rho_k^{\alpha_k} < \frac{1+r}{1-r}.$$

But $|\alpha| \geq 3$ implies

$$\|a_\alpha\| \prod_{k=2}^n \rho_k^{\alpha_k} < \left(\frac{1-r}{1+r}\right)^{|\alpha|/2-1} \to 0 \tag{4.5}$$

as $r \to 1^-$. Since the left-hand side of (4.5) is constant, $\|a_\alpha\| = 0$. Therefore K has no terms of degree ≥ 3. □

We now know the form of F.

Theorem 4.5. *If $\hat{u} = 0$, then F has the form*

$$F(z) = \frac{z}{1 - z_1} + K\left(\frac{\hat{z}}{1 - z_1}\right),$$

where $K : \mathbb{C}^{n-1} \to \mathbb{C}^n$ is a homogeneous polynomial of degree 2.

5. Mappings such that u fails to lie in the span of the singularity

In this more complicated situation, we restrict ourselves to dimension $n = 2$. As in the last section, assume $\operatorname{Im} u_1 \geq 0$. By composition of the form $U^* \circ F \circ U$, where $Uz = (z_1, \gamma z_2)$, $|\gamma| = 1$, we may assume $u_2 > 0$. The operator A is now a scalar $\lambda \in \mathbb{R}$. We shall prove the following.

Theorem 5.1. $\lambda = 0$.

With this in place, we have the following form for F.

Theorem 5.2. *If $n = 2$, $\operatorname{Im} u_1 \geq 0$, and $u_2 > 0$, then F has the form*

$$F(z) = \frac{z}{1 - z_1} - \frac{1}{2}\left(\frac{z_2}{1 - z_1}\right)^2 e_1 + K\left(\frac{z_1}{1 - z_1} - \frac{1}{2}\left(\frac{z_2}{1 - z_1}\right)^2 - \frac{u_1}{u_2}\frac{z_2}{1 - z_1}\right),$$

where K is holomorphic in a neighborhood of 0 in $\mathbb{C}$ with terms only of degree ≥ 2 in its expansion about 0.

Proof. Define $H : B \to \mathbb{C}^2$ by

$$H(z) = \frac{z}{1 - z_1} - \frac{1}{2}\left(\frac{z_2}{1 - z_1}\right)^2 e_1.$$

Since $\lambda = 0$, we have $a_1(t) = 2tu_1 + t^2u_2^2$, $a_2(t) = tu_2$, and $V(t) = 1$ for all $t \in \mathbb{R}$. This allows the calculation

$$\begin{aligned} H(\psi_t(z)) &= \left(\frac{z_1}{1 - z_1} + \frac{tu_2z_2}{1 - z_1} + tu_1 + \frac{t^2u_2^2}{2} - \frac{1}{2}\left(\frac{z_2}{1 - z_1} + tu_2\right)^2, \frac{z_2}{1 - z_1} + tu_2\right) \\ &= H(z) + tu \end{aligned}$$

for $z \in B$. Now H is clearly univalent, and so define $G = F \circ H^{-1}$. Writing $w = H(z)$, we have

$$G(w + tu) = F(\psi_t(z)) = F(z) + tu = G(w) + tu.$$

Define $K_0 : H(B) \to \mathbb{C}^2$ by $K_0(w) = G(w) - w$. Then in a neighborhood of 0, K_0 has terms only of degree ≥ 2. Implicitly define K_1 in a domain in $\mathbb{C}^2$ by the rule

$$K_1\left(w_1 - \frac{u_1w_2}{u_2}, w_2\right) = K_0(w).$$

Then

$$w + tu + K_1\left(w_1 - \frac{u_1w_2}{u_2}, w_2\right) = G(w + tu) = w + tu + K_1\left(w_1 - \frac{u_1w_2}{u_2}, w_2 + tu_2\right),$$

showing that K_1 is independent of its second component. Define a function K in a domain in $\mathbb{C}$ to satisfy the relationship

$$K\left(w_1 - \frac{u_1w_2}{u_2}\right) = K_1\left(w_1 - \frac{u_1w_2}{u_2}, w_2\right)$$

so that

$$G(w) = w + K\left(w_1 - \frac{u_1 w_2}{u_2}\right), \qquad w \in H(B), \tag{5.1}$$

as desired. □

We devote the remainder of the section to proving Theorem 5.1. Assume that $\lambda \neq 0$. Then for all $t \in \mathbb{R}$,

$$\begin{aligned} a_1(t) &= 2tu_1 - \frac{2u_2^2}{\lambda^2}(e^{-i\lambda t} + i\lambda t - 1), \\ a_2(t) &= -\frac{u_2}{i\lambda}(e^{-i\lambda t} - 1), \\ V(t) &= e^{-i\lambda t}. \end{aligned}$$

If $w \in \mathbb{C}$ satisfies $(1 - e^{-i\lambda t})w = a_2$, then $w = u_2/(i\lambda)$. The condition on a and V following (3.2) implies

$$0 \neq 2\overline{a}_2 w - \overline{a}_1 = 2tu_1 - \frac{2itu_2^2}{\lambda}, \qquad t \in \mathbb{R}.$$

It follows that $u_2^2 + i\lambda u_1 \neq 0$. Therefore define the constant

$$\rho = \frac{\lambda^2}{u_2^2 + i\lambda u_1}. \tag{5.2}$$

We shall now prove Theorem 5.1 through a sequence of lemmas.

Lemma 5.3. *Define $g : B \to \mathbb{C}$ by*

$$g(z) = \frac{z_1}{1 - z_1} + \frac{u_2}{i\lambda}\frac{z_2}{1 - z_1}.$$

Then for all $t \in \mathbb{R}$ and $z \in B$,

$$g(\psi_t(z)) = g(z) - \frac{i\lambda t}{\rho}.$$

Proof. The result follows from the simple calculation

$$\begin{aligned} g(\psi_t(z)) &= \frac{z_1}{1 - z_1} + \frac{V\overline{a}_2 z_2}{1 - z_1} + \frac{a_1}{2} + \frac{u_2}{i\lambda}\left(\frac{Vz_2}{1 - z_1} + a_2\right) \\ &= \frac{z_1}{1 - z_1} + \frac{u_2}{i\lambda}\frac{z_2}{1 - z_1} + tu_1 - \frac{itu_2^2}{\lambda} \\ &= g(z) - \frac{i\lambda t}{\rho}. \end{aligned}$$

□

As is the case with functions of the form given in Lemma 4.1, the function F given in the following lemma has the property that $F(B)$ is the union of lines parallel to u even if $\lambda \neq 0$. However, for F to be a convex mapping, Theorem 5.1 must hold.

Lemma 5.4. *Define $H : B \to \mathbb{C}^2$ by*

$$H(z) = \left(1 - \left[1 - \frac{i\lambda}{u_2}\frac{z_2}{1 - z_1}\right]e^{-\rho g(z)}\right)\left(\frac{u_2^2}{\lambda^2}, \frac{u_2}{i\lambda}\right) + \frac{i\rho g(z)}{\lambda}u.$$

Then

$$F(z) = H(z) + K\left(\frac{1}{\rho}\left(1 - \left[1 - \frac{i\lambda}{u_2}\frac{z_2}{1-z_1}\right]e^{-\rho g(z)}\right)\right),$$

where K is holomorphic in a domain in $\mathbb{C}$ and has terms only of degree ≥ 2 in its expansion about 0.

Proof. We calculate for $t \in \mathbb{R}$ and $z \in B$,

$$\begin{aligned} H(\psi_t(z)) &= \left(1 - \left[1 - \frac{i\lambda}{u_2}\left(\frac{e^{-i\lambda t}z_2}{1-z_1} - \frac{u_2}{i\lambda}(e^{-i\lambda t} - 1)\right)\right]e^{-\rho g(z)}e^{i\lambda t}\right)\left(\frac{u_2^2}{\lambda^2}, \frac{u_2}{i\lambda}\right) \\ &\quad + \frac{i\rho g(z)}{\lambda}u + tu \\ &= H(z) + tu. \end{aligned}$$

Note that $H(0) = 0$ and $DH(0) = I$. Furthermore, since $u_2^2 + i\lambda u_1 \neq 0$, the vectors u and $(u_2^2/\lambda^2, u_2/(i\lambda))$ are linearly independent. From that observation, it is easy to see that H is univalent.

Now define $G = F \circ H^{-1}$. By the same technique as in the proof of Theorem 5.2 leading to (5.1), we may write

$$G(w) = w + K\left(w_1 - \frac{u_1 w_2}{u_2}\right),$$

where $w = H(z)$ and K is analytic in a neighborhood of 0 in $\mathbb{C}$ and has terms only of degree ≥ 2 in its expansion about 0. The calculation

$$w_1 - \frac{u_1 w_2}{u_2} = \frac{1}{\rho}\left(1 - \left[1 - \frac{i\lambda}{u_2}\frac{z_2}{1-z_1}\right]e^{-\rho g(z)}\right), \qquad w = H(z),\ z \in B,$$

gives the result. □

Lemma 5.5. $\rho \geq 0$.

Proof. To the contrary, assume $\rho < 0$. Then $\lambda > 0$. We will begin by showing that K given in Lemma 5.4 must be entire. It suffices to prove

$$\left\{\left(1 - \frac{i\lambda}{u_2}\frac{z_2}{1-z_1}\right)e^{-\rho g(z)} : z \in B\right\} = \mathbb{C}. \tag{5.3}$$

To that end, write

$$z = \left(\frac{t+is}{1+t+is}, -\frac{i}{2}\frac{\sqrt{1+2t}}{1+t+is}\right).$$

Clearly $z \in B$ provided that $t > -1/2$ and $s \in \mathbb{R}$. With this substitution, the elements of the left-hand side of (5.3) have the form

$$\left(1 - \frac{\lambda\sqrt{1+2t}}{2u_2}\right)\exp\left(-\rho t + \frac{\rho u_2\sqrt{1+2t}}{2\lambda}\right)e^{-\rho is}. \tag{5.4}$$

The modulus of (5.4) can take on all nonnegative values, and the argument is unrestricted, proving that K is entire.

Suppose, for the moment, that $K \equiv 0$. For $r \in [0, 1)$, we have

$$F(r, 0) = H(r, 0) = \left(1 - \exp\frac{-\rho r}{1-r}\right)\left(\frac{u_2^2}{\lambda^2}, \frac{u_2}{i\lambda}\right) + \frac{i\rho r}{\lambda(1-r)}u.$$

Applying the growth inequality (1.1) and multiplying by $\exp(\rho r/(1-r))$ gives

$$\frac{r}{1-r}\exp\frac{\rho r}{1-r} \geq \left\| \left(\exp\frac{\rho r}{1-r} - 1\right)\left(\frac{u_2^2}{\lambda^2}, \frac{u_2}{i\lambda}\right) + \frac{i\rho r}{\lambda(1-r)}\exp\left(\frac{\rho r}{1-r}\right)u \right\|.$$

Taking the limit as $r \to 1^-$ contradicts the assumption that $u_2 > 0$. It follows that $K \not\equiv 0$.

We now have that K is nonzero and entire. Since $K(0) = 0$ and $K'(0) = 0$, K must contain terms only of degree ≥ 2. Observe that

$$\left\{\frac{1 - e^{-\rho t}e^{-\rho i s}}{\rho} : t \geq 0,\ s \in \mathbb{R}\right\} = \mathbb{C} \setminus D\left(\frac{1}{\rho}; \frac{1}{\rho}\right).$$

We can choose sequences $\{t_k\}$ and $\{s_k\}$, such that $t_k \geq 0$ and $s_k \in \mathbb{R}$ for all k, $t_k \to \infty$, and

$$\left\| K\left(\frac{1 - e^{-\rho t_k}e^{-i\rho s_k}}{\rho}\right)\right\| \geq c\left|\frac{1 - e^{-\rho t_k}e^{-i\rho s_k}}{\rho}\right|^2,$$

for some constant $c > 0$. Application of (1.1) gives

$$\begin{aligned}
\sqrt{t_k^2 + s_k^2} &\geq \left\| F\left(\frac{t_k + is_k}{1 + t_k + is_k}, 0\right)\right\| \\
&\geq \left\| K\left(\frac{1 - e^{-\rho t_k}e^{-i\rho s_k}}{\rho}\right)\right\| - \left\| H\left(\frac{t_k + is_k}{1 + t_k + is_k}, 0\right)\right\| \\
&\geq c\left|\frac{1 - e^{-\rho t_k}e^{-i\rho s_k}}{\rho}\right|^2 - \left\| (1 - e^{-\rho(t_k + is_k)})\left(\frac{u_2^2}{\lambda^2}, \frac{u_2}{i\lambda}\right) + \frac{i\rho(t_k + is_k)}{\lambda}u \right\|
\end{aligned}$$

for all k. Multiply through by $e^{2\rho t_k}$ and take the limit as $k \to \infty$ to get $0 \geq 1/\rho^2$, a contradiction. □

Lemma 5.6. $\rho = 0$.

Proof. Suppose $\rho > 0$. Set

$$E = \left\{ z \in B : z_2 = \frac{u_2(1 - z_1)}{i\lambda} \right\}.$$

Now $z \in E$ must satisfy

$$|1 - z_1|^2 \frac{u_2^2}{\lambda^2} < 1 - |z_1|^2.$$

It follows that

$$|z_1|^2\left(1 + \frac{u_2^2}{\lambda^2}\right) - \frac{2u_2^2}{\lambda^2}\operatorname{Re} z_1 < 1 - \frac{u_2^2}{\lambda^2},$$

and so

$$|z_1|^2 - \frac{2u_2^2}{\lambda^2 + u_2^2}\operatorname{Re} z_1 < \frac{\lambda^2 - u_2}{\lambda^2 + u_2}.$$

Complete the square to find

$$\left| z_1 - \frac{u_2^2}{\lambda^2 + u_2^2} \right| < \frac{\lambda^2}{\lambda^2 + u_2^2}, \qquad z \in E.$$

This shows that for $z \in E$, z_1 can take on any value in the disk centered at $u_2^2/(\lambda^2+u_2^2)$ tangent to the unit circle at the point 1. The image of this disk under $z_1 \mapsto z_1/(1-z_1)$ is the half-plane $\{\zeta \in \mathbb{C} : \operatorname{Re}\zeta > (u_2^2-\lambda^2)/(2\lambda^2)\}$. As a result,

$$F(E) = \left\{ \left(\frac{u_2^2}{\lambda^2}, \frac{u_2}{i\lambda}\right) + \frac{i\rho}{\lambda}\left(\zeta - \frac{u_2^2}{\lambda^2}\right)u + K\left(\frac{1}{\rho}\right) : \operatorname{Re}\zeta > \frac{u_2^2-\lambda^2}{2\lambda^2} \right\},$$

using Lemma 5.4. To simplify, $F(E)$ (and hence $F(B)$) contains a ray of the form $\{c + i\rho s u/\lambda : s > 0\}$ for some constant $c \in \mathbb{C}^2$. By convexity, $F(B)$ contains either $\{isu : s > 0\}$ or $\{isu : s < 0\}$ depending upon the sign of ρ/λ. However, since $\rho > 0$, $\lambda s > 0$ in either case. Since $F(B)$ contains $\{tu : t \in \mathbb{R}\}$, convexity implies that $\{\zeta u : \zeta \in P\} \subseteq F(B)$, where P is a neighborhood of either the upper or lower closed half-plane in $\mathbb{C}$.

With T given by (3.1), $z = e_1$, and $\varphi = \psi_t$, (3.2) becomes

$$T \circ F^{-1}(tu) = (1 + a_1(t), a_2(t)). \tag{5.5}$$

Since both sides of (5.5) are holomorphic in the real variable t,

$$T \circ F^{-1}(\zeta u) = (1 + a_1(\zeta), a_2(\zeta)) \tag{5.6}$$

must hold for all ζ in some neighborhood of 0 in $\mathbb{C}$. (We extend a_1 and a_2 to a complex variable in the natural way.) Now the right-hand side of (5.6) is entire in ζ, and the left-hand side is well defined for all $\zeta \in P$ by convexity. Hence (5.6) holds for all $\zeta \in P$ and takes on values in R. But if $\zeta = is$ (s as determined by P), then the first coordinate of the right-hand side of (5.6) becomes

$$1 + a_1(is) = 1 + 2isu_1 - \frac{2u_2^2}{\lambda^2}(e^{\lambda s} - \lambda s - 1). \tag{5.7}$$

Since $\lambda s > 0$, (5.7) becomes negative as $|s| \to \infty$, a contradiction. Hence $\rho = 0$. □

Clearly, as $\rho = 0$, $\lambda = 0$, completing the proof of Theorem 5.1.

6. Final remarks

It is interesting to note that the method of proof given in both Sections 4 and 5 is to define mappings of B that are such that $F(B)$ is the union of parallel lines and then to impose restrictions on F based on the hypothesis that $F(B)$ is convex. Hence we have constructed quite general mappings of B such that the image of B is the union of parallel lines but may not be convex.

Although we do not currently have a proof, we believe that Theorem 5.2 can be naturally extended to dimension $n > 2$. In fact, it seems reasonable to conjecture that, under the assumption that e_1 is the infinite boundary singularity on ∂B, if u is not in the span of e_1, $F(B)$ must still contain a line in the direction ie_1, and hence F has the concise form given in Theorem 4.5.

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