DIOPHANTINE APPROXIMATION BY ALGEBRAIC HYPERSURFACES AND VARIETIES

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Abstract. Questions on rational approximations to a real number can be generalized in two directions. On the one hand, we may ask about "approximation" to a point in $\mathbb{R}^n$ by hyperplanes defined over the rationals. That is, we seek hyperplanes with small distance from the given point. On the other hand, following Wirsing, we may ask about approximation to a real number by real algebraic numbers of degree at most $d$.

The present paper deals with a common generalization of both directions, namely with approximation to a point in $\mathbb{R}^n$ by algebraic hypersurfaces, or more generally algebraic varieties defined over the rationals.

1. Introduction

We will be dealing with approximation to points $\xi \in \mathbb{R}^n$ or $\mathbb{C}^n$. In order to treat both cases with a minimum of repetition, we set $\mathbb{C}_1 = \mathbb{R}$, $\mathbb{C}_2 = \mathbb{C}$. Throughout, $q$ will be 1 or 2, so that $\mathbb{C}_q$ will be $\mathbb{R}$ or $\mathbb{C}$. For $\eta = (\eta_1, \ldots, \eta_l) \in \mathbb{C}_q^l$ (with any $l$), let $|\eta|$ denote the maximum norm, i.e., $\max(|\eta_1|, \ldots, |\eta_l|)$. When $\xi \in \mathbb{C}^n$ and $A$ is a subset of $\mathbb{C}_q^n$, we define the distance $\delta(\xi, A)$ to be $\infty$ when $A$ is empty, and the infimum of $|\alpha - \xi|$ over points $\alpha \in A$ otherwise. We further put $\delta_q(\xi, A) = \delta(\xi, A \cap \mathbb{C}_q^n)$, and note that $\delta_2(\xi, A) = \delta(\xi, A)$.

How well can a given point $\xi \in \mathbb{C}_q^n$ be “approximated” by algebraic hypersurfaces, or more generally by algebraic varieties? More precisely, the question is about algebraic hypersurfaces or varieties $A$ with $\delta_q(\xi, A)$ small.

When $P(X) \in \mathbb{Z}[X] = \mathbb{Z}[X_1, \ldots, X_n]$ is a nonzero polynomial, let $A(P)$ be the hypersurface consisting of points $\alpha \in \mathbb{C}_q^n$ with $P(\alpha) = 0$. A goal will be to make $\delta_q(\xi, A(P))$ small, subject to certain conditions on $P$. In particular, this distance should be small in terms of the total degree of $P$ and the Height $H(P)$ of $P$. This Height is defined as the maximum modulus of the coefficients of $P$; for convenience of notation we will set $|P| := H(P)$.

Let $\mathcal{M}$ be a finite, nonempty set of monomials, i.e., expressions $X^\sigma = X_1^{\sigma_1} \cdots X_n^{\sigma_n}$ where $\sigma = (\sigma_1, \ldots, \sigma_n) \in \mathbb{N}_0^n$; here $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ where $\mathbb{N}$ consists of the positive integers. Let $\mathcal{P}(\mathcal{M})$ be the set of nonzero polynomials $P \in \mathbb{Z}[X]$ which are linear combinations of the monomials in $\mathcal{M}$. Also, let $|\mathcal{M}|$ be the cardinality of $\mathcal{M}$. Versions of the following lemma, which will be easily proved in Section 2, are well known.

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Lemma 1.1. Suppose $|\mathcal{M}| \geq 2$, $\xi \in \mathbb{C}_q^n$, and set

$$\langle \mathcal{M}(\xi) \rangle = \sum_{M \in \mathcal{M}} |M(\xi)|.$$ 

Then given $N \geq 1$, there is a polynomial $P \in \mathcal{P}(\mathcal{M})$ with

$$|P| \leq N \quad \text{and} \quad |P(\xi)| < 2\langle \mathcal{M}(\xi) \rangle N^{1-|\mathcal{M}|/q}. \tag{1.1}$$

In general it is not easy to deduce from an upper bound for $|P(\xi)|$ an upper bound for $\delta_1(\xi, A(P))$. When $V \subseteq \mathbb{C}^n$ is an algebraic set defined by polynomial equations $P_1 = \cdots = P_s = 0$, it is even harder to obtain a bound for $\max_i |P_i(\xi)|$ (or else a bound for a quantity $|V(\xi)|$ introduced by Nesterenko [10]) a bound for $\delta(\xi, V)$. Results in this direction are due, e.g., to Brownawell [4] and Philippon [11]. Amoroso [1, 2] proved “comparison theorems” involving the “size” of a polynomial to derive a metric theorem.

Given a polynomial $P$, the most common approach is to estimate $\delta(\xi, A(P))$ in terms of

$$|P(\xi)|/|\text{grad } P(\xi)|, \tag{1.2}$$

where grad $P(\xi)$ is the gradient vector of $P$ at $\xi$. As is well known, and will again be explained in Section 3, $\delta_2(\xi, A(P))$ is not larger than $d$ times (1.2), where $d$ is the total degree of $P$. This is not always true for $\delta_1(\xi, A(P))$. But in Section 3 we will explain that when $\xi \in \mathbb{R}^n$ and $P \in \mathbb{R}[X]$, then an estimate for (1.2) often yields an estimate for $\delta_1(\xi, A(P))$, or for $\delta_1(\xi, A(P|\sigma))$ for some partial derivative $P|\sigma$ of $P$.

A polynomial $P$ as in (1.1) with $|P| \leq N$ is likely to have $|P(\xi)|$ not much smaller than $N$, in fact to have $|\text{grad } P(\xi)|$ not much smaller than $N$. But then (1.1) will imply that (1.2) is not much larger than $N^{-|\mathcal{M}|/q} \leq |P|^{-|\mathcal{M}|/q}$, One may therefore ask whether, given $\xi \in \mathbb{C}_q^n$, there are infinitely many $P \in \mathcal{P}(\mathcal{M})$ with

$$\delta_2(\xi, A(P)) < c_1(\xi, \mathcal{M}) |P|^{-|\mathcal{M}|/q}, \tag{1.3}$$

where $c_1(\xi, \mathcal{M})$ depends on $\xi, \mathcal{M}$ only. In general, unless said otherwise, a constant $c(\phi, \Phi, \ldots)$ will depend only on quantities $\phi, \Phi, \ldots$.

The rather few instances when we can establish such a result will be described in Theorem 1.2. Monomials are contained in the multiplicative group of generalized monomials, i.e., expressions $X^u$ with $u \in \mathbb{Z}^n$. Let $\rho(\mathcal{M})$ be the rank of the subgroup generated by the quotients $M/M'$ with $M, M'$ in $\mathcal{M}$. Clearly

$$\rho(\mathcal{M}) \leq \min(n, |\mathcal{M}| - 1).$$

Theorem 1.2. Let $\mathcal{M}$ be a set of monomials with $\rho(\mathcal{M}) = n$.

(i) Suppose $n \geq q$, $|\mathcal{M}| = n + 1$, and $\xi \in \mathbb{C}_q^n$. Then given $N \geq 1$, there is a $P \in \mathcal{P}(\mathcal{M})$ with $|P| \leq N$ and

$$\delta_1(\xi, A(P)) < c_2 N^{1-|\mathcal{M}|/q} |P|^{-1}, \tag{1.4}$$

where $c_2 = c_2(\xi, \mathcal{M})$.

(ii) Suppose $q = 1$, $|\mathcal{M}| = n + 2$, and $\xi \in \mathbb{C}_1^n = \mathbb{R}^n$, but $\xi / A(P)$ for any $P \in \mathcal{P}(\mathcal{M})$. Then there are infinitely many polynomials $P \in \mathcal{P}(\mathcal{M})$ with

$$\delta_1(\xi, A(P)) < c_3 |P|^{-|\mathcal{M}|},$$

where $c_3 = c_3(\xi, \mathcal{M})$. 

Part (i) is a uniform result, since it asserts the existence of \( P \in \mathcal{P}(\mathcal{M}) \) with \( |P| \leq N \) and (1.4) for every \( N \geq 1 \). It is not much harder to prove than Dirichlet’s Theorem on linear forms, which yields the case \( q = 1 \), \( \mathcal{M} = \{1, X_1, \ldots, X_n\} \), and which may be interpreted as a result on approximation to \( \xi \) by hyperplanes. When \( \xi \notin A(P) \) for every \( P \in \mathcal{P}(\mathcal{M}) \), it implies the existence of infinitely many \( P \in \mathcal{P}(\mathcal{M}) \) with \( \delta_q(\xi, A(P)) < c_2 |P|^{-|\mathcal{M}|/q} \). Part (ii) is not uniform, and applies only to the real case. The important distinction of uniform and nonuniform results had also been pointed out in [5].

An example for part (ii) is when \( \mathcal{M} = \{1, X_1, \ldots, X_n, M\} \) where \( M \) is a monomial of total degree \( > 1 \). A special case is when \( n = 1 \) and \( \mathcal{M} = \{1, X, X^2\} \). In this case the assertion can be reformulated to say that for any \( \xi \in \mathbb{R} \) which is not rational or quadratic over \( \mathbb{Q} \), there are infinitely many \( \alpha \in \mathbb{R} \) of degree at most 2 over \( \mathbb{Q} \) with \( |\alpha - \xi| < c_4(\xi)H(\alpha)^{-3} \). Here \( H(\alpha) \) is the Height of the defining polynomial of \( \alpha \) over \( \mathbb{Z} \). This had been established by Davenport and Schmidt [6], and amounts to the quadratic case of a conjecture of Wirsing [14].

Our next theorem is a consequence of a special case of the important and deep work of Laurent and Roy [8]. It is essentially Theorem 2.3 of their paper, but with a more carefully worked out value of the exponent. Let \( \mathcal{M}(d) \) consist of all monomials of total degree at most \( d \), so that \( |\mathcal{M}(d)| \) equals

\[
(1.5) \quad m = m(n, d) = \binom{n + d}{d},
\]
and \( \mathcal{P}(d) := \mathcal{P}(\mathcal{M}(d)) \) consists of the nonzero polynomials in \( \mathbb{Z}[X] \) of total degree at most \( d \).

Set \( S_0 = \lceil \sqrt[q]{q(n+1)!} \rceil \), and

\[
(1.6) \quad \nu = ((m/q) - 1)/S_0.
\]

**Theorem 1.3.** Suppose \( n > 1 \). Also suppose \( \xi \in \mathbb{C}_q^n \) with \( |\xi| \leq \psi \), and \( P(\xi) \neq 0 \) for every \( P \in \mathcal{P}(d) \). Then there are infinitely many \( P \in \mathcal{P}(d) \) with

\[
(1.7) \quad \delta_q(\xi, A(P)) < c_5(n, d, \psi)|P|^{-\nu}.
\]

Now \( n > 1 \) yields \((n+1)! - qn! > 1 \), so that

\[
(d+1) \cdot (d+n) - qn! > d^n
\]
holds for \( d = 1 \), hence for \( d \geq 1 \), since the left-hand side increases faster in \( d \) than the right-hand side. As a consequence,

\[
(1.8) \quad (m/q) - 1 > d^n/qn!.
\]

It is easily seen that \( n > 1 \) gives \( S_0 \leq n + 1 \), except that \( S_0 = 4 \) when \( n = q = 2 \). Thus \( \nu > \nu' \) where

\[
(1.9) \quad \nu' = \begin{cases} \frac{d^2}{16} & \text{when } n = q = 2, \\ \frac{d^n}{q(n+1)!} & \text{otherwise}. \end{cases}
\]

Wirsing in pioneering work [13] had shown that when \( \xi \in \mathbb{C}_q \) is not algebraic of degree \( \leq d \) over \( \mathbb{Q} \), then there are infinitely many algebraic numbers \( \alpha \in \mathbb{C}_q \) of degree \( \leq d \) with

\[
(1.10) \quad |\alpha - \xi| < H(\alpha)^{-(d/(2q) - \gamma_0(d)}
\]
where \( H(\alpha) \) is the Height of the defining polynomial of \( \alpha \) over \( \mathbb{Z} \), and where \( \gamma_0(d) > 0 \) with \( \gamma_1(d) \to 2, \gamma_2(d) \to 1/4 \) as \( d \to \infty \). This has since been improved to \( \gamma_1(d) \to \)
Corollary 1.4. Suppose $n, d$ are positive integers, and define $\nu'$ by (1.9). If $\xi \in \mathbb{C}_q^n$, but $P(\xi) \neq 0$ for any $P \in \mathcal{P}(d)$, then there are infinitely many $P \in \mathcal{P}(d)$ with

$$\delta_q(\xi, A(P)) < |P|^{-\nu'}.$$ 

Again set

$$m = \left( \begin{array}{c} n + d \\ d \end{array} \right) = 1 + n + \left( \begin{array}{c} n + 1 \\ 2 \end{array} \right) + \cdots + \left( \begin{array}{c} n + d - 1 \\ d \end{array} \right),$$

and further put

$$m_0 = m_0(n, d) = 1 + n + \frac{1}{2} \left( \begin{array}{c} n + 1 \\ 2 \end{array} \right) + \cdots + \frac{1}{d} \left( \begin{array}{c} n + d - 1 \\ d \end{array} \right).$$

Theorem 1.5. (a) Suppose $\xi \in \mathbb{C}_q^n$ and $\psi \geq |\xi|$. Then given $N \geq 1$ there is a $P \in \mathcal{P}(d)$ with $|P| \leq N$ and

$$\delta_q(\xi, A(P)) < c_6(n, d, \psi)(N^{1-m/q}|P|^{-1})^{1/d}.$$

(b) Suppose $\xi \in \mathbb{R}^n = \mathbb{C}_1^n$ with $|\xi| \leq \psi$. Then given $N \geq 1$, there is an $\ell$ in $1 \leq \ell \leq d$, and a $P \in \mathcal{P}(d)$ with $|P| \leq N^{1/\ell}$ and

$$\delta_1(\xi, A(P)) < c_7(n, d, \psi)(N^{1-m}|P|^{-1})^{1/\ell}.$$ 

We cannot specify the value of $\ell$ in part (b), and there does not appear to be a clean version of part (b) for $\xi \in \mathbb{C}^n = \mathbb{C}_2^n$. The assertions of the theorem are uniform, since $|P|$ is under a bound involving $N$. The following is an immediate consequence.

Corollary 1.6. (a) Suppose $\xi \in \mathbb{C}_q^n$, $\psi \geq |\xi|$, and moreover $\xi \notin A(P)$ for any $P \in \mathcal{P}(d)$. Then there are infinitely many $P \in \mathcal{P}(d)$ with

$$\delta_q(\xi, A(P)) < c_8(n, d, \psi)|P|^{-m/(dq)}.$$ 

(b) Suppose $\xi \in \mathbb{R}^n$, $\psi \geq |\xi|$, and $\xi \notin A(P)$ for any $P \in \mathcal{P}(d)$. Then there are infinitely many $P \in \mathcal{P}(d)$ with

$$\delta_1(\xi, A(P)) < c_9(n, d, \psi)|P|^{1-m - 1/d}.$$ 

Since $m_0 - 1 > m/d$ when $d > 1$, $n > 1$, the inequality (1.13) is usually better than (1.12). But let us compare the exponent in (1.7) with the exponents in (1.12) or (1.13). Since $\nu = ((m/q) - 1)/S_0$ where $S_0 \leq n + 1$ when $n > 1$, (1.7) is better when $d$ is large compared to $n$. But (1.12), (1.13) are better when $n$ is large compared to $d$.

Not much is known about approximation by algebraic varieties of codimension $s > 1$. But see Philippon [11], where, e.g., the case $n = s = 2$ is treated, i.e., approximation to $\xi \in \mathbb{C}^2$ by algebraic points $\alpha$ of bounded degree. In [12], the Corollary to Theorem 15, using projective terminology, shows that when $\xi \in \mathbb{C}_q^n$ with $\xi \notin A$ for any linear manifold $A$ of codimension $s$ defined over $\mathbb{Q}$, then there are infinitely many such $A$ with $\delta_q(\xi, A) < c_8(\xi)H(A)^{-(n+1)/qs}$, and the exponent $-(n+1)/qs$ is best possible. Here we will derive results which are probably far from the full truth.
Suppose now that \( r > 0, s > 0, \)
\[
r + s = n.
\]
Let \( P = (P_1, \ldots, P_s) \) denote an \( s \)-tuple of polynomials in \( \mathbb{Z}[X] \). Given \( P \), write \( A(P) \) for the algebraic set consisting of points \( \alpha \in \mathbb{C}^n \) with \( P_i(\alpha) = \cdots = P_s(\alpha) = 0 \). When \( d = (d_1, \ldots, d_s) \in \mathbb{N}^s \), let \( \mathcal{P}(d) \) consist of \( s \)-tuples \( P \) where \( A(P) \) has dimension \( r \), and where for \( 1 \leq i \leq s \) we have \( \deg P_i \leq d_i \) and \( P_i = P_i(X_1, \ldots, X_{n-i+1}) \).

Set
\[
m_i = \left( \frac{n - i + 1 + d_i}{d_i} \right) \quad (i = 1, \ldots, s).
\]

**Theorem 1.7.** Suppose \( d \in \mathbb{N}^s \) is given, and \( m_i > q \) holds for \( i = 1, \ldots, s \).
There are constants \( \gamma_i = \gamma_i(n, d, \psi) \geq 1 \) \((i = 1, \ldots, s)\) with the following property. Suppose \( \xi \in \mathbb{C}_q^s \) with \( |\xi| \leq \psi \), and \( L \geq 1 \). Then after suitable ordering of the variables there is a \( P = (P_1, \ldots, P_s) \in \mathcal{P}(d) \) with
\[
|P_i| \leq (\gamma_i L)^{d_q(m_i - q)} \quad (i = 1, \ldots, s)
\]
and
\[
\delta_q(\xi, A(P)) < L^{-1}.
\]

No assertion is made for \( r = 0 \), since our arguments would give a poor result. In fact the theorem, as well as Theorem 1.8 below, are poor when the dimension \( r \) is small compared to \( n \).

By a variety we will understand an algebraic set \( V \subset \mathbb{C}^n \) which is defined over \( \mathbb{Q} \), and irreducible over \( \mathbb{Q} \). We define the Height of \( V \) to be \( H(V) = H(\hat{V}) \), where \( \hat{V} \) is the Zariski closure of \( V \) in projective space \( \mathbb{P}^n(\mathbb{C}) \), and \( H(\hat{V}) \) is the Height of a Chow form of \( \hat{V} \) (see [8, Section 4.1]), but observe that in [8] the logarithmic height \( h(\hat{V}) = \log H(\hat{V}) \) is used.

Suppose \( r, s, d \in \mathbb{N}^s \) and \( m_i \) \((1 \leq i \leq s)\) are as above, and that each \( m_i > q \). Furthermore, set \( d = d_1 \cdots d_s \),
\[
\Phi = d_q \left( \frac{1}{m_1 - q} + \cdots + \frac{1}{m_s - q} \right), \quad \Psi = 1/\Phi.
\]

**Theorem 1.8.** Suppose \( \xi \in \mathbb{C}_q^s \), and \( |\xi| \leq \psi \). Then for \( N \geq 1 \) there is a variety \( V \)
of dimension \( r \), degree \( \leq \delta \), and \( H(V) \leq N \) having
\[
\delta_q(\xi, V) < c_3(n, d, \psi)N^{-\Psi}.
\]

Observe that Theorems 1.7 and 1.8 are uniform. I have no intuition of what would be the best exponent on the right-hand side of (1.17). Suppose \( d = (a, \ldots, a) \), so that \( d = a^s \). Then for fixed \( a, s \) we have as \( n \to \infty \) that \( m_i = \left( \frac{n - i + 1 + a}{a} \right) \sim n^a/a! \) for \( 1 \leq i \leq s \), hence \( \Phi \sim a^s qsa! / n^a \), and
\[
\Psi \sim n^a/(qsa^a)! = n^{a^s}/(qsa)!.
\]

On the other hand, suppose \( s < n \) are fixed, and \( d \to \infty \). Say \( d_i = a^{u_i} \)
\((i = 1, \ldots, s)\) where \( u_i = u/(n - i + 1) \) and \( u = n(n - 1) \cdots (n - s + 1) \). Then as \( a \to \infty \), \( m_i \sim a^{u_i(n-s+1)}/(n-i+1)! = a^{u_i(n-i+1)} \), and
\[
\Phi \sim d_q a^{-u} (n! + \cdots + (n - s + 1)! \leq 2 d_q n! a^{-u},
\]
hence $\Psi \geq d^n/(2dqn!)$. But $u_1 + \cdots + u_s = u(\frac{1}{n} + \cdots + \frac{1}{n-s+1}) = uf(n, s)$, say, so that $d = a^s f(n, s)$, and setting $g(n, s) = f(n, s)^{-1} - 1$, e.g., $g(3, 2) = 1/5$, we have as $d \to \infty$,

$$\Psi \geq d^n/(2dqn!)$$

Also, $f(n, s) \leq s/(n-s)$, so that $g(n, s) \geq (n-s)-2$. It may be seen that $f(n, s) < 1$ when $s < (1-e^{-1})n$, and then $g(n, s) > 0$. But $f(n, s) > 1$ when $s > (1-e^{-1})(n+1)$, and then $g(n, s) < 0$ and our estimate is useless.

We will prove somewhat stronger theorems than those of the Introduction. See the definitions in Section 3 and Theorems 5.1, 7.2, 8.1, 9.1, 10.1. Our results could be restated in a projective setting, such as, e.g., in [8] or [12].

As already said, we set $\mathbb{N} = \{1, 2, \ldots \}$, $\mathbb{N}_0 = \{0, 1, 2, \ldots \}$. The degree of a polynomial will be understood to be the total degree. $X$ will denote an $n$-tuple of variables $(X_1, \ldots, X_n)$. A point $\xi$ or $\alpha$ or $\cdots$ in $\mathbb{C}^n$ will be understood to have coordinates $\xi_1, \ldots, \xi_n$ or $\alpha_1, \ldots, \alpha_n$ or $\cdots$. The notation $X^\alpha$ will signify $X_1^{\alpha_1} \cdots X_n^{\alpha_n}$. Besides the maximum norm $|\eta|$ for $\eta = (\eta_1, \ldots, \eta_k)$ introduced above, we will also use $|\eta| = |\eta_1| + \cdots + |\eta_k|$. The divided derivatives of a polynomial $P \in \mathbb{C}[X]$ are

$$P[\sigma] = \frac{1}{\sigma_1! \cdots \sigma_n!} \frac{\partial^{\sigma_1 + \cdots + \sigma_n} P}{\partial X_1^{\sigma_1} \cdots \partial X_n^{\sigma_n}}$$

where $\sigma \in \mathbb{N}_0^n$. The numbering of constants $c_1, c_2, \ldots$ will start anew in each section, and all these constants will be positive.

2. Proof of Lemma 1.1

This lemma (which corresponds to Lemma 6.6 in [8]) is an immediate consequence of the following.

Lemma 2.1. Let $\lambda \in \mathbb{C}_q^m$, where $m \geq 2$, and let $L$ be the linear form with $L(\alpha) = \lambda_1 \alpha_1 + \cdots + \lambda_m \alpha_m$. Then given $N \geq 1$, there is a nonzero integer point $\alpha \in \mathbb{Z}^m$ with $|\alpha| \leq N$ and

$$|L(\alpha)| < 2(\lambda)N^{1-m/q}.$$  

Proof. We may suppose by homogeneity that $|\lambda| = 1$. We also may suppose that $|\lambda_1| = \min |\lambda_i|$, so that $|\lambda_1| \leq |\lambda|/m = 1/m$. We will deal with the case $q = 2$; the case $q = 1$ can be done in a similar manner.

When $N^{m/2} < 4$, then $\alpha = (1, 0, \ldots, 0)$ has $|\alpha| = 1 \leq N$ and $|L(\alpha)| = |\lambda_1| \leq 1/m \leq 2N/4 < 2N^{1-m/2}$. When $N^{m/2} \geq 4$, we argue as follows. The number of integer points $b$ with $|b| \leq N/2$ is at least $N^m$. For such $b$ we have $|L(b)| \leq |\ell||b| \leq N/2$, so that $L(b)$ lies in a square in $\mathbb{C}$ of side $N$. Let $A$ be the integer with $N^{m/2} - 1 \leq A < N^{m/2}$; then $N^{m/2}/\sqrt{2} < A$ in view of $N^{m/2} \geq 4$. Divide our square into $A^2$ subsquares of side $N/A$. Since $A^2 < N^m$, two of our points $b$ will have $L(b)$ in the same subsquare. If $b_1, b_2$ are such points, set $a = b_1 - b_2$, so that $0 < |a| \leq N$ and $|L(a)| \leq \sqrt{2}N/A < 2N^{1-m/2}$.  

Observe that the lemma also follows from Minkowski’s theorem on linear forms.

3. Polynomials with small zeros

We will deal with polynomials in $\mathbb{R}[X]$ or $\mathbb{C}[X]$. The contents of this and the next section may be of independent interest.
Before going further, let us justify an assertion we made in the Introduction. When \( \xi \in \mathbb{C} \) and \( P \in \mathbb{C}[X] \) with \( P(\xi) \neq 0 \), \( P'(\xi) \neq 0 \), and roots \( \alpha_1, \ldots, \alpha_d \), then \( P'(\xi) = P(\xi) \sum_{i=1}^{d}(\xi - \alpha_i)^{-1} \), so that some \( \alpha_i \) has \( |\xi - \alpha_i| \leq d|P(\xi)|/|P'(\xi)| \), yielding \( \delta_2(\xi, A(P)) \leq d|P(\xi)|/|P'(\xi)| \). Since \( |\text{grad } P(\xi)| \) is defined in terms of the maximum norm, it follows that for \( \xi \in \mathbb{C}^n \), and \( P \in \mathbb{C}[X] \) with \( \text{grad } P(\xi) \neq 0 \), indeed, \( \delta_2(\xi, A(P)) \) is bounded by \( d \) times (1.2).

Suppose \( \xi \in \mathbb{C}^q_0 \) and \( A \in \mathbb{C}^q_0 \) is a manifold of dimension \( r \). We hope to not only have \( \delta_q(\xi, A) \) small, i.e., to have some \( a \in A \) with \( |a - \xi| < \delta \), but to have a "large" subset \( A' \) of \( A \) whose elements \( a \) have \( |a - \xi| < \delta \). We hope to have \( A' \) with reasonably large \( qr \)-dimensional volume, ideally of order of magnitude \( \gg \delta^{qr} \).

Suppose \( r, s \) are integers with
\[
3.1 \quad r + s = n \quad \text{and} \quad 0 \leq r < n, \quad 0 < s \leq n,
\]
and let \( \rho, \delta \) be reals with
\[
3.2 \quad 0 < \rho \leq \delta.
\]

Suppose at first that \( r > 0 \). Let \( C_q(r; \rho, \xi) \) consist of points \( z = (z_1, \ldots, z_r) \in \mathbb{C}^q_r \) with \( |z_i - \xi_i| < \rho \) \((i = 1, \ldots, r)\). Let \( C_q(s; \rho, \xi) \) consist of points \( w = (w_{s+1}, \ldots, w_s) \in \mathbb{C}^q_s \) with \( |w_j - \xi_j| < \delta \) \((j = r + 1, \ldots, n)\). An analytic \( (r; \rho, \delta)_{2}\)-piece for \( \xi \) will be understood to be a set of points
\[
\alpha = (z, w(z)) \in \mathbb{C}^n
\]
where \( z \) runs through \( C_2(r; \rho, \xi) \), where \( w(z) = w(z_1, \ldots, z_r) \) defines a map \( C_2(r; \rho, \xi) \rightarrow C_2(s; \rho, \xi) \), and where each \( w_j(z) \) is a regular function. An analytic \( (r; \rho, \delta)_{q}\)-piece for a point \( \xi \in \mathbb{C}^n = \mathbb{R}^n \) is an \( (r; \rho, \delta)_{q}\)-piece where each \( w_j(z) \) is a regular function, i.e., is real when \( z \in \mathbb{R}^r \). In this case the restriction of \( w(z) \) to \( z \in C_1(r; \rho, \xi) \) is a map into \( C_1(s; \rho, \xi) \). Observe that by (3.2) and our use of the maximum norm, every point \( \alpha \) of an analytic \( (r; \rho, \delta)_{q}\)-piece for \( \xi \) has \( |\alpha - \xi| < \delta \).

When \( r > 0 \), a set \( A \subset \mathbb{C}^q_0 \) will be said to be special \( (r; \rho, \delta)_{q}\)-close to \( \xi \) if it contains an analytic \( (r; \rho, \delta)_{q}\)-piece for \( \xi \). It will be said to be \( (r; \rho, \delta)_{q}\)-close if it is special \( (r; \rho, \delta)_{q}\)-close after suitable numbering of the variables, i.e., if \( T(A) \) is special \( (r; \rho, \delta)_{q}\)-close where \( T \) is a map \((x_1, \ldots, x_n) \mapsto (x_{i_1}, \ldots, x_{i_n}) \) for some permutation \((i_1, \ldots, i_n) \) of \((1, \ldots, n) \). When \( r = 0 \), special \((0; \rho, \delta)_{q}\)-closeness and \((0; \rho, \delta)_{q}\)-closeness will mean the same, namely that \( \delta_q(\xi, A) < \delta \). In this case \( \rho \) does not come into play! It will be important to observe that when \( A \) is \( (r; \rho, \delta)_{q}\)-close, and \( \rho' \leq \rho, \delta' \geq \delta \), then it is \( (r; \rho', \delta')_{q}\)-close, and the dilated set \( \lambda A \) is \( (r; \lambda \rho, \lambda \delta)_{q}\)-close.

A polynomial \( Q(X) \in \mathbb{C}_q[X] \) will be said to have the special \((\rho, \xi)_{q}\)-property if
\[
3.1 \quad \text{A(Q) is special (n - 1; \rho, (n + 1)\rho)_{q}\-close to } \xi,
\]
\[
3.2 \quad Q(\alpha_1, \ldots, \alpha_{n-1}, X_n) \text{ for any given } \alpha_1, \ldots, \alpha_{n-1} \text{ is a nonzero polynomial in } X_n. \text{ (When } n = 1, \text{ then } Q(X_1) \neq 0.\)
\]
It has the \((\rho, \xi)_{q}\)-property if after suitable numbering of the variables it has the special \((\rho, \xi)_{q}\)-property. The \((\rho, 0)_{q}\)-property will more simply be called \(\rho\)-property. Well, this should be the end of the definitions!

**Remark 3.1.** When \( Q \) has the \((\rho, \xi)_{q}\)-property, then \( \delta_q(\xi, A(Q)) < (n + 1)\rho \). Furthermore, when \( a \leq \rho \leq b \), then \( A(Q) \) is \((n - 1; a, (n + 1)b)_{q}\)-close to \( \xi \).
Remark 3.2. The set \( A \) of points \( x \) with \( x_{r+1} = \cdots = x_n = 0 \) is special \((r; 1, 1+|\xi|)_q\)-close to \( \xi \); this follows on setting \( w(z) = (w_{r+1}(z), \ldots, w_n(z)) = (0, \ldots, 0) \) for \( z = (z_1, \ldots, z_r) \).

Write
\[
Q(X) = Q_0(X) + \cdots + Q_d(X),
\]
where \( Q_j(X) \) for \( 0 \leq j \leq d \) is a homogeneous polynomial in \( \mathbb{C}_q[X] \) of degree \( j \).

Lemma 3.3. Let \( Q \) be given by (3.3), with \( Q_1 \neq 0 \),
\[
|Q_0|/|Q_1| \leq 1,
\]
\[
|Q_j|/|Q_1| \leq (4n^2)^{-j} \quad \text{for} \quad j = 2, 3, \ldots.
\]
Then \( Q \) has the \( 1_q \)-property.

Proof. We will suppose that \( n > 1 \), the case \( n = 1 \) being simpler. We may suppose that \( Q_1(X) = a_1X_1 + \cdots + a_nX_n \) with \( |Q_1| = |a_n| \). By homogeneity we may further suppose that \( a_n = 1 \), so that \( |Q_1|=1 \). Now \( Q_j \) is a sum of monomials, so that for any \( \alpha \),
\[
|Q_j(\alpha)| \leq |Q_j|(n|\alpha|)^j \quad (j = 1, \ldots, d).
\]
When \( |\alpha| \leq n+1 \), (3.5) yields
\[
|Q_j(\alpha)| \leq |Q_j|(2n^2)^j \leq 2^{-j},
\]
and therefore
\[
\sum_{j=2}^d |Q_j(\alpha)| \leq 1/2.
\]
Let \( z = (z_1, \ldots, z_{n-1}) \) be any element of \( \mathbb{C}_2(n-1, 1, 0) \), so that \( |z| < 1 \). Suppose \( w \in \mathbb{C} \) with \( |w| = n+1 \). Setting \( \alpha = (z, w) = (z_1, \ldots, z_{n-1}, w) \) we have
\[
|Q(\alpha) - w| = |Q(z, w) - w| \leq |Q_0| + \sum_{i=1}^{n-1} |a_i z_i| + \sum_{j=2}^d |Q_j(\alpha)|
\]
\[
\leq 1 + (n - 1) + 1/2 < |w|.
\]
Considering \( z \) fixed and \( Q(z, w) \) a polynomial in \( w \), we may infer from Rouche’s Theorem that there is a unique \( w = w(z) \in \mathbb{C} \) in the disk \( |w| < n+1 \) with \( Q(z, w) = 0 \). The map \( z \mapsto w(z) \) is a map \( \mathbb{C}_2(n-1, 1, 0) \to \mathbb{C}_2(1; n+1, 0) \).

Let \( z_0 \) be fixed for the moment and \( \bar{Q} \) an irreducible factor of \( Q \) with \( \bar{Q}(z_0, w(z_0)) = 0 \). By the “uniqueness” of \( w(z_0) \), \( \partial w/\partial w)\bar{Q} \neq 0 \) at \( (z_0, w(z_0)) \), and therefore there is a regular function \( f(z) \) defined in a neighborhood of \( z_0 \) with \( f(z_0) = w(z_0) \) and \( \bar{Q}(z, f(z)) = 0 \). By the uniqueness, \( f(z) = w(z) \). Then \( \bar{Q}(z, w(z)) = 0 \) with the same irreducible factor \( \bar{Q} \) of \( Q \), when \( z \) is in this neighborhood. It easily follows that \( \bar{Q}(z, w(z)) = 0 \) for \( z \in \mathbb{C}_2(n-1, 1, 0) \), and that \( w(z) \) is regular.

Since the points \( (z, w(z)) \) constructed form a special analytic \((n - 1; 1, n + 1)\) piece for \( 0 \), we see that \( Q \in \mathbb{C}_2[X] = \mathbb{C}[X] \) indeed has property \( 1_2 \). On the other hand, when \( Q \in \mathbb{C}_1[X] = \mathbb{R}[X] \), and if \( Q(z, w) = 0 \) with \( z \in C_1(n-1, 1, 0) \subset \mathbb{R}^{n-1} \), then also \( Q(z, \bar{w}) = 0 \) for the complex conjugate \( \bar{w} \) of \( w \), so that by uniqueness \( \bar{w} = w \), hence \( w \in \mathbb{R} \). In this case \( Q \) has property \( 1_1 \). □
Remark 3.4. It follows from the above argument that when $Q(X) = Q_1(X) \cdots Q_\ell(X)$, then $Q_i(z, w(z)) = 0$ for some fixed factor $Q_i$, and $Q_i$ has property $1_q$. More generally, when $Q$ has the $(\rho, \xi)_q$-property, then some $Q_i$ has this property.

Corollary 3.5. Suppose $Q \in \mathbb{C}_q[X]$ is given by (3.3), with $Q_1 \neq 0$ and

\begin{equation}
|Q_0|/|Q_1| \leq \lambda,
|Q_j|/|Q_1| \leq (4n^2)^{-j}\lambda^{j-1} \quad \text{for} \quad j = 2, 3, \ldots.
\end{equation}

for some $\lambda > 0$. Then $Q$ has the $\lambda_q$-property.

Now (3.7) means that $|Q(0)|/|\text{grad} \ Q(0)| \leq \lambda$, and the $\lambda_q$-property implies that $Q$ has a zero $\alpha \in \mathbb{C}_q^n$ with $|\alpha| < (n+1)\lambda$. Thus $\delta_q(0, A(Q)) < (n+1)\lambda$, justifying remarks we made on (1.2).

Proof. Setting $\tilde{Q}(X) = Q(\lambda X)$ we have $\tilde{Q} = \tilde{Q}_0 + \tilde{Q}_1 + \cdots$ with $|\tilde{Q}_j| = \lambda^j|Q_j|$. Therefore $|\tilde{Q}_0|/|\tilde{Q}_1| \leq 1$ and $|\tilde{Q}_j|/|\tilde{Q}_1| \leq (4n^2)^{-j}$ for $j = 2, 3, \ldots$. The assertion follows on applying Lemma 3.3 to $\tilde{Q}$, and observing that $A(Q) = \lambda A(\tilde{Q})$. \qed

Recall the definition of the norm $\langle \cdot \rangle$ and of divided derivatives. Observe that any divided derivative $Q^{[\sigma]}$ of a polynomial $Q$ of total degree at most $d$ has

\begin{equation}
|Q^{[\sigma]}| \leq 2^d|Q|.
\end{equation}

Lemma 3.6. Suppose $Q \in \mathbb{C}_q[X]$ is again given by (3.3), and that for some $\lambda > 0$ and $k \in \mathbb{N}$ we have $Q_k \neq 0$ and

\begin{equation}
|Q_{k-1}|/|Q_k| \leq \lambda,
|Q_{k+j}|/|Q_k| \leq (8n^2 \cdot 2^k)^{-1} (8n^2 \lambda)^{-j} \quad (j = 1, 2, \ldots).
\end{equation}

Then there is a $\sigma$ with $\langle \sigma \rangle = k-1$ such that $Q^{[\sigma]}$ has property $\lambda_q$.

Proof. Pick $\rho$ with $\langle \rho \rangle = k$ such that the coefficient $a_{\rho}$ of $X^\rho$ in $Q_k$ has $|a_{\rho}| = |Q_k|$. Pick $\sigma$ with $\langle \sigma \rangle = k-1$ and $\sigma_i \leq \rho_i$ ($i = 1, \ldots, n$); then $\rho = \sigma + e_\ell$ for some basis vector $e_\ell$. We have

\begin{equation}
Q^{[\sigma]} = Q^{[\sigma]}_{k-1} + Q^{[\sigma]}_k + Q^{[\sigma]}_{k+1} + \cdots = G_0 + G_1 + G_2 + \cdots,
\end{equation}

say, where $G_j$ is homogeneous of degree $j$. Writing $Q_{k-1} = \sum a_\sigma X^\tau$, we obtain $Q^{[\sigma]}_{k-1} = a_\sigma$, and therefore

\begin{equation}
|G_0| = |Q^{[\sigma]}_{k-1}| = |a_\sigma| \leq |Q_{k-1}|.
\end{equation}

Now $Q_k$ has the summand $a_\rho X^\rho = a_\rho X^{\sigma + e_\ell}$, so that $G_1 = Q^{[\sigma]}_k$ has the summand $(\sigma_\ell + 1)a_\rho X_\ell$, and

\begin{equation}
|G_1| \geq (\sigma_\ell + 1)|a_\rho| \geq |a_\rho| = |Q_k|.
\end{equation}

On the other hand, when $j > 0$,

\begin{equation}
|G_{j+1}| = |Q^{[\sigma]}_{k+j}| \leq 2^{k+j}|Q_{k+j}| \leq 2^{k+j} \cdot 2^{-k} (8n^2)^{-j-1} \lambda^{-j} |Q_k| < (4n^2)^{-j-1} \lambda^{-j} |Q_k|
\end{equation}

by (3.8) and our hypothesis. Thus $|G_j| < (4n^2)^{-j} \lambda^{j-1} |Q_k|$ for $j \geq 2$. Altogether,

\begin{equation}
|G_0|/|G_1| \leq |Q_{k-1}|/|Q_k| \leq \lambda \quad \text{and} \quad |G_j|/|G_1| \leq (4n^2)^{-j} \lambda^{j-1} \quad \text{for} \ j > 1.
\end{equation}

The lemma follows on applying Corollary 3.5 to $G = G_0 + G_1 + \cdots$. \qed
Lemma 3.7. Suppose \( \lambda > 0, \alpha \geq 1, \beta \geq 1 \), and set
\[
\lambda_t = 2^t(\alpha \beta)^{t+1/2}\lambda
\]
for \( t = 0, 1, \ldots \). Let \( q_0, q_1, \ldots \) be nonnegative reals with \( q_1 \neq 0 \) and \( q_0/q_1 \leq \lambda \). Then either there is a \( k > 0 \) with \( q_k \neq 0 \) and both
\begin{align*}
(3.13) & \quad q_{k-1}/q_k \leq \lambda_{k-1}, \\
(3.14) & \quad q_{k+j}/q_k \leq (1/\beta^2)^{(\alpha \lambda_{k-1})^{-j}} \text{ for } j = 1, 2, \ldots,
\end{align*}
or there are infinitely many \( k \) with \( q_k \neq 0 \) and (3.13).

Proof. We will suppose there is no \( k \) with both (3.13), (3.14). Since (3.13) is true by hypothesis for \( k = 1 \), it will suffice for us to show that for \( k \) with (3.13), there is a \( k' > k \) with \( q_{k'} \neq 0 \) and (3.13), i.e., with \( q_{k'-1}/q_{k'} \leq \lambda_{k'-1} \).

Since by our supposition (3.14) does not hold, there will be a \( j > 0 \) where (3.14) is violated. Let \( j \) be the least such number, and set \( k' = k + j \). Then
\[
q_{k'}/q_k > (1/\beta^2)^{(\alpha \lambda_{k-1})^{-k'}}
\]
but
\[
q_{k'-1}/q_k \leq (\alpha \lambda_{k-1})^{-k'+1}.
\]
Therefore
\[
q_{k'-1}/q_{k'} < \beta^2 \alpha \lambda_{k-1} = \lambda_k \leq \lambda_{k'-1}.
\]

Lemma 3.8. Let \( Q(X) \) be a polynomial of degree \( d \) as in (3.3), with \( Q_1 \neq 0 \) and \( |Q_0|/|Q_1| \leq \lambda \) where \( \lambda > 0 \). Set
\[
(3.15) \quad \lambda_t = 2^t(\alpha \beta)^{t+1/2}(2^6 n^4)^t\lambda
\]
for \( t = 0, 1, \ldots \). Then there is a \( t \) in \( 0 \leq t < d \) and a \( \sigma \) with \( \langle \sigma \rangle = t \) such that \( Q^{[\sigma]} \) has property \( (\lambda_t)_{q} \).

Proof. Set \( \alpha = \beta = 8n^2 \) and \( q_k = |Q_k| \) for \( 0 \leq k \leq d \), but \( q_k = 0 \) for \( k > d \). Since the second alternative of Lemma 3.7 cannot hold, there will be a \( k \) in \( 1 \leq k \leq d \) with (3.13), (3.14), i.e., with
\[
|Q_{k-1}|/|Q_k| \leq \lambda_{k-1},
|Q_{k+j}|/|Q_k| \leq (8n^2 2^k)^{-1}(8n^2 \lambda_{k-1})^{-j} \text{ for } j = 1, 2, \ldots .
\]

By Lemma 3.6 there is a \( \sigma \) with \( \langle \sigma \rangle = k - 1 \) such that \( Q^{[\sigma]} \) has property \( (\lambda_{k-1})_{q} \).

The desired conclusion holds with \( t = k - 1 \). \( \square \)

Suppose \( \xi \in \mathbb{C}_q^n \) and \( R \in \mathbb{C}_q[X] \). Observe that \( R \) has property \( (\rho, \xi)_q \) precisely if
\[
Q(X) := R(X + \xi)
\]
has property \( \rho_q \).

Lemma 3.9. Suppose \( \text{grad } R(\xi) \neq 0 \), and \( |R(\xi)|/|\text{grad } R(\xi)| \leq \lambda \) where \( \lambda > 0 \). Then there is a \( t \) in \( 0 \leq t < d \) and a \( \sigma \) with \( \langle \sigma \rangle = t \) such that \( Q^{[\sigma]} \) has property \( (\lambda_t, \xi)_q \).

\(^{+}\) There should be no confusion about property \( \lambda_q \) (\( q = 1 \) or 2) and the numbers \( \lambda_t \) (\( t = 0, 1, \ldots \)).
This lemma may be seen as a variation on our remarks on (1.2).

**Proof.** \(|Q(0)|/|\text{grad} Q(0)| \leq \lambda\), so by writing \(Q = Q_0 + Q_1 + \cdots\) we have \(|Q_0|/|Q_1| \leq \lambda\). By Lemma 3.8 there are \(t, \sigma\) with \(t = \langle \sigma \rangle\) such that \(Q^{(\sigma)}\) has property \((\lambda_t)_q\), hence \(R^{(\sigma)}\) has property \((\lambda_t, \xi)_q\).

\[\square\]

4. POLYNOMIALS WITH SMALL ZEROS, CONTINUED

The contents of this section will not be needed until Section 6.

**Lemma 4.1.** Suppose \(Q(X) \in C_q[X]\) is given by (3.3), and \(Q_\ell \neq 0\) with \(|Q_0|/|Q_\ell| \leq \kappa\) for some \(\ell > 0\), \(\kappa > 0\). Then there is a \(\tau\) and a \(t \in 0 \leq t \leq \langle \tau \rangle\) such that \(Q^{(\tau)}\) has property \((\lambda_t)_q\), where \(\lambda_t\) is given by (3.15) with \(\lambda = \kappa^{t/\ell}\).

**Proof.** There is a \(k \in 0 < k \leq \ell\) with \(Q_k \neq 0\) and \(|Q_k|/|Q_{k-1}| \leq \kappa^{1/\ell} = \lambda\). Pick \(\rho, \sigma\) as in the proof of Lemma 3.6 and set \(G = Q^{(\rho)}\). We again have (3.10), (3.11), (3.12), and therefore \(|G_0|/|G_1| \leq |Q_{k-1}|/|Q_k| \leq \lambda\). We now apply Lemma 3.8 to \(G\). Accordingly, there is a \(\phi\) such that \(G^{(\phi)}\) has property \((\lambda(t))_q\). Set \(\tau = \sigma + \phi\), \(t = \langle \phi \rangle\). Now \(Q^{(\tau)} = Q^{(\sigma)} + \phi\) is a multiple of \((Q^{(\sigma)})^{(\phi)} = G^{(\phi)}\) by a nonzero constant, so that indeed it has property \((\lambda_t)_q\).

\[\square\]

**Proposition 4.2.** Let \(Q \in C_q[X]\) be a nonzero polynomial, and \(\alpha \in A(Q)\). Suppose \(|\alpha| \leq \chi\). Then there is a \(\tau\) and a \(t \in 0 \leq t \leq \langle \tau \rangle\) such that \(Q^{(\tau)}\) has the \((\chi t)_q\)-property, where

\[\chi_t := 2^{(t^2 + 13t + 2)/2} 2^{4t + 1} \chi.\]

Observe that when \(q = 1\), so that \(C_1 = \mathbb{R}\), we still allow \(\alpha\) to be in \(C^n\), and yet get a conclusion on the \((\chi t)_1\)-property. In particular, since \(t \leq \langle \tau \rangle\), there will be a zero \(\beta \in \mathbb{R}^n\) of \(Q^{(\tau)}\) with

\[|\beta| \leq (n + 1)2^{((\tau)^2 + 13(\tau) + 2)/2} n^{(\tau) + 1} |\alpha|\]

In this connection, A. Schinzel kindly pointed out that the Grace–Heawood Theorem (see, e.g., [9]) already implies that if \(Q \in \mathbb{R}[X]\) is a polynomial of one variable of degree \(d > 0\), and \(\alpha\) is a complex root, then some derivative \(Q^{(\tau)}\) with \(0 \leq \tau < d\) has a real root \(\beta\) with \(|\beta| \leq c_1(d)|\alpha|\).

**Proof.** \(Q_0 + \sum_{j=1}^d Q_j(\alpha) = 0\), where \(d = \deg Q\). Since \(|Q_j(\alpha)| \leq |Q_j|/|Q_0| |(n\alpha)|^j\) \((j = 1, 2, \ldots)\) by (3.6), there will be an \(\ell > 0\) with \(|Q_0|/\ell \leq (2n|\alpha|)^\ell |Q_\ell| \leq (2n|\alpha|)^\ell |Q_0|/\ell\). Whether \(Q_0 = 0\) or not, there certainly will be an \(\ell > 0\) where this holds with \(Q_\ell = 0\). We therefore may apply Lemma 4.1 with \(\kappa = (2n|\alpha|)^\ell\). Then \(\lambda = 2n\chi\), so that by (3.15),

\[\lambda_\ell = 2n\chi \cdot 2^{((t^2 + t)/2} (2^n)^t = 2^{((t^2 + 13t + 2)/2} n^{4t + 1} \chi = \chi_t.\]

\[\square\]

**Lemma 4.3.** Suppose \(R \in C_q[X]\) has \(R^{(\rho)}(\xi) \neq 0\) with \(\ell = \langle \rho \rangle > 0\). Further, suppose that

\[\left|\frac{|R(\xi)|}{|R^{(\rho)}(\xi)|}\right|^{1/\ell} \leq \lambda\]

where \(\lambda > 0\). Then there is a \(\tau\) and a \(t \in 0 \leq t \leq \langle \tau \rangle\) such that \(R^{(\tau)}\) has property \((\lambda_t, \xi)_q\) with \(\lambda_t\) given by (3.15).

When \(\ell = 1\), Lemma 3.9 is a little stronger.
Proof:

\[ Q(X) := R(X + \xi) = R(\xi) + \sum_{j=1}^{d} \sum_{|\sigma|=j} R^{(\sigma)}(\xi)X^{\sigma}, \]

so that \( Q_0 = R(\xi) \) and \(|Q_t| \geq |R^{(\sigma)}(\xi)|\). By Lemma 4.1, there are \( \tau, t \) with \( 0 \leq t \leq \langle \tau \rangle \) such that \( Q^{(\tau)} \) has property \( (\lambda_1)_q \), hence \( R^{(\tau)} \) property \( (\lambda_1, \xi)_q \). □

Lemma 4.4. Suppose \( R \in \mathbb{C}_q[X], \xi \in \mathbb{C}_q^n \). Let \( \alpha \) be a zero of \( R \), and \(|\alpha - \xi| \leq \chi \). Then for some \( \tau, t \) with \( 0 \leq t \leq \langle \tau \rangle \), the polynomial \( R^{(\tau)} \) has the \((\chi_1, \xi)_q\)-property where \( \chi_t \) is given by (4.1).

Proof. \( \alpha' := \alpha - \xi \) is a zero of \( Q(X) = R(X + \xi) \). By Proposition 4.2 there are \( \tau, t \) with \( 0 \leq t \leq \langle \tau \rangle \) such that \( Q^{(\tau)} \) has the \((\chi_1)_q\)-property, hence \( R^{(\tau)} \) the \((\chi_1, \xi)_q\)-property.

Remark 4.5. Many results of Sections 3 and 4 can be generalized to rapidly converging power series \( Q \in \mathbb{C}_q[[X]] \), for instance, series \( Q = Q_0 + Q_1 + \cdots \) having \(|Q_j| \leq 2^{-c_2\lambda} \) for some \( c > 1/2 \). When \( Q \in \mathbb{R}[[X]] \) is such a series and \( \alpha \) a complex zero, then some \( Q^{(\tau)} \) has a real zero \( \beta \) with \(|\beta| \leq c_1(n, \langle \tau \rangle)\|\alpha\| \). The series has to converge very rapidly, for this clearly does not hold for \( e(X) = 1 + X + X^2/2! + \cdots \).

One also can show that if \( Q \in \mathbb{R}[[X]] \) is rapidly converging and \( 0 < |\beta| |Q_0|/|Q_1| < \lambda \), then some \( Q^{(\tau)} \neq 0 \) has a nonzero real root \( \beta \) with \(|\beta| \leq c_2(\tau)\lambda \).

5. Proof of Theorem 1.2

We will in fact prove a stronger version:

Theorem 5.1. Let \( M \) be as in Theorem 1.2, and set \( \mu = |M| \).

(i) When \( n \geq q, \mu = n + 1, \xi \in \mathbb{C}_q^n \) and \( N \geq 1 \), there is a \( P \in \mathcal{P}(M) \) for which \( A(P) \) is \((n - 1; N^{1-N/q}|P|^{-1}, c_1 N^{1-N/q}|P|^{-1})_{q, 1}\)-close to \( \xi \).

(ii) When \( \mu = n + 2, \xi \in \mathbb{R}^n \) but \( \xi \notin A(P) \) for any \( P \in \mathcal{P}(M) \), then there are infinitely many \( P \in \mathcal{P}(M) \) with \( A(P) \) being \((n - 1; |P|^{-1}, c_2 |P|^{-1})_{q, 1}\)-close to \( \xi \).

Constants \( c_1, c_2, \ldots \) of this section depend on \( \xi \) and \( M \). Now suppose \( M = \{M_1, \ldots, M_\mu\} \), and for \( u = (u_1, \ldots, u_\mu) \in \mathbb{Z}^\mu \) set

\[ Q(u, X) = u_1 M_1(X) + \cdots + u_\mu M_\mu(X), \]

\[ Q_j(u, X) = (\partial/\partial X_j)Q(u, X) \quad (j = 1, \ldots, n). \]

Let \( L_0, L_1, \ldots, L_n \) be the following linear forms in \( u \):

\[ L_0(u) = Q(u, \xi), \quad L_j(u) = Q_j(u, \xi) \quad (j = 1, \ldots, n). \]

Lemma 5.2. Suppose \( M \) is as in the theorem, and \( \xi \in \mathbb{C}_q^n \) with \( M_i(\xi) \neq 0 \) \((i = 1, \ldots, \mu) \). Then \( L_0, L_1, \ldots, L_n \) are linearly independent.

Proof. Suppose \( M_i(X) = X_1^{a_{1i}} \cdots X_n^{a_{ni}} = X^n \), so that \( \xi^n \neq 0 \) \((i = 1, \ldots, \mu) \). We need to consider the coefficient matrix of the linear forms \( L_0, L_1, \ldots, L_n \). After
multiplying the row coming from \( L_j \) by \( \xi_j \) for \( j = 1, \ldots, n \), and dividing the column of coefficients of \( u_i \) by \( \xi^a_i \) \((i = 1, \ldots, \mu)\), the matrix becomes

\[
\begin{pmatrix}
1 & \cdots & 1 \\
a_{11} & \cdots & a_{\mu 1} \\
\vdots & \ddots & \vdots \\
a_{1n} & \cdots & a_{\mu n}
\end{pmatrix}.
\]

If the forms were dependent, the matrix would have rank \( \leq n \), and there would be \( \mu - n \) independent linear dependence relations of the columns. (By our hypothesis, \( \mu - n \) is 1 or 2). Say \( b_{j1} + \cdots + b_{j\mu} = 0, b_{j1}a_1 + \cdots + b_{j\mu}a_{\mu} = 0 \) with vectors \( b_j = (b_{j1}, \ldots, b_{j\mu}) \) \((j = 1, \ldots, \mu - n)\). This gives \( \mu - n \) independent relations

\[
b_{j1}(a_1 - a_{\mu}) + \cdots + b_{j,\mu-1}(a_{\mu-1} - a_{\mu}) = 0 \quad (j = 1, \ldots, \mu - n)
\]

for \( a_1 - a_{\mu}, \ldots, a_{\mu-1} - a_{\mu} \). Therefore the group generated by all quotients \( M_i/M_{i_k} \) \((1 \leq i, k \leq \mu)\) would have rank \( \leq (\mu - 1) - (\mu - n) = n - 1 \), contradicting the hypothesis.

We now turn to part (i) of the theorem. Suppose first that each \( M_i(\xi) \neq 0 \), so that Lemma 5.2 applies. By Lemma 2.1, there is for \( H \geq 1 \) an \( u \neq 0 \) in \( \mathbb{Z}^\mu \) with \(|u| \leq H, |L_0(u)| \leq c_3 H^{1-\mu/q} \). Since \( L_0, \ldots, L_n \) are linearly independent, \( \max(|L_0(u)|, \ldots, |L_n(u)|) \geq c_4|u| > 0 \). When \( H \) is large, say \( H \geq c_5 \), then \(|L_0(u)| < c_4|u|\), so that \(|L_0(u)|\) may be deleted from the maximum. We then may infer that

\[
|Q(X) := Q(u, X)| \leq c_6|u| \leq c_6 H \quad \text{for} \quad |Q(\xi)| \leq c_3 H^{1-\mu/q}, \quad |\text{grad } Q(\xi)| \geq c_4|u| \geq c_7|Q|.
\]

Thus

\[
|Q(\xi)|/|\text{grad } Q(\xi)| \leq c_8 H^{1-\mu/q}/|Q| = \lambda,
\]

say. We may take \( c_5, c_6, c_8 \) to be \( \geq 1 \). By Lemma 3.9 there is a \( \sigma \) such that \( Q^{\sigma} \) has property \((\lambda, \xi, \xi)\). Since \(|Q^{\sigma}| \leq 2d|Q|\) where \( d \) is a bound on the degrees of the monomials in \( \mathcal{M} \), there is an \( r \in \mathbb{N} \) with \( 2d^{-1}|Q| < r|Q^{\sigma}| \leq 2d|Q| \). Setting \( P = rX^{\sigma}Q^{\sigma} \), we have \( P \in \mathcal{P}(\mathcal{M}) \) with \(|Q| < |P| \leq 2d|Q| \leq c_6 2^d H \) and

\[
H^{1-\mu/q}/|P| \leq H^{1-\mu/q}/|Q| \leq \lambda \leq \lambda(\sigma) \leq c_9 \lambda \leq c_{10} H^{1-\mu/q}/|P|.
\]

When \( N > c_5 c_6 2^d \), we apply the above with \( H = N/(c_6 2^d) > c_5 \). Then \(|P| \leq N \) and \( N^{1-\mu/q}/|P| \leq \lambda(\sigma) \leq c_{11} N^{1-\mu/q}/|P| \). So by Remark 3.1 we may conclude that \( A(P) \) is \((n - 1; N^{1-\mu/q}/|P|, c_1 N^{1-\mu/q}/|P|)\) \(q\)-close to \( \xi \), with \( c_1 = (n + 1)c_{11} \).

When \( 1 \leq N \leq c_5 c_6 2^d \), let \( M \in \mathcal{M} \) be nonconstant. Then \( P = M \) has \(|P| = 1 \leq N \), and \( A(P) \) contains a coordinate hyperplane, say \( x_n = 0 \). By Remark 3.2, \( A(P) \) is \((n - 1; 1, 1 + (\xi))\) \(q\)-close to \( \xi \), hence \((n - 1; N^{1-\mu/q}, c_1 N^{1-\mu/q})\) \(q\)-close, with \( c_1 = c_1(\xi, \mathcal{M}) \).

Finally, when some \( M_i(\xi) = 0 \), then some \( \xi_i = 0 \), say \( \xi_n = 0 \). The assertion holds with \( P = X_n \), since now \( A(P) \) is \((n - 1; 1, \rho)\) \(q\)-close to \( \xi \) for any \( \rho > 0 \).

Let us proceed to part (ii). Since \( \xi \notin A(P) \) for any \( P \in \mathcal{P}(\mathcal{M}) \), we have each \( M_i(\xi) \neq 0 \), so that \( L_0, \ldots, L_n \) are independent by Lemma 5.2. Since the number of variables \( u_1, \ldots, u_\mu \) is \( \mu = n + 2 \), we may apply the Theorem of [7] (with \( n, \mu \) playing the respective roles of \( m, n \) in [7]). Accordingly, there are infinitely many integer points \( u \neq 0 \) with

\[
|L_0(u)| \leq c_{12} \max(|L_1(u)|, \ldots, |L_n(u)|)|u|^{-\mu}.
\]
But now $Q(X) := Q(u, X)$ has $|Q| \leq c_{13}|u|$ and $|Q(ξ)| \leq c_{13}|\text{grad} \, Q(ξ)||u|^{-μ}$, hence

$$|Q(ξ)|/|\text{grad} \, Q(ξ)| ≤ c_{14}|Q|^{-μ} = \lambda,$$

say. We may suppose $c_{14} ≥ 1$. By Lemma 3.9, some $Q^{|σ|}$ has property $(λ_{|σ|}, ξ)_1$. Pick $P = rX^σQ^{|σ|}$ with $|Q| < |P| ≤ 2^{|σ|}|Q|$. Again, $P$ has property $(λ_{|σ|}, ξ)_1$, where now $|P|^{-μ} ≤ λ_{|σ|} ≤ c_{15}|P|^{-μ}$. Therefore $A(P)$ is $(n − 1; |P|^{-μ}, c_{12}2|P|^{-μ})_1$-close to $ξ$.

Our argument gives infinitely many distinct polynomials $Q$, and since $|P| > |Q|$, also infinitely many distinct polynomials $P$.

6. QUOTING A RESULT OF LAURENT AND ROY

Their work is done in a projective setting. When $θ, ρ$ are in $P^n_q = P^n(C_q)$, the symbols $θ, ρ$ will denote representatives $(θ_0, . . . , θ_n)$, $(ρ_0, . . . , ρ_n)$ in $C^{n+1}_q\setminus\{0\}$. When $P ∈ Z[X_0, . . . , X_n]$ is homogeneous, we will say that $θ$ is a zero if $P(θ) = 0$. The distance of $θ, ρ$ is defined to be

$$\text{dist} (θ, ρ) := |θ ∧ ρ|/|θ||ρ||$$

where $θ ∧ ρ$ is the wedge product, with components $θ_ιρ_ι − θ_ιρ_ι$ ($0 ≤ i < j ≤ n$).

We will be dealing with the case of Laurent and Roy’s work $[8]$ where $K = Q$, and $|\.|_w$ is the standard archimedean absolute value, so that $ε_w = λ_w = 1$ and $C_w = C$. Moreover, in our case $Z = P^n_q$, so that dim $Z = n$, deg $Z = 1$, $h(Z) = 0$. Furthermore, $S_n = S$ and $D_n = D$ will be fixed, and we will write $n, t, d$ in place of $m, n, D$ in $[8]$. Also, we will use the Height $h|P|$ rather than $h(P) = \log |P|$, and our $N_t$ corresponds to $\exp T_n$ in $[8]$.

By $P(d)$, where $d > 0$, we will denote the set of nonzero homogeneous polynomials in $Z[X_0, . . . , X_n]$ of degree $d$. We will use a rather special case of Theorem 2.2 of $[8]$, which may be formulated as follows.

**Proposition 6.1.** Suppose $ξ ∈ P^n(C)$, and $ξ$ is a representative with $|ξ| = 1$. Let $p, q, S, d$ be positive integers with $p < q$, $S ≤ d$. Suppose $N_t, V_t$ for $p ≤ t ≤ q$ are reals with $1 < N_p ≤ ⋯ ≤ N_q$ and $2S < V_p ≤ ⋯ ≤ V_q$. Set

$$η = d^{-1}\log m = d^{-1}\log \left(\frac{n + d}{d}\right),$$

$$a = (n + 1)(3\log(n + 1) + (n + 4)η + \log 4),$$

$$b = n + 1,$$

$$c = (n + 1)V_p/S + (n + 4)η + 2\log(n + 1),$$

and assume that

$$(6.1) \quad S^nV_q > (ad + b\log N_q + cS)d^n.$$

Suppose that for $t = p + 1, . . . , q$ there is a polynomial $P_t ∈ P(d)$ with $|P_t| ≤ N_t$ and

$$(6.2) \quad |P^{|σ|}_t(ξ)| ≤ |P_t| \exp(-V_t + (|σ|)/S)V_t$$

for every $σ ∈ N_q^{n+1}$ with $|σ| < S$.

Then for some $t$, $p < t ≤ q$, $P_t$ has a zero $θ ∈ P^n(C)$ with

$$\text{dist} (θ, ρ) ≤ \exp(-V_{t-1}/S).$$

The following already appears implicitly in the proof of Theorem 2.3 of $[8]$.
Corollary 6.2. Suppose $\zeta \in \mathbb{P}^n(\mathbb{C})$, with representative $\zeta$ having $|\zeta| = 1$. Let $p, S, d$ be positive integers with $S \leq d$. Further, suppose that $1 < N_p \leq N_{p+1} \leq \cdots$ and $2S < V_p \leq V_{p+1} \leq \cdots$, and that $V_t$ as well as

$$S^nV_t - (n + 1)d^n \log N_t$$

tends to infinity with $t$. Finally, suppose that for each $t > p$ there is a polynomial $\hat{P}_t \in \hat{P}(d)$ with $|\hat{P}_t| \leq N_t$ and

$$|\hat{P}_t(\zeta)| \leq |\hat{P}_1| \exp(-V_t).$$

Then for some $t > p$, $\hat{P}_t$ has a zero $\theta \in \mathbb{P}^n(\mathbb{C})$ with

$$\text{dist}(\theta, \zeta) < \exp((2d^2 - V_{t-1})/S).$$

Proof. Assume that this were false. Then for every zero $\zeta$ of $\hat{P}_t$, dist$(\theta, \zeta) \geq \rho_t$, where $\rho_t$ is the right-hand side of (6.4). Suppose $\sigma \in \mathbb{N}^{n+1}$, $0 < \langle \sigma \rangle < S$. By Lemma 6.8 of [5],

$$|\hat{P}_t(\zeta)| \leq \left( \frac{d}{\langle \sigma \rangle} \right) \left( \frac{\rho_t}{2} \right)^{-\langle \sigma \rangle} |\hat{P}_1(\zeta)|$$

$$< 2^{d+S}|\hat{P}_1| \exp(-V_t + (\langle \sigma \rangle/S)(V_{t-1} - 2d^2))$$

$$\leq |\hat{P}_1| \exp(-V_t + (\langle \sigma \rangle/S)V_t)$$

since $2^{d+S} \leq 2^{2d} < \exp(2d) \leq \exp(2d^2/S)$. So we have (6.2) for every $\sigma$ with $\langle \sigma \rangle < S$. The quantities $a, b, c$ of Proposition 6.1 are independent of $t$, so that

$$(ad + b \log N_t + cS)d^n = bd^n \log N_t + O(1) = (n + 1)d^n \log N_t + O(1)$$

as $t \to \infty$, and the divergence of (6.3) implies that (6.1) will hold for some $q > p$. By the Proposition, there will be a $t, p < t \leq q$, such that $\hat{P}_t$ has a zero $\theta \in \mathbb{P}^n_q$ with

$$\text{dist}(\theta, \zeta) \leq \exp(-V_{t-1}/S) < \exp((2d^2 - V_{t-1})/S).$$

So the desired conclusion is true after all.

As in (1.5), (1.6), set $m = \left( \frac{n + d}{d} \right)$ and $\nu = \langle (m/q) - 1 \rangle/S_\circ$ where $S_\circ = \lceil \sqrt[3]{q(n + 1)!} \rceil$.

Proposition 6.3. Suppose $n > 1$. Let $\zeta \in \mathbb{P}^n_q$ not be a zero of any $\hat{P} \in \hat{P}(d)$. Then there are infinitely many $\hat{P} \in \hat{P}(d)$ having a zero $\theta \in \mathbb{P}^n_q$ with

$$\text{dist}(\theta, \zeta) < c_1(n, d)|\hat{P}|^{-\nu}.$$
Setting $S = S_\circ$ we have for large $t$,

$$S^n V_t \geq q(n+1)!((d^n/qn!) + \varepsilon) \log N_t,$$

so that (6.3) tends to infinity with $t$. From Corollary 6.2 we may infer that for any $p \in \mathbb{N}$ there is a $t > p$ such that $\tilde{P}_t$ has a zero $\zeta \in \mathbb{P}^n(\mathbb{C})$ with

$$\text{dist} (\theta, \zeta) < \exp((2d^2 - V_{t-1})/S_\circ).$$

But

$$V_{t-1} - 2d^2 > V_t - c_3(n, d) \geq ((m/q) - 1) \log N_t - c_4(n, d),$$

yielding

$$\text{dist} (\theta, \zeta) < \exp(-\nu \log N_t + c_5(n, d)) < c_1(n, d)N_t^{-\nu} \leq c_1(n, d)|\tilde{P}_t|^{-\nu}.$$

The proposition follows. \qed

7. Proof of Theorem 1.3

Lemma 7.1. Suppose $n > 1$. Also suppose $\xi \in \mathbb{C}^n_q$, $|\xi| \leq \psi$, and $P(\xi) \neq 0$ for every $P \in \mathcal{P}(d)$. Then there are infinitely many polynomials $P \in \mathcal{P}(d)$ with

$$\delta(\xi, A(P)) < c_1(n, d, \psi)|P|^{-\nu},$$

where $\nu$ is given by (1.6).

When $q = 2$, so that $\delta(\xi, A(P)) = \delta_q(\xi, A(P))$, this is the assertion of Theorem 1.3. But when $q = 1$, $\xi \in \mathbb{C}^n_q = \mathbb{R}^n$, we can at this stage only conclude that there is an $\alpha \in \mathbb{C}^n$ with $P(\alpha) = 0$ and $|\alpha - \xi| < c_1(n, d, \psi)|P|^{-\nu}$, whereas the theorem asserts the existence of an $\alpha \in \mathbb{R}^n$ with this property.

Proof. Let $\zeta \in \mathbb{P}^n_q$ be the point represented by $\xi = (\zeta_0, \zeta_1, \ldots, \zeta_n) := (1, \xi_1, \ldots, \xi_n)$. Then $\tilde{P}(\zeta) \neq 0$ for $\tilde{P} \in \tilde{P}(d)$. By Proposition 6.3 there are infinitely many $\tilde{P} \in \tilde{P}(d)$ having a zero $\theta \in \mathbb{P}^n(\mathbb{C})$ with (6.5).

Let $\theta = (\alpha_0, \alpha_1, \ldots, \alpha_n)$ be a representative of $\theta$. Then

$$|\alpha_i \zeta_j - \alpha_j \zeta_i| < |\theta||\zeta||c_2(n, d)||\tilde{P}|^{-\nu} = |\theta|E,$$

say, for $0 \leq i < j \leq n$. Since $\zeta_0 = 1$, we have $|\alpha_j - \zeta_j \alpha_0| < |\theta|E$ for $1 \leq j \leq n$, hence $|\theta| < |\zeta_0||\alpha_0| + |\theta|E < (1 + \psi)|\alpha_0| + \frac{1}{2} |\theta|$ when $|\tilde{P}|$ is large, hence $E$ small. Then $|\theta| < 2(1 + \psi)|\alpha_0|$, so that $\alpha_0 \neq 0$. After renormalization we may suppose that $\alpha_0 = 1$, hence $|\theta| < 2 + 2\psi$, yielding

$$|\alpha_j - \xi_j| = |\alpha_i \zeta_j - \xi_j| < (2 + 2\psi)E \leq c_1(n, d, \psi)|\tilde{P}|^{-\nu} \quad (1 \leq j \leq n).$$

The polynomial $P(X) = \tilde{P}(1, X_1, \ldots, X_n)$ has the zero $\alpha = (\alpha_1, \ldots, \alpha_n)$ with $|\alpha - \xi| < c_1(n, d, \psi)|P|^{-\nu}$. \qed

A polynomial $P \in \mathbb{Z}[X]$ will be called irreducible if it is irreducible over $\mathbb{Q}$, i.e., if it cannot be written as the product of two nonconstant polynomials. This does not preclude the possibility that $P = pP_1$ where $p > 1$ is an integer and $P_1 \in \mathbb{Z}[X]$. We will establish Theorem 1.3 in the following stronger form.

Theorem 7.2. Suppose $n > 1$, and further $\xi \in \mathbb{C}^n_q$, $|\xi| \leq \psi$, with $P(\xi) \neq 0$ for any $P \in \mathcal{P}(d)$. Then there are infinitely many irreducible polynomials $P \in \mathcal{P}(d)$ for which $A(P)$ is $(n-1; |P|^{-\nu}, c_3|P|^{-\nu})_\circ$-close to $\xi$.

The constant $c_3$ and subsequent constants $c_4, \ldots$ depend at most on $n, d, \psi$. 

"
Proof: When \( Q = Q_1Q_2 \) is of degree at most \( d \), and \( Q_1, Q_2 \) in \( \mathbb{Z}[X] \), then it is well
known that \( |Q_i| \leq c_4|Q| \) (\( i = 1, 2 \)). Set \( c_5 = 2^d c_4 \). Let \( R \) be one of the infinitely
many polynomials in the conclusion of the preceding lemma, so that \( R \) has a zero \( \alpha \in \mathbb{C}^n \) with
\( |\alpha - \xi| < c_1 |R|^{-\nu} = \chi \), say. We may choose \( c_1 \) to be \( \geq 1 \). By Lemma
4.4, some \( R[\tau] \) has the \((\chi_t, \xi)_q\)-property where \( 0 \leq t \leq |\tau| \leq d \) and \( \chi_t \) is given by
(4.1). By Remark 3.4 some irreducible factor \( Q \) of \( R[\tau] \) also has this property. Here
\( |Q| \leq c_4|R[\tau]| \leq 2^d c_4|R| = c_5|R| \). For suitable \( r \in \mathbb{N} \) the polynomial \( P = rQ \) has
\( |R| < |P| < c_5|R| \), and still the \((\chi_t, \xi)_q\)-property. Here
\( |P|^{-\nu} < |R|^{-\nu} \leq \chi \leq c_6 \chi = c_1 c_6 |R|^{-\nu} < c_7 |P|^{-\nu} \).
By Remark 3.1, \( A(P) \) is indeed \((n - 1; |P|^{-\nu}, c_3 |P|^{-\nu})_q\)-close to \( \xi \).  \( \square \)

When \( n = 1 \), the present method would have given Theorem 1.3 with \( \nu = d/3 \)
when \( q = 1 \) and \( \nu = (d - 1)/8 \) when \( q = 2 \). But as pointed out in the Introduction,
Wirsing’s method gives better exponents.

8. Proof of Theorem 1.5

Given \( N \geq 1 \), set
\[
M = N^{(m/q) - 1}, \quad M_0 = N^{m_1 - 1}.
\]
We will establish the stronger

Theorem 8.1. (a) Suppose \( \xi \in \mathbb{C}^n, \psi \geq |\xi| \). Then for \( N \geq 1 \) there is an irreducible
\( P \in \mathcal{P}(d) \) with \( |P| \leq N \) and with \( A(P) \) \((n - 1; (M|P|)^{-1/d}, c_1 (M|P|)^{-1/d})_q\)-close to \( \xi \).

(b) Suppose \( \xi \in \mathbb{R}^n, \psi \geq |\xi| \). Then for \( N \geq 1 \) there is an \( \ell, 1 \leq \ell \leq d \), and
an irreducible \( P \in \mathcal{P}(d) \) with \( |P| \leq N^{1/\ell} \) such that \( A(P) \) is \((n - 1; (M_0|P|)^{-1/\ell}, c_2 (M_0|P|)^{-1/\ell})_1\)-close to \( \xi \).

The constants \( c_1, c_2, \ldots \) of this section depend only on \( n, d, \psi \). The polynomial in
(a) (respectively (b)) has \( \delta_\ell(\xi, A(P)) \leq c_3 (M|P|)^{-1/d} \) (respectively \( \delta_1(\xi, A(P)) \leq c_4 (M_0|P|)^{-1/\ell} \)), so that Theorem 1.5 follows.

Proof of Theorem 8.1. (a) Every factor \( Q \) of a derivative \( R[\tau] \) of a polynomial \( R \in
\mathcal{P}(d) \) has \( |Q| \leq c_5 |R[\tau]| \leq c_6 |R| \). Suppose \( N > c_5 \). By Lemma 1.1 there is an
\( R \in \mathcal{P}(d) \) with \( |R| \leq N/c_6 \) and
\[
|R(\xi)| \leq m \psi'^d (N/c_6)^{1-m/q}
\]
where \( \psi' = \max(1, \psi) \). Taylor expansion at \( \xi \) gives
\[
R(X) = \sum_\sigma (X - \xi)^\sigma R(\sigma)(\xi),
\]
yielding
\[
R[\tau](0) = \sum_\sigma' \left( \begin{array}{c} \sigma_1 \\ \tau_1 \end{array} \right) \cdots \left( \begin{array}{c} \sigma_n \\ \tau_n \end{array} \right) (-\xi)^{\sigma - \tau} R(\sigma)(\xi),
\]
where \( \sum' \) signifies a sum over \( \sigma \) with \( \sigma_i \geq \tau_i \) (\( i = 1, \ldots, n \)). We obtain
\[
|R| = \max_\tau |R[\tau](0)| \leq c_7 \max_\sigma |R(\sigma)(\xi)|.
\]
When \( N \) is so large that \( c_7m^{3/d}(N/c_6)^{1-m/q} < 1 \), there will be a \( \sigma \neq 0 \) with \(|R^{|\sigma|}(\xi)| \geq |R|/c_7\). Then setting \( \ell = (\sigma) \) we have

\[
(8.2) \quad |R(\xi)|/|R^{|\sigma|}(\xi)|^{1/\ell} \leq c_6(M|R|)^{-1/\ell} \leq c_8(M|R|)^{-1/d} = \lambda,
\]

say, where \( c_8 \) may be taken to be \( \geq 1 \). By Lemma 4.3 there are \( r, t \) such that \( R^{(|\sigma|)} \) has property \( (\lambda_t, \xi_t)^q \). By an argument used in the proof of Theorem 1.3, there is an irreducible polynomial \( P \in \mathcal{P}(d) \) with \(|R| < |P| \leq c_6|R| \leq N \) which also has property \( (\lambda_t, \xi_t)^q \). Now

\[
(M|R|)^{-1/d} \leq \lambda \leq c_9 \lambda \leq c_{10}(M|R|)^{-1/d},
\]

so that by Remark 3.1, \( A(P) \) is \((n - 1; (M|R|)^{-1/d}, c_1(M|R|)^{-1/d})_q\)-close to \( \xi \).

We are left with the case when \( N \) is small, say \( 1 \leq N \leq c_{11} \), hence \( 1 \leq M \leq c_{12} \). We set \( P(X) = X_n \), so that \(|P| = 1\). By Remark 3.2, \( A(P) \) is \((n - 1; 1, 1 + |\xi|)_q\)-close to \( \xi \), hence \((n - 1; M^{-1/d}, c_1M^{-1/d})_q\)-close.

(b) For \( R \in \mathcal{P}(d) \) set

\[
\omega_j(R) = \max_{(\sigma) = j} |R^{|\sigma|}(\xi)| \quad (1 \leq j \leq d).
\]

There are \( \left( \begin{array}{c} n + j - 1 \\ j \end{array} \right) \) \( n \)-tuples \( \sigma \) with \( (\sigma) = j \), so that

\[
\sum_{j=1}^{d} \sum_{(\sigma) = j} 1/j = m_v - 1.
\]

By Minkowski’s Theorem on linear forms, there is for \( N > 0 \) an \( R \in \mathcal{P}(d) \) with

\[
\omega_j(R) \leq N^{1/j} \quad (1 \leq j \leq d),
\]

\[
|R(\xi)| \leq c_{13}N^{1-m_v}.
\]

Since each coefficient of \( R \) is a linear combination of the \( R^{|\sigma|}(\xi) \), the Height \( |R| \leq c_{14} \max \{ |R(\xi)|, \omega_1(R), \ldots, \omega_d(R) \} \), where in view of (8.3) we may omit \(|R(\xi)|\) from this maximum when \( N \) is large. There is then an \( \ell, 1 \leq \ell \leq d \) with \(|R| \leq c_{14}\omega_\ell(R)\), say \(|R| \leq c_{14}|R^{|\sigma|}(\xi)| \leq c_{14}N^{1/\ell} \), and

\[
(|R(\xi)|/|R^{|\sigma|}(\xi)|)^{1/\ell} \leq c_{15}(M_v|R|)^{-1/\ell} =: \lambda'.
\]

From here on, the argument is very much as in (a).

One obtains an irreducible polynomial \( P \in \mathcal{P}(d) \) with \(|R| < |P| \leq c_6|R| \leq c_{16}N^{1/\ell} \) with property \( (\phi, \xi)_q \), where \((M_v|P|)^{-1/\ell} \leq \phi \leq c_{17}(M_v|P|)^{-1/\ell} \).

When \( N \geq (2c_{16})^d \), we repeat the above argument with \( N' = N/(2c_{16})^d \), to obtain an irreducible \( P \in \mathcal{P}(d) \) with \(|P| \leq N \) and property \( (\phi, \xi)_q \), where \((M_v|P|)^{-1/\ell} \leq \phi \leq c_{18}(M_v|P|)^{-1/\ell} \), so that \( A(P) \) is indeed \((n - 1; (M_v|P|)^{-1/\ell}, c_2(M_v|P|)^{-1/\ell})_q\)-close to \( \xi \). When \( N < (2c_{16})^d \), one reasons as at the end of part (a). \( \square \)

9. Proof of Theorem 1.7

Constants \( c_1, c_2, \ldots \) of this section will depend on \( n, d, \psi \). We will strengthen the theorem to

**Theorem 9.1.** There is an \( \varepsilon_\phi = \varepsilon_\phi(n, d, \psi) \), \( 0 < \varepsilon_\phi \leq 1 \), such that Theorem 1.7 holds with the stronger conclusion that \( A(P) \) is \((r; \varepsilon_\phi L^{-1}, L^{-1})_q\)-close to \( \xi \).
Proof: We begin with the case $s = 1$, and we write $d = d_1$, $m = m_1$. We start out as in the proof of Theorem 8.1(a). So for large $N$, there is an $R \in \mathcal{P}(d)$ with (8.2). But (8.2) also holds with $\lambda$ replaced by $\lambda' = c_1 M^{-1/d}$. Proceeding as with Theorem 8.1(a), we eventually obtain a $\lambda'_i$ with $M^{-1/d} \leq \lambda'_i \leq c_2 M^{-1/d}$, and a polynomial $P \in \mathcal{P}(d)$ with $|P| \leq N$ and property $(\lambda'_i, \xi)_q$. In particular, $A(P)$ is $(n-1; M^{-1/d}, c_3 M^{-1/d})_q$-close to $\xi$, where $c_3 = (n+1)c_2$. On the other hand, when $N \geq 1$ is small, then by Remark 3.2 this holds with $P = X_n$, provided $c_3$ is chosen sufficiently large.

Let $\phi \geq 1$ be a quantity to be chosen below. Given $L \geq 1$, let $M \geq 1$ be determined by $M^{1/d} = \phi L$, and $N$ by $N^{(m/q)-1} = M$. Then the polynomial $P$ above has $|P| \leq N = M^{q/(m-q)} = (\phi L)^{dq/(m-q)}$. Moreover, after suitable numbering of the variables,

(9i) $A(P)$ is special $(n-1; \phi^{-1} L^{-1}, c_3 \phi^{-1} L^{-1})_q$-close to $\xi$,
(9ii) $P(\alpha_1, \ldots, \alpha_{n-1}, X_n)$ for given $\alpha_1, \ldots, \alpha_{n-1}$ is a nonzero polynomial in $X_n$.

The assertion of Theorem 9.1 for $s = 1$ follows on setting $\phi = c_3$, and then $\gamma_1 = \phi, \epsilon_1 = \phi^{-1}$.

It remains for us to do the induction step from $s = 1$ to $s$, where $1 < s < n$. Suppose we already know that given $L \geq 1$, there is after suitable numbering of the variables a $P^* = (P_1, \ldots, P_{s-1})$ in $\mathcal{P}(d^*) = \mathcal{P}(d_1, \ldots, d_{s-1})$ with

\[
|P_i| \leq (\gamma_i L)^{d_i q/(m_i - q)} \quad (i = 1, \ldots, s - 1)
\]

such that

\[
A(P^*) \text{ is special } (r + 1; \varepsilon_{s-1} L^{-1}, L^{-1})_q \text{-close to } \xi.
\]

Moreover, our induction will be such that for $1 \leq i < s$, $P_i(\alpha_1, \ldots, \alpha_{n-i}, X_{n-i+1})$ for fixed $\alpha_1, \ldots, \alpha_{n-i}$ is a nonzero polynomial in $X_{n-i+1}$.

There is then a regular map $w^* : \mathcal{C}(r+1; \varepsilon_{s-1} L^{-1}, \xi) \to \mathcal{C}(s-1; L^{-1}, \xi)$ such that $P_i(w^*(z^*), w^*(z^*)) = 0 (i = 1, \ldots, s - 1)$ for $z^* = (z_1, \ldots, z_{s-1}) \in \mathcal{C}(r + 1; \varepsilon_{s-1} L^{-1}, \xi)$. When $\xi \in \mathcal{C}^n = \mathbb{R}^n$, the map is a regular map.

Suppose $\xi = (\xi_1, \ldots, \xi_n)$, and set $\tilde{\xi} = (\tilde{\xi}_1, \ldots, \tilde{\xi}_{s+1})$. We are going to apply the argument we gave for the case $s = 1$ to $\xi \in \mathcal{C}_q^{s+1}$. So given $L \geq 1$, there is a $P \in \mathcal{Z}[X_1, \ldots, X_{r+1}]$ of degree $\leq d_s$ with

\[
|P| \leq (\phi L)^{d_s q/(m_s - q)}
\]

which after suitable numbering of the variables $X_1, \ldots, X_{r+1}$ has the properties corresponding to (9i), (9ii). Thus

\[
A(\bar{P}) \subset \mathcal{C}_q^{r+1} \text{ is special } (r; \phi^{-1} L^{-1}, \tilde{c}_3 \phi^{-1} L^{-1})_q \text{-close to } \tilde{\xi},
\]

and $A(\alpha_1, \ldots, \alpha_r, X_{r+1})$ for given $\alpha_1, \ldots, \alpha_r$ is a nonzero polynomial in $X_{r+1}$. We now set $\phi = \tilde{c}_3 / \varepsilon_{s-1}$, and then

\[
A(\bar{P}) \text{ is special } (r; \varepsilon_s L^{-1}, \varepsilon_{s-1} L^{-1})_q \text{-close to } \tilde{\xi}
\]

with $\varepsilon_s = \varepsilon_{s-1} / \tilde{c}_3 \leq \varepsilon_{s-1}$. There is a regular map

\[
\bar{w} : \mathcal{C}(r; \varepsilon_s L^{-1}, \tilde{\xi}) \to \mathcal{C}(1; \varepsilon_{s-1} L^{-1}, \tilde{\xi})
\]

(the tildes referring to sets in $\mathcal{C}^{r+1}$) with $\bar{P}(\tilde{z}, \bar{w}(\tilde{z})) = 0$ for $\tilde{z} = (z_1, \ldots, z_r) \in \mathcal{C}(r; \varepsilon_s L^{-1}, \tilde{\xi})$. Moreover, $\bar{w}$ is real regular when $\xi \in \mathbb{R}^n$.

Let $w : \mathcal{C}(r; \varepsilon_s L^{-1}, \xi) \to \mathcal{C}(s, L^{-1}, \xi)$ be given by

\[
w(\tilde{z}) = (\bar{w}(\tilde{z}), w^*(z_1, \ldots, z_r, \bar{w}(\tilde{z}))).
Set $P_i(X) = \tilde{P}(X_1, \ldots, X_{r+1})$. Then indeed $P_i(z, \omega(z)) = 0$ for $1 \leq i \leq s$ and $z \in C(r; \varepsilon, L^{-1}, \xi)$. Therefore $A(P) = A(P_1, \ldots, P_s)$ is $(r; \varepsilon, L^{-1}, L^{-1})_q$-close to $\xi$.

By (9.1), (9.2), we have (1.16) for $i = 1, \ldots, s$ if we set $\gamma_s = \phi$.

For $1 \leq i \leq s$, and any $\alpha_1, \ldots, \alpha_{n-i}$, the polynomial $P_i(\alpha_1, \ldots, \alpha_{n-i}, X_{n-i+1})$ in $X_{n-i+1}$ is nonzero. Therefore $A(P)$ has codimension $\geq s$, hence dimension $\leq r$. But by the $(r; \varepsilon, L^{-1}, L^{-1})_q$-closeness, it has in fact dimension $r$, so that $P \in P(d)$.

10. Proof of Theorem 1.8

Again we will prove a stronger result:

**Theorem 10.1.** Let $r, s, d, d = d_1 \cdots d_s$, $m_i$ $(i = 1, \ldots, s)$ and $\Phi, \Psi$ be as in Theorem 1.8. Then when $\xi \in \mathbb{C}_n^r$, $|\xi| \leq \psi$ and $N \geq 1$, there is a variety $V$ of dimension $r$, degree $\leq d$, and Height $H(V) \leq N$ which is $(r, N^{-\Phi}, c_1 N^{-\Psi})_q$-close to $\xi$.

The constants $c_1, c_2, \ldots$ of this section, unless indicated otherwise, depend on $n, d, \psi$.

**Proof.** Given $L \geq 1$, we know from Theorem 9.1 that after suitable numbering of the coordinates there is a $P \in P(d)$ with (1.16), and with $A(P)$ being $(r; \varepsilon, L^{-1}, L^{-1})_q$-close to $\xi$. Here $P_i = P_i(X_1, \ldots, X_{n-i+1})$, and $P_i(\alpha_1, \ldots, \alpha_{n-i}, X_{n-i+1})$ for given $\alpha_1, \ldots, \alpha_{n-i}$ is a nonzero polynomial in $X_{n-i+1}$ $(1 \leq i \leq s)$.

Set $e_i = \deg P_i$, so that $e_i \leq d_i$ and let $\tilde{P}_i = X_0^{e_i} P_i(X_1/X_0, \ldots, X_{n-i+1}/X_0)$ be the homogenized version of $P_i$. Let $\div\tilde{P}_i$ be the divisor of $\tilde{P}_i$ in $\mathbb{P}^n(C)$, so that $\deg(\div\tilde{P}_i) = \deg P_i = e_i$. The intersection product $(\div\tilde{P}_i) \cdot (\div\tilde{P}_{i-1})$ is a divisor of degree $e_i e_{i-1}$, since $\tilde{P}_{i-1}$ does not vanish identically on any component of $\div\tilde{P}_i$, by the special properties of our polynomials. (For degrees of intersection products see [8] Lemma 4.2].) More generally, $\tilde{P}_i$ for $1 \leq i < s$ does not vanish identically on any component of $(\div\tilde{P}_i) \cdot \ldots \cdot (\div\tilde{P}_{i+1})$. One may conclude by downward induction on $i$ that

$$Z_i := (\div\tilde{P}_i) \cdot \ldots \cdot (\div\tilde{P}_i) \quad (1 \leq i \leq s)$$

has degree $e_i \cdots e_1$.

The height of a cycle $C = t_1 \tilde{V}_1 + \cdots + t_s \tilde{V}_s$ with projective varieties $\tilde{V}_i$ is $H(C) = H(\tilde{V}_1)^{t_1} \cdots H(\tilde{V}_s)^{t_s}$, and it is not very hard to see that $H(\div\tilde{P}_i) \leq c_2(n, d) H(\tilde{P}_i) = c_2(n, d)|P|$ for a polynomial $P$ of degree $\leq d$. The logarithmic height is $h = \log H$.

Since $Z_i = Z_{i+1} \cdot (\div\tilde{P}_i)$, we may infer, e.g., from [8] Proposition 4.9, that

$$\frac{h(Z_i)}{e_i \cdots e_1} \leq \frac{h(Z_{i+1})}{e_i \cdots e_{i+1}} + h(P_i) + c_3 \quad (1 \leq i < s)$$

where $c_3 = c_3(n, e_s, \ldots, e_i)$, so that $H(Z_s) \leq c_2|P_s|$, $H(Z_i) \leq c_4 H(Z_{i+1})^{e_i} |P_i|^{e_{i+1}}$ $(1 \leq i < s)$, and eventually

$$H(Z_1) \leq c_5 |P_1|^{e_{s-1} e_s} |P_2|^{e_{s-2} e_s e_1} \cdots |P_s|^{e_1} = c_6 L^\Psi.$$
Here $Z_1$ is a cycle of dimension $n - s = r$. Write $Z_1 = t_1 \hat{V}_1 + \cdots + t_\ell \hat{V}_\ell$ where
the $\hat{V}_j$ are projective varieties of dimension $r$, and let $V_j$ be the affine part of $\hat{V}_j$.

It is clear that $A(P) = V_1 \cup \cdots \cup V_\ell$. Let $w : C(r, \varepsilon_s L^{-1}, \xi) \to C'(s, L^{-1}, \xi)$ be
the regular map arising from the special $(r; \varepsilon_s L^{-1}, L^{-1})_q$-closeness to $\xi$ of $A(P)$.
Then $(z, w(z)) \in A(P)$ for every $z$ considered. Since the map is regular, there is
some $V_j$, say $V = V_1$, such that $(z, w(z)) \in V$ for every $z \in C(r, \varepsilon_s L^{-1}, \xi)$. We may
conclude that $V$ is $(r; \varepsilon_s L^{-1}, L^{-1})_q$-close to $\xi$. Here $\dim V = r$, $\deg V \leq \deg Z_1 =
\varepsilon_s \cdots c_1 \leq d$, and $H(V) \leq H(Z_1) \leq c_6 L^6$.

We may enlarge $c_6$ if necessary, to have $\varepsilon_s c_6^6 \geq 1$. Now if $N \geq c_6$, determine
$L \geq 1$ by $c_6 L^6 = N$. The above argument with this value of $L$ gives $V$ with
$H(V) \leq N$ which is $(r; \varepsilon_s c_6^6 N^{-\psi}, c_6^6 N^{-\psi})_q$-close to $\xi$, hence $(r; N^{-\psi}, c_1 N^{-\psi})_q$-

When $N \leq c_6 = c_6(n, d, \psi)$, let $V$ be given by $x_{r+1} = \cdots = x_n = 0$, so
that $\dim V = r$, $\deg V \leq 1$, $H(V) = 1$. By Remark 3.2, $V$ is special $(r; 1, 1 + |\xi|)_q$-close to $\xi$, hence $(r; N^{-\psi}, c_1 N^{-\psi})_q$-close with suitable $c_1$.

\[ \square \]

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