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ROTATION TOPOLOGICAL FACTORS OF MINIMAL $\mathbb{Z}^d$-ACTIONS ON THE CANTOR SET

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ABSTRACT. In this paper we study conditions under which a free minimal $\mathbb{Z}^d$-action on the Cantor set is a topological extension of the action of d rotations, either on the product $\mathbb{T}^d$ of d 1-tori or on a single 1-torus $\mathbb{T}^1$. We extend the notion of *linearly recurrent* systems defined for $\mathbb{Z}$-actions on the Cantor set to $\mathbb{Z}^d$-actions, and we derive in this more general setting a necessary and sufficient condition, which involves a natural combinatorial data associated with the action, allowing the existence of a rotation topological factor of one of these two types.

1. Introduction

Let $(X, \mathcal{A})$ be a $\mathbb{Z}^d$-action (by homeomorphisms) on a compact metric space X. The action is *free* if $\mathcal{A}(\bar{n}, x) = x$ for some $\bar{n} \in \mathbb{Z}^d$ and $x \in X$ implies $\bar{n} = 0$, and it is *minimal* if the orbit of any point $x \in X$, $O_{\mathcal{A}}(x) = \{\mathcal{A}(\bar{n}, x) : \bar{n} \in \mathbb{Z}^d\}$, is dense in X.

The simplest nontrivial examples of free minimal $\mathbb{Z}^d$-actions on a compact metric space are given by "rotation-type" actions on compact topological groups. This type of factor plays a central role in topological dynamics of $\mathbb{Z}^d$-actions since in particular they determine weak mixing property through the existence of continuous eigenvalues. In this paper, we focus on two kinds of "rotation-type" factors that we describe now.

- First consider the $\mathbb{Z}^d$-action generated by d rotations on the product d-torus $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d = \mathbb{T}^1 \times \cdots \times \mathbb{T}^1$, and let $\mathcal{A}^d_{\bar{\theta}} : \mathbb{Z}^d \times \mathbb{T}^d \to \mathbb{T}^d$ be the map defined by
$$\mathcal{A}^d_{\bar{\theta}}(\bar{n}, x) = x + [\bar{n}, \bar{\theta}] \mod \mathbb{Z}^d ,$$
for $\bar{n} = (n_1, \ldots, n_d) \in \mathbb{Z}^d$, $x \in \mathbb{T}^d$ and where $[\bar{n}, \bar{\theta}] = (n_1 \cdot \theta_1, \ldots, n_d \cdot \theta_d)$. This construction yields a minimal $\mathbb{Z}^d$-action $(\mathbb{O}^d, \mathcal{A}^d_{\bar{\theta}})$ on the closure $\mathbb{O}^d$ of the orbit of 0 in the d-torus $\mathbb{T}^d$. When the coordinates of $\bar{\theta}$ are rationally independent, the set $\mathbb{O}^d$ is the d-torus $\mathbb{T}^d$ and the action is free.
- The same $\bar{\theta}$ can be used to define a $\mathbb{Z}^d$-action on $\mathbb{T}^1$. Consider the map $\mathcal{A}^1_{\bar{\theta}} : \mathbb{Z}^d \times \mathbb{T}^1 \to \mathbb{T}^1$ given by
$$\mathcal{A}^1_{\bar{\theta}}(\bar{n}, t) = t + \langle \bar{n}, \bar{\theta} \rangle \mod \mathbb{Z},$$

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where $\langle \cdot, \cdot \rangle$ is the usual inner product in $\mathbb{R}^d$. The $\mathbb{Z}^d$-action $(\mathbb{O}^1, \mathcal{A}^1_{\bar\theta})$ on the closure $\mathbb{O}^1$ of the orbit of 0 in the 1-torus $\mathbb{T}^1$ is again minimal. When the coordinates of $\bar\theta$ are independent on $\mathbb{Q}$, the set $\mathbb{O}^1$ is the 1-torus $\mathbb{T}^1$ and the action is free.

Assume X is a Cantor set, *i.e.*, it has a countable basis of closed and open (clopen) sets and has no isolated points (or equivalently, it is a totally disconnected compact metric space with no isolated points).

The main question we address in this paper is to determine whether a free minimal $\mathbb{Z}^d$-action $\mathcal{A}$ on the Cantor set X is an extension of an action of type $(\mathbb{O}^d, \mathcal{A}^d_{\bar\theta})$ or $(\mathbb{O}^1, \mathcal{A}^1_{\bar\theta})$ for some $\bar\theta \in \mathbb{R}^d$.

Note that a complete combinatorial answer to this question is given in [BDM] in the particular case when the dimension $d = 1$ and when the free minimal $\mathbb{Z}$-action is *linearly recurrent*. The linear recurrence of a given $\mathbb{Z}$-action is a property that involves the combinatorics of *return times* associated with a nested sequence of clopen sets (for further references on linearly recurrent $\mathbb{Z}$-actions see [CDHM], [Du1] and [Du2]).

The notion of return time to a clopen set can be generalized to $\mathbb{Z}^d$-actions when $d \geq 2$. In this case, the combinatorics of the return times associated with a nested sequence of clopen sets inherits a richer structure than in the case $d = 1$. However, as for $d = 1$, there exists a natural definition of linearly recurrent $\mathbb{Z}^d$-action. These generalizations are developed in Section 2, which is devoted to the combinatorics of return times (for further references on the structure of return times associated with a $\mathbb{Z}^d$-action, see [BG], where the hierarchical ideas used in this paper are introduced; see also [S] and [SW] for related topics).

This combinatorial approach allows us to derive a necessary condition on the action to be an extension of an action of one of the two rotations described above. In the case of a linearly recurrent action this condition is sufficient. This result is given in Section 3 (Theorem 3.1) together with its proof.

2. Combinatorics of return times

Let us start this section with some general considerations.

Let $\mathbb{R}^d$ be the Euclidean d-space and $\|\cdot\|$ its Euclidean norm. Consider two positive numbers r and R. An (r,R)-*Delone set* is a subset $\mathcal{D}$ of the d-space $\mathbb{R}^d$ equipped with the Euclidean norm $\|\cdot\|$, which satisfies the following two properties:

(i) *Uniformly Discrete:* each open ball with radius r in $\mathbb{R}^d$ contains at most one point in $\mathcal{D}$.

(ii) *Relatively Dense:* each open ball with radius R contains at least one point in $\mathcal{D}$.

When the constants r and R are not explicitly used, we will say for short a *Delone set* for an (r,R)-Delone set. We refer to [LP] for a more detailed approach of the theory of Delone sets.

A *patch* of a Delone set $\mathcal{D}$ is a finite subset of $\mathcal{D}$. A Delone set is of *finite type* if for each $M > 0$, there exist only finitely many patches in $\mathcal{D}$ of diameter smaller than M up to translation. Finally, a Delone set of finite type is *repetitive* if for each patch P in $\mathcal{D}$, there exists $M > 0$ such that each ball with radius M in $\mathbb{R}^d$ contains a translated copy of P in $\mathcal{D}$.

Let x be a point of a Delone set $\mathcal{D}$. The *Voronoi cell* $\mathcal{V}_x$ associated with x is the convex closed set in $\mathbb{R}^d$ defined by

$$\mathcal{V}_x = \{y \in \mathbb{R}^d \ : \ \forall\, x' \in \mathcal{D},\ \|y-x\| \le \|y-x'\| \ \} \ .$$

The union $\bigcup_{x\in\mathcal{D}} \mathcal{V}_x$ is a cover of $\mathbb{R}^d$. We say that two points x and x' in $\mathcal{D}$ are *neighbors* if $\mathcal{V}_x \cap \mathcal{V}_{x'} \neq \emptyset$.

The set of return vectors associated with $\mathcal{D}$ is defined by

$$\vec{\mathcal{D}} = \{x - y \ : \ (x,y) \in \mathcal{D}\times\mathcal{D}\} \ .$$

Lemma 2.1. *Let $\mathcal{D}$ be a Delone set of finite type. Then, there exists a finite collection $\vec{\mathcal{F}}$ of vectors in $\vec{\mathcal{D}}$ such that:*

- *$\vec{\mathcal{F}} = -\vec{\mathcal{F}}$;*
- *any vector in $\vec{\mathcal{D}}$ is a linear combination with nonnegative integer coefficients of vectors in $\vec{\mathcal{F}}$.*

Proof. When $\mathcal{D}$ is a Delone set of finite type, the set of vectors

$$\vec{\mathcal{F}} = \bigcup_{(x,x')\in\mathcal{D}\times\mathcal{D},(x,x')\ \text{neighbors}} (x-x')$$

is finite, satisfies $\vec{\mathcal{F}} = -\vec{\mathcal{F}}$ and clearly any vector in $\vec{\mathcal{D}}$ is a linear combination with nonnegative integer coefficients of vectors in $\vec{\mathcal{F}}$. □

Given such a set $\vec{\mathcal{F}}$, we can define the $\vec{\mathcal{F}}$*-distance* $d_{\vec{\mathcal{F}}}(x,x')$ as the minimal number of vectors in $\vec{\mathcal{F}}$ (counted with multiplicity) needed to write $x-x'$ for $x,x' \in \mathcal{D}$. The $\vec{\mathcal{F}}$*-diameter* of a patch P, denoted by $diam_{\vec{\mathcal{F}}}(P)$, is the maximal $\vec{\mathcal{F}}$-distance of a pair of points in P.

Now consider a free minimal $\mathbb{Z}^d$-action $\mathcal{A}$ on the Cantor set X. Let $\mathcal{C}$ be a clopen set in X and y a point in $\mathcal{C}$. The set of *return times* of the orbit of y in $\mathcal{C}$ is defined by

$$\mathcal{R}_{\mathcal{C}}(y) = \{\bar{n} \in \mathbb{Z}^d \ : \ \mathcal{A}(\bar{n},y) \in \mathcal{C}\} \ .$$

Proposition 2.2. *The set of return times $\mathcal{R}_{\mathcal{C}}(y)$ is a repetitive Delone set of finite type in $\mathbb{Z}^d$. Furthermore, if y and y' are two points in $\mathcal{C}$, the sets $\mathcal{R}_{\mathcal{C}}(y)$ and $\mathcal{R}_{\mathcal{C}}(y')$ have the same patches up to translation.*

Proof. • *$\mathcal{R}_{\mathcal{C}}(y)$ is a Delone set of finite type.*

The minimality of the action implies that the orbit of any point in X visits $\mathcal{C}$. For each $x \in X$ let $\bar{n}_x \in \mathbb{Z}^d$ be such that $\mathcal{A}(\bar{n}_x,x)$ is in $\mathcal{C}$. Since $\mathcal{C}$ is open, there exists a small neighborhood U_x of x such that for any x' in U_x we also have $\mathcal{A}(\bar{n}_x,x') \in \mathcal{C}$. Therefore $\{U_x \ : \ x \in X\}$ is a cover of X. Since X is compact, we can extract a finite cover $\{U_{x_i} \ : \ i \in I\}$. Let us choose $R > \max_{i\in I} \|\bar{n}_{x_i}\|$. It is clear that any ball with radius R in $\mathbb{R}^d$ intersects $\mathcal{R}_{\mathcal{C}}(y)$. Thus, $\mathcal{R}_{\mathcal{C}}(y)$ is relatively dense. Since it is a subset of $\mathbb{Z}^d$, it is a Delone set of finite type.

• *$\mathcal{R}_{\mathcal{C}}(y)$ is repetitive.*[1]

Consider a patch P in $\mathcal{R}_{\mathcal{C}}(y)$, choose $\bar{n}_0$ in P and let $z = \mathcal{A}(\bar{n}_0,y) \in \mathcal{C}$. Now choose a clopen set $\mathcal{C}_z$ containing z, small enough so that for any z' in $\mathcal{C}_z$, $\mathcal{A}(\bar{n} - \bar{n}_0, z')$ is in $\mathcal{C}$ for each $\bar{n}$ in P. The set $\mathcal{R}_{\mathcal{C}_z}(z)$ is relatively dense, and let R_1 be its R-constant. Let M stand for the diameter of P and let us prove that any ball with

[1] The proof that minimality implies repetitivity is classical and works in a more general situation. However, for sake of completeness, we fix it here for our specific context.

radius $R_1 + M$ in $\mathbb{R}^d$ contains a translation of the patch P. Indeed, given such a ball B, choose an element $\bar{m} \in \mathcal{R}_{\mathcal{C}_z}(z)$ in the corresponding centered sub-ball of radius R_1. Then by construction $\bar{m} + P$ belongs to $\mathcal{R}_{\mathcal{C}}(y)$ and to the ball B.

• *$\mathcal{R}_{\mathcal{C}}(y)$ and $\mathcal{R}_{\mathcal{C}}(y')$ have the same patches up to translation.*

Let P be a patch of $\mathcal{R}_{\mathcal{C}}(y)$ and $\bar{n}_0$ be a point in P. The minimality of the action implies that the orbit of y' accumulates on $z = \mathcal{A}(\bar{n}_0, y)$. This means that there exists $\bar{n}_1 \in \mathbb{Z}^d$ such that $\mathcal{A}(\bar{n}_1 + \bar{n} - \bar{n}_0, y')$ is in $\mathcal{C}$ when $\bar{n}$ is in P. Thus a translation of the patch P is in $\mathcal{R}_{\mathcal{C}}(y')$. □

The set of *return vectors* associated with $\mathcal{C}$ is defined by

$$\vec{\mathcal{R}}_{\mathcal{C}} = \mathcal{R}_{\mathcal{C}}(y) - \mathcal{R}_{\mathcal{C}}(y) = \{\bar{n} - \bar{m} \ : \ (\bar{n}, \bar{m}) \in \mathcal{R}_{\mathcal{C}}(y) \times \mathcal{R}_{\mathcal{C}}(y)\} \ .$$

The fact that for any pair of points y and y' in $\mathcal{C}$, the patches of $\mathcal{R}_{\mathcal{C}}(y)$ and $\mathcal{R}_{\mathcal{C}}(y')$ fit up to translation, implies that $\vec{\mathcal{R}}_{\mathcal{C}}$ does not depend on y in $\mathcal{C}$, as suggested by the notation. Lemma 2.1 and Proposition 2.2 yield the following corollary.

Corollary 2.3. *There exists in $\vec{\mathcal{R}}_{\mathcal{C}}$ a finite collection of vectors $\vec{\mathcal{F}}_{\mathcal{C}}$ such that:*

- $\vec{\mathcal{F}}_{\mathcal{C}} = -\vec{\mathcal{F}}_{\mathcal{C}}$;
- *any vector in $\vec{\mathcal{R}}_{\mathcal{C}}$ is a linear combination with nonnegative integer coefficients of vectors in $\vec{\mathcal{F}}_{\mathcal{C}}$.*

Such a set $\vec{\mathcal{F}}_{\mathcal{C}}$ is called a *set of first return vectors* associated with $\mathcal{C}$.

We now construct a combinatorial data associated to a $\mathbb{Z}^d$-action. Let x be a point in X and consider a sequence of nested clopen sets $X = \mathcal{C}_0 \supseteq \mathcal{C}_1 \supseteq \cdots \supseteq \mathcal{C}_n \cdots$ such that

$$\bigcap_{n \geq 0} \mathcal{C}_n \ = \ \{x\} \ .$$

Also consider the associated sets of return times $\mathcal{R}_{\mathcal{C}_n}(x)$, of return vectors $\vec{\mathcal{R}}_{\mathcal{C}_n}$ and of first return vectors $\vec{\mathcal{F}}_{\mathcal{C}_n}$ that we denote respectively (for short) by $\mathcal{R}_n(x)$, $\vec{\mathcal{R}}_n$ and $\vec{\mathcal{F}}_n$.

Let us inductively construct a sequence of partitions of $\mathbb{Z}^d = \mathcal{R}_0(x)$:

- The first partition of $\mathbb{Z}^d$ is the trivial partition made by the patches $\mathcal{P}_0(\bar{m}) = \{\bar{m}\}$ for each $\bar{m}$ in $\mathbb{Z}^d$.
- We consider the Delone set $\mathcal{R}_1(x)$ and the Voronoi cell $\mathcal{V}_{\bar{m},1}$ associated with each point $\bar{m}$ in $\mathcal{R}_1(x)$. Up to translation there are only finitely many of these Voronoi cells. It may occasionally happen that a point $\bar{l}$ in $\mathcal{R}_0(x)$ belongs to more than one Voronoi cell $\mathcal{V}_{\bar{m},1}$. It this case, we make an arbitrary choice to exclude the point $\bar{l}$ from all the Voronoi cells that it belongs to but one. We can make this arbitrary choice in a coherent way, that is to say, so that if two Voronoi cells of the above decomposition are translated copies one of the other, and if there is such an ambiguous point in the first one, if we keep this point in the cell, we do the same for the translated Voronoi cell. This surgery done, the intersection of $\mathcal{V}_{\bar{m},1}$ with $\mathcal{R}_0(x) = \mathbb{Z}^d$ defines a patch $\mathcal{P}_1(\bar{m})$ which intersects $\mathcal{R}_1(x)$ at $\bar{m}$. The collection of patches $\{\mathcal{P}_1(\bar{m})\}_{\bar{m} \in \mathcal{R}_1(x)}$ realizes a new partition of $\mathbb{Z}^d$.
- We now consider the Delone set $\mathcal{R}_2(x)$ and the Voronoi cell $\mathcal{V}_{\bar{m},2}$ associated with each point $\bar{m}$ in $\mathcal{R}_2(x)$. Again, up to translation there are only finitely many of these Voronoi cells, and if necessary, we solve the problem of points of $\mathcal{R}_1(x)$ belonging to more than one Voronoi cell as for the $\mathcal{V}_{\bar{m},1}$'s. This

surgery done, we call $\mathcal{R}_{1,\bar{m}}(x)$ the subset of $\mathcal{R}_1(x)$ that remains in $\mathcal{V}_{\bar{m},2}$ after the surgery, and we define the patch $\mathcal{P}_2(\bar{m})$ as follows:

$$\mathcal{P}_2(\bar{m}) = \bigcup_{\bar{p}\in\mathcal{R}_{1,\bar{m}}(x)} \mathcal{P}_1(\bar{p}).$$

The collection of patches $\{\mathcal{P}_2(\bar{m})\}_{\bar{m}\in\mathcal{R}_2(x)}$ again realizes a partition of $\mathbb{Z}^d$.

- Finally, we assume that for $n > 1$, we have constructed a collection of patches $\{\mathcal{P}_n(\bar{m})\}_{\bar{m}\in\mathcal{R}_n(x)}$ that realizes a partition of $\mathbb{Z}^d$. Consider the Delone set $\mathcal{R}_{n+1}(x)$ and the Voronoi cell $\mathcal{V}_{\bar{m},n+1}$ associated with each point $\bar{m}$ in $\mathcal{R}_{n+1}(x)$. Again, up to translation there are only finitely many of these Voronoi cells, and if necessary, we solve the problem of points of $\mathcal{R}_n(x)$ belonging to more than one Voronoi cell as we did previously. This surgery done, we call $\mathcal{R}_{n,\bar{m}}(x)$ the subset of $\mathcal{R}_n(x)$ that remains in $\mathcal{V}_{\bar{m},n+1}$ after the surgery, and we define the patch $\mathcal{P}_{n+1}(\bar{m})$ as follows:

$$\mathcal{P}_{n+1}(\bar{m}) = \bigcup_{\bar{p}\in\mathcal{R}_{n,\bar{m}}(x)} \mathcal{P}_n(\bar{p}).$$

  The collection of patches $\{\mathcal{P}_{n+1}(\bar{m})\}_{\bar{m}\in\mathcal{R}_{n+1}(x)}$ again realizes a partition of $\mathbb{Z}^d$.

Clearly, for each $n \geq 0$ and each $\bar{m} \in \mathcal{R}_n(x)$, there exists $\bar{p} \in \mathcal{R}_{n+1}(x)$ such that $\mathcal{P}_n(\bar{m}) \subset \mathcal{P}_{n+1}(\bar{p})$.

For each $n \geq 0$ and each $\bar{m} \in \mathcal{R}_{n+1}(x)$, we set

$$\mathcal{P}^n_{n+1}(\bar{m}) = \mathcal{P}_{n+1}(\bar{m}) \cap \mathcal{R}_n(x).$$

Proposition 2.4. *For each $n \geq 0$, there exists a constant $k(n) > 0$ such that for each $\bar{m} \in \mathcal{R}_{n+1}(x)$, $diam_{\vec{\mathcal{F}}_n}(\mathcal{P}^n_{n+1}(\bar{m})) \leq k(n)$.*

Proof. Since $\mathcal{R}_{n+1}(x)$ and $\mathcal{R}_n(x)$ are repetitive Delone sets, the Euclidean diameters of the cells $\mathcal{V}_{\bar{m},n+1}$ are bounded independently of $\bar{m}$, and thus the $\vec{\mathcal{F}}_n$-diameters of the $\mathcal{P}^n_{n+1}(\bar{m})$'s are bounded independently of $\bar{m}$. □

We remark that Proposition 2.4 does not require any condition on the nested sequence of clopen sets. By considering a sequence of $\{\mathcal{C}_n\}_{n\geq 0}$ that converges fast enough to x, we can insure that for each $R > 0$ there exists $n \geq 0$ such that $\mathcal{P}_n(0)$ coincides with $\mathbb{Z}^d$ in the ball with center 0 and radius R in $\mathbb{R}^d$.

In this situation, the data $(x, \{\mathcal{C}_n\}_{n\geq 0}, \{\{\mathcal{P}_n(\bar{m})\}_{\bar{m}\in\mathcal{R}_n(x)}\}_{n\geq 0}, \{\vec{\mathcal{F}}_n\}_{n\geq 0})$ is called *a combinatorial data* associated with the action $(X, \mathcal{A})$.

For any $n_0 \geq 0$, consider a point $\bar{p}$ which is in $\mathcal{R}_{n_0}(x)$ and not in $\mathcal{R}_{n_0+1}(x)$. Let m_0 be the smallest $m > 0$ such that $\bar{p} \in \mathcal{P}_m(0)$; we set $\bar{p}_{m_0} = 0$. We construct the sequence $(\bar{p}_l)_{n_0\leq l\leq m_0}$ such that $\bar{p}_l$ is the unique point in $\mathcal{P}^l_{l+1}(\bar{p}_{l+1})$ such that $\bar{p} \in \mathcal{P}_l(\bar{p}_l)$. We have $\bar{p}_{m_0} = 0$ and $\bar{p}_{n_0} = \bar{p}$. This sequence is called the *hierarchical sequence* joining 0 to $\bar{p}$.

When there exists a combinatorial data

$$(x, \{\mathcal{C}_n\}_{n\geq 0}, \{\{\mathcal{P}_n(\bar{m})\}_{\bar{m}\in\mathcal{R}_n(x)}\}_{n\geq 0}, \{\vec{\mathcal{F}}_n\}_{n\geq 0})$$

such that the constant $k(n)$ in Proposition 2.4 is bounded independently on n, we say that the free minimal $\mathbb{Z}^d$-action $\mathcal{A}$ on the Cantor set X is *linearly recurrent*. In this case, the combinatorial data is said to be *adapted* to the action.

By better controlling the convergence of the sequence of $\{\mathcal{C}_n\}_{n\geq 0}$ to x, it is always possible to insure the following two extra properties for the combinatorial data:

(i) for each $n \geq 0$ and for each $\bar{m}$ in $\mathcal{R}_{n+1}(x)$,
$$\vec{\mathcal{F}}_n \subseteq \mathcal{P}^n_{n+1}(\bar{m}) - \mathcal{P}^n_{n+1}(\bar{m});$$

(ii) for each $n \geq 0$ and for each $\bar{m}$ in $\mathcal{R}_{n+2}(x)$, all the patches $\mathcal{P}_n(\bar{m})$ are identical up to translation.

In this case, we say that the combinatorial data
$$(x, \{\mathcal{C}_n\}_{n\geq 0}, \{\{\mathcal{P}_n(\bar{m})\}_{\bar{m}\in\mathcal{R}_n(x)}\}_{n\geq 0}, \{\vec{\mathcal{F}}_n\}_{n\geq 0})$$
is *well distributed*.

3. Main results

To each vector $\bar{\theta}$ in $\mathbb{R}^d$ we associate the linear maps $c^1_{\bar{\theta}} \in \mathcal{L}(\mathbb{Z}^d, \mathbb{T}^1)$ and $c^d_{\bar{\theta}} \in \mathcal{L}(\mathbb{Z}^d, \mathbb{T}^d)$ defined by
$$c^1_{\bar{\theta}}(\bar{p}) = \langle \bar{\theta}, \bar{p} \rangle \mod \mathbb{Z} \quad \text{and} \quad c^d_{\bar{\theta}}(\bar{p}) = [\bar{\theta}, \bar{p}] \mod \mathbb{Z}^d$$
for each $\bar{p}$ in $\mathbb{Z}^d$.

Consider a minimal free $\mathbb{Z}^d$-action $(X, \mathcal{A})$ on the Cantor set X and a combinatorial data $(x, \{\mathcal{C}_n\}_{n\geq 0}, \{\{\mathcal{P}_n(\bar{m})\}_{\bar{m}\in\mathcal{R}_n(x)}\}_{n\geq 0}, \{\vec{\mathcal{F}}_n\}_{n\geq 0})$ associated with this action. For any $n \geq 0$ and any $\bar{\theta} \in \mathbb{R}^d$ we define the *$\bar{\theta}$-length of $\vec{\mathcal{F}}_n$ of dimension* 1 *and* d respectively by
$$l^1_{n,\bar{\theta}} = \max_{r_n \in \vec{\mathcal{F}}_n} |||c^1_{\bar{\theta}}(r_n)||| \text{ and } l^d_{n,\bar{\theta}} = \max_{r_n \in \vec{\mathcal{F}}_n} |||c^d_{\bar{\theta}}(r_n)|||,$$
where $|||\cdot|||$ stands for the Euclidean distance to 0 on the k-torus, $\mathbb{T}^k = \mathbb{R}^k/\mathbb{Z}^k$, $k = 1$, d. The following theorem is the main result of this paper.

Theorem 3.1. *Let $(X, \mathcal{A})$ be a free minimal $\mathbb{Z}^d$-action on the Cantor set X, let $(x, \{\mathcal{C}_n\}_{n\geq 0}, \{\{\mathcal{P}_n(\bar{m})\}_{\bar{m}\in\mathcal{R}_n(x)}\}_{n\geq 0}, \{\vec{\mathcal{F}}_n\}_{n\geq 0})$ be an associated combinatorial data and let $k = 1$ or $k = d$.*

(i) *Assume that for some $\bar{\theta} \in \mathbb{R}^d$, $(X, \mathcal{A})$ is an extension of the action $(\mathbb{O}^k, \mathcal{A}^k_{\bar{\theta}})$. Assume furthermore that the combinatorial data is well distributed. Then the series $\sum_{n\geq 0} l^k_{n,\bar{\theta}}$ converges.*

(ii) *Conversely assume that the action is linearly recurrent, that the combinatorial data is adapted to the action and that, for some $\bar{\theta} \in \mathbb{R}^d$, the series $\sum_{n\geq 0} l^k_{n,\bar{\theta}}$ converges. Then $(X, \mathcal{A})$ is an extension of the action $(\mathbb{O}^k, \mathcal{A}^k_{\bar{\theta}})$.*

Remark 1. In the particular case when the $\mathbb{Z}^d$-action $\mathcal{A}$ is the product of d linearly recurrent $\mathbb{Z}$-actions on X, Theorem 3.1 for $k = d$ is a direct corollary of its $d = 1$ version proved in [BDM].

Remark 2. The Lie group structure of $\mathbb{T}^k$ allows us to construct a continuous surjective map $\phi : \mathbb{T}^d \to \mathbb{T}^1$ defined by $\phi(\alpha_1, \dots, \alpha_d) = \alpha_1 + \dots + \alpha_d$. Assume that $h : (X, \mathcal{A}) \to (\mathbb{O}^d, \mathcal{A}^d_{\bar{\theta}})$ is an extension; then the map $\phi \circ h : (X, \mathcal{A}) \to (\mathbb{O}^1, \mathcal{A}^1_{\bar{\theta}})$ is also an extension. This is coherent with the fact that the convergence of the series $\sum_{n\geq 0} l^d_{n,\bar{\theta}}$ implies the convergence of the series $\sum_{n\geq 0} l^1_{n,\bar{\theta}}$.

Proof of Theorem 3.1. The proofs of both assertions of Theorem 3.1 for $k = 1$ or $k = d$ follow the same scheme and will be gathered in a single demonstration. Let $\langle\langle\cdot,\cdot\rangle\rangle$ stand for $[\cdot,\cdot] \mod \mathbb{Z}^d$ when $k = d$ and for $\langle\cdot,\cdot\rangle \mod \mathbb{Z}$ when $k = 1$.

(i) Assume that the free minimal $\mathbb{Z}^d$-action $(X, \mathcal{A})$ is an extension of the action $\mathcal{A}^k_{\bar\theta}$ on the closure $\mathbb{O}^k$ of the orbit of the point 0 in the k-torus $\mathbb{T}^k$ for some $\bar\theta$ in $\mathbb{R}^d$. Let us denote by $h : X \to \mathbb{O}^k$ the extension. Choose a well-distributed associated combinatorial data

$$(x, \{\mathcal{C}_n\}_{n\geq 0}, \{\{\mathcal{P}_n(\bar m)\}_{\bar m\in\mathcal{R}_n(x)}\}_{n\geq 0}, \{\vec{\mathcal{F}}_n\}_{n\geq 0})$$

and fix $h(x) = 0 \in \mathbb{T}^k$.

For each $n \geq 0$ let v_n be the first return vector in $\vec{\mathcal{F}}_n$ such that

$$l^k_{n,\bar\theta} = \max_{u_n\in\vec{\mathcal{F}}_n} |||c^k_{\bar\theta}(u_n)||| = |||c^k_{\bar\theta}(v_n)|||.$$

The following observation is a direct consequence of the continuity of h.

Lemma 3.2. *The quantity $l^k_{n,\bar\theta}$ goes to* 0 *as n goes to ∞. Furthermore, for each $\epsilon > 0$ there exists $N > 0$ such that for any pair of points $(\bar n, \bar m)$ in $\mathcal{R}_N(x)\times\mathcal{R}_N(x)$, we have*

$$|||h(\mathcal{A}(\bar n, x)) - h(\mathcal{A}(\bar m, x))||| \leq \epsilon.$$

Let B be the open ball on the k-torus, centered at 0, with radius 1/4. Fix $0 < \epsilon < 1/4$ and let N verify the conclusion of Lemma 3.2 for this ϵ and such that $l^k_{n,\bar\theta} \leq \epsilon$ for $n \geq N$. The ball B is decomposed in 2^k sectors $S_{\epsilon_1,\ldots,\epsilon_k}$ with $\epsilon_i \in \{-1, 1\}$ for $i \in \{1,\ldots,k\}$ defined by

$$S_{\epsilon_1,\ldots,\epsilon_k} = \{(x_1,\ldots,x_k) \in B \;:\; x_i\cdot\epsilon_i \geq 0\;,\; \forall i \in \{1,\ldots,k\}\}\;.$$

Let $I_{\epsilon_1,\ldots,\epsilon_k}$ be the set of integers n such that $c^k_{\bar\theta}(v_n)$ is in $S_{\epsilon_1,\ldots,\epsilon_k}$, and let us prove that the series $\sum_{n\in I_{\epsilon_1,\ldots,\epsilon_k}} l^k_{n,\bar\theta}$ converges. Actually, we only need to prove that the series $\sum_{n\in I_{1,\ldots,1}} l^k_{n,\bar\theta}$ converges; a similar proof works for the other cases. This sum can be split into two parts:

$$\sum_{n\in I_{1,\ldots,1}} l^k_{n,\bar\theta} = \sum_{n\in I_{1,\ldots,1},\,\mathrm{even}} l^k_{n,\bar\theta} + \sum_{n\in I_{1,\ldots,1},\,\mathrm{odd}} l^k_{n,\bar\theta}.$$

Here again we only need to prove that the series $\sum_{n\in I_{1,\ldots,1},\,\mathrm{even}} l^k_{n,\bar\theta}$ converges; a similar proof also works for the case where n is odd. Observe that we are assuming $I_{1,\ldots,1}$ is infinite.

The proof splits in five steps:

Step 1 : Fix an even integer $N_0 > N$ in $I_{1,\ldots,1}$, and let $N < n_l < n_{l-1} < \cdots < n_1 < N_0$ be the ordered sequence of even integers between N and N_0 that belong to $I_{1,\ldots,1}$.

Step 2 : Since the combinatorial data is well distributed, we can choose two points $\bar m_1$ and $\bar p_1$ in $\mathcal{P}^{n_1}_{n_1+1}(0)$ such that

$$v_{n_1} = \bar p_1 - \bar m_1.$$

Furthermore the two patches $\mathcal{P}_{n_2+1}(\bar m_1)$ and $\mathcal{P}_{n_2+1}(\bar p_1)$ are identical up to translation, and there exists a pair of points $(\bar m_2, \bar m_2')$ in $\mathcal{P}^{n_2}_{n_2+1}(\bar m_1)\times\mathcal{P}^{n_2}_{n_2+1}(\bar m_1)$ such that

$$v_{n_2} = \bar m_2' - \bar m_2.$$

We define $\bar{p}_2$ in $\mathcal{P}^{n_2}_{n_2+1}(\bar{p}_1)$ by $\bar{p}_2 - \bar{p}_1 = \bar{m}_2 - \bar{m}_1 + v_{n_2}$. One has

$$\bar{p}_2 - \bar{m}_2 \;=\; v_{n_1} + v_{n_2}.$$

Step 3 : Since the combinatorial data is well distributed, the two patches $\mathcal{P}_{n_3+1}(\bar{m}_2)$ and $\mathcal{P}_{n_3+1}(\bar{p}_2)$ are identical up to translation, and there exists a pair of points $(\bar{m}_3, \bar{m}'_3)$ in $\mathcal{P}^{n_3}_{n_3+1}(\bar{m}_3) \times \mathcal{P}^{n_3}_{n_3+1}(\bar{m}_2)$ such that

$$v_{n_3} = \bar{m}'_3 - \bar{m}_3.$$

We define $\bar{p}_3$ in $\mathcal{P}_{n_2}(\bar{p}_2)$ by $\bar{p}_3 - \bar{p}_2 = \bar{m}_3 - \bar{m}_2 + v_{n_3}$. One has

$$\bar{p}_3 - \bar{m}_3 \;=\; v_{n_1} + v_{n_2} + v_{n_3}.$$

Step 4 : We iterate this construction until we get the points $\bar{m}_l$ and $\bar{p}_l$ which satisfy

$$\bar{p}_l - \bar{m}_l \;=\; \sum_{j=1}^{l} v_{n_j}.$$

Step 5 : We have

$$\begin{aligned} |||h(\mathcal{A}(\bar{p}_l, x)) \;-\; h(\mathcal{A}(\bar{m}_l, x))||| \;&= |||\langle\langle \sum_{j=1}^{l} v_{n_j}, \bar{\theta}\rangle\rangle||| \\ &= |||\sum_{j=1}^{l} c^k_{\bar{\theta}}(v_{n_j})|||. \end{aligned}$$

Since $\bar{p}_l$ and $\bar{m}_l$ are in $\mathcal{R}_N(x)$, Lemma 3.2 implies that

$$|||\sum_{j=1}^{l} c^k_{\bar{\theta}}(v_{n_j})||| \leq \epsilon \;.$$

Let $\pi : B \to B'$ be the canonical isometric identification of the ball B with the open ball B' in the Euclidean space $\mathbb{R}^d$ centered at 0 with radius 1/4. Through this identification, it is clear that for all x in B, $|||x||| = ||\pi(x)||$. Moreover for any pair of points x, x' in $S_{1,\ldots,1}$ such that $x + x'$ is also in $S_{1,\ldots,1}$, we have $\pi(x + x') = \pi(x) + \pi(x')$. It follows that

$$|||\sum_{j=1}^{l} c^k_{\bar{\theta}}(v_{n_j})||| \;=\; ||\sum_{j=1}^{l} \pi(c^k_{\bar{\theta}}(v_{n_j}))||.$$

Finally, since for $1 \leq j \leq l$, $c^k_{\bar{\theta}}(v_{n_j})$ is in $S_{1,\ldots,1}$, we have

$$\sum_{j=1}^{l} ||\pi(c^k_{\bar{\theta}}(v_{n_j}))|| \;\leq\; \sqrt{k} \cdot ||\sum_{j=1}^{l} \pi(c^k_{\bar{\theta}}(v_{n_j}))||,$$

which implies

$$\sum_{N \leq n,\, n \in I_{1,\ldots,1},\, \text{even}} l^k_{n_j,\bar{\theta}} \;\leq\; \sqrt{k} \cdot \epsilon \;.$$

This insures that the series $\sum_{n \in I_{1,\ldots,1},\, \text{even}} l^k_{n,\bar{\theta}}$ converges, and consequently the series $\sum_{n \geq 0} l^k_{n,\bar{\theta}}$ converges too.

(ii) Let $(X, \mathcal{A})$ be a linearly recurrent $\mathbb{Z}^d$-action on the Cantor set X. Assume that the combinatorial data is adapted to the action and that the series of $\bar{\theta}$-lengths

$\sum_{n\geq 0} l^k_{n,\bar\theta}$ converges for some $\bar\theta$ in $\mathbb{R}^d$. Fix $\epsilon>0$ and choose $n_1>0$ big enough so that

$$\sum_{n\geq n_1} l^k_{n,\bar\theta} < \epsilon.$$

Let us define the map h on the $\mathbb{Z}^d$-orbit of x by

$$h(\mathcal{A}(\bar n, x)) = \langle\langle \bar n, \bar\theta\rangle\rangle = \mathcal{A}^k_{\bar\theta}(\bar n, 0)$$

for each $\bar n$ in $\mathbb{Z}^d$. In order to prove that the map h extends to a continuous map on the closure $\mathbb{O}^k$ of the orbit of 0 in $\mathbb{T}^k$, it is enough to prove that h is uniformly continuous, which follows from the continuity of h at x. Choose a point $\bar p$ in $\mathcal{R}_{n_1}(x)$. We know that there exists $n_0\geq n_1$ such that $\bar p$ is in $\mathcal{R}_{n_0}(x)$ and is not in $\mathcal{R}_{n_0+1}(x)$, and a hierarchical sequence $(\bar p_l)_{n_0\leq l\leq m_0}$ such that $\bar p_{n_0}=\bar p$ and $p_{m_0}=0$.

Let us write

$$h(\mathcal{A}(\bar p, x)) = \sum_{l=n_0+1}^{m_0} (h(\mathcal{A}(\bar p_l, x)) - h(\mathcal{A}(\bar p_{l-1}, x))).$$

For any $n_0<l\leq m_0$ both points $\bar p_l$ and $\bar p_{l-1}$ are in $\mathcal{P}^{l-1}_l(\bar p_l)$. Consequently there exists a collection $\{v_{l,i}\}_{1\leq i\leq q_l}$ of vectors in $\vec{\mathcal{F}}_l$ such that:

- $q_l\leq k(l)$;
- the sequence of points $\{\bar p_{l-1,i}\}_{0\leq i\leq q_l}$ defined by:
  - $\bar p_{l-1,0}=\bar p_{l-1}$;
  - $\bar p_{l-1,i}=\bar p_{l-1,i-1}+v_{l,i}$ for $1\leq i\leq q_l$;
  - $\bar p_{l-1,q_l}=\bar p_l$,

belongs to $\mathcal{R}_l(x)$.

This yields

$$h(\mathcal{A}(\bar p, x)) = \sum_{l=n_0+1}^{m_0}\sum_{i=1}^{q_l}(h(\mathcal{A}(\bar p_{l-1,i}, x)) - h(\mathcal{A}(\bar p_{l-1,i-1}, x))).$$

Now we use the fact that the action is linearly recurrent and that the combinatorial data is adapted to this action. We denote by L a uniform upper bound for the sequence $\{k(n)\}_{n\geq 0}$. We get

$$|||h(\mathcal{A}(\bar p, x))||| \leq L\cdot \sum_{l=n_0}^{m_0} l^k_{m_0-l,\bar\theta} \leq L\cdot\sum_{n=n_0}^{\infty} l^k_{n,\bar\theta} \leq L\epsilon.$$

This proves the continuity of h at x. □

4. Example

Let $q>2$. We set $r=\frac{q}{2}-1$ if q is even, and $r=\frac{q-1}{2}$ if q is odd, and $l=q-r-1$. Define $D_0=\{\bar 0\}\subset\mathbb{Z}^d$ and, for $n>0$, the set D_n as the disjoint union of the sets $D_{n-1}+\bar m$, for $\bar m\in S_n$, where

$$S_n = q^{n-1}\{\bar z\in\mathbb{Z}^d : -l\leq z_i\leq r, \text{ for all } 1\leq i\leq d\}.$$

Let $\Sigma=\{1,\dots,q\}$ and $\tau:\Sigma\to\Sigma^{D_1}$ be an injective substitution such that for all $\sigma\in\Sigma$:

(1) $\tau(\sigma)_{\bar 0}=1$;
(2) $\forall\sigma'\in\Sigma \quad \exists\bar k\in D_1$ such that $\tau(\sigma)_{\bar k}=\sigma'$;
(3) $|\{\bar k\in D_1 : \tau(\sigma)_{\bar k}=1\}|=1$.

The substitution τ can be considered as a function from $\Sigma^{\mathbb{Z}^d}$ into itself by setting

$$\tau(x)_{\bar m} = (\tau(x_{\bar j}))_{\bar k}, \text{ where } \bar m = q\bar j + \bar k, \text{ with } \bar j \in \mathbb{Z}^d \text{ and } \bar k \in D_1.$$

In the same way, we can define $\tau : \Sigma^{D_n} \to \Sigma^{D_{n+1}}$ and then $\tau^n(\sigma) = \tau(\tau^{n-1}(\sigma))$ for all $\sigma \in \Sigma$ and $n > 0$.

From property (1), the set $\bigcap_{n\geq 0}\{x \in \Sigma^{\mathbb{Z}^d} : x_{\bar k} = \tau^n(1)_{\bar k}, \text{ for all } \bar k \in D_n\}$ is not empty and since $D_n \uparrow \mathbb{Z}^d$, there is only one element x^* in that set, and it is a fixed point of the substitution.

Let $\mathcal{A} : \mathbb{Z}^d \times \Sigma^{\mathbb{Z}^d} \to \Sigma^{\mathbb{Z}^d}$ be the shift action, i.e, $\mathcal{A}(\bar m, x) = \{x_{\bar k+\bar m}\}_{\bar k\in\mathbb{Z}^d}$ for all $\bar m \in \mathbb{Z}^d$ and $x \in \Sigma^{\mathbb{Z}^d}$. We define $X = \overline{\{\mathcal{A}(\bar m, x^*) : \bar m \in \mathbb{Z}^d\}}$ and $C_n = \bigcup_{\sigma\in\Sigma}\{x \in X : x_{\bar k} = \tau^n(\sigma)_{\bar k}, \text{ for all } \bar k \in D_n\}$. By (1), $\bigcap_{n\geq 0} C_n = \{x^*\}$ and $q^n\mathbb{Z}^d \subseteq \mathcal{R}_{C_n}(x^*)$, for all $n \geq 0$, which implies that x^* is uniformly recurrent and then $(X, \mathcal{A})$ is minimal. Since τ is injective, using (2) and (3), one gets $q^n\mathbb{Z}^d = \mathcal{R}_{C_n}(x^*)$ for all $n \geq 0$. Thus x^* is nonperiodic, and then $(X, \mathcal{A})$ is free.

For $\bar m \in \mathcal{R}_{C_n}(x^*)$ we set $\mathcal{P}_n(\bar m) = D_n + \bar m$ and $\vec{\mathcal{F}}_n = \{q^n e_i, -q^n e_i : 1 \leq i \leq d\}$ for all $n \geq 0$, where the e_i's are the canonical vectors of $\mathbb{Z}^d$. The data $(x^*, \{C_n\}_{n\geq 0}, \{\{\mathcal{P}_n(\bar m)\}_{\bar m\in\mathcal{R}_{C_n}(x^*)}\}_{n\geq 0}, \{\vec{\mathcal{F}}_n\}_{n\geq 0})$ is a well-distributed combinatorial data that satisfies $k(n) \leq q^d$ for all $n \geq 0$. Thus $(X, \mathcal{A})$ is linearly recurrent.

Claim. *$(X, \mathcal{A})$ is an extension of the action $(\mathbb{O}^k, \mathcal{A}^k_{\bar\theta})$ for $k = 1, d$ if and only if $\bar\theta \in E = \{\frac{1}{q^n}\bar m : \bar m \in \mathbb{Z}^d, \quad n \geq 0\}$.*

Proof. If $\bar\theta \in E$, then there exists $n \geq 0$ such that for all $m > n$, $c^1_{\bar\theta}(r_m) = 0$ and $c^d_{\bar\theta}(r_m) = \bar 0$ for all $r_m \in \vec{\mathcal{F}}_m$. Thus the series $\sum_{n\geq 0} l^k_{n,\bar\theta}$ converges, and we conclude that $(X, \mathcal{A})$ is an extension of the action $(\mathbb{O}^k, \mathcal{A}^k_{\bar\theta})$, for $k = 1, d$. Conversely, if $(X, \mathcal{A})$ is an extension of the action $(\mathbb{O}^k, \mathcal{A}^k_{\bar\theta})$, for $k = 1, d$, then $\sum_{n\geq 0} ||| \, c^k_{\bar\theta}(q^n e_i) \, |||$ converges for all $1 \leq i \leq d$. This implies that $\{q^n\theta_i\}_{n\geq 0}$ converges to $0 \mod \mathbb{Z}$, for all $1 \leq i \leq d$. Therefore, there exists $m_i \geq 0$ such that $q^{m_i}\theta_i = z_i + v_i$, where $z_i \in \mathbb{Z}$ and $v_i \in \mathbb{R}$ satisfies $q^n v_i \to 0$. Since the only v_i that verifies this condition is $v_i = 0$, we conclude that $\theta_i = \frac{z_i}{p_i^m}$. Setting $m = \max_{1\leq i\leq d} m_i$, one has $\bar\theta = \frac{\bar n}{q^m}$, for some $\bar n \in \mathbb{Z}^d$. □

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