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THIN STATIONARY SETS AND DISJOINT CLUB SEQUENCES

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ABSTRACT. We describe two opposing combinatorial properties related to adding clubs to ω_2: the existence of a thin stationary subset of $P_{\omega_1}(\omega_2)$ and the existence of a disjoint club sequence on ω_2. A special Aronszajn tree on ω_2 implies there exists a thin stationary set. If there exists a disjoint club sequence, then there is no thin stationary set, and moreover there is a fat stationary subset of ω_2 which cannot acquire a club subset by any forcing poset which preserves ω_1 and ω_2. We prove that the existence of a disjoint club sequence follows from Martin's Maximum and is equiconsistent with a Mahlo cardinal.

Suppose that S is a fat stationary subset of ω_2, that is, for every club set $C \subseteq \omega_2$, $S \cap C$ contains a closed subset with order type $\omega_1 + 1$. A number of forcing posets have been defined which add a club subset to S and preserve cardinals under various assumptions. Abraham and Shelah [1] proved that, assuming CH, the poset consisting of closed bounded subsets of S ordered by end-extension adds a club subset to S and is ω_1-distributive. S. Friedman [5] discovered a different poset for adding a club subset to a fat set $S \subseteq \omega_2$ with finite conditions.[1] This finite club poset preserves all cardinals provided that there exists a *thin stationary subset of* $P_{\omega_1}(\omega_2)$, that is, a stationary set $T \subseteq P_{\omega_1}(\omega_2)$ such that for all $\beta < \omega_2$, $|\{a \cap \beta : a \in T\}| \leq \omega_1$. This notion of stationarity appears in [9] and was discovered independently by Friedman. The question remained whether it is always possible to add a club subset to a given fat set and preserve cardinals, without any assumptions.

J. Krueger introduced a combinatorial principle on ω_2 which asserts the existence of a *disjoint club sequence*, which is a pairwise disjoint sequence $\langle \mathcal{C}_\alpha : \alpha \in A \rangle$ indexed by a stationary subset of $\omega_2 \cap \mathrm{cof}(\omega_1)$, where each $\mathcal{C}_\alpha$ is club in $P_{\omega_1}(\alpha)$. Krueger proved that the existence of such a sequence implies there is a fat stationary set $S \subseteq \omega_2$ which cannot acquire a club subset by any forcing poset which preserves ω_1 and ω_2.

We prove that a special Aronszajn tree on ω_2 implies there exists a thin stationary subset of $P_{\omega_1}(\omega_2)$. On the other hand assuming Martin's Maximum there exists a disjoint club sequence on ω_2. Moreover, we have the following equiconsistency result.

Theorem 0.1. *Each of the following statements is equiconsistent with a Mahlo cardinal:* (1) *There does not exist a thin stationary subset of* $P_{\omega_1}(\omega_2)$. (2) *There exists a disjoint club sequence on* ω_2. (3) *There exists a fat stationary set* $S \subseteq \omega_2$

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[1]A similar poset was defined independently by Mitchell [7].

such that any forcing poset which preserves ω_1 *and* ω_2 *does not add a club subset to* S.

Our proof of this theorem gives a totally different construction of the following result of Mitchell [8]: If κ is Mahlo in L, then there is a generic extension of L in which $\kappa = \omega_2$ and there is no special Aronszajn tree on ω_2. The consistency of Theorem 0.1(3) provides a negative solution to the following problem of Abraham and Shelah [1]: If $S \subseteq \omega_2$ is fat, does there exist an ω_1-distributive forcing poset which adds a club subset to S?

Section 1 outlines notation and background material. In Section 2 we discuss thin stationarity and prove that a special Aronszajn tree implies the existence of a thin stationary set. In Section 3 we introduce disjoint club sequences and prove that the existence of such a sequence implies there is a fat stationary set in ω_2 which cannot acquire a club subset by any forcing poset which preserves ω_1 and ω_2. In Section 4 we prove that Martin's Maximum implies there exists a disjoint club sequence. In Section 5 we construct a model in which there is a disjoint club sequence using an RCS iteration up to a Mahlo cardinal.

Sections 3 and 4 are due for the most part to J. Krueger. We would like to thank Boban Veličković and Mirna Džamonja for pointing out Theorem 2.3 to the authors.

1. Preliminaries

For a set X which contains ω_1, $P_{\omega_1}(X)$ denotes the collection of countable subsets of X. A set $C \subseteq P_{\omega_1}(X)$ is *club* if it is closed under unions of countable increasing sequences and is cofinal. A set $S \subseteq P_{\omega_1}(X)$ is *stationary* if it meets every club. If $C \subseteq P_{\omega_1}(X)$ is club, then there exists a function $F : X^{<\omega} \to X$ such that every a in $P_{\omega_1}(X)$ closed under F is in C. If $F : X^{<\omega} \to P_{\omega_1}(X)$ is a function and $Y \subseteq X$, we say that Y is *closed under* F if for all $\vec{\gamma}$ from $Y^{<\omega}$, $F(\vec{\gamma}) \subseteq Y$. A partial function $H : P_{\omega_1}(X) \to X$ is *regressive* if for all a in the domain of H, $H(a)$ is a member of a. *Fodor's Lemma* asserts that whenever $S \subseteq P_{\omega_1}(X)$ is stationary and $H : S \to X$ is a total regressive function, there is a stationary set $S^* \subseteq S$ and a set x in X such that for all a in S^*, $H(a) = x$.

If κ is a regular cardinal let $\mathrm{cof}(\kappa)$ (respectively, $\mathrm{cof}(< \kappa)$) denote the class of ordinals with cofinality κ (respectively, cofinality less than κ). If A is a cofinal subset of a cardinal λ and $\kappa < \lambda$, we write for example $A \cup \mathrm{cof}(\kappa)$ to abbreviate $A \cup (\lambda \cap \mathrm{cof}(\kappa))$.

A stationary set $S \subseteq \kappa$ is *fat* if for every club $C \subseteq \kappa$, $S \cap C$ contains closed subsets with arbitrarily large order types less than κ. If κ is the successor of a regular uncountable cardinal μ, this is equivalent to the statement that for every club $C \subseteq \kappa$, $S \cap C$ contains a closed subset with order type $\mu + 1$. In particular, if $A \subseteq \kappa^+ \cap \mathrm{cof}(\mu)$ is stationary, then $A \cup \mathrm{cof}(< \mu)$ is fat.

We write $\theta \gg \kappa$ to indicate θ is larger than $2^{2^{|H(\kappa)|}}$.

A tree $\mathcal{T}$ is a *special Aronszajn tree on* ω_2 if:

(1) $\mathcal{T}$ has height ω_2 and each level has size less than ω_2,

(2) each node in $\mathcal{T}$ is an injective function $f : \alpha \to \omega_1$ for some $\alpha < \omega_2$,

(3) the ordering on $\mathcal{T}$ is by extension of functions, and if f is in $\mathcal{T}$, then $f \restriction \beta$ is in $\mathcal{T}$ for all $\beta < \mathrm{dom}(f)$.

By [8] if there does not exist a special Aronszajn tree on ω_2, then ω_2 is a Mahlo cardinal in L.

If V is a transitive model of ZFC, we say that W is an *outer model of* V if W is a transitive model of ZFC such that $V \subseteq W$ and W has the same ordinals as V.

A forcing poset $\mathbb{P}$ is *κ-distributive* if forcing with $\mathbb{P}$ does not add any new sets of ordinals with size κ.

If $\mathbb{P}$ is a forcing poset, $\dot{a}$ is a $\mathbb{P}$-name, and G is a generic filter for $\mathbb{P}$, we write a for the set $\dot{a}^G$.

Martin's Maximum is the statement that whenever $\mathbb{P}$ is a forcing poset which preserves stationary subsets of ω_1, then for any collection $\mathcal{D}$ of dense subsets of $\mathbb{P}$ with $|\mathcal{D}| \leq \omega_1$, there is a filter $G \subseteq \mathbb{P}$ which intersects each dense set in $\mathcal{D}$.

A forcing poset $\mathbb{P}$ is *proper* if for all sufficiently large regular cardinals $\theta > 2^{|\mathbb{P}|}$, there is a club of countable elementary substructures N of $\langle H(\theta), \in \rangle$ such that for all p in $N \cap \mathbb{P}$, there is $q \leq p$ which is *generic for* N, i.e. q forces $N[\dot{G}] \cap \mathbf{On} = N \cap \mathbf{On}$. If $\mathbb{P}$ is proper, then $\mathbb{P}$ preserves ω_1 and preserves stationary subsets of $P_{\omega_1}(\lambda)$ for all $\lambda \geq \omega_1$. A forcing poset $\mathbb{P}$ is *semiproper* if the same statement holds as above except the requirement that q is generic is replaced by q being *semigeneric*, i.e. q forces $N[\dot{G}] \cap \omega_1 = N \cap \omega_1$. If $\mathbb{P}$ is semigeneric, then $\mathbb{P}$ preserves ω_1 and preserves stationary subsets of ω_1.

If $\mathbb{P}$ is ω_1-c.c. and N is a countable elementary substructure of $H(\theta)$, then $\mathbb{P}$ forces $N[\dot{G}] \cap \mathbf{On} = N \cap \mathbf{On}$; so every condition in $\mathbb{P}$ is generic for N.

We let ${}^{<\omega}\mathbf{On}$ denote the class of finite strictly increasing sequences of ordinals. If η and ν are in ${}^{<\omega}\mathbf{On}$, write $\eta \trianglelefteq \nu$ if η is an initial segment of ν, and write $\eta \triangleleft \nu$ if $\eta \trianglelefteq \nu$ and $\eta \neq \nu$. Let $l(\eta)$ denote the length of η. A set $T \subseteq {}^{<\omega}\mathbf{On}$ is a *tree* if for all η in T and $k < l(\eta)$, $\eta \restriction k$ is in T. A *cofinal branch of* T is a function $b : \omega \to \kappa$ such that for all $n < \omega$, $b \restriction n$ is in T.

Suppose I is an ideal on a set X. Then I^+ is the collection of subsets of X which are not in I. If S is in I^+ let $I \restriction S$ denote the ideal $I \cap \mathcal{P}(S)$. For example if $I = NS_\kappa$, the ideal of non-stationary subsets of κ, a set S is in I^+ iff S is stationary. In this case $NS_\kappa \restriction S$ is the ideal of non-stationary subsets of S, and $(NS_\kappa \restriction S)^+$ is the collection of stationary subsets of S.

If κ is regular and $\lambda \geq \kappa$ is a cardinal, then $\textsc{Coll}(\kappa, \lambda)$ is a forcing poset for collapsing λ to have cardinality κ: Conditions are partial functions $p : \kappa \to \lambda$ with size less than κ, ordered by an extension of functions.

2. Thin stationary sets

Let T be a cofinal subset of $P_{\omega_1}(\omega_2)$. We say that T is *thin* if for all $\beta < \omega_2$ the set $\{a \cap \beta : a \in T\}$ has size less than ω_2. Note that if CH holds, then $P_{\omega_1}(\omega_2)$ itself is thin. A set $S \subseteq P_{\omega_1}(\omega_2)$ is *closed under initial segments* if for all a in S and $\beta < \omega_2$, $a \cap \beta$ is in S.

Lemma 2.1. *If $S \subseteq P_{\omega_1}(\omega_2)$ is stationary and closed under initial segments, then for all uncountable $\beta < \omega_2$, the set $S \cap P_{\omega_1}(\beta)$ is stationary in $P_{\omega_1}(\beta)$.*

Proof. Consider $\beta < \omega_2$ and let $C \subseteq P_{\omega_1}(\beta)$ be a club set. Then the set $D = \{a \in P_{\omega_1}(\omega_2) : a \cap \beta \in C\}$ is a club subset of $P_{\omega_1}(\omega_2)$. Fix a in $S \cap D$. Since S is closed under initial segments, $a \cap \beta$ is in $S \cap C$. $\square$

Lemma 2.2. *If there exists a thin stationary subset of $P_{\omega_1}(\omega_2)$, then there is a thin stationary set S such that for all uncountable $\beta < \omega_2$, $S \cap P_{\omega_1}(\beta)$ is stationary in $P_{\omega_1}(\beta)$.*

Proof. Let T be a thin stationary set. Define $S = \{a \cap \beta : a \in T, \beta < \omega_2\}$. Then S is thin stationary and closed under initial segments. $\square$

A set $S \subseteq P_{\omega_1}(\omega_2)$ is a *local club* if there is a club set $C \subseteq \omega_2$ such that for all uncountable α in C, $S \cap P_{\omega_1}(\alpha)$ contains a club in $P_{\omega_1}(\alpha)$ (see [3]). Note that local clubs are stationary.

Theorem 2.3. *If there is a special Aronszajn tree on ω_2, then there is a thin local club subset of $P_{\omega_1}(\omega_2)$.*

Proof. Let $\mathcal{T}$ be a special Aronszajn tree on ω_2. For each f in $\mathcal{T}$ with $\mathrm{dom}(f) \geq \omega_1$, define $S_f = \{\{\alpha \in \mathrm{dom}(f) : f(\alpha) < i\} : i < \omega_1\}$. Note that S_f is a club subset of $P_{\omega_1}(\mathrm{dom} f)$. For each uncountable $\beta < \omega_2$ define $S_\beta = \bigcup\{S_f : f \in \mathcal{T}, \mathrm{dom}(f) = \beta\}$. Then S_β has size ω_1. Now define $S = \bigcup\{S_\beta : \omega_1 \leq \beta < \omega_2\}$. Clearly S is a local club. To show S is thin, it suffices to prove that whenever $\beta < \gamma$ are uncountable and a is in S_γ, then $a \cap \beta$ is in S_β. Fix f in $\mathcal{T}$ and $i < \omega_1$ such that $a = f^{-1}``i$. Then $f \restriction \beta$ is in $\mathcal{T}$, so $(f \restriction \beta)^{-1}``i$ is in $\mathcal{S}_\beta$. But $(f \restriction \beta)^{-1}``i = (f^{-1}``i) \cap \beta = a \cap \beta$. $\square$

In later sections of the paper we will construct models in which there does not exist a thin stationary subset of $P_{\omega_1}(\omega_2)$. Theorem 2.3 shows that in such a model there cannot exist a special Aronszajn tree on ω_2, so by [8] ω_2 is Mahlo in L. Mitchell [8] constructed a model in which there is no special Aronszajn tree on ω_2 by collapsing a Mahlo cardinal in L to become ω_2 with a proper forcing poset. However, in Mitchell's model the set $(P_{\omega_1}(\kappa))^L$ is a thin stationary subset of $P_{\omega_1}(\omega_2)$.

Lemma 2.4. *Suppose $S \subseteq P_{\omega_1}(\omega_2)$ is a local club. Then S is a local club in any outer model W with the same ω_1 and ω_2.*

Proof. Let C be a club subset of ω_2 such that for every uncountable α in C, $S \cap P_{\omega_1}(\alpha)$ contains a club in $P_{\omega_1}(\alpha)$. Then C remains club in W. For each uncountable α in C, fix a bijection $g_\alpha : \omega_1 \to \alpha$. Then $\{g_\alpha``i : i < \omega_1\}$ is a club subset of $P_{\omega_1}(\alpha)$. By intersecting this club with S, we get a club subset of $S \cap P_{\omega_1}(\alpha)$ of the form $\{a_i^\alpha : i < \omega_1\}$ which is increasing and continuous. Clearly this set remains a club subset of $P_{\omega_1}(\alpha)$ in W. $\square$

Proposition 2.5. (1) *Suppose there exists a thin local club in $P_{\omega_1}(\omega_2)$. Then there exists a thin local club in any outer model with the same ω_1 and ω_2.* (2) *Suppose κ is a cardinal such that for all $\mu < \kappa$, $\mu^\omega < \kappa$, and assume $\mathbb{P}$ is a proper forcing poset which collapses κ to become ω_2. Then $\mathbb{P}$ forces that there is a thin stationary subset of $P_{\omega_1}(\omega_2)$.*

Proof. (1) is immediate from Lemma 2.4 and the absoluteness of thinness. (2) Let G be generic for $\mathbb{P}$ over V and work in $V[G]$. Since $\mathbb{P}$ is proper, ω_1 is preserved and the set $S = (P_{\omega_1}(\kappa))^V$ is stationary in $P_{\omega_1}(\omega_2)$. We claim that S is thin. If $\beta < \omega_2$, then $\{a \cap \beta : a \in S\} = (P_{\omega_1}(\beta))^V$, and $|(P_{\omega_1}(\beta))^V| \leq |(P_{\omega_1}(\beta))^V|^V = |(\beta^\omega)^V|^V < \kappa$. $\square$

As we mentioned above, if CH holds, then the set $P_{\omega_1}(\omega_2)$ itself is thin. We show on the other hand that if CH fails, then no club subset of $P_{\omega_1}(\omega_2)$ is thin. The proof is due to Baumgartner and Taylor [2] who showed that for any club set $C \subseteq P_{\omega_1}(\omega_2)$, there is a countable set $A \subseteq \omega_2$ such that $C \cap \mathcal{P}(A)$ has size at least 2^ω. Their method of proof is described in the next lemma; we include the proof since we will use similar arguments later in the paper.

Lemma 2.6. *Suppose Z is a stationary subset of $\omega_2 \cap \mathrm{cof}(\omega)$ and for each α in Z, M_α is a countable cofinal subset of α. Then there is a sequence $\langle Z_s, \xi_s : s \in {}^{<\omega}2\rangle$ satisfying:*

(1) *each Z_s is a stationary subset of Z,*

(2) *if $s \trianglelefteq t$, then $Z_t \subseteq Z_s$,*

(3) *if α is in Z_s, then ξ_s is in M_α,*

(4) *if α is in $Z_{s^\frown 0}$ and β is in $Z_{s^\frown 1}$, then $\xi_{s^\frown 0}$ is not in M_β and $\xi_{s^\frown 1}$ is not in M_α.*

Proof. Let $Z_{\langle\rangle} = Z$ and let $\xi_{\langle\rangle} = 0$. Suppose Z_s is given. Define X_s as the set of ξ in ω_2 such that the set $\{\alpha \in Z_s : \xi \in M_\alpha\}$ is stationary. A straightforward argument using Fodor's Lemma shows that X_s is unbounded in ω_2. For each α in Z_s such that $X_s \cap \alpha$ has size ω_1, there exists $\xi < \alpha$ in X_s such that ξ is not in M_α. By Fodor's Lemma there is a stationary set $Z'_{s^\frown 1} \subseteq Z_s$ and $\xi_{s^\frown 0}$ in X_s such that for all α in $Z'_{s^\frown 1}$, $\xi_{s^\frown 0}$ is not in M_α. Let $Z'_{s^\frown 0}$ denote the set of α in Z_s such that $\xi_{s^\frown 0}$ is in M_α, which is stationary since $\xi_{s^\frown 0}$ is in X_s. Now define Y_s as the set of ξ in ω_2 such that $\{\alpha \in Z'_{s^\frown 1} : \xi \in M_\alpha\}$ is stationary. Then Y_s is unbounded in ω_2. So for each α in $Z'_{s^\frown 0}$ such that $Y_s \cap \alpha$ has size ω_1, there is $\xi < \alpha$ in Y_s which is not in M_α. By Fodor's Lemma there is $\xi_{s^\frown 1}$ in Y_s and $Z_{s^\frown 0} \subseteq Z'_{s^\frown 0}$ stationary such that for all α in $Z_{s^\frown 0}$, $\xi_{s^\frown 1}$ is not in M_α. Now define $Z_{s^\frown 1}$ as the set of α in $Z'_{s^\frown 1}$ such that $\xi_{s^\frown 1}$ is in M_α. $\square$

Theorem 2.7 (Baumgartner and Taylor)**.** *If $C \subseteq P_{\omega_1}(\omega_2)$ is club, then there is a countable set $A \subseteq \omega_2$ such that $C \cap \mathcal{P}(A)$ has size at least 2^ω. Hence if* CH *fails, then there does not exist a thin club subset of $P_{\omega_1}(\omega_2)$.*

Proof. Let $F : \omega_2^{<\omega} \to \omega_2$ be a function such that any a in $P_{\omega_1}(\omega_2)$ closed under F is in C. Let Z be the stationary set of α in $\omega_2 \cap \mathrm{cof}(\omega)$ closed under F. For each α in Z fix a countable set $M_\alpha \subseteq \alpha$ such that $\sup(M_\alpha) = \alpha$ and M_α is closed under F. Fix a sequence $\langle Z_s, \xi_s : s \in {}^{<\omega}2\}$ as described in Lemma 2.6.

For each function $f : \omega \to 2$ define $b_f = cl_F(\{\xi_{f\restriction n} : n < \omega\})$. Then b_f is in C. Note that if $n < \omega$ and α is in $Z_{f\restriction n}$, then $cl_F(\{\xi_{f\restriction m} : m \le n\}) \subseteq M_\alpha$. For by Lemma 2.6(2), for $m \le n$, $Z_{f\restriction n} \subseteq Z_{f\restriction m}$. So α is in $Z_{f\restriction m}$, and hence $\xi_{f\restriction m}$ is in M_α by (3). But M_α is closed under F.

Let $A = cl_F(\{\xi_s : s \in {}^{<\omega}2\})$. Since ${}^{<\omega}2$ has size ω, A is countable, and clearly each b_f is a subset of A. We claim that for distinct f and g, $b_f \neq b_g$. Let $n < \omega$ be least such that $f(n) \neq g(n)$. If $b_f = b_g$, then there is $k > n$ such that $\xi_{g\restriction(n+1)}$ is in $cl_F(\{\xi_{f\restriction m} : m \le k\})$. Fix α in $Z_{f\restriction k}$. By the last paragraph, $\xi_{g\restriction(n+1)}$ is in M_α. But α is in $Z_{f\restriction(n+1)}$ by (2), which contradicts (4). $\square$

Let κ be an uncountable cardinal. The *Weak Reflection Principle at κ* is the statement that whenever S is a stationary subset of $P_{\omega_1}(\kappa)$, there is a set Y in $P_{\omega_2}(\kappa)$ such that $\omega_1 \subseteq Y$ and $S \cap P_{\omega_1}(Y)$ is stationary in $P_{\omega_1}(Y)$. Martin's Maximum implies the Weak Reflection Principle holds for all uncountable cardinals κ [4]. The Weak Reflection Principle at ω_2 is equivalent to the statement that for every stationary set $S \subseteq P_{\omega_1}(\omega_2)$, there is a stationary set of uncountable $\beta < \omega_2$ such that $S \cap P_{\omega_1}(\beta)$ is stationary in $P_{\omega_1}(\beta)$. This is equivalent to the statement that every local club subset of $P_{\omega_1}(\omega_2)$ contains a club. The Weak Reflection Principle at ω_2 is equiconsistent with a weakly compact cardinal [3].

Corollary 2.8. *Suppose CH fails and there is a special Aronszajn tree on ω_2. Then the Weak Reflection Principle at ω_2 fails.*

Proof. By Theorems 2.3 and 2.7, there is a thin local club subset of $P_{\omega_1}(\omega_2)$ which is not club. Hence the Weak Reflection Principle at ω_2 fails. □

In Sections 4 and 5 we describe models in which there is no thin stationary subset of $P_{\omega_1}(\omega_2)$. On the other hand S. Friedman proved there always exists a thin cofinal set.

Theorem 2.9 (Friedman)**.** *There exists a thin cofinal subset of $P_{\omega_1}(\omega_2)$.*

Proof. We construct by induction a sequence $\langle S_\alpha : \omega_1 \leq \alpha < \omega_2 \rangle$ satisfying the properties: (1) each S_α is a cofinal subset of $P_{\omega_1}(\alpha)$ with size ω_1, (2) for uncountable $\beta < \gamma$, if a is in S_γ, then $a \cap \beta$ is in $\bigcup\{S_\alpha : \omega_1 \leq \alpha \leq \beta\}$, and (3) if $\beta < \gamma < \omega_2$, a is in $P_{\omega_1}(\gamma)$, and $a \cap \beta$ is in S_β, then there is b in S_γ such that $a \subseteq b$ and $a \cap \beta = b \cap \beta$.

Let $S_{\omega_1} = \omega_1$. Given S_α, let $S_{\alpha+1}$ be the collection $\{b \cup \{\alpha\} : b \in S_\alpha\}$. Conditions (1), (2), and (3) follow by induction. Suppose $\gamma < \omega_2$ is an uncountable limit ordinal and S_α is defined for all uncountable $\alpha < \gamma$. If $\mathrm{cf}(\gamma) = \omega_1$, then let $S_\gamma = \bigcup\{S_\alpha : \omega_1 \leq \alpha < \gamma\}$. The required conditions follow by induction.

Assume $\mathrm{cf}(\gamma) = \omega$. Fix an increasing sequence of uncountable ordinals $\langle \gamma_n : n < \omega \rangle$ unbounded in γ. Let T_γ be some cofinal subset of $P_{\omega_1}(\gamma)$ with size ω_1. Fix $n < \omega$. For each x in T_γ and a in S_{γ_n} define a set $b(a,x,n)$ in $P_{\omega_1}(\gamma)$ inductively as follows. Let $b(a,x,n) \cap \gamma_n = a$. Given $b(a,x,n) \cap \gamma_m$ in S_{γ_m} for some $m \geq n$, apply condition (3) to γ_m, γ_{m+1}, and the set

$$(b(a,x,n) \cap \gamma_m) \cup (x \cap [\gamma_m, \gamma_{m+1}))$$

to find y in $S_{\gamma_{m+1}}$ such that $y \cap \gamma_m = b(a,x,n) \cap \gamma_m$ and $x \cap [\gamma_m, \gamma_{m+1}) \subseteq y$. Let $b(a,x,n) \cap \gamma_{m+1} = y$. This completes the definition of $b(a,x,n)$. Clearly $b(a,x,n) \cap \gamma_n = a$, $x \setminus \gamma_n \subseteq b(a,x,n)$, and for all $k \geq n$, $b(a,x,n) \cap \gamma_k$ is in S_{γ_k}.

Now define $S_\gamma = \{b(a,x,n) : n < \omega, a \in S_{\gamma_n}, x \in T_\gamma\}$. We verify conditions (1), (2), and (3). Clearly S_γ has size ω_1. Let $\beta < \gamma$ and consider $b(a,x,n)$ in S_γ. Fix $k > n$ such that $\beta < \gamma_k$. Then $b(a,x,n) \cap \gamma_k$ is in S_{γ_k}. So by induction $b(a,x,n) \cap \beta$ is in $\bigcup\{S_\alpha : \omega_1 \leq \alpha \leq \beta\}$. Now assume a is in $P_{\omega_1}(\gamma)$, $\beta < \gamma$, and $a \cap \beta$ is in S_β. Choose x in T_γ such that $a \subseteq x$. Fix k such that $\beta < \gamma_k$. By the induction hypothesis there is a' in S_{γ_k} such that $a \cap \gamma_k \subseteq a'$ and $a' \cap \beta = a \cap \beta$. Let $c = b(a', x, k)$. Then c is in S_γ, $c \cap \beta = (c \cap \gamma_k) \cap \beta = a' \cap \beta = a \cap \beta$, and $a \subseteq c$.

To prove S_γ is cofinal consider a in $P_{\omega_1}(\gamma)$. Fix x in T_γ such that $a \subseteq x$. By induction S_{γ_0} is cofinal in $P_{\omega_1}(\gamma_0)$. So let y be in S_{γ_0} such that $x \cap \gamma_0 \subseteq y$. Then a is a subset of $b(y,x,0)$.

Now define $S = \bigcup\{S_\beta : \omega_1 \leq \beta < \omega_2\}$. Conditions (1) and (2) imply that S is thin and cofinal in $P_{\omega_1}(\omega_2)$. □

3. Disjoint club sequences

We introduce a combinatorial property of ω_2 which implies there does not exist a thin stationary subset of $P_{\omega_1}(\omega_2)$. This property follows from Martin's Maximum and is equiconsistent with a Mahlo cardinal. It implies there exists a fat stationary subset of ω_2 which cannot acquire a club subset by any forcing poset which preserves ω_1 and ω_2.

Definition 3.1. A disjoint club sequence on ω_2 is a sequence $\langle \mathcal{C}_\alpha : \alpha \in A \rangle$ such that A is a stationary subset of $\omega_2 \cap \mathrm{cof}(\omega_1)$, each $\mathcal{C}_\alpha$ is a club subset of $P_{\omega_1}(\alpha)$, and $\mathcal{C}_\alpha \cap \mathcal{C}_\beta$ is empty for all $\alpha < \beta$ in A.

Proposition 3.2. *Suppose there is a disjoint club sequence on ω_2. Then there does not exist a thin stationary subset of $P_{\omega_1}(\omega_2)$.*

Proof. Let $\langle \mathcal{C}_\alpha : \alpha \in A \rangle$ be a disjoint club sequence. Suppose for a contradiction there exists a thin stationary set. By Lemma 2.2 fix a thin stationary set $T \subseteq P_{\omega_1}(\omega_2)$ such that for all uncountable $\beta < \omega_2$, $T \cap P_{\omega_1}(\beta)$ is stationary in $P_{\omega_1}(\beta)$. Then for each β in A we can choose a set a_β in $\mathcal{C}_\beta \cap T$. Since $\mathrm{cf}(\beta) = \omega_1$, $\sup(a_\beta) < \beta$. By Fodor's Lemma there is a stationary set $B \subseteq A$ and a fixed $\gamma < \omega_2$ such that for all β in B, $\sup(a_\beta) = \gamma$. If $\alpha < \beta$ are in B, then $a_\alpha \neq a_\beta$ since $\mathcal{C}_\alpha \cap \mathcal{C}_\beta$ is empty. So the set $\{a_\beta : \beta \in B\}$ witnesses that T is not thin, which is a contradiction. $\square$

Lemma 3.3. *Suppose there is a disjoint club sequence $\langle \mathcal{C}_\alpha : \alpha \in A \rangle$ on ω_2. Let W be an outer model with the same ω_1 and ω_2 in which A is still stationary. Then there is a disjoint club sequence $\langle \mathcal{D}_\alpha : \alpha \in A \rangle$ in W.*

Proof. By the proof of Lemma 2.4, each $\mathcal{C}_\alpha$ contains a club set $\mathcal{D}_\alpha$ in W. Since ω_1 is preserved, each α in A still has cofinality ω_1. $\square$

Theorem 3.4. *Suppose $\langle \mathcal{C}_\alpha : \alpha \in A \rangle$ is a disjoint club sequence on ω_2. Then $A \cup \mathrm{cof}(\omega)$ does not contain a club.*

Proof. Suppose for a contradiction that $A \cup \mathrm{cof}(\omega)$ contains a club. Without loss of generality $2^{\omega_1} = \omega_2$. Otherwise work in a generic extension W by $\mathrm{Coll}(\omega_2, 2^{\omega_1})$: In W the set $A \cup \mathrm{cof}(\omega)$ contains a club, and by Lemma 3.3 there is a disjoint club sequence $\langle \mathcal{D}_\alpha : \alpha \in A \rangle$.

Since $2^{\omega_1} = \omega_2$, $H(\omega_2)$ has size ω_2. Fix a bijection $h : H(\omega_2) \to \omega_2$. Let $\mathcal{A}$ denote the structure $\langle H(\omega_2), \in, h \rangle$. Define B as the set of α in $\omega_2 \cap \mathrm{cof}(\omega_1)$ such that there exists an increasing and continuous sequence $\langle N_i : i < \omega_1 \rangle$ of countable elementary substructures of $\mathcal{A}$ such that:

(1) for $i < \omega_1$, N_i is in N_{i+1},

(2) the set $\{N_i \cap \omega_2 : i < \omega_1\}$ is club in $P_{\omega_1}(\alpha)$.

We claim that B is stationary in ω_2. To prove this let $C \subseteq \omega_2$ be club. Let $\mathcal{B}$ be the expansion of $\mathcal{A}$ by the function $\alpha \mapsto \min(C \setminus \alpha)$. Define by induction an increasing and continuous sequence $\langle N_i : i < \omega_1 \rangle$ of elementary substructures of $\mathcal{B}$ such that for all $i < \omega_1$, N_i is in N_{i+1}. Let $N = \bigcup\{N_i : i < \omega_1\}$. Then $\omega_1 \subseteq N$ so $N \cap \omega_2$ is an ordinal. Write $\alpha = N \cap \omega_2$. Then α is in C and $\{N_i \cap \omega_2 : i < \omega_1\}$ is club in $P_{\omega_1}(\alpha)$. So α is in $B \cap C$.

Since $A \cup \mathrm{cof}(\omega)$ contains a club, $A \cap B$ is stationary. For each α in $A \cap B$ fix a sequence $\langle N_i^\alpha : i < \omega_1 \rangle$ as described in the definition of B. Then $\{N_i^\alpha \cap \omega_2 : i < \omega_1\} \cap \mathcal{C}_\alpha$ is club in $P_{\omega_1}(\alpha)$. So there exists a club set $c_\alpha \subseteq \omega_1$ such that $\{N_i^\alpha \cap \omega_2 : i \in c_\alpha\}$ is club and is a subset of $\mathcal{C}_\alpha$. Write $i_\alpha = \min(c_\alpha)$ and let $d_\alpha = c_\alpha \setminus \{i_\alpha\}$.

Define $S = \{N_i^\alpha : \alpha \in A \cap B, i \in d_\alpha\}$. If N is in S, then there is a unique pair α in $A \cap B$ and i in d_α such that $N = N_i^\alpha$. For if $N = N_i^\alpha = N_j^\beta$, then $N \cap \omega_2$ is in $\mathcal{C}_\alpha \cap \mathcal{C}_\beta$, so $\alpha = \beta$. Clearly then $i = j$. Also note that if N_i^α is in S, then $N_{i_\alpha}^\alpha$ is in N_i^α. So the function $H : S \to H(\omega_2)$ defined by $H(N_i^\alpha) = N_{i_\alpha}^\alpha$ is well defined and regressive.

We claim that S is stationary in $P_{\omega_1}(H(\omega_2))$. To prove this let $F : H(\omega_2)^{<\omega} \to H(\omega_2)$ be a function. Define $G : \omega_2^{<\omega} \to \omega_2$ by letting $G(\alpha_0, \ldots, \alpha_n)$ be equal to $h(F(h^{-1}(\alpha_0), \ldots, h^{-1}(\alpha_n)))$. Let E be the club set of α in ω_2 closed under G. Fix α in $E \cap A \cap B$. Then there is i in d_α such that $N_i^\alpha \cap \omega_2$ is closed under G. We claim that N_i^α is closed under F. Given $a_0, \ldots, a_n$ in N_i^α, the ordinals $h(a_0), \ldots, h(a_n)$ are in $N_i^\alpha \cap \omega_2$. So $\gamma = G(h(a_0), \ldots, h(a_n)) = h(F(a_0, \ldots, a_n))$ is in $N_i^\alpha \cap \omega_2$. Therefore $h^{-1}(\gamma) = F(a_0, \ldots, a_n)$ is in N_i^α.

Since S is stationary and $H : S \to H(\omega_2)$ is regressive, there is a stationary set $S^* \subseteq S$ and a fixed N such that for all N_i^α in S^*, $H(N_i^\alpha) = N$. The set S^*, being stationary, must have size ω_2. So there are distinct α and β such that for some i in d_α and j in d_β, N_i^α and N_j^β are in S^*. Then $N = N_{i_\alpha}^\alpha = N_{i_\beta}^\beta$. So $N \cap \omega_2$ is in $\mathcal{C}_\alpha \cap \mathcal{C}_\beta$, which is a contradiction. □

Abraham and Shelah [1] asked the following question: Assume that A is a stationary subset of $\omega_2 \cap \mathrm{cof}(\omega_1)$. Does there exist an ω_1-distributive forcing poset which adds a club subset to $A \cup \mathrm{cof}(\omega)$? We answer this question in the negative.

Corollary 3.5. *Assume that $\langle \mathcal{C}_\alpha : \alpha \in A \rangle$ is a disjoint club sequence. Let W be an outer model of V with the same ω_1 and ω_2. Then in W, $A \cup \mathrm{cof}(\omega)$ does not contain a club subset.*

Proof. If A remains stationary in W, then by Lemma 3.3 there is a disjoint club sequence $\langle \mathcal{D}_\alpha : \alpha \in A \rangle$ in W. By Theorem 3.4 $A \cup \mathrm{cof}(\omega)$ does not contain a club in W. □

4. Martin's Maximum

In this section we prove that Martin's Maximum implies there exists a disjoint club sequence on ω_2. We apply MM to the poset for adding a Cohen real and then forcing a continuous ω_1-chain through $P_{\omega_1}(\omega_2) \setminus V$.

Theorem 4.1 (Krueger)**.** *Martin's Maximum implies there exists a disjoint club sequence on ω_2.*

We will use the following theorem from [1].

Theorem 4.2. *Suppose $\mathbb{P}$ is ω_1-c.c. and adds a real. Then $\mathbb{P}$ forces that $(P_{\omega_1}(\omega_2) \setminus V)$ is stationary in $P_{\omega_1}(\omega_2)$.*

Note: Gitik [6] proved that the conclusion of Theorem 4.2 holds for any outer model of V which contains a new real and computes the same ω_1.

Suppose that S is a stationary subset of $P_{\omega_1}(\omega_2)$. Following [3] we define a forcing poset $\mathbb{P}(S)$ which adds a continuous ω_1-chain through S. A condition in $\mathbb{P}(S)$ is a countable increasing and continuous sequence $\langle a_i : i \leq \beta \rangle$ of elements from S, where for each $i < \beta$, $a_i \cap \omega_1 < a_{i+1} \cap \omega_1$. The ordering on $\mathbb{P}(S)$ is by extension of sequences.

Proposition 4.3. *If $S \subseteq P_{\omega_1}(\omega_2)$ is stationary, then $\mathbb{P}(S)$ is ω-distributive.*

Proof. Suppose p forces $\dot{f} : \omega \to \mathbf{On}$. Let $\theta \gg \omega_2$ be a regular cardinal such that $\dot{f}$ is in $H(\theta)$. Since S is stationary, we can fix a countable elementary substructure N of the model

$$\langle H(\theta), \in, S, \mathbb{P}(S), p, \dot{f} \rangle$$

such that $N \cap \omega_2$ is in S. Let $\langle D_n : n < \omega \rangle$ be an enumeration of all the dense subsets of $\mathbb{P}(S)$ in N. Inductively define a decreasing sequence $\langle p_n : n < \omega \rangle$ of elements of $N \cap \mathbb{P}$ such that $p_0 = p$ and p_{n+1} is a refinement of p_n in $D_n \cap N$. Write $\bigcup\{p_n : n < \omega\} = \langle b_i : i < \gamma \rangle$. Clearly $\bigcup\{b_i : i < \gamma\} = N \cap \omega_2$. Since $N \cap \omega_2$ is in S, the sequence $\langle b_i : i < \gamma \rangle \cup \{\langle \gamma, N \cap \omega_2 \rangle\}$ is a condition below p which decides $\dot{f}(n)$ for all $n < \omega$. □

Theorem 4.4. *Suppose $\mathbb{P}$ is an ω_1-c.c. forcing poset which adds a real. Let $\dot{S}$ be a name such that $\mathbb{P}$ forces $\dot{S} = (P_{\omega_1}(\omega_2) \setminus V)$. Then $\mathbb{P} * \mathbb{P}(\dot{S})$ preserves stationary subsets of ω_1.*

Proof. By Theorem 4.2 and Proposition 4.3, the poset $\mathbb{P} * \mathbb{P}(\dot{S})$ preserves ω_1. Let A be a stationary subset of ω_1 in V. Suppose $p * \dot{q}$ is a condition in $\mathbb{P} * \mathbb{P}(\dot{S})$ which forces that $\dot{C}$ is a club subset of ω_1.

Let G be a generic filter for $\mathbb{P}$ over V which contains p. In $V[G]$ fix a regular cardinal $\theta \gg \omega_2$ and let

$$\mathcal{A} = \langle H(\theta), \in, A, S, q, \dot{C} \rangle.$$

Fix a Skolem function $F : H(\theta)^{<\omega} \to H(\theta)$ for $\mathcal{A}$. Define $F^* : \omega_2^{<\omega} \to P_{\omega_1}(\omega_2)$ by letting

$$F^*(\alpha_0, \ldots, \alpha_n) = cl_F(\{\alpha_0, \ldots, \alpha_n\}) \cap \omega_2.$$

Since $\mathbb{P}$ is ω_1-c.c. there is a function $H : \omega_2^{<\omega} \to P_{\omega_1}(\omega_2)$ in V such that for all $\vec{\alpha}$ in $\omega_2^{<\omega}$, $F^*(\vec{\alpha}) \subseteq H(\vec{\alpha})$. Let Z^* be the stationary set of α in $\omega_2 \cap \mathrm{cof}(\omega)$ closed under H.

Working in V, since A is stationary we can fix for each α in Z^* a countable cofinal set $M_\alpha \subseteq \alpha$ closed under H with $M_\alpha \cap \omega_1$ in A. By Fodor's Lemma there is $Z \subseteq Z^*$ stationary and δ in A such that for all α in Z, $M_\alpha \cap \omega_1 = \delta$. Fix a sequence $\langle \xi_s, Z_s : s \in {}^{<\omega}2 \rangle$ satisfying conditions (1)–(4) of Lemma 2.6.

Let $f : \omega \to 2$ be a function in $V[G] \setminus V$. For each $n < \omega$ let M_n denote $cl_H(\delta \cup \{\xi_{f \restriction m} : m \leq n\})$. Define $M = \bigcup\{M_n : n < \omega\}$. Note that M is closed under H and hence it is closed under F^*. Therefore $N = cl_F(M)$ is an elementary substructure of $\mathcal{A}$ such that $N \cap \omega_2 = M$.

As in the proof of Theorem 2.7, for all $n < \omega$, if α is in $Z_{f \restriction n}$ then $M_n \subseteq M_\alpha$. Note that $M \cap \omega_1 = \delta$. For if γ is in $M \cap \omega_1$, there is $n < \omega$ such that γ is in M_n. Fix α in $Z_{f \restriction n}$. Then γ is in $M_\alpha \cap \omega_1 = \delta$.

We prove that M is not in V by showing how to compute f by induction from M. Suppose $f \restriction n$ is known. Fix $j < 2$ such that $f(n) \neq j$. We claim that $\xi_{(f \restriction n)^\frown j}$ is not in M. Otherwise there is $k > n$ such that $\xi_{(f \restriction n)^\frown j}$ is in M_k. Fix α in $Z_{f \restriction k}$. Then $\xi_{(f \restriction n)^\frown j}$ is in M_α. But α is in $Z_{f \restriction (n+1)}$, contradicting Lemma 2.6(4). So $f(n)$ is the unique $i < 2$ such that $\xi_{(f \restriction n)^\frown i}$ is in M. This completes the definition of f from M. Since f is not in V, neither is M.

Let $\langle D_n : n < \omega \rangle$ enumerate the dense subsets of $\mathbb{P}(S)$ lying in N. Inductively define a decreasing sequence $\langle q_n : n < \omega \rangle$ in $N \cap \mathbb{P}(S)$ such that $q_0 = q$ and q_{n+1} is in $D_n \cap N$. Write $\bigcup\{q_n : n < \omega\} = \langle b_i : i < \gamma \rangle$. Clearly $\bigcup\{b_i : i < \gamma\} = N \cap \omega_2 = M$, and since M is not in V, $r = \langle b_i : i < \gamma \rangle \cup \{\langle \gamma, M \rangle\}$ is a condition in $\mathbb{P}(S)$. By an easy density argument, r forces that $N \cap \omega_1 = \delta$ is a limit point of $\dot{C}$, and hence is in $\dot{C}$. Let $\dot{r}$ be a name for r. Then $p * \dot{r} \leq p * \dot{q}$ and $p * \dot{r}$ forces that δ is in $A \cap \dot{C}$. □

Now we are ready to prove that MM implies there exists a disjoint club sequence on ω_2.

Proof of Theorem 4.1. Assume Martin's Maximum. Inductively define A and $\langle \mathcal{C}_\alpha : \alpha \in A \rangle$ as follows. Suppose α is in $\omega_2 \cap \mathrm{cof}(\omega_1)$ and $A \cap \alpha$ and $\langle \mathcal{C}_\beta : \beta \in A \cap \alpha \rangle$ are defined. Let α be in A iff the set $\bigcup \{\mathcal{C}_\beta : \beta \in A \cap \alpha\}$ is non-stationary in $P_{\omega_1}(\alpha)$. If α is in A, then choose a club set $\mathcal{C}_\alpha \subseteq P_{\omega_1}(\alpha)$ with size ω_1 which is disjoint from this union.

This completes the definition. We prove that A is stationary. Then clearly $\langle \mathcal{C}_\alpha : \alpha \in A \rangle$ is a disjoint club sequence. Fix a club set $C \subseteq \omega_2$.

Let Add denote the forcing poset for adding a single Cohen real with finite conditions and let $\dot{S}$ be an Add-name for the set $(P_{\omega_1}(\omega_2) \setminus V)$. By Theorem 4.4 the poset $\text{Add} * \mathbb{P}(\dot{S})$ preserves stationary subsets of ω_1. We will apply Martin's Maximum to this poset after choosing a suitable collection of dense sets.

For each $\alpha < \omega_2$ fix a surjection $f_\alpha : \omega_1 \to \alpha$. If β is in A enumerate $\mathcal{C}_\beta$ as $\langle a_i^\beta : i < \omega_1 \rangle$. For every quadruple i, j, k, l of countable ordinals let $D(i, j, k, l)$ denote the set of conditions $p * \dot{q}$ such that:

(1) p forces that i and j are in the domain of $\dot{q}$, and for some β_i and β_j, p forces $\beta_i = \sup(\dot{q}(i))$ and $\beta_j = \sup(\dot{q}(j))$,

(2) there is some $\zeta < \omega_1$ such that p forces ζ is the least element in $\mathrm{dom}(\dot{q})$ such that $f_{\beta_i}(j) \in \dot{q}(\zeta)$,

(3) there is ξ in C larger than β_i and β_j such that p forces ξ is the supremum of the maximal set in $\dot{q}$,

(4) if $f_{\beta_j}(k) = \gamma$ is in A, then there is z such that p forces z is in the symmetric difference $\dot{q}(i) \triangle a_l^\gamma$.

It is routine to check that $D(i, j, k, l)$ is dense.

Let $G * H$ be a filter on $\text{Add} * \mathbb{P}(\dot{S})$ intersecting each $D(i, j, k, l)$. For $i < \omega_1$ define a_i as the set of β for which there exists some $p * \dot{q}$ in $G * H$ such that p forces $i \in \mathrm{dom}(\dot{q})$ and p forces β is in $\dot{q}(i)$. The definition of the dense sets implies that $\langle a_i : i < \omega_1 \rangle$ is increasing, continuous, and cofinal in $P_{\omega_1}(\alpha)$ for some α in $C \cap \mathrm{cof}(\omega_1)$. By (4), for each γ in $A \cap \alpha$, $\{a_i : i < \omega_1\}$ is disjoint from $\mathcal{C}_\gamma$. Therefore $\bigcup \{\mathcal{C}_\gamma : \gamma \in A \cap \alpha\}$ is non-stationary in $P_{\omega_1}(\alpha)$, hence by the definition of A, α is in $A \cap C$. So A is stationary. $\square$

5. The equiconsistency result

We now prove Theorem 0.1 establishing the consistency strength of each of the following statements to be exactly a Mahlo cardinal: (1) There does not exist a thin stationary subset of $P_{\omega_1}(\omega_2)$. (2) There exists a disjoint club sequence on ω_2. (3) There exists a fat stationary set $S \subseteq \omega_2$ such that any forcing poset which preserves ω_1 and ω_2 does not add a club subset to S.

By [5] if there exists a thin stationary subset of $P_{\omega_1}(\omega_2)$, then for any fat stationary set $S \subseteq \omega_2$, there is a forcing poset which preserves cardinals and adds a club subset to S. So (2) and (3) both imply (1), which in turn implies there is no special Aronszajn tree on ω_2. So ω_2 is Mahlo in L by [8].

In the other direction assume that κ is a Mahlo cardinal. We will define a revised countable support iteration which collapses κ to become ω_2 and adds a disjoint club sequence on ω_2. At individual stages of the iteration we force with either a collapse forcing or the poset $\text{Add} * \mathbb{P}(\dot{S})$ from the previous section. To ensure that ω_1 is not collapsed we verify that $\mathrm{ADD} * \mathbb{P}(\dot{S})$ satisfies an iterable condition known as the $\mathbb{I}$-*universal property*. Our description of this construction is self-contained, except for the proof of Theorem 5.9 which summarizes the relevant properties of the RCS iteration. For more information on such iterations and the $\mathbb{I}$-universal property see [10].

Definition 5.1. A pair $\langle T, \mathbf{I}\rangle$ is a tagged tree if:
(1) $T \subseteq {}^{<\omega}\mathbf{On}$ is a tree such that each η in T has at least one successor,
(2) $\mathbf{I} : T \to V$ is a partial function such that each $\mathbf{I}(\eta)$ is an ideal on some set X_η and for each η in the domain of $\mathbf{I}$, the set $\{\alpha : \eta^\frown\alpha \in T\}$ is in $(\mathbf{I}(\eta))^+$,
(3) for each cofinal branch b of T, there are infinitely many $n < \omega$ such that $b \restriction n$ is in the domain of $\mathbf{I}$.

If η is in the domain of $\mathbf{I}$, we say that η is a *splitting point of* T. It follows from (1) and (3) that for every η in T there is $\eta \lhd \nu$ which is a splitting point.

Definition 5.2. Let $\mathbb{I}$ be a family of ideals and $\langle T, \mathbf{I}\rangle$ a tagged tree. Then $\langle T, \mathbf{I}\rangle$ is an $\mathbb{I}$-tree if for each splitting point η in T, $\mathbf{I}(\eta)$ is in $\mathbb{I}$.

Suppose $T \subseteq {}^{<\omega}\mathbf{On}$ is a tree. If η is in T, let $T^{[\eta]}$ denote the tree $\{\nu \in T : \nu \trianglelefteq \eta \text{ or } \eta \trianglelefteq \nu\}$. A set $J \subseteq T$ is called a *front* if for distinct nodes in J, neither is an initial segment of the other, and for any cofinal branch b of T there is η in J which is an initial segment of b.

Definition 5.3. Suppose $\langle T, \mathbf{I}\rangle$ is tagged tree. Let θ be a regular cardinal such that $\langle T, \mathbf{I}\rangle$ is in $H(\theta)$, and let $<_\theta$ be a well-ordering of $H(\theta)$. A sequence $\langle N_\eta : \eta \in T\rangle$ is a tree of models for θ provided that:
(1) each N_η is a countable elementary substructure of $\langle H(\theta), \in, <_\theta, \langle T, \mathbf{I}\rangle\rangle$,
(2) if $\eta \lhd \nu$ in T, then $N_\eta \prec N_\nu$,
(3) for each η in T, η is in N_η.

Definition 5.4. Suppose $\langle T, \mathbf{I}\rangle$ is an $\mathbb{I}$-tree, and θ is a regular cardinal such that $H(\theta)$ contains $\langle T, \mathbf{I}\rangle$ and $\mathbb{I}$. A sequence $\langle N_\eta : \eta \in T\rangle$ is an $\mathbb{I}$-suitable tree of models for θ if it is a tree of models for θ, and for every η in T and I in $\mathbb{I} \cap N_\eta$, the set

$$\{\nu \in T^{[\eta]} : \nu \text{ is a splitting point and } \mathbf{I}(\nu) = I\}$$

contains a front in $T^{[\eta]}$.

Definition 5.5. Let $\langle T, \mathbf{I}\rangle$, $\mathbb{I}$, and θ be as in Definition 5.4. A sequence $\langle N_\eta : \eta \in T\rangle$ is an ω_1-strictly $\mathbb{I}$-suitable tree of models for θ if it is an $\mathbb{I}$-suitable tree of models for θ and there exists $\delta < \omega_1$ such that for all η in T, $N_\eta \cap \omega_1 = \delta$.

If $\langle N_\eta : \eta \in T\rangle$ is a tree of models and b is a cofinal branch of T, write N_b for the set $\bigcup\{N_{b\restriction n} : n < \omega\}$. Note that if $\langle N_\eta : \eta \in T\rangle$ is an ω_1-strictly $\mathbb{I}$-suitable tree of models for θ, then for any cofinal branch b of T, $N_b \cap \omega_1 = N_{\langle\rangle} \cap \omega_1$.

Lemma 5.6. *Let $\langle T, \mathbf{I}\rangle$, $\mathbb{I}$, and θ be as in Definition 5.4, and let $\langle N_\eta : \eta \in T\rangle$ be an ω_1-strictly $\mathbb{I}$-suitable tree of models for θ. Suppose $\eta \lhd \nu$ in T and $(N_\nu \cap \omega_2) \setminus N_\eta$ is non-empty. Let γ be the minimum element of $(N_\nu \cap \omega_2) \setminus N_\eta$. Then $\gamma \geq \sup(N_\eta \cap \omega_2)$.*

Proof. Otherwise there is β in $N_\eta \cap \omega_2$ such that $\gamma < \beta$. By elementarity, there is a surjection $f : \omega_1 \to \beta$ in N_η. So $f^{-1}(\gamma) \in N_\nu \cap \omega_1 = N_\eta \cap \omega_1$. Hence $f(f^{-1}(\gamma)) = \gamma$ is in N_η, which is a contradiction. $\square$

Let $\mathbb{I}$ be a family of ideals. We say that $\mathbb{I}$ is *restriction-closed* if for all I in $\mathbb{I}$, for any set A in I^+, the ideal $I \restriction A$ is in $\mathbb{I}$. If μ is a regular uncountable cardinal, we say that $\mathbb{I}$ is *μ-complete* if each ideal in $\mathbb{I}$ is μ-complete.

Definition 5.7. Suppose that $\mathbb{I}$ is a non-empty restriction-closed ω_2-complete family of ideals and let $\mathbb{P}$ be a forcing poset. Then $\mathbb{P}$ satisfies the $\mathbb{I}$-universal property if for all sufficiently large regular cardinals θ with $\mathbb{I}$ in $H(\theta)$, if $\langle N_\eta : \eta \in T\rangle$ is an ω_1-strictly $\mathbb{I}$-suitable tree of models for θ, then for all p in $N_{\langle\rangle} \cap \mathbb{P}$ there is $q \leq p$ such that q forces that there is a cofinal branch b of T such that $N_b[\dot{G}] \cap \omega_1 = N_{\langle\rangle} \cap \omega_1$.

Definition 5.7 is Shelah's characterization of the $\mathbb{I}$-universal property given in [10], Chapter XV 2.11, 2.12, and 2.13. Note that in the definition, q is semigeneric for $N_{\langle\rangle}$. In 2.12 Shelah proves that there are stationarily many structures N for which $N = N_{\langle\rangle}$ for some ω_1-strictly $\mathbb{I}$-suitable tree of models $\langle N_\eta : \eta \in T\rangle$. So by standard arguments if $\mathbb{P}$ satisfies the $\mathbb{I}$-universal property, then $\mathbb{P}$ preserves ω_1 and preserves stationary subsets of ω_1. Note that any semiproper forcing poset satisfies the $\mathbb{I}$-universal property.

Theorem 5.8. *Let $\mathbb{I}$ be the family of ideals of the form $NS_{\omega_2} \restriction A$, where A is a stationary subset of $\omega_2 \cap \mathrm{cof}(\omega)$. Let $\dot{S}$ be an* ADD*-name for the set $(P_{\omega_1}(\omega_2) \setminus V)$. Then* ADD $* \mathbb{P}(\dot{S})$ *satisfies the $\mathbb{I}$-universal property.*

Proof. Fix a regular cardinal $\theta \gg \omega_2$ and let $\langle N_\eta : \eta \in T\rangle$ be an ω_1-strictly $\mathbb{I}$-suitable tree of models for θ. Let $p * \dot{q}$ be a condition in $(\text{ADD} * \mathbb{P}(\dot{S})) \cap N_{\langle\rangle}$. We find a refinement of $p * \dot{q}$ which forces that there is a cofinal branch b of T such that $N_b[\dot{G} * \dot{H}] \cap \omega_1 = N_{\langle\rangle} \cap \omega_1$.

We use an argument similar to the proof of Lemma 2.6 to define a sequence $\langle \eta_s, \xi_s : s \in {}^{<\omega}2\rangle$ satisfying:

(1) each η_s is in T, each ξ_s is in $N_{\eta_s} \cap \omega_2$, and $s \lhd t$ implies $\eta_s \lhd \eta_t$,

(2) if $s^\frown 0 \unlhd t$, then $\xi_{s^\frown 1}$ is not in N_{η_t}, and if $s^\frown 1 \unlhd u$, then $\xi_{s^\frown 0}$ is not in N_{η_u}.

Let $\eta_{\langle\rangle} = \langle\rangle$ and $\xi_{\langle\rangle} = 0$. Suppose η_s is defined. Choose a splitting point ν_s in T above η_s. Let Z denote the set of $\alpha < \omega_2$ such that $\nu_s{}^\frown\alpha$ is in T. Since ν_s is a splitting point, by the definition of $\mathbb{I}$ the set Z is a stationary subset of $\omega_2 \cap \mathrm{cof}(\omega)$. For each α in Z, α is in $N_{(\nu_s{}^\frown\alpha)}$ and has cofinality ω, so $N_{(\nu_s{}^\frown\alpha)} \cap \alpha$ is a countable cofinal subset of α. Define X_s as the set of ξ in ω_2 such that the set

$$\{\alpha \in Z : \xi \in N_{(\nu_s{}^\frown\alpha)} \cap \alpha\}$$

is stationary. An easy argument using Fodor's Lemma shows that X_s is unbounded in ω_2. For all large enough α in Z, the set $(X_s \setminus \sup(N_{\nu_s} \cap \omega_2)) \cap \alpha$ has size ω_1. So there is a stationary set $Z_1' \subseteq Z$ and an ordinal $\xi_{s^\frown 0}$ in X_s such that $\xi_{s^\frown 0}$ is larger than $\sup(N_{\nu_s} \cap \omega_2)$ and for all α in Z_1', $\xi_{s^\frown 0}$ is not in $N_{(\nu_s{}^\frown\alpha)} \cap \alpha$. Let Z_0' be the stationary set of α in Z such that $\xi_{s^\frown 0}$ is in $N_{(\nu_s{}^\frown\alpha)} \cap \alpha$. Now define Y_s as the set of ξ in ω_2 such that the set

$$\{\alpha \in Z_1' : \xi \in N_{(\nu_s{}^\frown\alpha)} \cap \alpha\}$$

is stationary. Again we can find $Z_0 \subseteq Z_0'$ stationary and $\xi_{s^\frown 1}$ in Y_s such that $\xi_{s^\frown 1}$ is larger than $\sup(N_{\nu_s} \cap \omega_2)$ and for all α in Z_0, $\xi_{s^\frown 1}$ is not in $N_{(\nu_s{}^\frown\alpha)} \cap \alpha$. Let Z_1 be the stationary set of α in Z_1' such that $\xi_{s^\frown 1}$ is in $N_{(\nu_s{}^\frown\alpha)} \cap \alpha$.

Now define $\eta_{s^\frown 0}$ to be equal to $\nu_s{}^\frown\alpha$ for some α in Z_0 larger than $\xi_{s^\frown 1}$, and define $\eta_{s^\frown 1}$ to be $\nu_s{}^\frown\beta$ for some β in Z_1 larger than $\xi_{s^\frown 0}$. By definition $\xi_{s^\frown 0}$ is in $N_{\eta_{s^\frown 0}}$ and $\xi_{s^\frown 1}$ is in $N_{\eta_{s^\frown 1}}$.

We claim that if $\eta_{s^\frown 0} \trianglelefteq \nu$ in T, then $\xi_{s^\frown 1}$ is not in N_ν. Since α is in Z_0, $\xi_{s^\frown 1}$ is not in $N_{(\eta_{s^\frown 0})} \cap \alpha$. But $\xi_{s^\frown 1} < \alpha$, so $\xi_{s^\frown 1}$ is not in $N_{(\eta_{s^\frown 0})}$. By Lemma 5.6 the minimum element of $N_\nu \cap \omega_2$ which is not in $N_{(\eta_{s^\frown 0})}$, if such an ordinal exists, is at least $\sup(N_{(\eta_{s^\frown 0})} \cap \omega_2) \geq \alpha > \xi_{s^\frown 1}$. So $\xi_{s^\frown 1}$ is not in N_ν. Similarly if $\eta_{s^\frown 1} \trianglelefteq \nu$ in T, then $\xi_{s^\frown 0}$ is not in N_ν. This completes the definition. Conditions (1) and (2) are now easily verified.

Since $\mathbb{P}$ is ω_1-c.c., the condition p itself is generic for each N_η. Let G be a generic filter for ADD over V which contains p. Then for all η in T, $N_\eta[G] \cap \omega_2 = N_\eta \cap \omega_2$. So for any cofinal branch b of T in $V[G]$, $N_b[G] \cap \omega_2 = \bigcup\{N_{b \restriction n} \cap \omega_2 : n < \omega\}$; in particular, $N_b[G] \cap \omega_1 = N_{\langle\rangle} \cap \omega_1$.

Let $f : \omega \to 2$ be a function in $V[G] \setminus V$. Define $b_f = \bigcup\{\eta_{f \restriction n} : n < \omega\}$. We prove that $N_{b_f} \cap \omega_2$ is not in V by showing how to define f inductively from this set. Suppose $f \restriction n$ is known. Fix $j < 2$ such that $f(n) \neq j$. We claim that $\xi^* = \xi_{(f \restriction n)^\frown j}$ is not in $N_{b_f} \cap \omega_2$. Otherwise there is $k > n$ such that ξ^* is in $N_{\eta_{f \restriction k}}$. But $f \restriction (n+1) \trianglelefteq f \restriction k$. So by condition (2), ξ^* is not in $N_{\eta_{f \restriction k}}$, which is a contradiction. So $f(n)$ is the unique $i < 2$ such that $\xi_{(f \restriction n)^\frown i}$ is in $N_{b_f} \cap \omega_2$.

Let $\langle D_n : n < \omega\rangle$ enumerate all the dense subsets of $\mathbb{P}(S)$ lying in $N_{b_f}[G]$. Inductively define a sequence $\langle q_n : n < \omega\rangle$ by letting $q_0 = q$ and choosing q_{n+1} to be a refinement of q in $D_n \cap N_{b_f}[G]$. Let $\langle b_i : i < \gamma\rangle = \bigcup\{q_n : n < \omega\}$. Clearly $\bigcup\{b_i : i < \gamma\} = N_{b_f} \cap \omega_2$. Since $N_{b_f} \cap \omega_2$ is not in V, $r = \langle b_i : i < \gamma\rangle^\frown(N_{b_f} \cap \omega_2)$ is a condition in $\mathbb{P}(S)$ below q and r is generic for $N_{b_f}[G]$. So r forces $N_{b_f}[G][\dot{H}] \cap \omega_1 = N_{b_f}[G] \cap \omega_1 = N_{\langle\rangle} \cap \omega_1$. Let $\dot{r}$ be a name for r. Then $p * \dot{r} \leq p * \dot{q}$ is as required. $\square$

We state without proof the facts concerning RCS iterations which we shall use. These facts follow immediately from [10] Chapter XI 1.13 and Chapter XV 4.15.

Theorem 5.9. *Suppose $\langle \mathbb{P}_i, \mathbb{Q}_j : i \leq \alpha, j < \alpha\rangle$ is an RCS iteration. Then $\mathbb{P}_\alpha$ preserves ω_1 if the iteration satisfies the following properties:*

(1) *for each $i < \alpha$ there is $n < \omega$ such that $\mathbb{P}_{i+n} \Vdash |\mathbb{P}_i| \leq \omega_1$,*

(2) *for each $i < \alpha$ there is an uncountable regular cardinal κ_i and a $\mathbb{P}_i$-name $\dot{\mathbb{I}}_i$ such that $\mathbb{P}_i$ is κ_i-c.c. and $\mathbb{P}_i$ forces $\dot{\mathbb{I}}_i$ is a non-empty restriction-closed κ_i-complete family of ideals such that $\dot{\mathbb{Q}}_i$ satisfies the $\dot{\mathbb{I}}_i$-universal property.*

Theorem 5.10. *Let α be a strongly inaccessible cardinal. Suppose that $\langle \mathbb{P}_i, \mathbb{Q}_j : i \leq \alpha, j < \alpha\rangle$ is a revised countable support iteration such that $\mathbb{P}_\alpha$ preserves ω_1 and for all $i < \alpha$, $|\mathbb{P}_i| < \alpha$. Then $\mathbb{P}_\alpha$ is α-c.c.*

Suppose κ is a Mahlo cardinal and let A be the stationary set of strongly inaccessible cardinals below κ. Define an RCS iteration $\langle \mathbb{P}_i, \dot{\mathbb{Q}}_j : i \leq \kappa, j < \kappa\rangle$ by recursion as follows. Our recursion hypotheses will include that each $\mathbb{P}_\alpha$ preserves ω_1, and is α-c.c. if α is in A.

Suppose $\mathbb{P}_\alpha$ is defined. If α is not in A, then let $\dot{\mathbb{Q}}_\alpha$ be a name for COLL$(\omega_1, |\mathbb{P}_\alpha|)$. Suppose α is in A. By the recursion hypotheses $\mathbb{P}_\alpha$ forces $\alpha = \omega_2$. Let $\dot{\mathbb{Q}}_\alpha$ be a name for the poset ADD $* \mathbb{P}(\dot{S})$.

If α is not in A, then choose some regular cardinal κ_α larger than $|\mathbb{P}_\alpha|$, and let $\dot{\mathbb{I}}_\alpha$ be a name for some non-empty restriction-closed κ_α-complete family of ideals on κ_α. Then $\mathbb{P}_\alpha$ is κ_α-c.c., and since $\dot{\mathbb{Q}}_\alpha$ is proper, $\mathbb{P}_\alpha$ forces $\dot{\mathbb{Q}}_\alpha$ satisfies the $\dot{\mathbb{I}}_\alpha$-universal property. Suppose α is in A. Then let $\alpha = \kappa_\alpha$ and define $\dot{\mathbb{I}}_\alpha$ as a name for the family of ideals on ω_2 as described in Theorem 5.8. Then $\mathbb{P}_\alpha$ is κ_α-c.c. and forces that $\dot{\mathbb{Q}}_\alpha$ satisfies the $\dot{\mathbb{I}}_\alpha$-universal property.

Suppose $\beta \leq \kappa$ is a limit ordinal and $\mathbb{P}_\alpha$ is defined for all $\alpha < \beta$. Define $\mathbb{P}_\beta$ as the revised countable support limit of $\langle \mathbb{P}_\alpha : \alpha < \beta \rangle$. By Theorem 5.9 and the recursion hypotheses, $\mathbb{P}_\beta$ preserves ω_1. Hence if β is in $A \cup \{\kappa\}$, then $\mathbb{P}_\beta$ is β-c.c. by Theorem 5.10.

This completes the definition. Let G be generic for $\mathbb{P}_\kappa$. The poset $\mathbb{P}_\kappa$ is κ-c.c. and preserves ω_1, so in $V[G]$ we have that $\kappa = \omega_2$ and A is a stationary subset of $\omega_2 \cap \mathrm{cof}(\omega_1)$. For each α in A let $\mathcal{C}_\alpha$ be the club on $P_{\omega_1}(\alpha)$ introduced by $\mathbb{Q}_\alpha$. If $\alpha < \beta$ are in A, then $\mathcal{C}_\alpha$ and $\mathcal{C}_\beta$ are disjoint since $\mathcal{C}_\beta$ is disjoint from $V[G \restriction \beta]$. So $\langle \mathcal{C}_\alpha : \alpha \in A \rangle$ is a disjoint club sequence on ω_2 in $V[G]$.

We conclude the paper with several questions.

(1) Assuming Martin's Maximum, the poset $\textsc{Add} * \mathbb{P}(\dot{S})$ is semiproper. Is this poset semiproper in general?

(2) Is it consistent that there exists a stationary set $A \subseteq \omega_2 \cap \mathrm{cof}(\omega_1)$ such that neither $A \cup \mathrm{cof}(\omega)$ nor $\omega_2 \setminus A$ can acquire a club subset in an ω_1 and ω_2 preserving extension?

(3) To what extent can the results of this paper be extended to cardinals greater than ω_2? For example, is it consistent that there is a fat stationary subset of ω_3 which cannot acquire a club subset by any forcing poset which preserves ω_1, ω_2, and ω_3?

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