α-CONTINUITY PROPERTIES OF THE SYMMETRIC α-STABLE PROCESS

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Abstract. Let $D$ be a domain of finite Lebesgue measure in $\mathbb{R}^d$ and let $X^D_t$ be the symmetric $\alpha$-stable process killed upon exiting $D$. Each element of the set $\{\lambda^\alpha_{i} \}_{i=1}^\infty$ of eigenvalues associated to $X^D_t$, regarded as a function of $\alpha \in (0,2)$, is right continuous. In addition, if $D$ is Lipschitz and bounded, then each $\lambda^\alpha_{i}$ is continuous in $\alpha$ and the set of associated eigenfunctions is precompact.

1. Introduction

Let $X_t$ be a $d$-dimensional symmetric $\alpha$-stable process of order $\alpha \in (0,2]$. The process $X_t$ has stationary independent increments and its transition density $p^{\alpha}(t, z, w) = f^{\alpha}_t(z - w)$ is determined by its Fourier transform

$$\exp(-t|z|^\alpha) = \int_{\mathbb{R}^d} e^{iz \cdot w} f^{\alpha}_t(w) dw.$$ 

These processes have right continuous sample paths and their transition densities satisfy the scaling property

$$p^{\alpha}(t, x, y) = t^{-d/\alpha} p^{\alpha}(1, t^{-1/\alpha}x, t^{-1/\alpha}y).$$

When $\alpha = 2$ the process $X_t$ is a $d$-dimensional Brownian motion running at twice the usual speed. The nonlocal operator associated to $X_t$ is $(-\Delta)^{\alpha/2}$ where $\Delta$ is the Laplace operator in $\mathbb{R}^d$.

Let $D$ be an open set in $\mathbb{R}^d$ and let $X^D_t$ be the symmetric $\alpha$-stable process killed upon leaving $D$. We write $p^{\alpha}_{X^D}(t, x, y)$ for the transition density of $X^D_t$ and $H_\alpha$ for its associated nonlocal self-adjoint positive operator. It is well known that if $D$ has finite Lebesgue measure, then the spectrum of $H_\alpha$ is discrete. Let

$$0 < \lambda^\alpha_1(D) \leq \lambda^\alpha_2(D) \leq \lambda^\alpha_3(D) \leq \cdots$$

be the eigenvalues of $H_\alpha$, and let

$$\varphi^\alpha_1, \varphi^\alpha_2, \varphi^\alpha_3, \cdots$$

be the corresponding sequence of orthonormal $L^2(D)$ eigenfunctions. Also, $\varphi^\alpha_\alpha$ is chosen so as to be positive on $D$. Note that if $\alpha < 2$, then $\lambda^\alpha_1(D) < \lambda^\alpha_2(D)$, but this need not be true if $\alpha = 2$ (unless $D$ is connected).

Several authors have studied properties of the eigenvalues and eigenfunctions of $H_\alpha$. One common theme has been to extend results on Brownian motion ($\alpha = 2$) to
analogous results for symmetric $\alpha$-stable processes. For example, R. M. Blumenthal and R. K. Getoor \cite{8} have shown Weyl’s asymptotic law holds: if $D$ is a bounded open set and $N(\lambda)$ is the number of eigenvalues less than or equal to $\lambda$, then there exists a constant $C_{d,\alpha}$, depending only on $d$ and $\alpha$, such that

$$N(\lambda) \approx C_{d,\alpha} \frac{m(D)}{\Gamma(d/\alpha + 1)} \lambda^{d/\alpha}$$

as $\lambda \to \infty$, provided $m(\partial D) = 0$, where $m$ is Lebesgue measure.

If $D \subseteq \mathbb{R}^d$ is a domain, define the inner radius $R_D$ to be the supremum of the radii of all balls contained in $D$. R. Bañuelos et al. \cite{6} and P. Mendoza-Hernández \cite{20} have shown that if $D$ is a convex domain with finite inner radius $R_D$ and $I_D$ is the interval $(-R_D, R_D)$, then

$$\lambda^\alpha_{1}(I_D) \leq \lambda^\alpha_{1}(D).$$

Moreover, if $D \subseteq \mathbb{R}^d$ has finite volume and $D^*$ is a ball in $\mathbb{R}^d$ with the same volume as $D$, then it was proved in \cite{6} that the Faber-Krahn inequality holds:

$$\lambda^\alpha_{1}(D^*) \leq \lambda^\alpha_{1}(D).$$

Another line of inquiry taken by those authors was to consider the eigenvalues as a function of the index $\alpha$. For instance, if $D$ is a convex domain with finite inner radius $R_D$, then

$$\frac{2^\alpha \Gamma(1 + \frac{\alpha}{2}) \Gamma(1 + \frac{d}{2})}{\Gamma(d/2) R_D^d} \leq \lambda^\alpha_{1}(D) \leq \lambda^\alpha_{1}(B_{R_D}),$$

where $B_{R_D}$ is a ball in $\mathbb{R}^d$ of radius $R_D$. They also proved that if $D \subseteq \mathbb{R}^d$ has finite volume, then

$$\lambda^\alpha_{1}(D) \leq \left[\mu_1(D)\right]^{\alpha/2},$$

where $\mu_1(D)$ is the first Dirichlet eigenvalue of $-\Delta$ on $D$.

For the Cauchy process, i.e. $\alpha = 1$, and bounded Lipshitz domains, R. Bañuelos and T. Kulczycki \cite{5} extended \eqref{1.1} to

$$\lambda^1_{1}(D) \leq \left[\mu_i(D)\right]^{1/2}, \quad i = 1, 2, \ldots,$$

where

$$0 < \mu_1(D) < \mu_2(D) \leq \cdots$$

are all the Dirichlet eigenvalues of $-\Delta$ on $D$. Their proof of \eqref{1.2} is based on a variational formula for $\lambda^1_{1}(D)$ that they developed from a connection with the Steklov problem for the Laplacian. They also obtained many detailed properties of the eigenfunctions $\varphi^1_{i}$ for the Cauchy process.

By finding a connection with the symmetric stable process with rational index $\alpha$ and PDEs of order higher than 2, R. D. DeBlassie \cite{14} derived a variational formula for the eigenvalues which led to the following extension of \eqref{1.1} and \eqref{1.2}:

$$\lambda^\alpha_{1}(D) \leq \left[\mu_i(D)\right]^{\alpha/2}, \quad i = 1, 2, \ldots,$$

for all rational $\alpha \in (0, 2)$ and certain bounded domains $D \subseteq \mathbb{R}^d$. The class of admissible domains includes convex polyhedra, Lipschitz domains with sufficiently small Lipschitz constant and $C^1$ domains. Please note there is an error in \cite{14} in the derivation of \eqref{1.3} above, as pointed out by the referee of the present article. See \cite{15} for a correction. Also, we have learned of a recent preprint of Z.-Q. Chen
and R. Song [11] which extends (1.3) to all indices $\alpha \in (0, 2)$ and domains $D$ of finite Lebesgue measure.

In this article, we study the eigenvalues and eigenfunctions regarded as functions of the index $\alpha$. Our first result concerns continuity of the eigenvalues.

**Theorem 1.1.** Let $D$ be a domain of finite Lebesgue measure. Then, as a function of $\alpha \in (0, 2)$, $\lambda_\alpha^i$ is right continuous for each positive integer $i$.

In order to prove Theorem 1.1 we need the following interesting monotonicity property extending (1.3) above. It is due to Z.-Q. Chen and R. Song [11]; see their Example 5.4.

**Theorem 1.2.** Let $D$ be an open set of finite Lebesgue measure in $\mathbb{R}^d$. If $0 < \alpha < \beta \leq 2$, then for all positive integers $i$,

$$[\lambda_\alpha^i(D)]^{1/\alpha} \leq [\lambda_\beta^i(D)]^{1/\beta}.$$  

Even though those authors consider bounded open sets $D$, it is clear their argument works for open sets of finite Lebesgue measure.

By requiring more regularity of $\partial D$, we can prove the following extension of Theorem 1.1.

**Theorem 1.3.** Let $D$ be a bounded Lipschitz domain. Then, as a function of $\alpha \in (0, 2)$, $\lambda_\alpha^i$ is continuous for each positive integer $i$.

We will obtain Theorem 1.3 from the following result that we believe is of independent interest.

**Theorem 1.4.** Let $D$ be a bounded Lipschitz domain. If $\alpha_m$ converges to $\alpha \in (0, 2)$, then for each positive integer $i$, $\{\varphi_{\alpha_m}^i : m \geq 1\}$ is precompact in $C(\overline{D})$ equipped with the sup norm. Moreover, if $\lambda_{\alpha_m}^i$ converges to $\lambda$, then any limit point of $\{\varphi_{\alpha_m}^i : m \geq 1\}$ is an eigenfunction of $H_\alpha$ and $\lambda$ is the corresponding eigenvalue.

As a corollary of the proof of the last theorem, we obtain continuity of the first eigenfunction as a function of $\alpha$.

**Theorem 1.5.** If $D$ is a bounded Lipschitz domain and $\alpha_m$ converges to $\alpha \in (0, 2)$, then $\varphi_1^{\alpha_m}$ converges uniformly to $\varphi_1^\alpha$ on $D$.

The article is organized as follows. In Section 2 we present some results needed in the proof of Theorem 1.1. In Section 3 we establish Theorem 1.1 by proving upper semicontinuity and right lower semicontinuity of the eigenvalues via Dirichlet forms. Lower semicontinuity of the eigenvalues, for Lipschitz domains, is proved in section 4 using Theorem 1.4. This will yield Theorems 1.3 and 1.5. Section 5 deals with certain weak convergence results needed to prove Theorem 1.4. Finally, in Section 6 we prove Theorem 1.4.

2. **Preliminary results**

Throughout this section we will assume the domain $D$ has finite Lebesgue measure. We denote by $C_c^\infty(D)$ the set of $C^\infty$ functions with compact support in $D$. The inner product and the norm in $L^2(D)$ will be denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|_2$, respectively.
For any domain $D \subseteq \mathbb{R}^d$, we define $\tau_D$ to be the first exit time of $X_t$ from $D$, i.e.,

$$\tau_D = \inf \{ t > 0 : X_t \notin D \}.$$ 

Let

$$F_\alpha = \{ \varphi \in L^2(\mathbb{R}^d) : \int \int \frac{[\varphi(y) - \varphi(x)]^2}{|y-x|^{d+\alpha}} dydx < \infty \}.$$ 

The Dirichlet form $(E_\alpha, F_\alpha)$ associated to $X_t$ is given by

$$E_\alpha(\psi, \varphi) = A(d, \alpha) \int \int \frac{[\psi(y) - \psi(x)] [\varphi(y) - \varphi(x)]}{|y-x|^{d+\alpha}} dydx,$$

for all $\psi, \varphi \in F_\alpha$, where

$$A(d, \alpha) = \frac{\Gamma\left(\frac{d+\alpha}{2}\right)}{2^\alpha \pi^{d/2} \Gamma\left(\frac{\alpha}{2}\right)}.$$ 

It is well known that the Dirichlet form corresponding to $X_t^D$ is given by $(E_\alpha, F_{\alpha,D})$, where

$$F_{\alpha,D} = \{ u \in F_\alpha : u \text{ is continuous on } (0,2) \}. $$

Recall that for all $\psi, \varphi$ in the domain of $H_\alpha$ we have

$$E_\alpha(\psi, \varphi) = \langle \psi, H_\alpha \varphi \rangle.$$ 

As seen in Theorem 4.4.3 of [17], $F_{\alpha,D}$ is the closure of $C_c^\infty(D)$ in $F_\alpha$ with respect to the norm

$$\| \varphi \|_\alpha = \sqrt{E_\alpha(\varphi, \varphi)} + \| \varphi \|_2.$$ 

**Lemma 2.1.** Let $\varphi, \psi \in C_c^\infty(D)$. Then the function

$$E_\alpha(\varphi, \psi) : (0, 2) \to \mathbb{R}$$

is continuous on $(0, 2)$.

**Proof.** Let $\varphi, \psi \in C_c^\infty(D)$, and let $\beta \in (\alpha - \delta, \alpha + \delta)$, where $\delta = \frac{1}{2} \min \{ 2 - \alpha, \alpha \}$. Then there exists a constant $C > 0$, depending only on $\varphi$ and $\psi$, such that

$$\frac{|\psi(y) - \psi(x)|}{|y-x|^{d+\beta}} \leq \frac{C}{|y-x|^{d+\beta-2}} \leq C \max \left\{ \frac{1}{|y-x|^{d+\alpha-\delta-2}}, \frac{1}{|y-x|^{d+\alpha+\delta-2}} \right\}.$$ 

Since $D$ has finite measure, a simple computation using polar coordinates shows that

$$\max \left\{ \frac{1}{|y-x|^{d+\alpha-\delta-2}}, \frac{1}{|y-x|^{d+\alpha+\delta-2}} \right\}$$

is integrable in both $(\text{supp}(\varphi) \cup \text{supp}(\psi)) \times D$ and $D \times (\text{supp}(\varphi) \cup \text{supp}(\psi))$. The result immediately follows from the dominated convergence theorem. 

We end this section with some basic estimates on $L^2$ norms to be used in the next section. Suppose $k$ is a positive integer, $0 < \epsilon < 1$, and $\varphi_1, \ldots, \varphi_k \in L^2(D)$ satisfy

$$|\langle \varphi_i, \varphi_j \rangle| < \frac{\epsilon}{4k^2}, \quad i \neq j,$$

$$\left( 1 - \frac{\epsilon}{4k^2} \right) < \| \varphi_i \|_2^2 < \left( 1 + \frac{\epsilon}{4k^2} \right),$$

for all $i, j$. The proof follows from the Cauchy-Schwarz inequality and the choice of $\epsilon$. 

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for all $1 \leq i, j \leq k$. If $\psi = \sum_{i=1}^{k} a_i \varphi_i$ with $\|\psi\|_2 = 1$, then we show that

$$\frac{1}{1 + \epsilon/2} \leq \sum_{i=1}^{k} a_i^2 \leq \frac{1}{1 - \epsilon/2}$$

and $\varphi_1, \ldots, \varphi_k$ are linearly independent.

For the proof, note that we have

$$1 = \langle \psi, \psi \rangle = \sum_{i=1}^{k} a_i^2 \|\varphi_i\|_2^2 + 2 \sum_{i=1}^{k} \sum_{j>i} a_i a_j \langle \varphi_i, \varphi_j \rangle$$

$$\geq \sum_{i=1}^{k} a_i^2 \left(1 - \frac{\epsilon}{4k^2}\right) - 2 \sum_{i=1}^{k} \sum_{j>i} |a_i| |a_j| \frac{\epsilon}{4k^2}$$

$$\geq \sum_{i=1}^{k} a_i^2 \left(1 - \frac{\epsilon}{4k^2}\right) - (k^2 - k) \sum_{i=1}^{k} a_i^2 \frac{\epsilon}{4k^2}$$

$$\geq (1 - \epsilon/2) \sum_{i=1}^{k} a_i^2,$$

and we conclude that

$$\sum_{i=1}^{k} a_i^2 \leq \frac{1}{1 - \epsilon/2}.$$ 

Similar computations give the remaining assertions.

3. Proof of Theorem

We will use the following well-known result; see [12].

**Theorem 3.1.** Let $H$ be a nonnegative self-adjoint unbounded operator with discrete spectrum $\{\lambda_i\}_{i=1}^{\infty}$ and domain $\text{Dom}(H)$. Then for $i \geq 1$

$$\lambda_i = \inf \{\lambda(L) : L \subseteq \text{Dom}(H), \dim(L) = i\},$$

where

$$\lambda(L) = \sup \{\langle Hf, f \rangle : f \in L, \|f\|_2 = 1\},$$

and $L$ is a vector subspace of $\text{Dom}(H)$ of dimension $i$.

We will prove the right continuity of the $k$th eigenvalue in two steps.

**Proposition 3.2.** Let $D$ be a domain of finite Lebesgue measure. Then for all $k \geq 1$

$$\lim_{\beta \to \alpha} \sup_{\beta} \lambda_{\beta}^k(D) \leq \lambda_{\alpha}^k(D).$$

**Proof.** Let $0 < \epsilon < 1$ and $k \geq 1$. Recall $C_c^\infty(D)$ is dense in $\text{Dom}(H_\alpha)$ under the norm $\|\cdot\|_\alpha$. Then for all $\alpha \in (0, 2)$, there exist $\varphi_1, \ldots, \varphi_k \in C_c^\infty(D)$ such that

$$\langle \varphi_1^0, \varphi_j^0 \rangle - \langle \varphi_1, \varphi_j \rangle < \frac{\epsilon}{8k^2}$$

and

$$|E_\alpha(\varphi_1^0, \varphi_j^0) - E_\alpha(\varphi_1, \varphi_j)| < \frac{\epsilon}{8k^2},$$

for all $1 \leq i, j \leq k$. 

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Thanks to Lemma 2.1 there exists $\eta_0$ such that for all $\beta \in (\alpha - \eta_0, \alpha + \eta_0)$

$$|E_\alpha(\varphi_i, \varphi_j) - E_\beta(\varphi_i, \varphi_j)| < \frac{\epsilon}{8k^2}. \quad (3.5)$$

Notice that (3.3) implies

$$|\langle \varphi_i, \varphi_j \rangle| < \frac{\epsilon}{8k^2}, \quad i \neq j,$$

and

$$1 - \frac{\epsilon}{8k^2} < \|\varphi_i\|^2 < 1 + \frac{\epsilon}{8k^2},$$

for all $1 \leq i, j \leq k$. Then by the comments at the end of Section 2, we know $\varphi_1, \ldots, \varphi_k$ are linearly independent.

Theorem 3.1 implies

$$\lambda_\beta^k(D) \leq \lambda_\beta(L_k),$$

where $L_k = \text{span}\{\varphi_1, \ldots, \varphi_k\}$ and

$$\lambda_\beta(L_k) = \sup \{\langle H_\beta f, f \rangle : f \in L_k, \|f\|_2 = 1\}.$$

Take $\psi = \sum_{i=1}^k a_i \varphi_i \in L_k$ such that

$$\lambda_\beta(L_k) \leq E_\beta(\psi, \psi) + \epsilon/4$$

and

$$\|\psi\|_2 = 1.$$

Thanks to (2.1), with $\epsilon$ there replaced by $\epsilon/2$, we have

$$\sum_{i=1}^k a_i^2 \leq 2.$$

Then since

$$|E_\beta(\psi, \psi) - E_\alpha(\psi, \psi)| \leq \sum_{i=1}^k \sum_{j=1}^k |a_i a_j| \cdot |E_\beta(\varphi_i, \varphi_j) - E_\alpha(\varphi_i, \varphi_j)|,$$

(3.5) implies

$$|E_\beta(\psi, \psi) - E_\alpha(\psi, \psi)| < \frac{\epsilon}{4}. \quad (3.7)$$

Thus

$$\lambda_\beta^k(D) \leq E_\alpha(\psi, \psi) + \epsilon/2. \quad (3.8)$$

Consider $\psi_0 = \sum_{i=1}^k a_i \varphi_i^0$. By (2.1) we have

$$\frac{1}{1 + \epsilon/4} \leq \|\psi_0\|^2 = \sum_{i=1}^k a_i^2 \leq \frac{1}{1 - \epsilon/4}.$$

Following the argument used to obtain (3.7), one easily proves (3.4) implies

$$|E_\alpha(\psi_0, \psi_0) - E_\alpha(\psi, \psi)| < \frac{\epsilon}{4}. \quad (3.4)$$
Hence
\[ \lambda_k^\beta(D) \leq E_\alpha(\psi, \psi) + \epsilon/2 \]
\[ \leq E_\alpha(\psi_0, \psi_0) + 3\epsilon/4 \]
\[ = \sum_{i=1}^k a_i^2 \alpha(D) + 3\epsilon/4 \]
\[ \leq \lambda_k^\beta(D) \sum_{i=1}^k a_i^2 + 3\epsilon/4 \]
\[ \leq \frac{1}{1 - \epsilon/4} \lambda_k^\beta(D) + 3\epsilon/4, \]
and the result immediately follows. \[ \square \]

**Proposition 3.3.** Let \( D \) be a domain of finite Lebesgue measure. Then for all \( k \geq 1 \)
\[ \liminf_{\beta \to \alpha^+} \lambda_k^\beta(D) \geq \lambda_k^\alpha(D). \]

**Proof.** By Theorem 1.2,
\[ \lambda_k^\alpha(D) \leq [\lambda_k^{\alpha+\epsilon}(D)]^{\alpha/(\alpha+\epsilon)} \cdot \]
Now let \( \epsilon \to 0 \) to get the desired \( \liminf \) behavior. \[ \square \]

Combining Propositions 3.2 and 3.3, we get Theorem 1.1.

4. **PROOF OF THEOREMS 1.3 AND 1.5**

We now show how Theorem 1.4 implies Theorem 1.3. In order to simplify the notation, throughout this section we will write \( \lambda_k^\alpha \) for \( \lambda_k^\alpha(D) \) and \( \mu_k \) for \( \mu_k(D) \).

**Proof of Theorem 1.3.** We proceed by induction on \( i \). For \( i = 1 \), let \( \{\alpha_m\}_{m=1}^\infty \) be a sequence converging to \( \alpha \) in \((0, 2)\). Consider any subsequence \( \beta_{r_m} = \alpha_m \). Theorem 1.2 implies the sequence \( \{\lambda_1^{\alpha_m}\}_{m=1}^\infty \) is bounded, and so there is a subsequence \( \gamma_{t_m} = \beta_{r_m} \) such that \( \lambda_1^{\gamma_{t_m}} \) converges as \( t_m \to \infty \), say to \( \lambda \). Thanks to Theorem 1.4 we can choose a subsequence \( \eta_p = \gamma_{t_p} \) such that \( \varphi_1^{\eta_p} \) converges uniformly to \( \varphi \) an eigenfunction of \( H_\alpha \) with eigenvalue \( \lambda \). Since \( \varphi_1^{\eta_p} \) is nonnegative, so is \( \varphi \). But the only nonnegative eigenfunction of \( H_\alpha \) is \( \varphi_1^\alpha \). Thus \( \lambda = \lambda_1^\alpha \) and \( \varphi = \varphi_1^\alpha \). Hence we have shown any subsequence of \( \lambda_1^{\alpha_m} \) contains a further subsequence converging to \( \lambda_1^\alpha \). We conclude that
\[ \lim_{m \to \infty} \lambda_1^{\alpha_m} = \lambda_1^\alpha. \]

Note this also proves Theorem 1.5.

Next, assume the theorem is true for \( j \leq i \). We verify it is true for \( j = i + 1 \). We will show
\[ \liminf_{\beta \to \alpha} \lambda_i^\beta(D) \geq \lambda_i^{\alpha} \]
Combined with the \( \limsup \) behavior from Proposition 3.2, we conclude the desired result
\[ \lim_{\beta \to \alpha} \lambda_i^{\beta} = \lambda_i^{\alpha}. \]
To get the lim inf behavior, by way of contradiction, assume \( \lambda = \liminf_{\beta \to \alpha} \lambda_{i+1}^\beta < \lambda_{i+1}^\alpha \). Let \( \{\alpha_m\}_{m=1}^\infty \) be a sequence converging to \( \alpha \) with
\[
\lim_{m \to \infty} \lambda_{i+1}^{\alpha_m} = \lambda.
\]
By the induction hypothesis, \( \lambda_{j+1}^{\alpha_m} \) converges to \( \lambda_{j+1}^\alpha \) for \( j \leq i \). Then Theorem 1.4 implies we can choose a subsequence \( \beta_r = \alpha_{m_r} \) such that:

- For each \( j, 1 \leq j \leq i \), \( \lambda_j^{\beta_r} \) converges to \( \lambda_j^\alpha \) and \( \varphi_j^{\beta_r} \) converges uniformly to an eigenfunction \( \varphi_j \) of \( H_\alpha \) with corresponding eigenvalue \( \lambda_j^\alpha \).

- The limit \( \lambda \) from (4.2) is an eigenvalue of \( H_\alpha \), and \( \varphi_{i+1}^{\beta_r} \) converges uniformly to an eigenfunction \( \varphi_{i+1} \) of \( H_\alpha \) with eigenvalue \( \lambda \).

Since \( \lambda \) is an eigenvalue strictly less than \( \lambda_{i+1}^\alpha \), we can choose positive integers \( \ell \) and \( m \) such that \( \ell \leq m \leq i \), \( \lambda_m^\alpha = \lambda \) and
\[
\lambda_{i+1}^\alpha < \lambda_1^\alpha = \cdots = \lambda_{\ell}^\alpha < \lambda_{\ell+1}^\alpha < \cdots < \lambda_{m-1}^\alpha < \lambda_{m}^\alpha \leq \cdots \leq \lambda_{i}^\alpha.
\]
(Here we take \( \lambda_0^\alpha := 0 \).) In particular, if \( E \) is the eigenspace corresponding to \( \lambda = \lambda_m^\alpha \), then
\[
\dim(E) = m - \ell + 1.
\]
On the other hand, the uniform convergence implies for \( j_1, j_2 \in \{1, \ldots, i+1\} \)
\[
\delta_{j_1,j_2} = \int_D \varphi_{j_1}^{\beta_r} \varphi_{j_2}^{\beta_r} \, dx \quad \text{converges to} \quad \int_D \varphi_{j_1} \varphi_{j_2} \, dx.
\]
Thus \( \{\varphi_1, \ldots, \varphi_{i+1}\} \) is an orthonormal set, and so \( \{\varphi_1, \ldots, \varphi_m\} \cup \{\varphi_{i+1}\} \) is an orthonormal subset of \( E \). This forces \( \dim(E) \geq m - \ell + 2 \), which contradicts (4.2).

We conclude (4.1) holds.

5. Weak Convergence Results

Let \( D[0, \infty) \) be the space of right continuous functions \( \omega: [0, \infty) \to \mathbb{R}^d \) with left limits. That is, \( \omega(t^+) = \lim_{s \to t^+} \omega(s) = \omega(t) \) and \( \omega(t^-) = \lim_{s \to t^-} \omega(s) \) exists. The usual convention is \( \omega(0^-) := \omega(0) \). Let \( X_t(\omega) = \omega(t) \) be the coordinate process and let \( \mathcal{F}_t \) be the \( \sigma \)-algebra generated by the cylindrical sets. We equip \( D[0, \infty) \) with the Skorohod topology. Our main reference is Chapter 3 in Ethier and Kurtz [16]. Let \( P_\alpha \) denote the law on \( D[0, \infty) \) of the symmetric \( \alpha \)-stable process started at \( x \); the corresponding expectation will be denoted by \( E_x^\alpha \).

**Lemma 5.1.** If \( (x_n, \alpha_n) \) converges to \( (x, \alpha) \) in \( \mathbb{R}^n \times (0, 2) \), then \( P_{x_n}^{\alpha_n} \) converges weakly to \( P_x^\alpha \) in \( D[0, \infty) \).

**Proof.** Using characteristic functions it is easy to show the corresponding finite dimensional distributions converge. Thus, by Theorem 7.8 on page 131 in [16], it suffices to show \( \{P_{x_n}^{\alpha_n} : n \geq 1\} \) is tight on \( D[0, \infty) \).

By Theorem 7.2 and Remark 7.3 of [16] and Theorem 15.2 of [7], it suffices to show that for each \( t_0 \geq 0 \), \( \{P_{x_n}^{\alpha_n} : n \geq 1\} \) is tight on \( D[0, t_0] \). For this, we proceed as in the proof of Proposition 3.2 in [3], using a theorem of Aldous [2]. For the convenience of the reader, we state the theorem using our notation.

For each \( n \geq 1 \) let \( \tau_n \) be a stopping time with finitely many values, and let \( \delta_n \geq 0 \) converge to 0. Aldous’s Theorem is the following. Suppose for any \( \eta > 0 \),
\[
P_{x_n}^{\alpha_n}(\{|X(\tau_n + \delta_n) - X(\tau_n)| \geq \eta\}) \to 0
\]

as \( n \to \infty \). If for each \( t \in [0, t_0] \) the collection
\[
\{ P_{x_n}^\alpha \circ X_t : n \geq 1 \}
\]
is tight on \( \mathbb{R}^d \), then \( \{ P_{x_n}^\alpha : n \geq 1 \} \) is tight on \( \mathbb{D}[0, t_0] \).

Even though the theorem is stated for dimension one, the argument also works for higher dimensions.

We now verify the conditions of the theorem. First, note that for \( \beta = \alpha \) or \( \alpha_n \) and \( y = x \) or \( x_n \), \( P_y^\beta \) solves the martingale problem:
\begin{enumerate}[(a)]  
    \item \( P_y^\beta(X_0 = y) = 1 \),  
    \item for each \( f \in C_0^2(\mathbb{R}^d) \),
\end{enumerate}
\[
f(X_t) - f(X_0) - \int_0^t \mathcal{L}_\beta f(X_s)ds
\]
is a \( P_y^\beta \)-martingale, where \( C_0^2(\mathbb{R}^d) \) is the space of functions with bounded continuous derivatives up to and including order 2 and
\[
\mathcal{L}_\beta f(x) = A(d, \alpha) \int_{\mathbb{R}^d \setminus \{x\}} \frac{f(y) - f(x) - \nabla f(x) \cdot (y - x) I(|y - x| < 1)}{|y - x|^{d+\alpha}} dy;
\]
see Section 2 of [4]. It is easy to show that for any \( f \in C_0^2(\mathbb{R}^d) \) there exists \( C_f > 0 \) independent of \( \alpha_n \) and \( x_n \) such that \( f(X_t) - f(X_0) - C_f t \) is a \( P_{x_n}^\alpha \)-supermartingale. Then we can argue as in the proof of Proposition 3.2 in [3] to get formula (3.1) from that article: for any bounded stopping time \( \tau_n \),
\[
P_{x_n}^\alpha (\sup_{\tau_n \leq s \leq \tau_n + \delta} |X_s - X_{\tau_n}| \geq \eta) \leq \frac{c \delta}{\eta^2},
\]
where \( c \) is independent of \( n, \delta \) and \( \eta \). Even though the formula from that article is stated for one dimension and \( x_n \equiv x \), the proof works in higher dimensions with \( x_n \) converging to \( x \), since the constant \( C_f \) is independent of \( n \). Note too the requirement there that \( \delta < 1 \) can be dropped.

Replacing \( \delta \) by \( \delta_n \to 0 \) as \( n \to \infty \), upon letting \( n \to \infty \) in (5.2), we obtain condition (5.1). Next we handle tightness of \( \{ P_{x_n}^\alpha \circ X_t : n \geq 1 \} \) on \( \mathbb{R}^d \) for \( t \in [0, t_0] \). If \( n \) is large, say \( n \geq N \), then \( |x_n| < |x| + 1 \). Thus, if \( \lambda > |x| + 1 \), then for \( t \in [0, t_0] \), by (5.2) with \( \tau_n \equiv 0 \) and \( \delta = t_0 \)
\[
\sup_{n \geq N} P_{x_n}^\alpha (|X_t| \geq \lambda) \leq \sup_{n \geq N} P_{x_n}^\alpha (|X_t - x_n| + |x_n| \geq \lambda) \\
\leq \sup_{n \geq N} P_{x_n}^\alpha (|X_t - x_n| \geq \lambda - |x| - 1) \\
\leq \sup_{n \geq N} P_{x_n}^\alpha (\sup_{s \leq t_0} |X_s - x_n| \geq \lambda - |x| - 1) \\
\leq \frac{c t_0}{(\lambda - |x| - 1)^2}.
\]
This gives the desired tightness.

The next step is to show for each \( T > 0 \) that the distribution of \( X_{T \wedge \tau_D} \) under \( P_{x_n}^\alpha \) converges to that under \( P_{x}^\alpha \) as \( (x_n, \alpha_n) \) converges to \( (x, \alpha) \). To this end, define
\[
\mathcal{A}_D = \{ \omega \in \mathbb{D}[0, \infty) : d(X[0, \tau_D(\omega) - r], D^r) > 0 \text{ for all rational } 0 < r < \tau_D(\omega) \},
\]
Lemma 5.2. For open $D \subseteq \mathbb{R}^d$, $\tau_D$ is continuous on $C_D$.

Proof. Let $\omega \in C_D$ and suppose $\omega_n$ converges to $\omega$ in $\mathbb{D}[0, \infty)$. We will show that $\tau_D(\omega_n)$ converges to $\tau_D(\omega)$. Let

$$
\Lambda' = \{ \lambda : [0, \infty) \to [0, \infty) \mid \lambda \text{ is strictly increasing and surjective} \}.
$$

Proposition 5.3 (a) and (c), on page 119 in [16], implies that for each $T > 0$ there exist $\{ \lambda_n \} \subseteq \Lambda'$ such that

\begin{align}
& \lim_{n \to \infty} \sup_{0 \leq t \leq T} |\lambda_n(t) - t| = 0, \\
& \lim_{n \to \infty} \sup_{0 \leq t \leq T} |\omega_n(t) - \omega(\lambda_n(t))| = 0.
\end{align}

First we show that

$$
\liminf_{n \to \infty} \tau_D(\omega_n) \geq \tau_D(\omega).
$$

Let $\delta \in (0, \tau_D(\omega)/2)$ be rational and set

$$
\varepsilon = d(\omega[0, \tau_D(\omega) - \delta], D^c).
$$

Since $\omega \in C_D$, we have $\varepsilon > 0$. Using $T = \tau_D(\omega)$ in (5.3)–(5.4), there exists $N$ such that for $n \geq N$,

\begin{equation}
\Bigg\{ \begin{array}{l}
\frac{t - \delta}{T} < \lambda_n(t) < t + \delta \quad \text{for all } \frac{t}{T} \leq T, \\
\sup_{0 \leq t \leq T} |\omega_n(t) - \omega(\lambda_n(t))| < \frac{\varepsilon}{2}.
\end{array} \tag{5.6} \end{equation}

In particular, for all $t \leq \tau_D(\omega) - 2\delta$ and $n \geq N$

$$
\lambda_n(t) < t + \delta \leq \tau_D(\omega) - \delta < T.
$$

Thus $\omega(\lambda_n(t)) \in D$ and $d(\omega(\lambda_n(t)), D^c) \geq \varepsilon$. Therefore,

$$
\omega_n(t) \in B(\omega(\lambda_n(t)), \varepsilon/2) \subseteq D,
$$

for all $t \leq \tau_D(\omega) - 2\delta$ and $n \geq N$. This implies $\tau_D(\omega_n) > \tau_D(\omega) - 2\delta$ for $n \geq N$. Take the lim inf as $n \to \infty$ and then let $\delta \to 0$ to get (5.5).

To finish, we show that

$$
\limsup_{n \to \infty} \tau_D(\omega_n) \leq \tau_D(\omega).
$$

Given that $\omega \in C_D$ and $\omega$ is right continuous, we can choose $\delta > 0$ such that

$$
\varepsilon := d(\omega[\tau_D(\omega), \tau_D(\omega) + 2\delta], D) > 0.
$$

Using $T = \tau_D(\omega) + 2\delta$ in (5.3)–(5.4) we can choose $N$ such that for $n \geq N$, (5.6) holds for this choice of $\delta, \varepsilon$ and $T$. In particular, for $n \geq N$,

$$
\tau_D(\omega) < \lambda_n(\tau_D(\omega) + \delta) < \tau_D(\omega) + 2\delta
$$

and

$$
|\omega_n(\tau_D(\omega) + \delta) - \omega(\lambda_n(\tau_D(\omega) + \delta))| < \frac{\varepsilon}{2}.
$$

Together these imply

$$
d(\omega_n(\tau_D(\omega) + \delta), D) > 0, \quad n \geq N,
$$

and

$$
C_D = \{ \omega \in \mathbb{D}[0, \infty) : X(\tau_D(\omega)) \in D \} \cap A_D.
$$

Here $X[0, t] = \{ X_s : 0 \leq s \leq t \}$ and $d(A, B)$ is the distance between $A$ and $B$. 


which in turn yields
\[ \tau_D(\omega_n) \leq \tau_D(\omega) + \delta, \quad n \geq N. \]
Taking the lim sup as \( n \to \infty \) and then letting \( \delta \to 0 \) yields (5.7). \( \square \)

**Lemma 5.3.** If \( D \) is a bounded domain that satisfies an exterior cone condition or if \( D \) is a cone, then for all \( x \in D \) and \( 0 < \alpha < 2 \),

\[ P_x^\alpha(C_D \cap \{ X(\tau_D) \in D \}) = 1. \]

**Proof.** If \( D \) is bounded and satisfies a uniform exterior cone condition, it is known that
\[ (5.8) \quad P^\alpha_x( X(\tau_D) \in \partial D ) = 0; \]
see Lemma 6 in [10]. If \( D \) is a cone, we can apply Lemma 6 in [10] to \( D \cap B_M(0) \) and letting \( M \to \infty \), we get (5.8).

The proof of Theorem 2 in [18] implies
\[ (5.9) \quad P^\alpha_x( X(\tau_D) \in \partial D, X(\tau_D) \in E ) = 0, \quad E \subseteq \overline{E} \subseteq \overline{D}^c \]
(see the lines before the footnote on page 89). Combined with (5.8),
\[ P^\alpha_x( X(\tau_D) \in \overline{D}^c, X(\tau_D) \in D ) = 1. \]
Thus to prove the lemma we need to show that
\[ (5.10) \quad P^\alpha_x( d(X[0, \tau_D - r], D^c) > 0 \text{ for all rational } r < \tau_D ) = 1. \]
Let
\[ D_n = \left\{ x \in D : d(x, D^c) > \frac{1}{n} \right\} \]
and observe \( \tau_{D_n} \leq \tau_D \) increases to some limit \( L \leq \tau_D \). By quasi-left continuity, \( X(\tau_{D_n}) \to X(L) \) almost surely. One easily sees \( X(L) \notin D \), i.e., \( \tau_D \leq L \). Hence \( \tau_D = L \), and the increasing limit of \( \tau_{D_n} \) is \( \tau_D \).

If for some rational \( r < \tau_D \) we have
\[ d(X[0, \tau_D - r], D^c) = 0, \]
then for some sequence \( s_n \leq \tau_D - r \),
\[ d( X_{s_n}, D^c ) \to 0. \]
It is no loss to assume \( s_n \) converges, say to \( s \). Choose \( N \) such that for all \( n \geq N \)
\[ \tau_D - r < \tau_{D_n} \leq \tau_D. \]
Given \( n \geq N \), choose \( M_n \) such that for all \( m \geq M_n \),
\[ d( X_{s_m}, D^c ) < \frac{1}{2n}. \]
Then for such \( m \), \( X_{s_m} \in D_n^c \), which forces
\[ \tau_{D_n} \leq s_m \leq \tau_D - r. \]
Let \( m \to \infty \) to get \( \tau_{D_n} \leq s \leq \tau_D - r \), then let \( n \to \infty \) to get \( \tau_D = \lim_{n \to \infty} \tau_{D_n} \leq \tau_D - r; \) a contradiction. Thus (5.10) holds. \( \square \)

We will need the following elementary result shortly.

**Lemma 5.4.** Let \( a_n \) and \( b_n \) be nonnegative sequences such that \( a_n \wedge b_n \to 0 \) as \( n \to \infty \). Then for some \( n_k \), either \( a_{n_k} \to 0 \) or \( b_{n_k} \to 0 \) as \( k \to \infty \).
Proof. Suppose \( \liminf_{n \to \infty} a_n = a > 0 \). Choose a subsequence \( a_{n_k} \) such that \( a_{n_k} \geq \frac{a}{2} \) for all \( k \). Then
\[
a_{n_k} \wedge b_{n_k} \geq \frac{a}{2} \wedge b_{n_k} \geq 0.
\]
Since \( a_{n_k} \wedge b_{n_k} \to 0 \) as \( k \to \infty \), we must have \( \frac{a}{2} \wedge b_{n_k} \to 0 \), which in turn forces \( b_{n_k} \to 0 \). \( \square \)

Lemma 5.5. Let \( D \) be a bounded domain that satisfies an uniform exterior cone condition, and let \( f \) be a bounded continuous function on \( \mathbb{R}^d \). If \((x_n, \alpha_n)\) converges to \((x, \alpha)\) in \( D \times (0, 2) \), then for each \( T > 0 \),
\[
E_{x_n}^\alpha[f(X_{T \wedge \tau_D})] \to E_x^\alpha[f(X_{T \wedge \tau_D})].
\]

Proof. Let
\[
C_D(T) = C_D \cap \{X(\tau_D) \in D\} \cap \{\tau_D \neq T\} \cap \{\lim_{s \to T} X_s = X_T\}.
\]
Recall that the symmetric \( \alpha \)-stable process has no fixed discontinuities. Then by the eigenfunction expansion of \( P_x^\alpha(\tau_D > t) \),
\[
P_x^\alpha(\tau_D \neq T, \lim_{s \to T} X_s = X_T) = 1.
\]

Thanks to Lemma 5.3
\[
P_x^\alpha(C_D(T)) = 1.
\]
If we can show that
\[
\omega \in C_D(T) \to \omega(T \wedge \tau_D(\omega))
\]
is continuous, then by an extension of the continuous mapping theorem (Theorem 5.1 in [7]), the desired conclusion will follow.

Let \( \omega \in C_D(T) \) and suppose \( \omega_n \in C_D(T) \) converges to \( \omega \) in \( D[0, \infty) \). Define
\[
t_n = T \wedge \tau_D(\omega_n),
\]
\[
t = T \wedge \tau_D(\omega).
\]
To show that
\[
\lim_{n \to \infty} \omega_n(T \wedge \tau_D(\omega_n)) = \omega(T \wedge \tau_D(\omega)),
\]
we show for every subsequence \( \omega_{n_k} \) there is a further subsequence \( \omega_{n_{k_l}} \) such that
\[
\lim_{l \to \infty} \omega_{n_{k_l}}(T \wedge \tau_D(\omega_{n_{k_l}})) = \omega(T \wedge \tau_D(\omega)).
\]

By Lemma 5.2 \( \lim_{n \to \infty} t_n = t \). Applying Proposition 6.5(a) on page 125 in [16],
\[
\lim_{n \to \infty} |\omega_n(t_n) - \omega(t)| \wedge |\omega_n(t_n) - \omega(t^-)| = 0.
\]

Hence by Lemma 5.3 for any subsequence \( \omega_{n_k} \) there is a further subsequence \( \omega_{n_{k_l}} \) such that either
\[
\lim_{l \to \infty} |\omega_{n_{k_l}}(t_{n_{k_l}}) - \omega(t)| = 0
\]
or
\[
(5.11) \quad \lim_{l \to \infty} |\omega_{n_{k_l}}(t_{n_{k_l}}) - \omega(t^-)| = 0.
\]
In the first case, clearly
\[
\lim_{l \to \infty} \omega_{n_{k_l}}(T \wedge \tau_D(\omega_{n_{k_l}})) = \omega(T \wedge \tau_D(\omega)),
\]
as desired.
In the second case \([5.11]\), we distinguish two cases: \(T > \tau_D(\omega)\) and \(T < \tau_D(\omega)\) (recall \(\omega \in C_D(T)\) implies \(\tau_D(\omega) \neq T\)).

Let us first assume \(T > \tau_D(\omega)\). Since \(\tau_D(\omega_{n_k})\) converges to \(\tau_D(\omega)\) by Lemma \(6.2\), \(t_{n_k} = \tau_D(\omega_{n_k})\) for large \(l\). Hence by \([5.11]\)

\[
\lim_{l \to \infty} \omega_{n_k}(\tau_D(\omega_{n_k})) = \lim_{l \to \infty} \omega_{n_k}(t_{n_k}) = \omega(T^-) = \omega(\tau_D(\omega)^-).
\]

Notice that if \(\lim_{l \to \infty} \omega_{n_k}(\tau_D(\omega_{n_k})) = y\) exists, then \(y \in D^c\). But then \(\omega \in C_D(T)\) implies \(y = \omega(\tau_D(\omega)^-) \in D\); a contradiction. Thus \(T > \tau_D(\omega)\) is not possible.

Finally, if \(T < \tau_D(\omega)\), then by Lemma \(6.2\)

\[
T < \tau_D(\omega_{n_k}),
\]

for \(l\) large. Since \(\omega \in C_D(T)\), \([5.11]\) becomes

\[
\omega_{n_k}(T) \to \omega(T^-) = \omega(T).
\]

We conclude that

\[
\lim_{l \to \infty} \omega_{n_k}(T \land \tau_D(\omega_{n_k})) = \lim_{l \to \infty} \omega_{n_k}(T) = \omega(T^-) = \omega(T) = \omega(T \land \tau_D(\omega)).
\]

In any event, we get the desired continuity. \(\square\)

6. Proof of Theorem \(1.4\)

We will need the following lemma; it is formula (2.7) in [5]. Although the authors do not mention the statement concerning continuity in \(\alpha\), it is possible to trace back through the literature they cite to see the statement holds.

**Lemma 6.1.** If \(D \subseteq \mathbb{R}^d\) is a bounded Lipschitz domain, then for some positive continuous functions \(C(\alpha)\) and \(\beta(\alpha)\),

\[
E_x^\alpha(\tau_D) \leq C(\alpha) \delta_D^\beta(\alpha)(x), \quad \text{for all } x \in D.
\]

The next result immediately follows.

**Corollary 6.2.** Given a bounded Lipschitz domain \(D\) and compact \(K \times [a, b] \subseteq \overline{D} \times (0, 2)\),

\[
\sup \{ E_x^\alpha(\tau_D) : (x, \alpha) \in K \times [a, b] \} < \infty.
\]

Corollary \(6.2\) will allow us to get equicontinuity of the eigenfunctions near \(\partial D\). For the interior of \(D\) we need the following Krylov–Safanov type of theorem. Let

\[
G_0^\alpha g(x) = E_x^\alpha \left[ \int_0^{\tau_D} g(X_t) \, dt \right]
\]

be the 0-resolvent of the killed symmetric \(\alpha\)-stable process in \(D\).

**Lemma 6.3.** Suppose \(g\) is bounded with support in \(\overline{D}\). Then for each \(x \in D\) there exist positive continuous functions \(C(\alpha)\) and \(\beta(\alpha)\), independent of \(g\), such that for all \(y \in D\)

\[
|G_0^\alpha g(x) - G_0^\alpha g(y)| \leq C(\alpha) \left[ \sup |G_0^\alpha g| + \sup |g| \right] |x - y|^{\beta(\alpha)}.
\]
Proof. This theorem is essentially due to Bass and Levin [4] (see their Proposition 4.2 on page 387). While they consider the 0-resolvent
\[ S_0g(x) = E_x \left[ \int_0^\infty g(X_t)dt \right], \]
their proof also works for the killed resolvent because their crucial formula
\[ S_0g(y) = E_y \left[ \int_{\tau_{B(x,r)}}^\infty g(X_t)dt \right] + E_y \left[ S_0g(X_{\tau_{B(x,r)}}) \right] \]
holds when \( S_0 \) is replaced by \( G_0^\alpha \) and \( E_y \) is replaced by \( E_\alpha y \), where \( r > 0 \) is such that \( B(x, r) \subset D \). Since we are restricted to \( D \) instead of \( \mathbb{R}^d \), the numbers \( C(\alpha) \) and \( \beta(\alpha) \) depend on \( x \), in contrast to the case treated by Bass and Levin. Moreover, it is a simple matter to go through their proof and see that the numbers \( C(\alpha) \) and \( \beta(\alpha) \) can be chosen to depend continuously on \( \alpha \). □

Corollary 6.4. Assume \( D \) is bounded and Lipschitz. Then for each \( x \in D \) and \( [a, b] \subseteq (0, 2) \), there exist positive \( C \) and \( r \) such that
\[ |G_0^\alpha g(x) - G_0^\alpha g(y)| \leq C|x - y|^r \sup |g| \]
for all \( y \in D \), \( \alpha \in [a, b] \) and bounded \( g \) with support in \( D \).

Proof. By Corollary 6.2
\[ \sup |G^\alpha g| \leq \sup |g| \cdot \sup_x E_x^\alpha (\tau_D) \]
\[ \leq \sup |g| \cdot C \]
where \( C \) is independent of \( \alpha \in [a, b] \) and \( g \). The result follows from this and the continuity of \( C(\alpha) \) and \( \beta(\alpha) \) from Lemma 6.3. □

At last we can prove Theorem 1.4. It is well known that
\[ 0 \leq p_D^\alpha(t, x, y) \leq p^\beta(t, x, y). \]
Moreover,
\[ p^\alpha(t, x, y) \leq C(\alpha)t^{-d/\alpha} \]
where \( C(\alpha) \) is continuous in \( \alpha \) (see (2.1) in [9]).

Let \( \{\alpha_m\}_{m=1}^\infty \subseteq (0, 2) \) be a sequence converging to \( \alpha \in (0, 2) \). Recall that for all \( \beta \in (0, 2) \), \( \{\varphi_\beta^m\}_{m=1}^\infty \) is an orthonormal set. Then thanks to the symmetry of the heat kernel and \( \varphi_\beta \)
\[ \varphi_\beta^m(x) = e^{\lambda^x t} \int_D p_D^\beta(t, x, y)\varphi_\beta^m(y)dy \]
\[ \leq e^{\lambda^x t} \sqrt{\int_D \left[ p_D^\beta(t, x, y) \right]^2 dy} \]
\[ = e^{\lambda^x t} \sqrt{p_D^\beta(2t, x, x)} \]
\[ \leq e^{\lambda^x t} \sqrt{\frac{C(\beta)}{(2t)^{d/\beta}}}. \]
In particular, taking $t = 1$ and using Theorem 1.2,

$$\sup_{x \in D, m \geq 1} \varphi_{i,m}^\alpha(x) \leq \sup_{m \geq 1} e^{\lambda_m^\alpha} \sqrt{\frac{C_1}{2^d \alpha_m}} < \infty. \tag{6.3}$$

Thus for each $i \geq 1$, the sequence $\{\varphi_{i,m}^\alpha\}_{m=1}^\infty$ is uniformly bounded. Next we show the sequence $\{\varphi_{i,m}^\alpha\}_{i=1}^\infty$ is pointwise equicontinuous on $\overline{D}$. Indeed, since

$$\varphi_i^\beta = \lambda_i^\beta G_0^\beta \varphi_i^\beta, \tag{6.4}$$

Corollary [6.4] implies that for each $x \in D$ there exist $C$ and $r$ such that

$$|\varphi_{i,m}^\alpha(x) - \varphi_{i,m}^\alpha(y)| = \lambda_i^\alpha |G_0^\alpha \varphi_{i,m}^\alpha(x) - G_0^\alpha \varphi_{i,m}^\alpha(y)| \leq C \left[ \sup_{m \geq 1} [\mu_i^\alpha]^{\alpha_m/2} \right] \left[ \sup_{y \in D, m \geq 1} |\varphi_{i,m}^\alpha(y)| \right] |x - y|^r$$

for all $m \geq 1$ and $y \in D$. Thanks to (6.3) we get the desired equicontinuity for $x \in D$.

As for $x \in \partial D$, first notice (6.4) and Lemma 6.1 imply there are $r$ and $C$ independent of $m$ such that for each $z \in D$

$$|\varphi_{i,m}^\alpha(z)| \leq \left[ \sup_{m \geq 1} [\mu_i^\alpha]^{\alpha_m/2} \right] \left[ \sup_{y \in D, m \geq 1} |\varphi_{i,m}^\alpha(y)| \right] E_{i,m}^\alpha(\partial D) \leq C \left[ \delta_D(z) \right]^r.$$

Thus $\varphi_{i,m}^\alpha$ is continuous on $\overline{D}$ with boundary value 0. Hence if $x \in \partial D$, then

$$|\varphi_{i,m}^\alpha(x) - \varphi_{i,m}^\alpha(y)| = |\varphi_{i,m}^\alpha(y)| \leq C |\delta_D(y)|^r \leq C|x - y|^r.$$

By Ascoli’s Theorem, the sequence $\{\varphi_{i,m}^\alpha\}_{m=1}^\infty$ is precompact in $C(\overline{D})$.

Next assume $\{\lambda_i^\alpha\}_{i=1}^\infty$ converges to $\lambda$. We show any limit point $\varphi$ of the sequence $\{\varphi_{i,m}^\alpha\}_{i=1}^\infty$ is an eigenfunction of $H_\alpha$ and the corresponding eigenvalue is $\lambda$. Choose a subsequence $\beta_i = \alpha_{m_i}$ such that, as $r \to \infty$, $\varphi_{i,r}^\beta$ converges uniformly to $\varphi$ on $\overline{D}$. Since $\varphi_{i,r}^\beta$ and $\varphi$ are 0 on $\partial D$, we can extend them to all of $\mathbb{R}^d$ by taking them to be 0 outside $D$. Then

$$E_x^{\beta_r} \left[ \varphi_{i,r}^\beta(X_{t \wedge \tau_D}) \right] = E_x^{\beta_r} \left[ \varphi_{i,r}^\beta(X_t) I_{\tau_D > t} \right],$$

$$E_x^{\beta_r} \left[ \varphi(X_{t \wedge \tau_D}) \right] = E_x^{\beta_r} \left[ \varphi(X_t) I_{\tau_D > t} \right],$$

and $\varphi_{i,r}^\beta$ converges to $\varphi$ uniformly on $\mathbb{R}^d$. Thus we have

$$e^{-\lambda_i^\beta t} \varphi_{i,r}^\beta(x) = \int_{\partial D} D^\beta_{D}(t, x, y) \varphi_{i,r}^\beta(y) dy \tag{6.5}$$

$$= E_x^{\beta_r} \left[ \varphi_{i,r}^\beta(X_{t \wedge \tau_D}) \right] - E_x^{\beta_r} \left[ \varphi_{i,r}^\beta(X_{t \wedge \tau_D}) - \varphi(X_{t \wedge \tau_D}) \right] + E_x^{\beta_r} \left[ \varphi(X_{t \wedge \tau_D}) \right].$$
Lemma 5.5 and the uniform convergence of $\varphi_\alpha^{\alpha m}$ to $\varphi$ imply
\[
\lim_{r \to \infty} E_\beta^\beta \left[ \varphi_\alpha^{\alpha m}(X_{t \wedge \tau_D}) - \varphi(X_{t \wedge \tau_D}) \right] + E_\beta^\beta \left[ \varphi(X_{t \wedge \tau_D}) \right] = E_\alpha^\alpha \left[ \varphi(X_{t \wedge \tau_D}) \right].
\]
Since the left-hand side of (6.5) converges to $e^{-\lambda t} \varphi(x)$, we conclude that
\[
e^{-\lambda t} \varphi(x) = E_\alpha^\alpha \left[ \varphi(X_{t \wedge \tau_D}) \right] = E_\alpha^\alpha \left[ \varphi(X_t) I_{\tau_D > t} \right] = \int_D p_D^\alpha(t, x, y) \varphi(y) dy.
\]
Hence $\varphi$ is an eigenfunction of $H_\alpha$, and the corresponding eigenvalue is $\lambda$. \hfill \Box

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