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DIOPHANTINE APPROXIMATION BY ALGEBRAIC HYPERSURFACES AND VARIETIES

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Abstract. Questions on rational approximations to a real number can be generalized in two directions. On the one hand, we may ask about "approximation" to a point in $\mathbb{R}^n$ by hyperplanes defined over the rationals. That is, we seek hyperplanes with small distance from the given point. On the other hand, following Wirsing, we may ask about approximation to a real number by real algebraic numbers of degree at most d.

The present paper deals with a common generalization of both directions, namely with approximation to a point in $\mathbb{R}^n$ by algebraic hypersurfaces, or more generally algebraic varieties defined over the rationals.

1. Introduction

We will be dealing with approximation to points $\boldsymbol{\xi}$ in $\mathbb{R}^n$ or $\mathbb{C}^n$. In order to treat both cases with a minimum of repetition, we set $\mathbb{C}_1 = \mathbb{R}$, $\mathbb{C}_2 = \mathbb{C}$. Throughout, q will be 1 or 2, so that $\mathbb{C}_q$ will be $\mathbb{R}$ or $\mathbb{C}$. For $\boldsymbol{\eta} = (\eta_1, \dots, \eta_\ell) \in \mathbb{C}_q^\ell$ (with any ℓ), let $|\boldsymbol{\eta}|$ denote the maximum norm, i.e., $\max(|\eta_1|, \dots, |\eta_\ell|)$. When $\boldsymbol{\xi} \in \mathbb{C}^n$ and A is a subset of $\mathbb{C}^n$, we define the distance $\delta(\boldsymbol{\xi}, A)$ to be ∞ when A is empty, and the infimum of $|\boldsymbol{\alpha} - \boldsymbol{\xi}|$ over points $\boldsymbol{\alpha} \in A$ otherwise. We further put $\delta_q(\boldsymbol{\xi}, A) = \delta(\boldsymbol{\xi}, A \cap \mathbb{C}_q^n)$, and note that $\delta_2(\boldsymbol{\xi}, A) = \delta(\boldsymbol{\xi}, A)$.

How well can a given point $\boldsymbol{\xi} \in \mathbb{C}_q^n$ be "approximated" by algebraic hypersurfaces, or more generally by algebraic varieties? More precisely, the question is about algebraic hypersurfaces or varieties A with $\delta_q(\boldsymbol{\xi}, A)$ small.

When $P(\mathbf{X}) \in \mathbb{Z}[\mathbf{X}] = \mathbb{Z}[X_1, \dots, X_n]$ is a nonzero polynomial, let $A(P)$ be the hypersurface consisting of points $\boldsymbol{\alpha} \in \mathbb{C}_q^n$ with $P(\boldsymbol{\alpha}) = 0$. A goal will be to make $\delta_q(\boldsymbol{\xi}, A(P))$ small, subject to certain conditions on P. In particular, this distance should be small in terms of the total degree of P and the Height $H(P)$ of P. This Height is defined as the maximum modulus of the coefficients of P; for convenience of notation we will set $|P| := H(P)$.

Let $\mathcal{M}$ be a finite, nonempty set of monomials, i.e., expressions $\mathbf{X}^{\boldsymbol{\sigma}} = X_1^{\sigma_1} \cdots X_n^{\sigma_n}$ where $\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_n) \in \mathbb{N}_\circ^n$; here $\mathbb{N}_\circ = \mathbb{N} \cup \{0\}$ where $\mathbb{N}$ consists of the positive integers. Let $\mathcal{P}(\mathcal{M})$ be the set of nonzero polynomials $P \in \mathbb{Z}[\mathbf{X}]$ which are linear combinations of the monomials in $\mathcal{M}$. Also, let $|\mathcal{M}|$ be the cardinality of $\mathcal{M}$. Versions of the following lemma, which will be easily proved in Section 2, are well known.

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Lemma 1.1. *Suppose* $|\mathcal{M}| \geq 2$, $\boldsymbol{\xi} \in \mathbb{C}_q^n$, *and set*

$$\langle \mathcal{M}(\boldsymbol{\xi}) \rangle = \sum_{M \in \mathcal{M}} |M(\boldsymbol{\xi})|.$$

Then given $N \geq 1$, *there is a polynomial* $P \in \mathcal{P}(\mathcal{M})$ *with*

$$|P| \leq N \quad \text{and} \quad |P(\boldsymbol{\xi})| < 2\langle \mathcal{M}(\boldsymbol{\xi}) \rangle N^{1-|\mathcal{M}|/q}. \tag{1.1}$$

In general it is not easy to deduce from an upper bound for $|P(\boldsymbol{\xi})|$ an upper bound for $\delta_q(\boldsymbol{\xi}, A(P))$. When $V \subset \mathbb{C}^n$ is an algebraic set defined by polynomial equations $P_1 = \cdots = P_s = 0$, it is even harder to obtain from a bound for $\max_i |P_i(\boldsymbol{\xi})|$ (or else a bound for a quantity $|V(\boldsymbol{\xi})|$ introduced by Nesterenko [10]) a bound for $\delta(\boldsymbol{\xi}, V)$. Results in this direction are due, e.g., to Brownawell [4] and Philippon [11]. Amoroso [1], [2] proved "comparison theorems" involving the "size" of a polynomial to derive a metric theorem.

Given a polynomial P, the most common approach is to estimate $\delta(\boldsymbol{\xi}, A(P))$ in terms of

$$|P(\boldsymbol{\xi})|/|\operatorname{grad} P(\boldsymbol{\xi})|, \tag{1.2}$$

where $\operatorname{grad} P(\boldsymbol{\xi})$ is the gradient vector of P at $\boldsymbol{\xi}$. As is well known, and will again be explained in Section 3, $\delta_2(\boldsymbol{\xi}, A(P))$ is not larger than d times (1.2), where d is the total degree of P. This is not always true for $\delta_1(\boldsymbol{\xi}, A(P))$. But in Section 3 we will explain that when $\boldsymbol{\xi} \in \mathbb{R}^n$ and $P \in \mathbb{R}[\mathbf{X}]$, then an estimate for (1.2) often yields an estimate for $\delta_1(\boldsymbol{\xi}, A(P))$, or for $\delta_1(\boldsymbol{\xi}, A(P^{[\boldsymbol{\sigma}]}))$ for some partial derivative $P^{[\boldsymbol{\sigma}]}$ of P.

A polynomial P as in (1.1) with $|P| \leq N$ is likely to have $|P(\boldsymbol{\xi})|$ not much smaller than N, in fact to have $|\operatorname{grad} P(\boldsymbol{\xi})|$ not much smaller than N. But then (1.1) will imply that (1.2) is not much larger than $N^{-|\mathcal{M}|/q} \leq |P|^{-|\mathcal{M}|/q}$. One may therefore ask whether, given $\boldsymbol{\xi} \in \mathbb{C}_q^n$, there are infinitely many $P \in \mathcal{P}(\mathcal{M})$ with

$$\delta_q(\boldsymbol{\xi}, A(P)) < c_1(\boldsymbol{\xi}, \mathcal{M})|P|^{-|\mathcal{M}|/q} \tag{1.3}$$

where $c_1(\boldsymbol{\xi}, \mathcal{M})$ depends on $\boldsymbol{\xi}, \mathcal{M}$ only. In general, unless said otherwise, a constant $c(\phi, \Phi, \dots)$ will depend only on quantities $\phi, \Phi, \dots$.

The rather few instances when we can establish such a result will be described in Theorem 1.2. Monomials are contained in the multiplicative group of generalized monomials, i.e., expressions $\mathbf{X}^{\mathbf{u}}$ with $\mathbf{u} \in \mathbb{Z}^n$. Let $\rho(\mathcal{M})$ be the rank of the subgroup generated by the quotients M/M' with M, M' in $\mathcal{M}$. Clearly

$$\rho(\mathcal{M}) \leq \min(n, |\mathcal{M}| - 1).$$

Theorem 1.2. *Let* $\mathcal{M}$ *be a set of monomials with* $\rho(\mathcal{M}) = n$.

(i) *Suppose* $n \geq q$, $|\mathcal{M}| = n + 1$, *and* $\boldsymbol{\xi} \in \mathbb{C}_q^n$. *Then given* $N \geq 1$, *there is a* $P \in \mathcal{P}(\mathcal{M})$ *with* $|P| \leq N$ *and*

$$\delta_q(\boldsymbol{\xi}, A(P)) < c_2 N^{1-|\mathcal{M}|/q}|P|^{-1}, \tag{1.4}$$

where $c_2 = c_2(\boldsymbol{\xi}, \mathcal{M})$.

(ii) *Suppose* $q = 1$, $|\mathcal{M}| = n + 2$, *and* $\boldsymbol{\xi} \in \mathbb{C}_1^n = \mathbb{R}^n$, *but* $\boldsymbol{\xi} \notin A(P)$ *for any* $P \in \mathcal{P}(\mathcal{M})$. *Then there are infinitely many polynomials* $P \in \mathcal{P}(\mathcal{M})$ *with*

$$\delta_1(\boldsymbol{\xi}, A(P)) < c_3|P|^{-|\mathcal{M}|},$$

where $c_3 = c_3(\boldsymbol{\xi}, \mathcal{M})$.

Part (i) is a *uniform* result, since it asserts the existence of $P \in \mathcal{P}(\mathcal{M})$ with $|P| \le N$ and (1.4) for every $N \ge 1$. It is not much harder to prove than Dirichlet's Theorem on linear forms, which yields the case $q = 1$, $\mathcal{M} = \{1, X_1, \dots, X_n\}$, and which may be interpreted as a result on approximation to $\boldsymbol{\xi}$ by hyperplanes. When $\boldsymbol{\xi} \notin A(P)$ for every $P \in \mathcal{P}(\mathcal{M})$, it implies the existence of infinitely many $P \in \mathcal{P}(\mathcal{M})$ with $\delta_q(\boldsymbol{\xi}, A(P)) < c_2|P|^{-|\mathcal{M}|/q}$. Part (ii) is not uniform, and applies only to the real case. The important distinction of uniform and nonuniform results had also been pointed out in [5].

An example for part (ii) is when $\mathcal{M} = \{1, X_1, \dots, X_n, M\}$ where M is a monomial of total degree > 1. A special case is when $n = 1$ and $\mathcal{M} = \{1, X, X^2\}$. In this case the assertion can be reformulated to say that for any $\xi \in \mathbb{R}$ which is not rational or quadratic over $\mathbb{Q}$, there are infinitely many $\alpha \in \mathbb{R}$ of degree at most 2 over $\mathbb{Q}$ with $|\alpha - \xi| < c_4(\xi)H(\alpha)^{-3}$. Here $H(\alpha)$ is the Height of the defining polynomial of α over $\mathbb{Z}$. This had been established by Davenport and Schmidt [6], and amounts to the quadratic case of a conjecture of Wirsing [14].

Our next theorem is a consequence of a special case of the important and deep work of Laurent and Roy [8]. It is essentially Theorem 2.3 of their paper, but with a more carefully worked out value of the exponent. Let $\mathcal{M}(d)$ consist of all monomials of total degree at most d, so that $|\mathcal{M}(d)|$ equals

$$m = m(n, d) = \binom{n+d}{d}, \tag{1.5}$$

and $\mathcal{P}(d) := \mathcal{P}(\mathcal{M}(d))$ consists of the nonzero polynomials in $\mathbb{Z}[\mathbf{X}]$ of total degree at most d.

Set $S_\circ = \lceil \sqrt[n]{q(n+1)!} \rceil$, and

$$\nu = ((m/q) - 1)/S_\circ. \tag{1.6}$$

Theorem 1.3. *Suppose $n > 1$. Also suppose $\boldsymbol{\xi} \in \mathbb{C}_q^n$ with $|\boldsymbol{\xi}| \le \psi$, and $P(\boldsymbol{\xi}) \ne 0$ for every $P \in \mathcal{P}(d)$. Then there are infinitely many $P \in \mathcal{P}(d)$ with*

$$\delta_q(\boldsymbol{\xi}, A(P)) < c_5(n, d, \psi)|P|^{-\nu}. \tag{1.7}$$

Now $n > 1$ yields $(n+1)! - qn! > 1$, so that

$$(d+1)\cdots(d+n) - qn! > d^n$$

holds for $d = 1$, hence for $d \ge 1$, since the left-hand side increases faster in d than the right-hand side. As a consequence,

$$(m/q) - 1 > d^n/qn!. \tag{1.8}$$

It is easily seen that $n > 1$ gives $S_\circ \le n + 1$, *except* that $S_\circ = 4$ when $n = q = 2$. Thus $\nu > \nu'$ where

$$\nu' = \begin{cases} d^2/16 & \text{when } n = q = 2, \\ d^n/q(n+1)! & \text{otherwise.} \end{cases} \tag{1.9}$$

Wirsing in pioneering work [14] had shown that when $\xi \in \mathbb{C}_q$ is not algebraic of degree $\le d$ over $\mathbb{Q}$, then there are infinitely many algebraic numbers $\alpha \in \mathbb{C}_q$ of degree $\le d$ with

$$|\alpha - \xi| < H(\alpha)^{-(d/(2q) - \gamma_q(d)} \tag{1.10}$$

where $H(\alpha)$ is the Height of the defining polynomial of α over $\mathbb{Z}$, and where $\gamma_q(d) > 0$ with $\gamma_1(d) \to 2$, $\gamma_2(d) \to 1/4$ as $d \to \infty$. This has since been improved to $\gamma_1(d) \to$

3 by Bernik and Tishchenko [3], and $\gamma_2(d) \to 3/2$ by Tishchenko [13]. When P is the defining polynomial of α over $\mathbb{Z}$, then (1.10) certainly gives $\delta_q(\xi, A(P)) < |P|^{-d/2q}$. We obtain the following

Corollary 1.4. *Suppose n, d are positive integers, and define ν' by* (1.9). *If $\boldsymbol{\xi} \in \mathbb{C}_q^n$, but $P(\boldsymbol{\xi}) \neq 0$ for any $P \in \mathcal{P}(d)$, then there are infinitely many $P \in \mathcal{P}(d)$ with*

$$\delta_q(\boldsymbol{\xi}, A(P)) < |P|^{-\nu'}.$$

Again set

$$m = \binom{n+d}{d} = 1 + n + \binom{n+1}{2} + \cdots + \binom{n+d-1}{d},$$

and further put

$$m_\circ = m_\circ(n,d) = 1 + n + \frac{1}{2}\binom{n+1}{2} + \cdots + \frac{1}{d}\binom{n+d-1}{d}.$$

Theorem 1.5. (a) *Suppose $\boldsymbol{\xi} \in \mathbb{C}_q^n$ and $\psi \geq |\boldsymbol{\xi}|$. Then given $N \geq 1$ there is a $P \in \mathcal{P}(d)$ with $|P| \leq N$ and*

$$\delta_q(\boldsymbol{\xi}, A(P)) < c_6(n,d,\psi)(N^{1-m/q}|P|^{-1})^{1/d}.$$

(b) *Suppose $\boldsymbol{\xi} \in \mathbb{R}^n = \mathbb{C}_1^n$ with $|\boldsymbol{\xi}| \leq \psi$. Then given $N \geq 1$, there is an ℓ in $1 \leq \ell \leq d$, and a $P \in \mathcal{P}(d)$ with $|P| \leq N^{1/\ell}$ and*

$$\delta_1(\boldsymbol{\xi}, A(P)) < c_7(n,d,\psi)(N^{1-m_\circ}|P|^{-1})^{1/\ell}. \tag{1.11}$$

We cannot specify the value of ℓ in part (b), and there does not appear to be a clean version of part (b) for $\boldsymbol{\xi} \in \mathbb{C}^n = \mathbb{C}_2^n$. The assertions of the theorem are uniform, since $|P|$ is under a bound involving N. The following is an immediate consequence.

Corollary 1.6. (a) *Suppose $\boldsymbol{\xi} \in \mathbb{C}_q^n$, $\psi \geq |\boldsymbol{\xi}|$, and moreover $\boldsymbol{\xi} \notin A(P)$ for any $P \in \mathcal{P}(d)$. Then there are infinitely many $P \in \mathcal{P}(d)$ with*

$$\delta_q(\boldsymbol{\xi}, A(P)) < c_6(n,d,\psi)|P|^{-m/(dq)}. \tag{1.12}$$

(b) *Suppose $\boldsymbol{\xi} \in \mathbb{R}^n$, $\psi \geq |\boldsymbol{\xi}|$, and $\boldsymbol{\xi} \notin A(P)$ for any $P \in \mathcal{P}(d)$. Then there are infinitely many $P \in \mathcal{P}(d)$ with*

$$\delta_1(\boldsymbol{\xi}, A(P)) < c_7(n,d,\psi)|P|^{1-m_\circ-1/d}. \tag{1.13}$$

Since $m_\circ - 1 > m/d$ when $d > 1$, $n > 1$, the inequality (1.13) is usually better than (1.12). But let us compare the exponent in (1.7) with the exponents in (1.12) or (1.13). Since $\nu = ((m/q) - 1)/S_\circ$ where $S_\circ \leq n+1$ when $n > 1$, (1.7) is better when d is large compared to n. But (1.12), (1.13) are better when n is large compared to d.

Not much is known about approximation by algebraic varieties of codimension $s > 1$. But see Philippon [11] where, e.g., the case $n = s = 2$ is treated, i.e., approximation to $\boldsymbol{\xi} \in \mathbb{C}^2$ by algebraic points $\boldsymbol{\alpha}$ of bounded degree. In [12], the Corollary to Theorem 15, using projective terminology, shows that when $\boldsymbol{\xi} \in \mathbb{C}_q^n$ with $\boldsymbol{\xi} \notin A$ for any linear manifold A of codimension s defined over $\mathbb{Q}$, then there are infinitely many such A with $\delta_q(\boldsymbol{\xi}, A) < c_8(\boldsymbol{\xi})H(A)^{-(n+1)/qs}$, and the exponent $-(n+1)/qs$ is best possible. Here we will derive results which are probably far from the full truth.

Suppose now that $r > 0$, $s > 0$,

$$r + s = n. \tag{1.14}$$

Let $\mathbf{P} = (P_1, \dots, P_s)$ denote an s-tuple of polynomials in $\mathbb{Z}[\mathbf{X}]$. Given $\mathbf{P}$, write $A(\mathbf{P})$ for the algebraic set consisting of points $\boldsymbol{\alpha} \in \mathbb{C}^n$ with $P_1(\boldsymbol{\alpha}) = \cdots = P_s(\boldsymbol{\alpha}) = 0$. When $\mathbf{d} = (d_1, \dots, d_s) \in \mathbb{N}^s$, let $\mathcal{P}(\mathbf{d})$ consist of s-tuples $\mathbf{P}$ where $A(\mathbf{P})$ has dimension r, and where for $1 \le i \le s$ we have $\deg P_i \le d_i$ and $P_i = P_i(X_1, \dots, X_{n-i+1})$. Set

$$m_i = \binom{n - i + 1 + d_i}{d_i} \quad (i = 1, \dots, s). \tag{1.15}$$

Theorem 1.7. *Suppose $\mathbf{d} \in \mathbb{N}^s$ is given, and $m_i > q$ holds for $i = 1, \dots, s$. There are constants $\gamma_i = \gamma_i(n, \mathbf{d}, \psi) \ge 1$ $(i = 1, \dots, s)$ with the following property. Suppose $\boldsymbol{\xi} \in \mathbb{C}_q^n$ with $|\boldsymbol{\xi}| \le \psi$, and $L \ge 1$. Then after suitable ordering of the variables there is a $\mathbf{P} = (P_1, \dots, P_s) \in \mathcal{P}(\mathbf{d})$ with*

$$|P_i| \le (\gamma_i L)^{d_i q/(m_i - q)} \quad (i = 1, \dots, s) \tag{1.16}$$

and

$$\delta_q(\boldsymbol{\xi}, A(\mathbf{P})) < L^{-1}.$$

No assertion is made for $r = 0$, since our arguments would give a poor result. In fact the theorem, as well as Theorem 1.8 below, are poor when the dimension r is small compared to n.

By a *variety* we will understand an algebraic set $V \subset \mathbb{C}^n$ which is defined over $\mathbb{Q}$, and irreducible over $\mathbb{Q}$. We define the Height of V to be $H(V) = H(\widehat{V})$, where $\widehat{V}$ is the Zariski closure of V in projective space $\mathbb{P}^n(\mathbb{C})$, and $H(\widehat{V})$ is the Height of a Chow form of $\widehat{V}$ (see [8, Section 4.1], but observe that in [8] the logarithmic height $h(\widehat{V}) = \log H(\widehat{V})$ is used).

Suppose r, s, $\mathbf{d} \in \mathbb{N}^s$ and m_i $(1 \le i \le s)$ are as above, and that each $m_i > q$. Furthermore, set $d = d_1 \cdots d_s$,

$$\Phi = dq \left(\frac{1}{m_1 - q} + \cdots + \frac{1}{m_s - q} \right), \quad \Psi = 1/\Phi.$$

Theorem 1.8. *Suppose $\boldsymbol{\xi} \in \mathbb{C}_q^n$, and $|\boldsymbol{\xi}| \le \psi$. Then for $N \ge 1$ there is a variety V of dimension r, degree $\le d$ and Height $H(V) \le N$ having*

$$\delta_q(\boldsymbol{\xi}, V) < c_9(n, \mathbf{d}, \psi) N^{-\Psi}. \tag{1.17}$$

Observe that Theorems 1.7 and 1.8 are uniform. I have no intuition of what would be the best exponent on the right-hand side of (1.17). Suppose $\mathbf{d} = (a, \dots, a)$, so that $d = a^s$. Then for fixed a, s we have as $n \to \infty$ that $m_i = \binom{n - i + 1 + a}{a} \sim n^a/a!$ for $1 \le i \le s$, hence $\Phi \sim a^s qsa!/n^a$, and

$$\Psi \sim n^a/(qsa^s a!) = n^{d^{1/s}}/(qsda!).$$

On the other hand, suppose $s < n$ are fixed, and $d \to \infty$. Say $d_i = a^{u_i}$ $(i = 1, \dots, s)$ where $u_i = u/(n - i + 1)$ and $u = n(n-1)\cdots(n - s + 1)$. Then as $a \to \infty$, $m_i \sim a^{u_i(n-i+1)}/(n - i + 1)! = a^u/(n - i + 1)!$, and

$$\Phi \sim dqa^{-u}(n! + \cdots + (n - s + 1)!) \le 2dqn!a^{-u},$$

hence $\Psi \gtrsim a^u/(2dqn!)$. But $u_1 + \cdots + u_s = u(\frac{1}{n} + \cdots + \frac{1}{n-s+1}) = uf(n,s)$, say, so that $d = a^{uf(n,s)}$, and setting $g(n,s) = f(n,s)^{-1} - 1$, e.g., $g(3,2) = 1/5$, we have as $d \to \infty$,

$$\Psi \gtrsim d^{g(n,s)}/(2qn!).$$

Also, $f(n,s) \le s/(n-s)$, so that $g(n,s) \ge (n/s)-2$. It may be seen that $f(n,s) < 1$ when $s < (1-e^{-1})n$, and then $g(n,s) > 0$. But $f(n,s) > 1$ when $s > (1-e^{-1})(n+1)$, and then $g(n,s) < 0$ and our estimate is useless.

We will prove somewhat stronger theorems than those of the Introduction. See the definitions in Section 3 and Theorems 5.1, 7.2, 8.1, 9.1, 10.1. Our results could be restated in a projective setting, such as, e.g., in [8] or [12].

As already said, we set $\mathbb{N} = \{1, 2, \dots\}$, $\mathbb{N}_\circ = \{0, 1, 2, \dots\}$. The degree of a polynomial will be understood to be the total degree. $\mathbf{X}$ will denote an n-tuple of variables $(X_1, \dots, X_n)$. A point $\boldsymbol{\xi}$ or $\boldsymbol{\alpha}$ or $\cdots$ in $\mathbb{C}^n$ will be understood to have coordinates $\xi_1, \dots, \xi_n$ or $\alpha_1, \dots, \alpha_n$ or $\cdots$. The notation $\mathbf{X}^{\boldsymbol{\rho}}$ will signify $X_1^{\rho_1} \cdots X_n^{\rho_n}$. Besides the maximum norm $|\boldsymbol{\eta}|$ for $\boldsymbol{\eta} = (\eta_1, \dots, \eta_\ell)$ introduced above, we will also use $\langle \boldsymbol{\eta} \rangle = |\eta_1| + \cdots + |\eta_\ell|$. The divided derivatives of a polynomial $P \in \mathbb{C}[\mathbf{X}]$ are

$$P^{[\boldsymbol{\sigma}]} = \frac{1}{\sigma_1! \cdots \sigma_n!} \frac{\partial^{\sigma_1 + \cdots + \sigma_n}}{\partial X_1^{\sigma_1} \cdots \partial X_n^{\sigma_n}} P$$

where $\boldsymbol{\sigma} \in \mathbb{N}_\circ^n$. The numbering of constants $c_1, c_2, \dots$ will start anew in each section, and all these constants will be positive.

2. Proof of Lemma 1.1

This lemma (which corresponds to Lemma 6.6 in [8]) is an immediate consequence of the following.

Lemma 2.1. *Let $\boldsymbol{\lambda} \in \mathbb{C}_q^m$, where $m \ge 2$, and let L be the linear form with $L(\mathbf{a}) = \lambda_1 a_1 + \cdots + \lambda_m a_m$. Then given $N \ge 1$, there is a nonzero integer point $\mathbf{a} \in \mathbb{Z}^m$ with $|\mathbf{a}| \le N$ and*

$$|L(\mathbf{a})| < 2\langle \boldsymbol{\lambda} \rangle N^{1-m/q}.$$

Proof. We may suppose by homogeneity that $\langle \boldsymbol{\lambda} \rangle = 1$. We also may suppose that $|\lambda_1| = \min_i |\lambda_i|$, so that $|\lambda_1| \le \langle \boldsymbol{\lambda} \rangle / m = 1/m$. We will deal with the case $q = 2$; the case $q = 1$ can be done in a similar manner.

When $N^{m/2} < 4$, then $\mathbf{a} = (1, 0, \dots, 0)$ has $|\mathbf{a}| = 1 \le N$ and$|L(\mathbf{a})| = |\lambda_1| \le 1/m \le 2N/4 < 2N^{1-m/2}$. When $N^{m/2} \ge 4$, we argue as follows. The number of integer points $\mathbf{b}$ with $|\mathbf{b}| \le N/2$ is at least N^m. For such $\mathbf{b}$ we have $|L(\mathbf{b})| \le \langle \boldsymbol{\ell} \rangle |\mathbf{b}| \le N/2$, so that $L(\mathbf{b})$ lies in a square in $\mathbb{C}$ of side N. Let A be the integer with $N^{m/2} - 1 \le A < N^{m/2}$; then $N^{m/2}/\sqrt{2} < A$ in view of $N^{m/2} \ge 4$. Divide our square into A^2 subsquares of side N/A. Since $A^2 < N^m$, two of our points $\mathbf{b}$ will have $L(\mathbf{b})$ in the same subsquare. If $\mathbf{b}_1, \mathbf{b}_2$ are such points, set $\mathbf{a} = \mathbf{b}_1 - \mathbf{b}_2$, so that $0 < |\mathbf{a}| \le N$ and $|L(\mathbf{a})| \le \sqrt{2}N/A < 2N^{1-m/2}$. □

Observe that the lemma also follows from Minkowski's theorem on linear forms.

3. Polynomials with small zeros

We will deal with polynomials in $\mathbb{R}[\mathbf{X}]$ or $\mathbb{C}[\mathbf{X}]$. The contents of this and the next section may be of independent interest.

Before going further, let us justify an assertion we made in the Introduction. When $\xi \in \mathbb{C}$ and $P \in \mathbb{C}[X]$ with $P(\xi) \neq 0$, $P'(\xi) \neq 0$, and roots $\alpha_1, \ldots, \alpha_d$, then $P'(\xi) = P(\xi) \sum_{i=1}^{d} (\xi - \alpha_i)^{-1}$, so that some α_i has $|\xi - \alpha_i| \leq d|P(\xi)|/|P'(\xi)|$, yielding $\delta_2(\xi, A(P)) \leq d|P(\xi)|/|P'(\xi)|$. Since $|\operatorname{grad} P(\boldsymbol{\xi})|$ is defined in terms of the maximum norm, it follows that for $\boldsymbol{\xi} \in \mathbb{C}^n$, and $P \in \mathbb{C}[\mathbf{X}]$ with $\operatorname{grad} P(\boldsymbol{\xi}) \neq 0$, indeed, $\delta_2(\boldsymbol{\xi}, A(P))$ is bounded by d times (1.2).

Suppose $\boldsymbol{\xi} \in \mathbb{C}_q^n$ and $A \subset \mathbb{C}_q^n$ is a manifold of dimension r. We hope to not only have $\delta_q(\boldsymbol{\xi}, A)$ small, i.e., to have some $\boldsymbol{\alpha} \in A$ with $|\boldsymbol{\alpha} - \boldsymbol{\xi}|$ small, say $|\boldsymbol{\alpha} - \boldsymbol{\xi}| < \delta$, but to have a "large" subset A' of A whose elements $\boldsymbol{\alpha}$ have $|\boldsymbol{\alpha} - \boldsymbol{\xi}| < \delta$. We hope to have A' with reasonably large qr-dimensional volume, ideally of order of magnitude $\gg \delta^{qr}$.

Suppose r, s are integers with

$$r + s = n \quad \text{and} \quad 0 \leq r < n, \quad 0 < s \leq n, \tag{3.1}$$

and let ρ, δ be reals with

$$0 < \rho \leq \delta. \tag{3.2}$$

Suppose at first that $r > 0$. Let $\mathcal{C}_q(r; \rho, \boldsymbol{\xi})$ consist of points $\mathbf{z} = (z_1, \ldots, z_r) \in \mathbb{C}_q^r$ with $|z_i - \xi_i| < \rho$ $(i = 1, \ldots, r)$. Let $\mathcal{C}_q'(s; \delta, \boldsymbol{\xi})$ consist of points $\mathbf{w} = (w_{r+1}, \ldots, w_n) \in \mathbb{C}_q^s$ with $|w_j - \xi_j| < \delta$ $(j = r+1, \ldots, n)$. An *analytic* $(r; \rho, \delta)_2$*-piece for* $\boldsymbol{\xi}$ will be understood to be a set of points

$$\boldsymbol{\alpha} = (\mathbf{z}, \mathbf{w}(\mathbf{z})) \in \mathbb{C}^n$$

where $\mathbf{z}$ runs through $\mathcal{C}_2(r; \rho, \boldsymbol{\xi})$, where $\mathbf{w}(\mathbf{z}) = \mathbf{w}(z_1, \ldots, z_r)$ defines a map $\mathcal{C}_2(r; \rho, \boldsymbol{\xi}) \to \mathcal{C}_2'(s; \delta, \boldsymbol{\xi})$, and where each $w_j(\mathbf{z})$ is a regular function. An analytic $(r; \rho, \delta)_1$-piece for a point $\boldsymbol{\xi} \in \mathbb{C}_1^n = \mathbb{R}^n$ is an $(r; \rho, \delta)_2$-piece where each $w_j(\mathbf{z})$ is a real regular function, i.e., is real when $\mathbf{z} \in \mathbb{R}^r$. In this case the restriction of $\mathbf{w}(\mathbf{z})$ to $\mathbf{z} \in \mathcal{C}_1(r; \rho, \boldsymbol{\xi})$ is a map into $\mathcal{C}_1'(s; \delta, \boldsymbol{\xi})$. Observe that by (3.2) and our use of the maximum norm, every point $\boldsymbol{\alpha}$ of an analytic $(r; \rho, \delta)_q$-piece for $\boldsymbol{\xi}$ has $|\boldsymbol{\alpha} - \boldsymbol{\xi}| < \delta$.

When $r > 0$, a set $A \subset \mathbb{C}_q^n$ will be said to be *special* $(r; \rho, \delta)_q$*-close* to $\boldsymbol{\xi}$ if it contains an analytic $(r; \rho, \delta)_q$-piece for $\boldsymbol{\xi}$. It will be said to be $(r; \rho, \delta)_q$*-close* if it is special $(r; \rho, \delta)_q$-close after suitable numbering of the variables, i.e., if $T(A)$ is special $(r; \rho, \delta)_q$-close where T is a map $(x_1, \ldots, x_n) \mapsto (x_{i_1}, \ldots, x_{i_n})$ for some permutation $(i_1, \ldots, i_n)$ of $(1, \ldots, n)$. When $r = 0$, special $(0; \rho, \delta)_q$-closeness and $(0; \rho, \delta)_q$-closeness will mean the same, namely that $\delta_q(\boldsymbol{\xi}, A) < \delta$. In this case ρ does not come into play! It will be important to observe that when A is $(r; \rho, \delta)_q$-close, and $\rho' \leq \rho$, $\delta' \geq \delta$, then it is $(r; \rho', \delta')_q$-close, and the dilated set λA is $(r; \lambda\rho, \lambda\delta)_q$-close.

A polynomial $Q(\mathbf{X}) \in \mathbb{C}_q[\mathbf{X}]$ will be said to have the *special* $(\rho, \boldsymbol{\xi})_q$*-property* if

(3i) *$A(Q)$ is special $(n-1; \rho, (n+1)\rho)_q$-close to $\boldsymbol{\xi}$,*

(3ii) *$Q(\alpha_1, \ldots, \alpha_{n-1}, X_n)$ for any given $\alpha_1, \ldots, \alpha_{n-1}$ is a nonzero polynomial in X_n. (When $n = 1$, then $Q(X_1) \neq 0$.)*

It has the $(\rho, \boldsymbol{\xi})_q$*-property* if after suitable numbering of the variables it has the special $(\rho, \boldsymbol{\xi})_q$-property. The $(\rho, \mathbf{0})_q$-property will more simply be called ρ_q-property. Well, this should be the end of the definitions!

Remark 3.1. When Q has the $(\rho, \boldsymbol{\xi})_q$-property, then $\delta_q(\boldsymbol{\xi}, A(Q)) < (n+1)\rho$. Furthermore, when $a \leq \rho \leq b$, then $A(Q)$ is $(n-1; a, (n+1)b)_q$-close to $\boldsymbol{\xi}$.

Remark 3.2. The set A of points $\mathbf{x}$ with $x_{r+1} = \cdots = x_n = 0$ is special $(r; 1, 1+|\boldsymbol{\xi}|)_q$-close to $\boldsymbol{\xi}$: this follows on setting $\mathbf{w}(\mathbf{z}) = (w_{r+1}(\mathbf{z}), \ldots, w_n(\mathbf{z})) = (0, \ldots, 0)$ for $\mathbf{z} = (z_1, \ldots, z_r)$.

Write

$$Q(\mathbf{X}) = Q_0(\mathbf{X}) + \cdots + Q_d(\mathbf{X}), \tag{3.3}$$

where $Q_j(\mathbf{X})$ for $0 \le j \le d$ is a homogeneous polynomial in $\mathbb{C}_q[\mathbf{X}]$ of degree j.

Lemma 3.3. *Let Q be given by (3.3), with $Q_1 \neq 0$,*

$$|Q_0|/|Q_1| \le 1, \tag{3.4}$$

$$|Q_j|/|Q_1| \le (4n^2)^{-j} \quad \textit{for} \quad j = 2, 3, \ldots\ . \tag{3.5}$$

Then Q has the 1_q-property.

Proof. We will suppose that $n > 1$, the case $n = 1$ being simpler. We may suppose that $Q_1(\mathbf{X}) = a_1X_1 + \cdots + a_nX_n$ with $|Q_1| = |a_n|$. By homogeneity we may further suppose that $a_n = 1$, so that $|Q_1| = 1$. Now Q_j is a sum of

$$\binom{n-1+j}{j} = \frac{n-1+j}{j} \cdot \frac{n-1+j-1}{j-1} \cdots \frac{n}{1} \le n^j$$

monomials, so that for any $\boldsymbol{\alpha}$,

$$|Q_j(\boldsymbol{\alpha})| \le |Q_j|(n|\boldsymbol{\alpha}|)^j \quad (j = 1, \ldots, d). \tag{3.6}$$

When $|\boldsymbol{\alpha}| \le n+1$, (3.5) yields

$$|Q_j(\boldsymbol{\alpha})| \le |Q_j|(2n^2)^j \le 2^{-j},$$

and therefore

$$\sum_{j=2}^{d} |Q_j(\boldsymbol{\alpha})| \le 1/2.$$

Let $\mathbf{z} = (z_1, \ldots, z_{n-1})$ be any element of $\mathcal{C}_2(n-1, 1, \mathbf{0})$, so that $|\mathbf{z}| < 1$. Suppose $w \in \mathbb{C}$ with $|w| = n+1$. Setting $\boldsymbol{\alpha} = (\mathbf{z}, w) = (z_1, \ldots, z_{n-1}, w)$ we have

$$\begin{aligned} |Q(\boldsymbol{\alpha}) - w| = |Q(\mathbf{z}, w) - w| &\le |Q_0| + \sum_{i=1}^{n-1} |a_i z_i| + \sum_{j=2}^{d} |Q_j(\boldsymbol{\alpha})| \\ &\le 1 + (n-1) + 1/2 < |w|. \end{aligned}$$

Considering $\mathbf{z}$ fixed and $Q(\mathbf{z}, w)$ a polynomial in w, we may infer from Rouche's Theorem that there is a unique $w = w(\mathbf{z}) \in \mathbb{C}$ in the disk $|w| < n+1$ with $Q(\mathbf{z}, w) = 0$. The map $\mathbf{z} \mapsto w(\mathbf{z})$ is a map $\mathcal{C}_2(n-1; 1, \mathbf{0}) \to \mathcal{C}_2'(1; n+1, \mathbf{0})$.

Let $\mathbf{z}_\circ$ be fixed for the moment and $\widetilde{Q}$ an irreducible factor of Q with $\widetilde{Q}(\mathbf{z}_\circ, w(\mathbf{z}_\circ)) = 0$. By the "uniqueness" of $w(\mathbf{z}_\circ)$, $(\partial/\partial w)\widetilde{Q} \neq 0$ at $(\mathbf{z}_\circ, w(\mathbf{z}_\circ))$, and therefore there is a regular function $f(\mathbf{z})$ defined in a neighborhood of $\mathbf{z}_\circ$ with $f(\mathbf{z}_\circ) = w(\mathbf{z}_\circ)$ and $\widetilde{Q}(\mathbf{z}, f(\mathbf{z})) = 0$. By the uniqueness, $f(\mathbf{z}) = w(\mathbf{z})$. Then $\widetilde{Q}(\mathbf{z}, w(\mathbf{z})) = 0$ with the same irreducible factor $\widetilde{Q}$ of Q, when $\mathbf{z}$ is in this neighborhood. It easily follows that $\widetilde{Q}(\mathbf{z}, w(\mathbf{z})) = 0$ for $\mathbf{z} \in \mathcal{C}_2(n-1; 1, \mathbf{0})$, and that $w(\mathbf{z})$ is regular.

Since the points $(\mathbf{z}, w(\mathbf{z}))$ constructed form a special analytic $(n-1; 1, n+1)_2$-piece for $\mathbf{0}$, we see that $Q \in \mathbb{C}_2[\mathbf{X}] = \mathbb{C}[\mathbf{X}]$ indeed has property 1_2. On the other hand, when $Q \in \mathbb{C}_1[\mathbf{X}] = \mathbb{R}[\mathbf{X}]$, and if $Q(\mathbf{z}, w) = 0$ with $\mathbf{z} \in \mathcal{C}_1(n-1; 1, \mathbf{0}) \subset \mathbb{R}^{n-1}$, then also $Q(\mathbf{z}, \bar{w}) = 0$ for the complex conjugate $\bar{w}$ of w, so that by uniqueness $\bar{w} = w$, hence $w \in \mathbb{R}$. In this case Q has property 1_1. $\square$

Remark 3.4. It follows from the above argument that when $Q(\mathbf{X}) = Q_1(\mathbf{X}) \cdots Q_\ell(\mathbf{X})$, then $Q_i(\mathbf{z}, w(\mathbf{z})) = 0$ for some fixed factor Q_i, and Q_i has property 1_q. More generally, when Q has the $(\rho, \boldsymbol{\xi})_q$-property, then some Q_i has this property.

Corollary 3.5. *Suppose $Q \in \mathbb{C}_q[\mathbf{X}]$ is given by* (3.3)*, with $Q_1 \neq 0$ and*

$$\begin{aligned} &|Q_0|/|Q_1| \leq \lambda, \\ &|Q_j|/|Q_1| \leq (4n^2)^{-j}\lambda^{1-j} \quad \textit{for} \quad j = 2, 3, \dots \end{aligned} \tag{3.7}$$

for some $\lambda > 0$. Then Q has the λ_q-property.

Now (3.7) means that $|Q(\mathbf{0})|/|\operatorname{grad} Q(\mathbf{0})| \leq \lambda$, and the λ_q-property implies that Q has a zero $\boldsymbol{\alpha} \in \mathbb{C}_q^n$ with $|\boldsymbol{\alpha}| < (n+1)\lambda$. Thus $\delta_q(\mathbf{0}, A(Q)) < (n+1)\lambda$, justifying remarks we made on (1.2).

Proof. Setting $\widetilde{Q}(\mathbf{X}) = Q(\lambda \mathbf{X})$ we have $\widetilde{Q} = \widetilde{Q}_0 + \widetilde{Q}_1 + \cdots$ with $|\widetilde{Q}_j| = \lambda^j |Q_j|$. Therefore $|\widetilde{Q}_0|/|\widetilde{Q}_1| \leq 1$ and $|\widetilde{Q}_j|/|\widetilde{Q}_1| \leq (4n^2)^{-j}$ for $j = 2, 3, \dots$. The assertion follows on applying Lemma 3.3 to $\widetilde{Q}$, and observing that $A(Q) = \lambda A(\widetilde{Q})$. □

Recall the definition of the norm $\langle \cdot \rangle$ and of divided derivatives. Observe that any divided derivative $Q^{[\boldsymbol{\sigma}]}$ of a polynomial Q of total degree at most d has

$$|Q^{[\boldsymbol{\sigma}]}| \leq 2^d |Q|. \tag{3.8}$$

Lemma 3.6. *Suppose $Q \in \mathbb{C}_q[\mathbf{X}]$ is again given by* (3.3)*, and that for some $\lambda > 0$ and $k \in \mathbb{N}$ we have $Q_k \neq 0$ and*

$$\begin{aligned} &|Q_{k-1}|/|Q_k| \leq \lambda, \\ &|Q_{k+j}|/|Q_k| \leq (8n^2 \cdot 2^k)^{-1}(8n^2\lambda)^{-j} \ (j = 1, 2, \dots). \end{aligned} \tag{3.9}$$

Then there is a $\boldsymbol{\sigma}$ with $\langle \boldsymbol{\sigma} \rangle = k - 1$ such that $Q^{[\boldsymbol{\sigma}]}$ has property λ_q.

Proof. Pick $\boldsymbol{\rho}$ with $\langle \boldsymbol{\rho} \rangle = k$ such that the coefficient $a_{\boldsymbol{\rho}}$ of $\mathbf{X}^{\boldsymbol{\rho}}$ in Q_k has $|a_{\boldsymbol{\rho}}| = |Q_k|$. Pick $\boldsymbol{\sigma}$ with $\langle \boldsymbol{\sigma} \rangle = k - 1$ and $\sigma_i \leq \rho_i$ $(i = 1, \dots, n)$; then $\boldsymbol{\rho} = \boldsymbol{\sigma} + \mathbf{e}_\ell$ for some basis vector $\mathbf{e}_\ell$. We have

$$Q^{[\boldsymbol{\sigma}]} = Q_{k-1}^{[\boldsymbol{\sigma}]} + Q_k^{[\boldsymbol{\sigma}]} + Q_{k+1}^{[\boldsymbol{\sigma}]} + \cdots = G_0 + G_1 + G_2 + \cdots, \tag{3.10}$$

say, where G_j is homogeneous of degree j. Writing $Q_{k-1} = \sum_{\boldsymbol{\tau}} a_{\boldsymbol{\tau}} \mathbf{X}^{\boldsymbol{\tau}}$, we obtain $Q_{k-1}^{[\boldsymbol{\sigma}]} = a_{\boldsymbol{\sigma}}$, and therefore

$$|G_0| = |Q_{k-1}^{[\boldsymbol{\sigma}]}| = |a_{\boldsymbol{\sigma}}| \leq |Q_{k-1}|. \tag{3.11}$$

Now Q_k has the summand $a_{\boldsymbol{\rho}} \mathbf{X}^{\boldsymbol{\rho}} = a_{\boldsymbol{\rho}} \mathbf{X}^{\boldsymbol{\sigma} + \mathbf{e}_\ell}$, so that $G_1 = Q_k^{[\boldsymbol{\sigma}]}$ has the summand $(\sigma_\ell + 1) a_{\boldsymbol{\rho}} X_\ell$, and

$$|G_1| \geq (\sigma_\ell + 1)|a_{\boldsymbol{\rho}}| \geq |a_{\boldsymbol{\rho}}| = |Q_k|. \tag{3.12}$$

On the other hand, when $j > 0$,

$$|G_{j+1}| = |Q_{k+j}^{[\boldsymbol{\sigma}]}| \leq 2^{k+j}|Q_{k+j}| \leq 2^{k+j} \cdot 2^{-k}(8n^2)^{-j-1}\lambda^{-j}|Q_k| < (4n^2)^{-j-1}\lambda^{-j}|Q_k|$$

by (3.8) and our hypothesis. Thus $|G_j| < (4n^2)^{-j}\lambda^{1-j}|Q_k|$ for $j \geq 2$. Altogether,

$$|G_0|/|G_1| \leq |Q_{k-1}|/|Q_k| \leq \lambda \text{ and } |G_j|/|G_1| \leq (4n^2)^{-j}\lambda^{1-j} \text{ for } j > 1.$$

The lemma follows on applying Corollary 3.5 to $G = G_0 + G_1 + \cdots$. □

Lemma 3.7. *Suppose $\lambda > 0$, $\alpha \geq 1$, $\beta \geq 1$, and set*

$$\lambda_t = 2^{t(t+1)/2}(\alpha\beta)^t\lambda$$

for $t = 0, 1, \ldots$.[+)] *Let $q_0, q_1, \ldots$ be nonnegative reals with $q_1 \neq 0$ and $q_0/q_1 \leq \lambda$. Then either there is a $k > 0$ with $q_k \neq 0$ and both*

$$q_{k-1}/q_k \leq \lambda_{k-1}, \tag{3.13}$$

$$q_{k+j}/q_k \leq (1/\beta 2^k)(\alpha\lambda_{k-1})^{-j} \text{ for } j = 1, 2, \ldots, \tag{3.14}$$

or there are infinitely many k with $q_k \neq 0$ and (3.13).

Proof. We will suppose there is no k with both (3.13), (3.14). Since (3.13) is true by hypothesis for $k = 1$, it will suffice for us to show that for k with (3.13), there is a $k' > k$ with $q_{k'} \neq 0$ and (3.13), i.e., with $q_{k'-1}/q_{k'} \leq \lambda_{k'-1}$.

Since by our supposition (3.14) does not hold, there will be a $j > 0$ where (3.14) is violated. Let j be the least such number, and set $k' = k + j$. Then

$$q_{k'}/q_k > (1/\beta 2^k)(\alpha\lambda_{k-1})^{k-k'},$$

but

$$q_{k'-1}/q_k \leq (\alpha\lambda_{k-1})^{k-k'+1}.$$

Therefore

$$q_{k'-1}/q_{k'} < \beta 2^k \alpha\lambda_{k-1} = \lambda_k \leq \lambda_{k'-1}.$$

□

Lemma 3.8. *Let $Q(\mathbf{X})$ be a polynomial of degree d as in* (3.3)*, with $Q_1 \neq 0$ and $|Q_0|/|Q_1| \leq \lambda$ where $\lambda > 0$. Set*

$$\lambda_t = 2^{t(t+1)/2}(2^6 n^4)^t\lambda \tag{3.15}$$

for $t = 0, 1, \ldots$. Then there is a t in $0 \leq t < d$ and a $\boldsymbol{\sigma}$ with $\langle\boldsymbol{\sigma}\rangle = t$ such that $Q^{[\boldsymbol{\sigma}]}$ has property $(\lambda_t)_q$.

Proof. Set $\alpha = \beta = 8n^2$ and $q_k = |Q_k|$ for $0 \leq k \leq d$, but $q_k = 0$ for $k > d$. Since the second alternative of Lemma 3.7 cannot hold, there will be a k in $1 \leq k \leq d$ with (3.13), (3.14), i.e., with

$$|Q_{k-1}|/|Q_k| \leq \lambda_{k-1},$$

$$|Q_{k+j}|/|Q_k| \leq (8n^2 2^k)^{-1}(8n^2\lambda_{k-1})^{-j} \text{ for } j = 1, 2, \ldots.$$

By Lemma 3.6 there is a $\boldsymbol{\sigma}$ with $\langle\boldsymbol{\sigma}\rangle = k - 1$ such that $Q^{[\boldsymbol{\sigma}]}$ has property $(\lambda_{k-1})_q$. The desired conclusion holds with $t = k - 1$. □

Suppose $\boldsymbol{\xi} \in \mathbb{C}_q^n$ and $R \in \mathbb{C}_q[\mathbf{X}]$. Observe that R has property $(\rho, \boldsymbol{\xi})_q$ precisely if

$$Q(\mathbf{X}) := R(\mathbf{X} + \boldsymbol{\xi})$$

has property ρ_q.

Lemma 3.9. *Suppose $\operatorname{grad} R(\boldsymbol{\xi}) \neq \mathbf{0}$, and*

$$|R(\boldsymbol{\xi})|/|\operatorname{grad} R(\boldsymbol{\xi})| \leq \lambda$$

where $\lambda > 0$. Then there is a t in $0 \leq t < d$ and a $\boldsymbol{\sigma}$ with $\langle\boldsymbol{\sigma}\rangle = t$ such that $Q^{[\boldsymbol{\sigma}]}$ has property $(\lambda_t, \boldsymbol{\xi})_q$.

[+)] There should be no confusion about property λ_q ($q = 1$ or 2) and the numbers λ_t ($t = 0, 1, \ldots$).

This lemma may be seen as a variation on our remarks on (1.2).

Proof. $|Q(\mathbf{0})|/|\operatorname{grad} Q(\mathbf{0})| \le \lambda$, so by writing $Q = Q_0 + Q_1 + \cdots$ we have $|Q_0|/|Q_1| \le \lambda$. By Lemma 3.8 there are $t, \boldsymbol{\sigma}$ with $t = \langle \boldsymbol{\sigma} \rangle$ such that $Q^{[\boldsymbol{\sigma}]}$ has property $(\lambda_t)_q$, hence $R^{[\boldsymbol{\sigma}]}$ has property $(\lambda_t, \boldsymbol{\xi})_q$. □

4. Polynomials with small zeros, continued

The contents of this section will not be needed until Section 6.

Lemma 4.1. *Suppose $Q(\mathbf{X}) \in \mathbb{C}_q[\mathbf{X}]$ is given by (3.3), and $Q_\ell \ne 0$ with $|Q_0|/|Q_\ell| \le \kappa$ for some $\ell > 0$, $\kappa > 0$. Then there is a $\boldsymbol{\tau}$ and a t in $0 \le t \le \langle \boldsymbol{\tau} \rangle$ such that $Q^{[\boldsymbol{\tau}]}$ has property $(\lambda_t)_q$, where λ_t is given by (3.15) with $\lambda = \kappa^{1/\ell}$.*

Proof. There is a k in $0 < k \le \ell$ with $Q_k \ne 0$ and $|Q_{k-1}|/|Q_k| \le \kappa^{1/\ell} = \lambda$. Pick $\boldsymbol{\rho}, \boldsymbol{\sigma}$ as in the proof of Lemma 3.6 and set $G = Q^{[\boldsymbol{\sigma}]}$. We again have (3.10), (3.11), (3.12), and therefore $|G_0|/|G_1| \le |Q_{k-1}|/|Q_k| \le \lambda$. We now apply Lemma 3.8 to G. Accordingly, there is a $\boldsymbol{\phi}$ such that $G^{[\boldsymbol{\phi}]}$ has property $(\lambda_{\langle \boldsymbol{\phi} \rangle})_q$. Set $\boldsymbol{\tau} = \boldsymbol{\sigma} + \boldsymbol{\phi}$, $t = \langle \boldsymbol{\phi} \rangle$. Now $Q^{[\boldsymbol{\tau}]} = Q^{[\boldsymbol{\sigma}+\boldsymbol{\phi}]}$ is a multiple of $(Q^{[\boldsymbol{\sigma}]})^{[\boldsymbol{\phi}]} = G^{[\boldsymbol{\phi}]}$ by a nonzero constant, so that indeed it has property $(\lambda_t)_q$. □

Proposition 4.2. *Let $Q \in \mathbb{C}_q[\mathbf{X}]$ be a nonzero polynomial, and $\boldsymbol{\alpha} \in A(Q)$. Suppose $|\boldsymbol{\alpha}| \le \chi$. Then there is a $\boldsymbol{\tau}$ and a t in $0 \le t \le \langle \boldsymbol{\tau} \rangle$ such that $Q^{[\boldsymbol{\tau}]}$ has the $(\chi_t)_q$-property, where*

$$\chi_t := 2^{(t^2+13t+2)/2} 2^{4t+1} \chi. \tag{4.1}$$

Observe that when $q = 1$, so that $\mathbb{C}_1 = \mathbb{R}$, we still allow $\boldsymbol{\alpha}$ to be in $\mathbb{C}^n$, and yet get a conclusion on the $(\chi_t)_1$-property. In particular, since $t \le \langle \boldsymbol{\tau} \rangle$, there will be a zero $\boldsymbol{\beta} \in \mathbb{R}^n$ of $Q^{[\boldsymbol{\tau}]}$ with

$$|\boldsymbol{\beta}| \le (n+1) 2^{(\langle \boldsymbol{\tau} \rangle^2 + 13\langle \boldsymbol{\tau} \rangle + 2)/2} n^{4\langle \boldsymbol{\tau} \rangle + 1} |\boldsymbol{\alpha}|.$$

In this connection, A. Schinzel kindly pointed out that the Grace–Heawood Theorem (see, e.g., [9]) already implies that if $Q \in \mathbb{R}[X]$ is a polynomial in one variable of degree $d > 0$, and α is a complex root, then some derivative $Q^{[\tau]}$ with $0 \le \tau < d$ has a real root β with $|\beta| \le c_1(d)|\alpha|$.

Proof. $Q_0 + \sum_{j=1}^d Q_j(\boldsymbol{\alpha}) = 0$, where $d = \deg Q$. Since $|Q_j(\boldsymbol{\alpha})| \le |Q_j|(n|\boldsymbol{\alpha}|)^j$ $(j = 1, 2, \dots)$ by (3.6), there will be an $\ell > 0$ with $|Q_0| \le (2n|\boldsymbol{\alpha}|)^\ell |Q_\ell| \le (2n\chi)^\ell |Q_\ell|$. Whether Q_0 is 0 or not, there certainly will be an $\ell > 0$ where this holds with $Q_\ell \ne 0$. We therefore may apply Lemma 4.1 with $\kappa = (2n\chi)^\ell$. Then $\lambda = 2n\chi$, so that by (3.15),

$$\lambda_t = 2n\chi \cdot 2^{(t^2+t)/2} (2^6 n^4)^t = 2^{(t^2+13t+2)/2} n^{4t+1} \chi = \chi_t.$$

□

Lemma 4.3. *Suppose $R \in \mathbb{C}_q[\mathbf{X}]$ has $R^{[\boldsymbol{\rho}]}(\boldsymbol{\xi}) \ne 0$ with $\ell = \langle \boldsymbol{\rho} \rangle > 0$. Further, suppose that*

$$(|R(\boldsymbol{\xi})|/|R^{[\boldsymbol{\rho}]}(\boldsymbol{\xi})|^{1/\ell} \le \lambda$$

where $\lambda > 0$. Then there is a $\boldsymbol{\tau}$ and a t in $0 \le t \le \langle \boldsymbol{\tau} \rangle$ such that $R^{[\boldsymbol{\tau}]}$ has property $(\lambda_t, \boldsymbol{\xi})_q$ with λ_t given by (3.15).

When $\ell = 1$, Lemma 3.9 is a little stronger.

Proof.

$$Q(\mathbf{X}) := R(\mathbf{X}+\boldsymbol{\xi}) = R(\boldsymbol{\xi}) + \sum_{j=1}^{d} \sum_{\langle \boldsymbol{\sigma}\rangle = j} R^{[\boldsymbol{\sigma}]}(\boldsymbol{\xi})\mathbf{X}^{\boldsymbol{\sigma}},$$

so that $Q_0 = R(\boldsymbol{\xi})$ and $|Q_\ell| \geq |R^{[\boldsymbol{\rho}]}(\boldsymbol{\xi})|$. By Lemma 4.1, there are $\boldsymbol{\tau}, t$ with $0 \leq t \leq \langle \boldsymbol{\tau}\rangle$ such that $Q^{[\boldsymbol{\tau}]}$ has property $(\lambda_t)_q$, hence $R^{[\boldsymbol{\tau}]}$ property $(\lambda_t, \boldsymbol{\xi})_q$. □

Lemma 4.4. *Suppose $R \in \mathbb{C}_q[\mathbf{X}]$, $\boldsymbol{\xi} \in \mathbb{C}_q^n$. Let $\boldsymbol{\alpha}$ be a zero of R, and $|\boldsymbol{\alpha} - \boldsymbol{\xi}| \leq \chi$. Then for some $\boldsymbol{\tau}, t$ with $0 \leq t \leq \langle \boldsymbol{\tau}\rangle$, the polynomial $R^{[\boldsymbol{\tau}]}$ has the $(\chi_t, \boldsymbol{\xi})_q$-property where χ_t is given by* (4.1).

Proof. $\boldsymbol{\alpha}' := \boldsymbol{\alpha} - \boldsymbol{\xi}$ is a zero of $Q(\mathbf{X}) = R(\mathbf{X}+\boldsymbol{\xi})$. By Proposition 4.2 there are $\boldsymbol{\tau}, t$ with $0 \leq t \leq \langle \boldsymbol{\tau}\rangle$ such that $Q^{[\boldsymbol{\tau}]}$ has the $(\chi_t)_q$-property, hence $R^{[\boldsymbol{\tau}]}$ the $(\chi_t, \boldsymbol{\xi})_q$-property. □

Remark 4.5. Many results of Sections 3 and 4 can be generalized to rapidly converging power series $Q \in \mathbb{C}_q[[\mathbf{X}]]$, for instance, series $Q = Q_0 + Q_1 + \cdots$ having $|Q_j| \ll 2^{-cj^3}$ for some $c > 1/2$. When $Q \in \mathbb{R}[[\mathbf{X}]]$ is such a series and $\boldsymbol{\alpha}$ a complex zero, then some $Q^{[\boldsymbol{\tau}]}$ has a real zero $\boldsymbol{\beta}$ with $|\boldsymbol{\beta}| \leq c_1(n, \langle \boldsymbol{\tau}\rangle)|\boldsymbol{\alpha}|$. The series has to converge very rapidly, for this clearly does not hold for $e(X) = 1 + X + X^2/2! + \cdots$. One also can show that if $Q \in \mathbb{R}[[X]]$ is rapidly converging and $0 < |Q_0|/|Q_1| < \lambda$, then some $Q^{[\tau]} \neq 0$ has a *nonzero* real root β with $|\beta| \leq c_2(\tau)\lambda$.

5. Proof of Theorem 1.2

We will in fact prove a stronger version:

Theorem 5.1. *Let $\mathcal{M}$ be as in Theorem* 1.2*, and set $\mu = |\mathcal{M}|$.*

(i) *When $n \geq q$, $\mu = n+1$, $\boldsymbol{\xi} \in \mathbb{C}_q^n$ and $N \geq 1$, there is a $P \in \mathcal{P}(\mathcal{M})$ for which $A(P)$ is $(n-1; N^{1-\mu/q}|P|^{-1}, c_1 N^{1-\mu/q}|P|^{-1})_q$-close to $\boldsymbol{\xi}$.*

(ii) *When $\mu = n+2$, $\boldsymbol{\xi} \in \mathbb{R}^n$ but $\boldsymbol{\xi} \notin A(P)$ for any $P \in \mathcal{P}(\mathcal{M})$, then there are infinitely many $P \in \mathcal{P}(\mathcal{M})$ with $A(P)$ being $(n-1; |P|^{-\mu}, c_2|P|^{-\mu})_1$-close to $\boldsymbol{\xi}$.*

Constants $c_1, c_2, \ldots$ of this section depend on $\boldsymbol{\xi}$ and $\mathcal{M}$. Now suppose $\mathcal{M} = \{M_1, \ldots, M_\mu\}$, and for $\mathbf{u} = (u_1, \ldots, u_\mu) \in \mathbb{Z}^\mu$ set

$$Q(\mathbf{u}, \mathbf{X}) = u_1 M_1(\mathbf{X}) + \cdots + u_\mu M_\mu(\mathbf{X}),$$
$$Q_j(\mathbf{u}, \mathbf{X}) = (\partial/\partial X_j) Q(\mathbf{u}, \mathbf{X}) \quad (j = 1, \ldots, n).$$

Let $L_0, L_1, \ldots, L_n$ be the following linear forms in $\mathbf{u}$:

$$L_0(\mathbf{u}) = Q(\mathbf{u}, \boldsymbol{\xi}), \qquad L_j(\mathbf{u}) = Q_j(\mathbf{u}, \boldsymbol{\xi}) \quad (j = 1, \ldots, n).$$

Lemma 5.2. *Suppose $\mathcal{M}$ is as in the theorem, and $\boldsymbol{\xi} \in \mathbb{C}_q^n$ with $M_i(\boldsymbol{\xi}) \neq 0$ $(i = 1, \ldots, \mu)$. Then $L_0, L_1, \ldots, L_n$ are linearly independent.*

Proof. Suppose $M_i(\mathbf{X}) = X_1^{a_{i1}} \cdots X_n^{a_{in}} = \mathbf{X}^{\mathbf{a}_i}$, so that $\boldsymbol{\xi}^{\mathbf{a}_i} \neq 0$ $(i = 1, \ldots, \mu)$. We need to consider the coefficient matrix of the linear forms $L_0, L_1, \ldots, L_n$. After

multiplying the row coming from L_j by ξ_j for $j = 1, \dots, n$, and dividing the column of coefficients of u_i by $\boldsymbol{\xi}^{\mathbf{a}_i}$ $(i = 1, \dots, \mu)$, the matrix becomes

$$\begin{pmatrix} 1 & \cdots & 1 \\ a_{11} & \cdots & a_{\mu 1} \\ \cdots & & \cdots \\ a_{1n} & \cdots & a_{\mu n} \end{pmatrix}.$$

If the forms were dependent, the matrix would have rank $\le n$, and there would be $\mu - n$ independent linear dependence relations of the columns. (By our hypothesis, $\mu - n$ is 1 or 2). Say $b_{j1} + \cdots + b_{j\mu} = 0$, $b_{j1}\mathbf{a}_1 + \cdots + b_{j\mu}\mathbf{a}_\mu = \mathbf{0}$ with vectors $\mathbf{b}_j = (b_{j1}, \dots, b_{j\mu})$ $(j = 1, \dots, \mu - n)$. This gives $\mu - n$ independent relations

$$b_{j1}(\mathbf{a}_1 - \mathbf{a}_\mu) + \cdots + b_{j,\mu-1}(\mathbf{a}_{\mu-1} - \mathbf{a}_\mu) = \mathbf{0} \quad (j = 1, \dots, \mu - n)$$

for $\mathbf{a}_1 - \mathbf{a}_\mu, \dots, \mathbf{a}_{\mu-1} - \mathbf{a}_\mu$. Therefore the group generated by $M_1/M_\mu, \dots, M_{\mu-1}/M_\mu$, hence the group generated by all quotients M_i/M_k $(1 \le i, k \le \mu)$ would have rank $\le (\mu - 1) - (\mu - n) = n - 1$, contradicting the hypothesis. □

We now turn to part (i) of the theorem. Suppose first that each $M_i(\boldsymbol{\xi}) \ne 0$, so that Lemma 5.2 applies. By Lemma 2.1, there is for $H \ge 1$ an $\mathbf{u} \ne 0$ in $\mathbb{Z}^\mu$ with $|\mathbf{u}| \le H$, $|L_0(\mathbf{u})| \le c_3 H^{1-\mu/q}$. Since $L_0, \dots, L_n$ are linearly independent, $\max(|L_0(\mathbf{u})|, \dots, |L_n(\mathbf{u})|) \ge c_4|\mathbf{u}| > 0$. When H is large, say $H \ge c_5$, then $|L_0(\mathbf{u})| < c_4|\mathbf{u}|$, so that $|L_0(\mathbf{u})|$ may be deleted from the maximum. We then may infer that $Q(\mathbf{X}) := Q(\mathbf{u}, \mathbf{X})$ has $|Q| \le c_6|\mathbf{u}| \le c_6 H$ and

$$|Q(\boldsymbol{\xi})| \le c_3 H^{1-\mu/q}, \quad |\operatorname{grad} Q(\boldsymbol{\xi})| \ge c_4|\mathbf{u}| \ge c_7|Q|.$$

Thus

$$|Q(\boldsymbol{\xi})|/|\operatorname{grad} Q(\boldsymbol{\xi})| \le c_8 H^{1-\mu/q}/|Q| = \lambda,$$

say. We may take c_5, c_6, c_8 to be ≥ 1. By Lemma 3.9 there is a $\boldsymbol{\sigma}$ such that $Q^{[\boldsymbol{\sigma}]}$ has property $(\lambda_{\langle\boldsymbol{\sigma}\rangle}, \boldsymbol{\xi})_q$. Since $|Q^{[\boldsymbol{\sigma}]}| \le 2^d|Q|$ where d is a bound on the degrees of the monomials in $\mathcal{M}$, there is an $r \in \mathbb{N}$ with $2^{d-1}|Q| < r|Q^{[\boldsymbol{\sigma}]}| \le 2^d|Q|$. Setting $P = r\mathbf{X}^{\boldsymbol{\sigma}} Q^{[\boldsymbol{\sigma}]}$, we have $P \in \mathcal{P}(\mathcal{M})$ with $|Q| < |P| \le 2^d|Q| \le c_6 2^d H$ and

$$H^{1-\mu/q}/|P| \le H^{1-\mu/q}/|Q| \le \lambda \le \lambda_{\langle\boldsymbol{\sigma}\rangle} \le c_9\lambda \le c_{10} H^{1-\mu/q}/|P|.$$

When $N > c_5 c_6 2^d$, we apply the above with $H = N/(c_6 2^d) > c_5$. Then $|P| \le N$ and $N^{1-\mu/q}/|P| \le \lambda_{\langle\boldsymbol{\sigma}\rangle} \le c_{11} N^{1-\mu/q}/|P|$. So by Remark 3.1 we may conclude that $A(P)$ is $(n-1; N^{1-\mu/q}/|P|, c_1 N^{1-\mu/q}/|P|)_q$-close to $\boldsymbol{\xi}$, with $c_1 = (n+1)c_{11}$.

When $1 \le N \le c_5 c_6 2^d$, let $M \in \mathcal{M}$ be nonconstant. Then $P = M$ has $|P| = 1 \le N$, and $A(P)$ contains a coordinate hyperplane, say $x_n = 0$. By Remark 3.2, $A(P)$ is $(n-1; 1, 1 + (\boldsymbol{\xi}|)_q$-close to $\boldsymbol{\xi}$, hence $(n-1; N^{1-\mu/q}, c_1 N^{1-\mu/q})_q$-close, with $c_1 = c_1(\boldsymbol{\xi}, \mathcal{M})$.

Finally, when some $M_i(\boldsymbol{\xi}) = 0$, then some $\xi_j = 0$, say $\xi_n = 0$. The assertion holds with $P = X_n$, since now $A(P)$ is $(n-1; \rho, \rho)_q$-close to $\boldsymbol{\xi}$ for any $\rho > 0$.

Let us proceed to part (ii). Since $\boldsymbol{\xi} \notin A(P)$ for any $P \in \mathcal{P}(\mathcal{M})$, we have each $M_i(\boldsymbol{\xi}) \ne 0$, so that $L_0, \dots, L_n$ are independent by Lemma 5.2. Since the number of variables $u_1, \dots, u_\mu$ is $\mu = n + 2$, we may apply the Theorem of [7] (with n, μ playing the respective roles of m, n in [7]). Accordingly, there are infinitely many integer points $\mathbf{u} \ne \mathbf{0}$ with

$$|L_0(\mathbf{u})| \le c_{12} \max(|L_1(\mathbf{u})|, \dots, |L_n(\mathbf{u})|)|\mathbf{u}|^{-\mu}.$$

But now $Q(\mathbf{X}) := Q(\mathbf{u}, \mathbf{X})$ has $|Q| \le c_{13}|\mathbf{u}|$ and $|Q(\boldsymbol{\xi})| \le c_{12}|\operatorname{grad} Q(\boldsymbol{\xi})||\mathbf{u}|^{-\mu}$, hence

$$|Q(\boldsymbol{\xi})|/|\operatorname{grad} Q(\boldsymbol{\xi})| \le c_{14}|Q|^{-\mu} = \lambda,$$

say. We may suppose $c_{14} \ge 1$. By Lemma 3.9, some $Q^{[\boldsymbol{\sigma}]}$ has property $(\lambda_{\langle\boldsymbol{\sigma}\rangle}, \boldsymbol{\xi})_1$. Pick $P = r\mathbf{X}^{\boldsymbol{\sigma}}Q^{[\boldsymbol{\sigma}]}$ with $|Q| < |P| \le 2^d|Q|$. Again, P has property $(\lambda_{\langle\boldsymbol{\sigma}\rangle}, \boldsymbol{\xi})_1$, where now $|P|^{-\mu} \le \lambda_{\langle\boldsymbol{\sigma}\rangle} \le c_{15}|P|^{-\mu}$. Therefore $A(P)$ is $(n-1; |P|^{-\mu}, c_2|P|^{-\mu})_1$-close to $\boldsymbol{\xi}$.

Our argument gives infinitely many distinct polynomials Q, and since $|P| > |Q|$, also infinitely many distinct polynomials P.

6. Quoting a result of Laurent and Roy

Their work is done in a projective setting. When θ, ζ are in $\mathbb{P}_q^n = \mathbb{P}^n(\mathbb{C}_q)$, the symbols $\boldsymbol{\theta}, \boldsymbol{\zeta}$ will denote representatives $(\theta_0, \dots, \theta_n)$, $(\zeta_0, \dots, \zeta_n)$ in $\mathbb{C}_q^{n+1}\backslash\{\mathbf{0}\}$. When $\widehat{P} \in \mathbb{Z}[X_0, \dots, X_n]$ is homogeneous, we will say that θ is a zero if $\widehat{P}(\boldsymbol{\theta}) = 0$. The distance of θ, ζ is defined to be

$$\operatorname{dist}(\theta, \zeta) := |\boldsymbol{\theta} \wedge \boldsymbol{\zeta}|/(|\boldsymbol{\theta}||\boldsymbol{\zeta}|)$$

where $\boldsymbol{\theta} \wedge \boldsymbol{\zeta}$ is the wedge product, with components $\theta_i\zeta_j - \theta_j\zeta_i$ $(0 \le i < j \le n)$.

We will be dealing with the case of Laurent and Roy's work [8] where $K = \mathbb{Q}$, and $|\cdot|_w$ is the standard archimedean absolute value, so that $\varepsilon_w = \lambda_w = 1$ and $\mathbb{C}_w = \mathbb{C}$. Moreover, in our case $Z = \mathbb{P}_q^n$, so that $\dim Z = n$, $\deg Z = 1$, $h(Z) = 0$. Furthermore, $S_n = S$ and $D_n = D$ will be fixed, and we will write n, t, d in place of m, n, D in [8]. Also, we will use the Height $|P|$ rather than $h(P) = \log|P|$, and our N_t corresponds to $\exp T_n$ in [8].

By $\widehat{\mathcal{P}}(d)$, where $d > 0$, we will denote the set of nonzero homogeneous polynomials in $\mathbb{Z}[X_0, \dots, X_n]$ of degree d. We will use a rather special case of Theorem 2.2 of [8], which may be formulated as follows.

Proposition 6.1. *Suppose $\zeta \in \mathbb{P}^n(\mathbb{C})$, and $\boldsymbol{\zeta}$ is a representative with $|\boldsymbol{\zeta}| = 1$. Let p, q, S, d be positive integers with $p < q$, $S \le d$. Suppose N_t, V_t for $p \le t \le q$ are reals with $1 < N_p \le \dots \le N_q$ and $2S < V_p \le \dots \le V_q$. Set*

$$\begin{aligned}
\eta &= d^{-1}\log m = d^{-1}\log\binom{n+d}{d},\\
a &= (n+1)(3\log(n+1) + (n+4)\eta + \log 4),\\
b &= n+1,\\
c &= (n+1)V_p/S + (n+4)\eta + 2\log(n+1),
\end{aligned}$$

and assume that

$$S^nV_q > (ad + b\log N_q + cS)d^n. \tag{6.1}$$

Suppose that for $t = p+1, \dots, q$ there is a polynomial $\widehat{P}_t \in \widehat{\mathcal{P}}(d)$ with $|\widehat{P}_t| \le N_t$ and

$$|\widehat{P}_t^{[\boldsymbol{\sigma}]}(\boldsymbol{\zeta})| \le |\widehat{P}_t|\exp(-V_t + (\langle\boldsymbol{\sigma}\rangle/S)V_t) \tag{6.2}$$

for every $\boldsymbol{\sigma} \in \mathbb{N}_\circ^{n+1}$ with $\langle\boldsymbol{\sigma}\rangle < S$.

Then for some t, $p < t \le q$, $\widehat{P}_t$ has a zero $\theta \in \mathbb{P}^n(\mathbb{C})$ with

$$\operatorname{dist}(\theta, \zeta) \le \exp(-V_{t-1}/S).$$

The following already appears implicitly in the proof of Theorem 2.3 of [8].

Corollary 6.2. *Suppose $\zeta \in \mathbb{P}^n(\mathbb{C})$, with representative $\boldsymbol{\zeta}$ having $|\boldsymbol{\zeta}| = 1$. Let p, S, d be positive integers with $S \le d$. Further, suppose that $1 < N_p \le N_{p+1} \le \cdots$ and $2S < V_p \le V_{p+1} \le \cdots$, and that V_t as well as*

$$S^n V_t - (n+1)d^n \log N_t \tag{6.3}$$

tends to infinity with t. Finally, suppose that for each $t > p$ there is a polynomial $\widehat{P}_t \in \widehat{\mathcal{P}}(d)$ with $|\widehat{P}_t| \le N_t$ and

$$|\widehat{P}_t(\boldsymbol{\zeta})| \le |\widehat{P}_t| \exp(-V_t).$$

Then for some $t > p$, $\widehat{P}_t$ has a zero $\theta \in \mathbb{P}^n(\mathbb{C})$ with

$$\operatorname{dist}(\theta, \zeta) < \exp((2d^2 - V_{t-1})/S). \tag{6.4}$$

Proof. Assume that this were false. Then for every zero ζ of $\widehat{P}_t$, $\operatorname{dist}(\theta, \zeta) \ge \rho_t$, where ρ_t is the right-hand side of (6.4). Suppose $\boldsymbol{\sigma} \in \mathbb{N}_\circ^{n+1}$, $0 < \langle \boldsymbol{\sigma} \rangle < S$. By Lemma 6.8 of [8],

$$\begin{aligned} |\widehat{P}_t^{[\boldsymbol{\sigma}]}(\boldsymbol{\zeta})| &\le \binom{d}{\langle \boldsymbol{\sigma} \rangle} \left(\frac{\rho_r}{2}\right)^{-\langle \boldsymbol{\sigma} \rangle} |\widehat{P}_t(\boldsymbol{\zeta})| \\ &< 2^{d+S} |\widehat{P}_t| \exp(-V_t + (\langle \boldsymbol{\sigma} \rangle / S)(V_{t-1} - 2d^2)) \\ &\le |\widehat{P}_t| \exp(-V_t + (\langle \boldsymbol{\sigma} \rangle / S) V_t) \end{aligned}$$

since $2^{d+S} \le 2^{2d} < \exp(2d) \le \exp(2d^2/S)$. So we have (6.2) for every $\boldsymbol{\sigma}$ with $\langle \boldsymbol{\sigma} \rangle < S$. The quantities a, b, c of Proposition 6.1 are independent of t, so that

$$(ad + b \log N_t + cS)d^n = bd^n \log N_t + O(1) = (n+1)d^n \log N_t + O(1)$$

as $t \to \infty$, and the divergence of (6.3) implies that (6.1) will hold for some $q > p$. By the Proposition, there will be a t, $p < t \le q$, such that $\widehat{P}_t$ has a zero $\theta \in \mathbb{P}_q^n$ with

$$\operatorname{dist}(\theta, \zeta) \le \exp(-V_{t-1}/S) < \exp((2d^2 - V_{t-1})/S).$$

So the desired conclusion is true after all. □

As in (1.5), (1.6), set $m = \binom{n+d}{d}$ and $\nu = ((m/q) - 1)/S_\circ$ where $S_\circ = \lceil \sqrt[n]{q(n+1)!} \rceil$.

Proposition 6.3. *Suppose $n > 1$. Let $\zeta \in \mathbb{P}_q^n$ not be a zero of any $\widehat{P} \in \widehat{\mathcal{P}}(d)$. Then there are infinitely many $\widehat{P} \in \widehat{\mathcal{P}}(d)$ having a zero $\theta \in \mathbb{P}_q^n$ with*

$$\operatorname{dist}(\theta, \zeta) < c_1(n, d) |\widehat{P}|^{-\nu}. \tag{6.5}$$

This is essentially Theorem 2.3 of [8] in a very special case, but with a slightly better exponent in the estimate.

Proof. Set $N_t = e^t$. Since polynomials in $\widehat{\mathcal{P}}(d)$ are linear combinations of m monomials, we see from Lemma 1.1 that for a given representative $\boldsymbol{\zeta}$ of ζ with $|\boldsymbol{\zeta}| = 1$, there is for each $t > 0$ a $\widehat{P} \in \widehat{\mathcal{P}}(d)$ with $|\widehat{P}| \le N_t$ and

$$|\widehat{P}_t(\boldsymbol{\zeta})| < 2m N_t^{-((m/q)-1)} < |\widehat{P}_t| \exp(-V_t)$$

where $V_t = ((m/q) - 1) \log N_t - 2m$, so that by (1.8), $V_t \ge ((d^n/qn!) + \varepsilon) \log N_t$ for some $\varepsilon > 0$ and for $t > c_2(n, d, \varepsilon)$. Observe that

$$V_t - V_{t-1} = ((m/q) - 1) < m.$$

Setting $S = S_\circ$ we have for large t,

$$S^n V_t \geq q(n+1)!((d^n/qn!) + \varepsilon) \log N_t,$$

so that (6.3) tends to infinity with t. From Corollary 6.2 we may infer that for any $p \in \mathbb{N}$ there is a $t > p$ such that $\widehat{P}_t$ has a zero $\zeta \in \mathbb{P}^n(\mathbb{C})$ with

$$\operatorname{dist}(\theta, \zeta) < \exp((2d^2 - V_{t-1})/S_\circ).$$

But

$$V_{t-1} - 2d^2 > V_t - c_3(n,d) \geq ((m/q) - 1) \log N_t - c_4(n,d),$$

yielding

$$\operatorname{dist}(\theta, \zeta) < \exp(-\nu \log N_t + c_5(n,d)) < c_1(n,d) N_t^{-\nu} \leq c_1(n,d) |\widehat{P}_t|^{-\nu}.$$

The proposition follows. □

7. Proof of Theorem 1.3

Lemma 7.1. *Suppose $n > 1$. Also suppose $\boldsymbol{\xi} \in \mathbb{C}_q^n$, $|\boldsymbol{\xi}| \leq \psi$, and $P(\boldsymbol{\xi}) \neq 0$ for every $P \in \mathcal{P}(d)$. Then there are infinitely many polynomials $P \in \mathcal{P}(d)$ with*

$$\delta(\boldsymbol{\xi}, A(P)) < c_1(n,d,\psi)|P|^{-\nu},$$

where ν is given by (1.6).

When $q = 2$, so that $\delta(\boldsymbol{\xi}, A(P)) = \delta_q(\boldsymbol{\xi}, A(P))$, this is the assertion of Theorem 1.3. But when $q = 1$, $\boldsymbol{\xi} \in \mathbb{C}_1^n = \mathbb{R}^n$, we can at this stage only conclude that there is an $\boldsymbol{\alpha} \in \mathbb{C}^n$ with $P(\boldsymbol{\alpha}) = 0$ and $|\boldsymbol{\alpha} - \boldsymbol{\xi}| < c_1(n,d,\psi)|P|^{-\nu}$, whereas the theorem asserts the existence of an $\boldsymbol{\alpha} \in \mathbb{R}^n$ with this property.

Proof. Let $\zeta \in \mathbb{P}_q^n$ be the point represented by $\boldsymbol{\zeta} = (\zeta_0, \zeta_1, \ldots, \zeta_n) := (1, \xi_1, \ldots, \xi_n)$. Then $\widehat{P}(\zeta) \neq 0$ for $\widehat{P} \in \widehat{\mathcal{P}}(d)$. By Proposition 6.3 there are infinitely many $\widehat{P} \in \widehat{\mathcal{P}}(d)$ having a zero $\theta \in \mathbb{P}^n(\mathcal{C})$ with (6.5).

Let $\boldsymbol{\theta} = (\alpha_0, \alpha_1, \ldots, \alpha_n)$ be a representative of θ. Then

$$|\alpha_i \zeta_j - \alpha_j \zeta_i| < |\boldsymbol{\theta}||\boldsymbol{\zeta}| c_2(n,d) |\widehat{P}|^{-\nu} = |\boldsymbol{\theta}| E,$$

say, for $0 \leq i < j \leq n$. Since $\zeta_0 = 1$, we have $|\alpha_j - \zeta_j \alpha_0| < |\boldsymbol{\theta}| E$ for $1 \leq j \leq n$, hence $|\boldsymbol{\theta}| < |\boldsymbol{\zeta}||\alpha_0| + |\boldsymbol{\theta}| E < (1+\psi)|\alpha_0| + \frac{1}{2}|\boldsymbol{\theta}|$ when $|\widehat{P}|$ is large, hence E small. Then $|\boldsymbol{\theta}| < 2(1+\psi)|\alpha_0|$, so that $\alpha_0 \neq 0$. After renormalization we may suppose that $\alpha_0 = 1$, hence $|\boldsymbol{\theta}| < 2 + 2\psi$, yielding

$$|\alpha_j - \xi_j| = |\alpha_j - \zeta_j| < (2 + 2\psi) E \leq c_1(n,d,\psi) |\widehat{P}|^{-\nu} \quad (1 \leq j \leq n).$$

The polynomial $P(\mathbf{X}) = \widehat{P}(1, X_1, \ldots, X_n)$ has the zero $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$ with $|\boldsymbol{\alpha} - \boldsymbol{\xi}| < c_1(n,d,\psi)|P|^{-\nu}$. □

A polynomial $P \in \mathbb{Z}[\mathbf{X}]$ will be called *irreducible* if it is irreducible over $\mathbb{Q}$, i.e., if it cannot be written as the product of two nonconstant polynomials. This does not preclude the possibility that $P = pP_1$ where $p > 1$ is an integer and $P_1 \in \mathbb{Z}[\mathbf{X}]$. We will establish Theorem 1.3 in the following stronger form.

Theorem 7.2. *Suppose $n > 1$, and further $\boldsymbol{\xi} \in \mathbb{C}_q^n$, $|\boldsymbol{\xi}| \leq \psi$, with $P(\boldsymbol{\xi}) \neq 0$ for any $P \in \mathcal{P}(d)$. Then there are infinitely many irreducible polynomials $P \in \mathcal{P}(d)$ for which $A(P)$ is $(n-1; |P|^{-\nu}, c_3|P|^{-\nu})_q$-close to $\boldsymbol{\xi}$.*

The constant c_3 and subsequent constants $c_4, \ldots$ depend at most on n, d, ψ.

Proof. When $Q = Q_1Q_2$ is of degree at most d, and Q_1, Q_2 in $\mathbb{Z}[\mathbf{X}]$, then it is well known that $|Q_i| \le c_4|Q|$ $(i = 1, 2)$. Set $c_5 = 2^d c_4$. Let R be one of the infinitely many polynomials in the conclusion of the preceding lemma, so that R has a zero $\boldsymbol{\alpha} \in \mathbb{C}^n$ with $|\boldsymbol{\alpha} - \boldsymbol{\xi}| < c_1|R|^{-\nu} = \chi$, say. We may choose c_1 to be ≥ 1. By Lemma 4.4, some $R^{[\boldsymbol{\tau}]}$ has the $(\chi_t, \boldsymbol{\xi})_q$-property where $0 \le t \le \langle \boldsymbol{\tau} \rangle \le d$ and χ_t is given by (4.1). By Remark 3.4 some irreducible factor Q of $R^{[\boldsymbol{\tau}]}$ also has this property. Here $|Q| \le c_4|R^{[\boldsymbol{\tau}]}| \le 2^d c_4|R| = c_5|R|$. For suitable $r \in \mathbb{N}$ the polynomial $P = rQ$ has $|R| < |P| < c_5|R|$, and still the $(\chi_t, \boldsymbol{\xi})_q$-property. Here

$$|P|^{-\nu} < |R|^{-\nu} \le \chi \le \chi_t \le c_6\chi = c_1c_6|R|^{-\nu} < c_7|P|^{-\nu}.$$

By Remark 3.1, $A(P)$ is indeed $(n-1; |P|^{-\nu}, c_3|P|^{-\nu})_q$-close to $\boldsymbol{\xi}$. □

When $n = 1$, the present method would have given Theorem 1.3 with $\nu = d/3$ when $q = 1$ and $\nu = (d-1)/8$ when $q = 2$. But as pointed out in the Introduction, Wirsing's method gives better exponents.

8. Proof of Theorem 1.5

Given $N \ge 1$, set

$$M = N^{(m/q)-1}, \qquad M_\circ = N^{m_\circ - 1}. \tag{8.1}$$

We will establish the stronger

Theorem 8.1. (a) *Suppose* $\boldsymbol{\xi} \in \mathbb{C}_q^n$, $\psi \ge |\boldsymbol{\xi}|$. *Then for* $N \ge 1$ *there is an irreducible* $P \in \mathcal{P}(d)$ *with* $|P| \le N$ *and with* $A(P)$ $(n-1; (M|P|)^{-1/d}, c_1(M|P|)^{-1/d})_q$*-close to* $\boldsymbol{\xi}$.

(b) *Suppose* $\boldsymbol{\xi} \in \mathbb{R}^n$, $\psi \ge |\boldsymbol{\xi}|$. *Then for* $N \ge 1$ *there is an* ℓ, $1 \le \ell \le d$, *and an irreducible* $P \in \mathcal{P}(d)$ *with* $|P| \le N^{1/\ell}$ *such that* $A(P)$ *is* $(n-1; (M_\circ|P|)^{-1/\ell}$, $c_2(M_\circ|P|)^{-1/\ell})_1$*-close to* $\boldsymbol{\xi}$.

The constants $c_1, c_2, \ldots$ of this section depend only on n, d, ψ. The polynomial in (a) (respectively (b)) has $\delta_q(\boldsymbol{\xi}, A(P)) \le c_3(M|P|)^{-1/d}$ (respectively $\delta_1(\boldsymbol{\xi}, A(P)) \le c_4(M_\circ|P|)^{-1/\ell}$), so that Theorem 1.5 follows.

Proof of Theorem 8.1. (a) Every factor Q of a derivative $R^{[\boldsymbol{\tau}]}$ of a polynomial $R \in \mathcal{P}(d)$ has $|Q| \le c_5|R^{[\boldsymbol{\tau}]}| \le c_6|R|$. Suppose $N > c_6$. By Lemma 1.1 there is an $R \in \mathcal{P}(d)$ with $|R| \le N/c_6$ and

$$|R(\boldsymbol{\xi})| \le m\psi'^d (N/c_6)^{1-m/q}$$

where $\psi' = \max(1, \psi)$. Taylor expansion at $\boldsymbol{\xi}$ gives

$$R(\mathbf{X}) = \sum_{\boldsymbol{\sigma}} (\mathbf{X} - \boldsymbol{\xi})^{\boldsymbol{\sigma}} R^{[\boldsymbol{\sigma}]}(\boldsymbol{\xi}),$$

yielding

$$R^{[\boldsymbol{\tau}]}(\mathbf{0}) = \sum_{\boldsymbol{\sigma}}{}' \binom{\sigma_1}{\tau_1} \cdots \binom{\sigma_n}{\tau_n} (-\boldsymbol{\xi})^{\boldsymbol{\sigma}-\boldsymbol{\tau}} R^{[\boldsymbol{\sigma}]}(\boldsymbol{\xi}),$$

where $\sum'$ signifies a sum over $\boldsymbol{\sigma}$ with $\sigma_i \ge \tau_i$ $(i = 1, \ldots, n)$. We obtain

$$|R| = \max_{\boldsymbol{\tau}} |R^{[\boldsymbol{\tau}]}(\mathbf{0})| \le c_7 \max_{\boldsymbol{\sigma}} |R^{[\boldsymbol{\sigma}]}(\boldsymbol{\xi})|.$$

When N is so large that $c_7 m\psi'^d(N/c_6)^{1-m/q} < 1$, there will be a $\boldsymbol{\sigma} \neq \mathbf{0}$ with $|R^{[\boldsymbol{\sigma}]}(\boldsymbol{\xi})| \geq |R|/c_7$. Then setting $\ell = \langle \boldsymbol{\sigma} \rangle$ we have

$$(|R(\boldsymbol{\xi})|/|R^{[\boldsymbol{\sigma}]}(\boldsymbol{\xi})|)^{1/\ell} \leq c_8(M|R|)^{-1/\ell} \leq c_8(M|R|)^{-1/d} = \lambda, \tag{8.2}$$

say, where c_8 may be taken to be ≥ 1. By Lemma 4.3 there are $\boldsymbol{\tau}, t$ such that $R^{[\boldsymbol{\tau}]}$ has property $(\lambda_t, \boldsymbol{\xi})_q$. By an argument used in the proof of Theorem 1.3, there is an irreducible polynomial $P \in \mathcal{P}(d)$ with $|R| < |P| \leq c_6|R| \leq N$ which also has property $(\lambda_t, \boldsymbol{\xi})_q$. Now

$$(M|P|)^{-1/d} \leq \lambda \leq \lambda_t \leq c_9\lambda \leq c_{10}(M|P|)^{-1/d},$$

so that by Remark 3.1, $A(P)$ is $(n-1; (M|P|)^{-1/d}, c_1(M|P|)^{-1/d})_q$-close to $\boldsymbol{\xi}$.

We are left with the case when N is small, say $1 \leq N \leq c_{11}$, hence $1 \leq M \leq c_{12}$. We set $P(\mathbf{X}) = X_n$, so that $|P| = 1$. By Remark 3.2, $A(P)$ is $(n-1; 1, 1+|\boldsymbol{\xi}|)_q$-close to $\boldsymbol{\xi}$, hence $(n-1; M^{-1/d}, c_1M^{-1/d})_q$-close.

(b) For $R \in \mathcal{P}(d)$ set

$$\omega_j(R) = \max_{\langle \boldsymbol{\sigma} \rangle = j} |R^{[\boldsymbol{\sigma}]}(\boldsymbol{\xi})| \quad (1 \leq j \leq d).$$

There are $\binom{n+j-1}{j}$ n-tuples $\boldsymbol{\sigma}$ with $\langle \boldsymbol{\sigma} \rangle = j$, so that

$$\sum_{j=1}^{d} \sum_{\langle \boldsymbol{\sigma} \rangle = j} 1/j = m_\circ - 1.$$

By Minkowski's Theorem on linear forms, there is for $N > 0$ an $R \in \mathcal{P}(d)$ with

$$\begin{aligned} &\omega_j(R) \leq N^{1/j} \quad (1 \leq j \leq d), \\ &|R(\boldsymbol{\xi})| \leq c_{13}N^{1-m_0}. \end{aligned} \tag{8.3}$$

Since each coefficient of R is a linear combination of the $R^{[\boldsymbol{\sigma}]}(\boldsymbol{\xi})$, the Height $|R| \leq c_{14} \max(|R(\boldsymbol{\xi})|, \omega_1(R), \ldots, \omega_d(R))$, where in view of (8.3) we may omit $|R(\boldsymbol{\xi})|$ from this maximum when N is large. There is then an ℓ, $1 \leq \ell \leq d$ with $|R| \leq c_{14}\omega_\ell(R)$, say $|R| \leq c_{14}|R^{[\boldsymbol{\sigma}]}(\boldsymbol{\xi})| \leq c_{14}N^{1/\ell}$, and

$$(|R(\boldsymbol{\xi})|/|R^{[\boldsymbol{\sigma}]}(\boldsymbol{\xi})|)^{1/\ell} \leq c_{15}(M_\circ|R|)^{-1/\ell} =: \lambda.$$

From here on, the argument is very much as in (a).

One obtains an irreducible polynomial $P \in \mathcal{P}(d)$ with $|R| < |P| \leq c_6|R| \leq c_{16}N^{1/\ell}$ with property $(\phi, \boldsymbol{\xi})_q$, where $(M_\circ|P|)^{-1/\ell} \leq \phi \leq c_{17}(M_\circ|P|)^{-1/\ell}$.

When $N \geq (2c_{16})^d$, we repeat the above argument with $N' = N/(2c_{16})^d$, to obtain an irreducible $P \in \mathcal{P}(d)$ with $|P| \leq N$ and property $(\phi, \boldsymbol{\xi})_q$, where $(M_\circ|P|)^{-1/\ell} \leq \phi \leq c_{18}(M_\circ|P|)^{-1/\ell}$, so that $A(P)$ is indeed $(n-1; (M_\circ|P|)^{-1/\ell}, c_2(M_\circ|P|)^{-1/\ell})_q$-close to $\boldsymbol{\xi}$. When $N < (2c_{16})^d$, one reasons as at the end of part (a). □

9. Proof of Theorem 1.7

Constants $c_1, c_2, \ldots$ of this section will depend on $n, \mathbf{d}, \psi$. We will strengthen the theorem to

Theorem 9.1. *There is an $\varepsilon_s = \varepsilon_s(n, \mathbf{d}, \psi)$, $0 < \varepsilon_s \leq 1$, such that Theorem 1.7 holds with the stronger conclusion that $A(\mathbf{P})$ is $(r; \varepsilon_s L^{-1}, L^{-1})_q$-close to $\boldsymbol{\xi}$.*

Proof. We begin with the case $s = 1$, and we write $d = d_1$, $m = m_1$. We start out as in the proof of Theorem 8.1(a). So for large N, there is an $R \in \mathcal{P}(d)$ with (8.2). But (8.2) also holds with λ replaced by $\lambda' = c_1 M^{-1/d}$. Proceeding as with Theorem 8.1(a), we eventually obtain a λ'_t with $M^{-1/d} \le \lambda'_t \le c_2 M^{-1/d}$, and a polynomial $P \in \mathcal{P}(d)$ with $|P| \le N$ and property $(\lambda'_t, \boldsymbol{\xi})_q$. In particular, $A(P)$ is $(n-1; M^{-1/d}, c_3 M^{-1/d})_q$-close to $\boldsymbol{\xi}$, where $c_3 = (n+1)c_2$. On the other hand, when $N \ge 1$ is small, then by Remark 3.2 this holds with $P = X_n$, provided c_3 is chosen sufficiently large.

Let $\phi \ge 1$ be a quantity to be chosen below. Given $L \ge 1$, let $M \ge 1$ be determined by $M^{1/d} = \phi L$, and N by $N^{(m/q)-1} = M$. Then the polynomial P above has $|P| \le N = M^{q/(m-q)} = (\phi L)^{dq/(m-q)}$. Moreover, after suitable numbering of the variables,

(9i) *$A(P)$ is special $(n-1; \phi^{-1}L^{-1}, c_3\phi^{-1}L^{-1})_q$-close to $\boldsymbol{\xi}$,*

(9ii) *$P(\alpha_1, \dots, \alpha_{n-1}, X_n)$ for given $\alpha_1, \dots, \alpha_{n-1}$ is a nonzero polynomial in X_n.*

The assertion of Theorem 9.1 for $s = 1$ follows on setting $\phi = c_3$, and then $\gamma_1 = \phi$, $\varepsilon_1 = \phi^{-1}$.

It remains for us to do the induction step from $s-1$ to s, where $1 < s < n$. Suppose we already know that given $L \ge 1$, there is after suitable numbering of the variables a $\mathbf{P}^* = (P_1, \dots, P_{s-1})$ in $\mathcal{P}(\mathbf{d}^*) = \mathcal{P}(d_1, \dots, d_{s-1})$ with

$$|P_i| \le (\gamma_i L)^{d_i q/(m_i - q)} \qquad (i = 1, \dots, s-1) \tag{9.1}$$

such that

$$A(\mathbf{P}^*) \text{ is special } (r+1; \varepsilon_{s-1}L^{-1}, L^{-1})_q\text{-close to } \boldsymbol{\xi}.$$

Moreover, our induction will be such that for $1 \le i < s$, $P_i(\alpha_1, \dots, \alpha_{n-i}, X_{n-i+1})$ for fixed $\alpha_1, \dots, \alpha_{n-i}$ is a nonzero polynomial in X_{n-i+1}.

There is then a regular map $\mathbf{w}^* : \mathcal{C}(r+1; \varepsilon_{s-1}L^{-1}, \boldsymbol{\xi}) \to \mathcal{C}'(s-1; L^{-1}, \boldsymbol{\xi})$ such that $P_i(\mathbf{z}^*, \mathbf{w}^*(\mathbf{z}^*)) = 0$ $(i = 1, \dots, s-1)$ for $\mathbf{z}^* = (z_1, \dots, z_{r+1}) \in \mathcal{C}(r+1; \varepsilon_{s-1}L^{-1}, \boldsymbol{\xi})$. When $\boldsymbol{\xi} \in \mathbb{C}_1^n = \mathbb{R}^n$, the map is a real regular map.

Suppose $\boldsymbol{\xi} = (\xi_1, \dots, \xi_n)$, and set $\tilde{\boldsymbol{\xi}} = (\xi_1, \dots, \xi_{r+1})$. We are going to apply the argument we gave for the case $s = 1$ to $\tilde{\boldsymbol{\xi}} \in \mathbb{C}_q^{r+1}$. So given $L \ge 1$, there is a $\widetilde{P} \in \mathbb{Z}[X_1, \dots, X_{r+1}]$ of degree $\le d_s$ with

$$|\widetilde{P}| \le (\phi L)^{d_s q/(m_s - q)} \tag{9.2}$$

which after suitable numbering of the variables $X_1, \dots, X_{r+1}$ has the properties corresponding to (9i), (9ii). Thus

$$A(\widetilde{P}) \subset \mathbb{C}_q^{r+1} \text{ is special } (r; \phi^{-1}L^{-1}, \tilde{c}_3\phi^{-1}L^{-1})_q\text{-close to } \tilde{\boldsymbol{\xi}},$$

and $\widetilde{P}(\alpha_1, \dots, \alpha_r, X_{r+1})$ for given $\alpha_1, \dots, \alpha_r$ is a nonzero polynomial in X_{r+1}. We now set $\phi = \tilde{c}_3/\varepsilon_{s-1}$, and then

$$A(\widetilde{P}) \text{ is special } (r; \varepsilon_s L^{-1}, \varepsilon_{s-1}L^{-1})_q\text{-close to } \tilde{\boldsymbol{\xi}}$$

with $\varepsilon_s = \varepsilon_{s-1}/\tilde{c}_3 \le \varepsilon_{s-1}$. There is a regular map

$$\tilde{w} : \widetilde{\mathcal{C}}(r; \varepsilon_s L^{-1}, \tilde{\boldsymbol{\xi}}) \to \widetilde{\mathcal{C}}'(1; \varepsilon_{s-1}L^{-1}, \tilde{\boldsymbol{\xi}})$$

(the tildes referring to sets in $\mathbb{C}^{r+1}$) with $\widetilde{P}(\tilde{\mathbf{z}}, \tilde{w}(\tilde{\mathbf{z}})) = 0$ for $\tilde{\mathbf{z}} = (z_1, \dots, z_r) \in \widetilde{\mathcal{C}}(r; \varepsilon_s L^{-1}, \tilde{\boldsymbol{\xi}})$. Moreover, $\tilde{w}$ is real regular when $\boldsymbol{\xi} \in \mathbb{R}^n$.

Let $\mathbf{w} : \mathcal{C}(r; \varepsilon_s L^{-1}, \boldsymbol{\xi}) \to \mathcal{C}'(s, L^{-1}, \boldsymbol{\xi})$ be given by

$$\mathbf{w}(\mathbf{z}) = (\tilde{w}(\tilde{\mathbf{z}}), \mathbf{w}^*(z_1, \dots, z_r, \tilde{w}(\tilde{\mathbf{z}})).$$

Set $P_s(\mathbf{X}) = \widetilde{P}(X_1, \dots, X_{r+1})$. Then indeed $P_i(\mathbf{z}, \mathbf{w}(\mathbf{z})) = 0$ for $1 \le i \le s$ and $\mathbf{z} \in \mathcal{C}(r; \varepsilon_s L^{-1}, \boldsymbol{\xi})$. Therefore $A(\mathbf{P}) = A(P_1, \dots, P_s)$ is $(r; \varepsilon_s L^{-1}, L^{-1})_q$-close to $\boldsymbol{\xi}$. By (9.1), (9.2), we have (1.16) for $i = 1, \dots, s$ if we set $\gamma_s = \phi$.

For $1 \le i \le s$, and any $\alpha_1, \dots, \alpha_{n-i}$, the polynomial $P_i(\alpha_1, \dots, \alpha_{n-i}, X_{n-i+1})$ in X_{n-i+1} is nonzero. Therefore $A(\mathbf{P})$ has codimension $\ge s$, hence dimension $\le r$. But by the $(r; \varepsilon_s L^{-1}, L^{-1})_q$-closeness, it has in fact dimension r, so that $\mathbf{P} \in \mathcal{P}(\mathbf{d})$. $\square$

10. Proof of Theorem 1.8

Again we will prove a stronger result:

Theorem 10.1. *Let $r, s, \mathbf{d}$, $d = d_1 \cdots d_s$, m_i $(i = 1, \dots, s)$ and Φ, Ψ be as in Theorem 1.8. Then when $\boldsymbol{\xi} \in \mathbb{C}_q^n$, $|\boldsymbol{\xi}| \le \psi$ and $N \ge 1$, there is a variety V of dimension r, degree $\le d$, and Height $H(V) \le N$ which is $(r, N^{-\Psi}, c_1 N^{-\Psi})_q$-close to $\boldsymbol{\xi}$.*

The constants $c_1, c_2, \dots$ of this section, unless indicated otherwise, depend on $n, \mathbf{d}, \psi$.

Proof. Given $L \ge 1$, we know from Theorem 9.1 that after suitable numbering of the coordinates there is a $\mathbf{P} \in \mathcal{P}(\mathbf{d})$ with (1.16), and with $A(\mathbf{P})$ being $(r; \varepsilon_s L^{-1}, L^{-1})_q$-close to $\boldsymbol{\xi}$. Here $P_i = P_i(X_1, \dots, X_{n-i+1})$, and $P_i(\alpha_1, \dots, \alpha_{n-i}, X_{n-i+1})$ for given $\alpha_1, \dots, \alpha_{n-i}$ is a nonzero polynomial in X_{n-i+1} $(1 \le i \le s)$.

Set $e_i = \deg P_i$, so that $e_i \le d_i$, and let $\widehat{P}_i = X_0^{e_i} P_i(X_1/X_0, \dots, X_{n-i+1}/X_0)$ be the homogenized version of P_i. Let $\operatorname{div} \widehat{P}_i$ be the divisor of $\widehat{P}_i$ in $\mathbb{P}^n(\mathbb{C})$, so that $\deg(\operatorname{div} \widehat{P}_i) = \deg P_i = e_i$. The intersection product $(\operatorname{div} \widehat{P}_s) \cdot (\operatorname{div} \widehat{P}_{s-1})$ is a divisor of degree $e_s e_{s-1}$, since $\widehat{P}_{s-1}$ does not vanish identically on any component of $\operatorname{div} \widehat{P}_s$ by the special properties of our polynomials. (For degrees of intersection products see [8, Lemma 4.2].) More generally, $\widehat{P}_i$ for $1 \le i < s$ does not vanish identically on any component of $(\operatorname{div} \widehat{P}_s) \cdot \ \cdots \ \cdot (\operatorname{div} \widehat{P}_{i+1})$. One may conclude by downward induction on i that

$$Z_i := (\operatorname{div} \widehat{P}_s) \cdot \ \cdots \ \cdot (\operatorname{div} \widehat{P}_i) \quad (1 \le i \le s)$$

has degree $e_s \cdots e_i$.

The Height of a cycle $C = t_1 \widehat{V}_1 + \cdots + t_\ell \widehat{V}_\ell$ with projective varieties $\widehat{V}_j$ is $H(C) = H(\widehat{V}_1)^{t_1} \cdots H(\widehat{V}_\ell)^{t_\ell}$, and it is not very hard to see that $H(\operatorname{div} \widehat{P}) \le c_2(n, d) H(\widehat{P}) = c_2(n, d)|P|$ for a polynomial P of degree $\le d$. The logarithmic height is $h = \log H$. Since $Z_i = Z_{i+1} \cdot (\operatorname{div} \widehat{P}_i)$, we may infer, e.g., from [8, Proposition 4.9], that

$$\frac{h(Z_i)}{e_s \cdots e_i} \le \frac{h(Z_{i+1})}{e_s \cdots e_{i+1}} + \frac{h(P_i)}{e_i} + c_3 \quad (1 \le i < s)$$

where $c_3 = c_3(n, e_s, \dots, e_i)$, so that $H(Z_s) \le c_2 |P_s|$,

$$H(Z_i) \le c_4 H(Z_{i+1})^{e_i} |P_i|^{e_s \cdots e_{i+1}} \quad (1 \le i < s),$$

and eventually

$$H(Z_1) \le c_5 |P_1|^{e_s \cdots e_2} |P_2|^{e_s \cdots e_3 e_1} \cdots |P_s|^{e_{s-1} \cdots e_1}.$$

In view of (1.16) this gives

$$H(Z_1) \le c_6 L^{d_1 \cdots d_s q((m_1 - q)^{-1} + \cdots + (m_s - q)^{-1})} = c_6 L^{\Phi}.$$

Here Z_1 is a cycle of dimension $n-s=r$. Write $Z_1 = t_1\widehat{V}_1 + \cdots + t_\ell \widehat{V}_\ell$ where the $\widehat{V}_j$ are projective varieties of dimension r, and let V_j be the affine part of $\widehat{V}_j$.

It is clear that $A(\mathbf{P}) = V_1 \cup \cdots \cup V_\ell$. Let $\mathbf{w} : \mathcal{C}(r, \varepsilon_s L^{-1}, \boldsymbol{\xi}) \to \mathcal{C}'(s, L^{-1}, \boldsymbol{\xi})$ be the regular map arising from the special $(r; \varepsilon_s L^{-1}, L^{-1})_q$-closeness to $\boldsymbol{\xi}$ of $A(\mathbf{P})$. Then $(\mathbf{z}, \mathbf{w}(\mathbf{z})) \in A(\mathbf{P})$ for every $\mathbf{z}$ considered. Since the map is regular, there is some V_j, say $V = V_1$, such that $(\mathbf{z}, \mathbf{w}(\mathbf{z})) \in V$ for every $\mathbf{z} \in \mathcal{C}(r, \varepsilon_s L^{-1}, \boldsymbol{\xi})$. We may conclude that V is $(r; \varepsilon_s L^{-1}, L^{-1})_q$-close to $\boldsymbol{\xi}$. Here $\dim V = r$, $\deg V \le \deg Z_1 = e_s \cdots e_1 \le d$, and $H(V) \le H(Z_1) \le c_6 L^{\Phi}$.

We may enlarge c_6 if necessary, to have $\varepsilon_s c_6^{\Psi} \ge 1$. Now if $N \ge c_6$, determine $L \ge 1$ by $c_6 L^{\phi} = N$. The above argument with this value of L gives V with $H(V) \le N$ which is $(r; \varepsilon_s c_6^{\Psi} N^{-\Psi}, c_6^{\Psi} N^{-\Psi})_q$-close to $\boldsymbol{\xi}$, hence $(r; N^{-\Psi}, c_1 N^{-\Psi})_q$-close with $c_1 = c_6^{\Psi}$.

When $N \le c_6 = c_6(n, \mathbf{d}, \psi)$, let V be given by $x_{r+1} = \cdots = x_n = 0$, so that $\dim V = r$, $\deg V = 1$, $H(V) = 1$. By Remark 3.2, V is special $(r; 1, 1 + |\boldsymbol{\xi}|)_q$-close to $\boldsymbol{\xi}$, hence $(r; N^{-\Psi}, c_1 N^{-\Psi})_q$-close with suitable c_1. $\square$

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