TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 359, Number 6, June 2007, Pages 2559–2575
S 0002-9947(07)03953-0
Article electronically published on January 19, 2007

$L^p - L^{p'}$ ESTIMATES FOR OVERDETERMINED RADON TRANSFORMS

LUCA BRANDOLINI, ALLAN GREENLEAF, AND GIANCARLO TRAVAGLINI

ABSTRACT. We prove several variations on the results of F. Ricci and G. Travaglini (2001), concerning $L^p - L^{p'}$ bounds for convolution with all rotations of arc length measure on a fixed convex curve in $\mathbb{R}^2$. Estimates are obtained for averages over higher-dimensional convex (nonsmooth) hypersurfaces, smooth k-dimensional surfaces, and nontranslation-invariant families of surfaces. We compare Ricci and Travaglini's approach, based on average decay of the Fourier transform, with an approach based on L^2 boundedness of Fourier integral operators, and show that essentially the same geometric condition arises in proofs using the two techniques.

§1. INTRODUCTION

Our starting point is the following result from [RT].

Theorem 1. *Let $\Gamma \subset \mathbb{R}^2$ be a compact, convex curve with arc length measure μ. Let μ_θ denote the rotation of μ by $\theta \in \mathbb{S}^1$. Then*

$$\left(\int_{\mathbb{S}^1}\int_{\mathbb{R}^2} |f * \mu_\theta(x)|^3 dx d\theta\right)^{\frac{1}{3}} \lesssim \left(\int_{\mathbb{R}^2} |f(y)|^{\frac{3}{2}} dy\right)^{\frac{2}{3}}. \tag{1.1}$$

Thus, the $L^{\frac{3}{2}}(\mathbb{R}^2) \longrightarrow L^3(\mathbb{R}^2)$ estimate that holds for curves in the plane with nonzero curvature ([L], [Str]) generalizes to arbitrary (i.e., not necessarily smooth) convex curves when averaged over all rotations. The goal here is to extend this in several ways: to averages over k-dimensional surfaces in $\mathbb{R}^n$; to more general transformations than rotations; and to nontranslation-invariant averaging operators. In doing so, we will primarily use two techniques: estimates for average decay of the Fourier transform of surface measure (as in [RT]), and L^2 regularity of nondegenerate Fourier integral operators. Although these methods appear to be different, the geometric assumptions needed to use them are essentially the same.

To start with, one can extend Theorem 1 to rotations of hypersurfaces in higher dimensions with the same convexity assumption. For $\theta \in SO(n)$ and μ a measure on $\mathbb{R}^n$, let μ_θ be defined by $\langle f, \mu_\theta\rangle = \langle f(\theta^{-1}\cdot), \mu\rangle$, so that $\hat{\mu}_\theta(\xi) = \hat{\mu}(\theta\xi)$.

Received by the editors December 16, 2003 and, in revised form, February 7, 2005.
2000 *Mathematics Subject Classification.* Primary 42B10, 44A12.
Key words and phrases. Radon transform, averages over curves, L^p improving.
The second author was partially supported by a grant from the National Science Foundation.

Theorem 2. *Let $S \subset \mathbb{R}^n$ be a compact, convex hypersurface with induced measure μ. Then,*

$$\left(\int_{SO(n)} \int_{\mathbb{R}^n} |f * \mu_\theta(x)|^{n+1} dx d\theta\right)^{\frac{1}{n+1}} \lesssim \left(\int_{\mathbb{R}^n} |f(y)|^{\frac{n+1}{n}} dy\right)^{\frac{n}{n+1}}. \tag{1.2}$$

(Here, and throughout, we use $\lesssim$ to denote $\leq c\cdot$, with c dependent only on the operator in question.) Theorem 1 can also be modified to cover all rotations of a surface in $\mathbb{R}^n$ of arbitrary dimension, under a smoothness assumption.

Theorem 3. *Let $S \subset \mathbb{R}^n$ be a smooth k-dimensional surface, $1 \leq k \leq n-1, n \geq 2$, and μ a smooth, compactly supported multiple of induced surface measure on S. Then,*

$$\left(\int_{SO(n)} \int_{\mathbb{R}^n} |f * \mu_\theta(x)|^{\frac{2n-k}{n-k}} dx d\theta\right)^{\frac{n-k}{2n-k}} \lesssim \left(\int_{\mathbb{R}^n} |f(y)|^{\frac{2n-k}{n}} dy\right)^{\frac{n}{2n-k}}. \tag{1.3}$$

A crucial ingredient in the proof of Theorem 1 was the L^2 average decay of the Fourier transform $\hat{\mu}$ from [P]. Theorems 2 and 3 follow immediately by replacing Podkorytov's estimate in the argument of [RT] by the results of [BHI] and Proposition 1 below, respectively.

To obtain the optimal $L^{\frac{2n-k}{n}} \longrightarrow L^{\frac{2n-k}{n-k}}$ boundedness, we do not actually need to use all rotations of the surface, or even linear transformations for that matter, nor does the operator need to be translation-invariant. To start with, we keep the translation-invariance, but allow nonlinear transformations to act on the surface. Let $T_s : \mathbb{R}^n \to \mathbb{R}^n$ be a smooth family of transformation of $\mathbb{R}^n$ parametrized by $s \in K \subset\subset \mathbb{R}^m$, $\quad m \geq n-1$, let $\gamma : A \subset \mathbb{R}^{n-1} \to \mathbb{R}^n$ be a C^2 parametrized convex hypersurface in $\mathbb{R}^n$ (if $n = 2$ we merely need to assume that γ is convex) and set $\gamma_s(t) = T_s(\gamma(t))$. We are interested in the operator

$$Rf(x,s) = \int f(x - \gamma_s(t))\, \chi(t) dt = (f * \mu_s)(x)$$

where μ_s is the measure defined by

$$\int f(x)\, d\mu_s = \int f(\gamma_s(t))\, \chi(t) dt,$$

with $\chi \in C_0^\infty(\mathbb{R}^{n-1})$ is a fixed cutoff function. Denote by $J_{T_s}(x)$ and $J_{\frac{\partial T_s}{\partial \theta_k}}(x)$ the Jacobian matrices at x of the maps T_s and $\frac{\partial T_s}{\partial s_k}$, respectively. We have

Theorem 4. *Let γ be a convex curve if $n = 2$ and a C^2 convex hypersurface if $n \geq 3$. Assume that for every unit vector $\Xi \in \mathbb{S}^{n-1}$ and for every $s \in K$ the matrix*

$$C = \begin{bmatrix} \Xi^t J_{T_s}(x) \\ \Xi^t J_{\frac{\partial T_s}{\partial s_1}}(x) \\ \vdots \\ \Xi^t J_{\frac{\partial T_s}{\partial s_{n-1}}}(x) \end{bmatrix} \tag{1.4}$$

has rank n. Then

$$\left\{\int_K \int_{\mathbb{R}^n} |Rf(x,s)|^{n+1} dx ds\right\}^{1/(n+1)} \lesssim \left\{\int_{\mathbb{R}^n} |f(x)|^{(n+1)/n} dx\right\}^{n/(n+1)}. \tag{1.5}$$

To consider nontranslation-invariant operators, we next take

$$\gamma : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^k \longrightarrow \mathbb{R}^n$$

to be a C^∞ map, with $D_t\gamma(x,s,t)$ injective, so that each $\gamma_{x,s} := \{\gamma(x,s,t) : t \in \mathbb{R}^k\}$ is a smooth immersed k-surface in $\mathbb{R}^n$. Define the (overdetermined) generalized Radon transform $R : C^\infty(\mathbb{R}^n) \longrightarrow C^\infty(\mathbb{R}^n \times \mathbb{R}^m)$ by

$$Rf(x,s) = \int_{\mathbb{R}^k} f(\gamma(x,s,t))\chi(t)dt, \quad \chi \in C_0^\infty(\mathbb{R}^k). \tag{1.6}$$

Then (see [Gu], [GuSt]) the Schwartz kernel of R is a smooth density δ_Z supported on the incidence relation

$$Z = \{(x,s,y) : y \in \gamma_{x,s}\} \subset \mathbb{R}^{n+m} \times \mathbb{R}^n,$$

which is codimension $n-k$ in $\mathbb{R}^{n+m} \times \mathbb{R}^n$. If

$$Z = \{F_1(x,s,y) = \cdots = F_{n-k}(x,s,y) = 0\}$$

locally, with $\{dF_1, \ldots, dF_{n-k}\}$ linearly independent, then δ_Z has the oscillatory representation

$$\delta_Z = \int_{\mathbb{R}^{n-k}} e^{i\sum_{j=1}^{n-k} F_j(x,s,y)\theta_j} a(x,s,y;\theta)d\theta \tag{1.7}$$

in the sense of [H1], with $a(x,s,y;\theta)$ a symbol of order 0 (essentially $\equiv 1$ in θ). In general, δ_Z is a locally finite sum of such expressions. Thus, δ_Z is a Fourier integral distribution on $\mathbb{R}^{n+m} \times \mathbb{R}^n$ associated to the conormal bundle of Z,

$$N^*Z = \Big\{(x,s,\xi,\sigma;y,\eta) \in (T^*\mathbb{R}^{n+m} \times T^*\mathbb{R}^n)\backslash 0 : (x,s,y) \in Z, (\xi,\sigma,\eta) \perp TZ\Big\},$$

and hence R is a Fourier integral operator,

$$R \in I^r(\mathbb{R}^{n+k}, \mathbb{R}^n; C),$$

where

$$C = N^*Z' = \Big\{(x,s,\xi,\sigma;y,\eta) : (x,s,\xi,\sigma;y,-\eta) \in N^*Z\Big\}$$

is a canonical relation, i.e., a lagrangian submanifold for the difference symplectic form $\omega_{T^*\mathbb{R}^{n+m}} - \omega_{T^*\mathbb{R}^n}$ on $T^*\mathbb{R}^{n+m} \times T^*\mathbb{R}^n$, and the order r is calculated by

$$\begin{aligned} r &= (\text{order of } a) + \frac{\text{number of phase variables}}{2} - \frac{\text{number of spatial variables}}{4} \\ &= 0 + \frac{n-k}{2} - \frac{n+m+n}{4} = -\frac{k}{2} - \frac{m}{4}. \end{aligned}$$

L^2 estimates for Fourier integral operators associated with a canonical relation $C \subset T^*X \times T^*Y$ depend on the structure of the projections $\pi_R : C \longrightarrow T^*Y$ and $\pi_L : C \longrightarrow T^*X$. The optimal L^2 estimates for an operator $F \in I^r(X,Y;C)$ hold under the assumption that π_R is a submersion (which guarantees that π_L is an immersion), together with the mild requirement that the spatial projections $\pi_X : C \longrightarrow X$ and $\pi_Y : C \longrightarrow Y$ are submersions [H1], [H2]; such canonical relations C are called *nondegenerate*. Substituting L^2 estimates for such operators in place of the average decay estimates for the Fourier transform of surface-carried measures, we can show

Theorem 5. *Let R be a generalized Radon transform as in* (1.6) *such that the associated canonical relation C is nondegenerate. Then,*

$$Rf : L^{\frac{2n-k}{n}}_{comp}(\mathbb{R}^n) \longrightarrow L^{\frac{2n-k}{n-k}}_{loc}(\mathbb{R}^{n+m}). \tag{1.8}$$

Letting $\gamma_0 : \mathbb{R}^k_t \longrightarrow \mathbb{R}^n$ be a local parametrization of a smooth k-surface $S \subset \mathbb{R}^n$, $m = \frac{n(n-1)}{2}$, $\theta : \mathbb{R}^m_s \longrightarrow SO(n)$ a coordinate chart and $\gamma(x,s,t) = x - \theta(s)(\gamma_0(t))$, we see that Theorem 5 extends Theorem 3. It is also possible to use proper subgroups of $SO(n)$ and obtain the same estimates. These and other particular cases of Theorem 5 will be discussed in §5 below.

All of these results involve estimates on the line of duality. Via interpolation with the $L^1 - L^1$ and $L^\infty - L^\infty$ bounds, we find that the type sets of the operators contain certain closed triangles, symmetric about the line of duality. For general hypersurfaces, this is sharp, as the example of the unit sphere shows, with rotation not producing any additional L^p improvement. The emphasis here is on the extension of these estimates to low regularity and variable coefficient settings. For higher codimension surfaces, the results here fail to be sharp. For example, Drury [D] (see also Christ [C1]) has shown that the X-ray transform on $\mathbb{R}^n$ maps $L^{\frac{n+1}{2}}(\mathbb{R}^n)$ to $L^{n+1}(M_{1,n})$, where $M_{1,n}$ is the Grassmannian of affine lines in $\mathbb{R}^n$, and this then implies an improvement of Theorem 3 for S a line segment. Also, these results have a somewhat different character then those of, for example, [O], [PhS], [GSW], [S], [C2], [Bu], [BuC], [TaW], where the specific geometry of the curve or family of curves determines a more complicated type set in the absence of rotations.

Finally, mixed norm estimates are possible for certain model surfaces in $\mathbb{R}^n$, just as in [RT] for model curves in $\mathbb{R}^2$. Writing $x = (x', x_n)$, consider the hypersurface

$$S_\beta = \left\{ (x', x_n) \in \mathbb{R}^n : x_n = |x'|^\beta \right\}$$

where $\beta > 2$. Let $d\mu_\beta$ be the induced measure on S_β, multiplied by a C^∞ function on $\mathbb{R}^n$ with compact support, identically equal to 1 in a neighborhood of the origin, and $\mu_{\beta,\theta}$ its rotation by $\theta \in SO(n)$. We have

Theorem 6. *Let $R_\beta f(x,\theta) = f * \mu_{\beta,\theta}(x)$. Then,*

(i) $$\|R_\beta f\|_{L^{\frac{p'(n-1)(\beta-1)}{(p'-2)\beta-2(n-1)}}\left(SO(n); L^{p'}(\mathbb{R}^n)\right)} \lesssim \|f\|_{L^p}, \quad \frac{n+1}{n} < p < \frac{2\beta + 2\,(n-1)}{\beta + 2\,(n-1)},$$

(ii) $$\|R_\beta f\|_{L^\infty\left(SO(n); L^{p'}(\mathbb{R}^n)\right)} \lesssim \|f\|_{L^p}, \quad p > \frac{2\beta + 2\,(n-1)}{\beta + 2\,(n-1)}.$$

Theorems 2 and 3, which are based on L^2 average decay properties of Fourier transforms of surface-carried measure, are proved in §2. Theorem 6, which uses L^p average decay properties, is proved in §3, and Theorem 4, which still concerns translation-invariant operators and thus can be treated using Fourier transform estimates, is treated in §4. In §5, we prove Theorem 5 and discuss geometric criteria for nondegeneracy of the canonical relation.

§2. Euclidian motions of a fixed surface

We begin by considering averages over all translations and rotations of a fixed k-dimensional surface in $\mathbb{R}^n$.

Proposition 1. *Let $\Phi : \Omega \subseteq \mathbb{R}^k \to \mathbb{R}^n$ be a parameterization of a C^∞ k-dimensional surface $S \subset \mathbb{R}^n$ and let $\mu = \Phi_*(\chi(u)du)$, where Φ_* denotes pushforward and χ is a suitable cut-off function on $\mathbb{R}^k$, so that*

$$\widehat{\mu}(\rho\omega) = \int_\Omega e^{-2\pi i\rho\omega\cdot\Phi(u)}\eta(u)\,du \tag{2.1}$$

for $\omega \in \mathbb{S}^{n-1}$. Then

$$\left\{\int_{\mathbb{S}^{n-1}} |\widehat{\mu}(\rho\omega)|^2\,d\omega\right\}^{\frac{1}{2}} \lesssim \rho^{-\frac{k}{2}}. \tag{2.2}$$

Proof. We can change parameterization and choose coordinates in $\mathbb{R}^n$ so that

$$\Phi(u) = (u, \Psi(u)).$$

We can also assume that $\Phi(0) = 0$ and $\nabla\Psi(0) = 0$. We have

$$\widehat{\mu}(\rho\omega) = \int_\Omega e^{-2\pi i\rho\omega\cdot(u,\Psi(u))}J(u)\,du,$$

where $J(u)$ is a suitable function that takes into account the change of parameterization. Let $\psi(\omega) = \psi(\omega_1, \ldots, \omega_k)$ be a cut-off function supported on $|\omega_j| \leqslant \frac{1}{10}$ for $j = 1, \ldots, k$. Then

$$\begin{aligned}
&\int_{\mathbb{S}^{n-1}} |\widehat{\mu}|^2(\rho\omega)\,\psi(\omega)\,d\omega \\
&\quad= \int_\Omega\int_\Omega\int_{\mathbb{S}^{n-1}} e^{-2\pi i\rho\omega\cdot(u-v,\Psi(u)-\Psi(v))}\psi(\omega)\,d\omega\,J(u)\,J(v)\,du\,dv \\
&\quad= \int_\Omega\int_\Omega \frac{1}{(1+\rho|u-v|)^N}\,J(u)\,J(v)\,du\,dv \\
&\quad\lesssim \int_\Omega\int_{\mathbb{R}^k} \frac{1}{(1+\rho|u-v|)^N}\,du\,J(v)\,dv \\
&\quad= \int_\Omega\int_{\mathbb{R}^k} \frac{1}{(1+\rho|u|)^N}\,du\,J(v)\,dv = c\rho^{-k}\int_\Omega J(v)\,dv \\
&\quad\lesssim \rho^{-k}|\Omega|
\end{aligned}$$

for $N > k$. Here we used the fact that ω and $(u-v, \Psi(u)-\Psi(v))$ are almost orthogonal on the support of ψ, and we can evaluate the integral on $\mathbb{S}^{n-1}$ integrating by parts N times.

Let ω be in the support of $1-\psi$. Then

$$\widehat{\mu}(\rho\omega) = \int_\Omega e^{-2\pi i\rho\omega\cdot(u,\Psi(u))}J(u)\,du.$$

Since $|\nabla_u(\omega\cdot(u,\Psi(u)))| > c > 0$, integrating by parts N times gives

$$|\widehat{\mu}(\rho\omega)| \lesssim \rho^{-N},$$

finishing the proof of Proposition 1. □

Remark. Related results for curves in $\mathbb{R}^n$ can be found in [M].

Proofs of Theorems 2 *and* 3. By decomposing S into a finite number of pieces we can assume that S is defined by

$$\left\{(x, x') \in \mathbb{R}^k \times \mathbb{R}^{n-k} : x' = \Phi(x)\right\}$$

and that the Jacobian of Φ has bounded entries. Also observe that the tangent spaces to S do not contain any line parallel to $\{0\}\times\mathbb{R}^{n-k}$.

Let i_z be the distribution defined by

$$\langle i_z,\varphi\rangle=\frac{1}{\Gamma(z)}\int_0^\infty\varphi(t)\,t^{z-1}dt, \tag{2.3}$$

for test functions $\varphi\in C_0^\infty(\mathbb{R})$, and for $\theta\in SO(n)$ let the distribution μ_θ^z be defined by

$$\widehat{\mu_\theta^z}(\xi)=\widehat{\mu_\theta}(\xi)\;\widehat{i_z}\left((\theta\xi)_{n-k+1}\right)\;\cdots\;\widehat{i_z}\left((\theta\xi)_n\right), \tag{2.4}$$

where

$$\left((\theta\xi)_{n-k+1},\dots,(\theta\xi)_n\right)$$

denotes the last k components of $\theta\xi$. Introducing the analytic family of operators

$$T^z f(x,\theta)=(f*\mu_\theta^z)(x),\quad z\in\mathbb{C}, \tag{2.5}$$

the proof now follows exactly as in [RT]: using either [BHI] for Theorem 2 or Proposition 1 for Theorem 3, one shows that

$$T^{-\frac{k}{2(n-k)}+i\sigma}:L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n\times SO(n)),\quad \sigma\in\mathbb{R}, \tag{2.6}$$

and by (2.4),

$$T^{1+i\sigma}:L^1(\mathbb{R}^n)\to L^\infty(\mathbb{R}^n\times SO(n)),\quad \sigma\in\mathbb{R}. \tag{2.7}$$

Analytic interpolation then yields that

$$T^0:L^{\frac{2n-k}{n-k}}(\mathbb{R}^n)\longrightarrow L^{\frac{2n-k}{n}}(\mathbb{R}^n\times SO(n)),$$

which is (1.2) (for $k=n-1$) and (1.3). $\square$

§3. Mixed norm estimates for model surfaces

Consider the hypersurface

$$S_\beta=\left\{(x',x_n)\in\mathbb{R}^n: x_n=|x'|^\beta\right\}$$

for values of $\beta>2$. Let μ_β be the measure induced by the Lebesgue measure on S_β, multiplied by $\widetilde{\psi}\in C_0^\infty(\mathbb{R}^n)$, identically equal to 1 in a neighborhood of the origin. We are first interested in the decay at infinity of the Fourier transform of this measure,

$$\widehat{\mu_\beta}(\xi)=\int_{\mathbb{R}^{n-1}}e^{-2\pi i\left(\xi'\cdot x'+\xi_n|x'|^\beta\right)}\psi(x')\,dx'.$$

Lemma 1. *We have*

$$\left|\widehat{\mu_\beta}(\xi)\right|\lesssim\left|\xi'\right|^{-\frac{(n-1)(\beta-2)}{2(\beta-1)}}|\xi_n|^{-\frac{n-1}{2(\beta-1)}} \tag{3.1}$$

and

$$\left|\widehat{\mu_\beta}(\xi)\right|\lesssim|\xi|^{-\frac{n-1}{\beta}}. \tag{3.2}$$

Proof. To prove (3.1) let $\varphi(x') = \psi(x') - \psi(2x')$. Then

$$\begin{aligned}
\widehat{\mu_\beta}(\xi) &= \sum_{j=1}^{+\infty} \int_{\mathbb{R}^{n-1}} e^{-2\pi i\left(\xi'\cdot x' + \xi_n |x'|^\beta\right)} \varphi\left(2^j x'\right) dx' \\
&= \sum_{j=1}^{+\infty} 2^{-(n-1)j} \int_{\mathbb{R}^{n-1}} e^{-2\pi i\left(2^{-j}\xi'\cdot u + 2^{-\beta j}\xi_n |u|^\beta\right)} \varphi(u)\, du \qquad (3.3)\\
&= \sum_{j=1}^{+\infty} 2^{-(n-1)j} I\left(2^{-j}\xi', 2^{-\beta j}\xi_n\right),
\end{aligned}$$

where

$$I(\xi) = \int_{\mathbb{R}^{n-1}} e^{-2\pi i\left(2^{-j}\xi'\cdot u + 2^{-\beta j}\xi_n |u|^\beta\right)} \varphi(u)\, du.$$

Since ψ is identically equal to 1 in a neighborhood of the origin, φ is supported away from the origin. Therefore, for I we have the estimate

$$|I(\xi)| \lesssim |\xi|^{-\frac{n-1}{2}},$$

since S_β has strictly positive Gaussian curvature away from the origin. It follows that

$$|\widehat{\mu_\beta}(\xi)| \lesssim \sum_{j=1}^{+\infty} 2^{-(n-1)j} \left|\left(2^{-j}\xi', 2^{-\beta j}\xi_n\right)\right|^{-\frac{n-1}{2}}.$$

Let $j_o = \left[\frac{1}{\beta-1}\log_2\left(\frac{|\xi_n|}{|\xi'|}\right)\right]$ where $[\cdot]$ denotes the integral part. Observe that when $j \leqslant j_o$ we have

$$\left|\left(2^{-j}\xi', 2^{-\beta j}\xi_n\right)\right| \approx 2^{-\beta j}|\xi_n|$$

while for $j > j_o$ we have

$$\left|\left(2^{-j}\xi', 2^{-\beta j}\xi_n\right)\right| \approx 2^{-j}|\xi'|.$$

Splitting the above series yields

$$\begin{aligned}
|\widehat{\mu_\beta}(\xi)| &\lesssim \sum_{j=1}^{j_o} 2^{-(n-1)j}\left(2^{-\beta j}|\xi_n|\right)^{-\frac{n-1}{2}} + \sum_{j=j_o+1}^{+\infty} 2^{-(n-1)j}\left(2^{-j}|\xi'|\right)^{-\frac{n-1}{2}} \\
&\lesssim |\xi'|^{-\frac{(\beta-2)(n-1)}{2(\beta-1)}} |\xi_n|^{-\frac{n-1}{2(\beta-1)}}.
\end{aligned}$$

To prove (3.2) we observe that it is enough to consider the case $|\xi'| < c|\xi_n|$. From (3.3) we get

$$\begin{aligned}
|\widehat{\mu_\beta}(\xi)| &\lesssim \sum_{j=1}^{+\infty} 2^{-(n-1)j} I\left(2^{-j}\xi', 2^{-\beta j}\xi_n\right) \\
&\lesssim \sum_{j=1}^{+\infty} 2^{-(n-1)j}\left(1 + 2^{-\beta j}|\xi_n|\right)^{-\frac{n-1}{2}} \\
&= \sum_{j=1}^{j_0} 2^{-(n-1)j}\left(1 + 2^{-\beta j}|\xi_n|\right)^{-\frac{n-1}{2}} + \sum_{j=j_0+1}^{+\infty} 2^{-(n-1)j}\left(1 + 2^{-\beta j}|\xi_n|\right)^{-\frac{n-1}{2}},
\end{aligned}$$

where $j_0 = \left[\frac{1}{\beta}\log|\xi_n|\right]$. Therefore,

$$\left|\widehat{d\mu}\,(\xi)\right| \lesssim \sum_{j=1}^{j_0} 2^{-(n-1)j} 2^{\beta j \frac{n-1}{2}} \left|\xi_n\right|^{-\frac{n-1}{2}} + c \sum_{j=j_0+1}^{+\infty} 2^{-(n-1)j}$$
$$\lesssim |\xi_n|^{-\frac{n-1}{\beta}} \lesssim |\xi|^{-\frac{n-1}{\beta}} .$$

□

Lemma 1 allows us to obtain L^p average decay of $\widehat{\mu_\beta}$, extending a result in [BRT].

Proposition 2. *We have the following estimates:*

$$\left\{\int_{\mathbb{S}^{n-1}} \left|\widehat{\mu_\beta}\,(\rho\omega)\right|^p d\omega\right\}^{\frac{1}{p}} \lesssim \begin{cases} \rho^{-\frac{n-1}{2}}, & p < \frac{2(\beta-1)}{\beta-2}, \\ \rho^{-\frac{n-1}{2}} \log(\rho)^{\frac{(\beta-2)(n-1)}{2(\beta-1)}}, & p = \frac{2(\beta-1)}{\beta-2}, \\ \rho^{-(n-1)\left(\frac{1}{p}+\frac{1}{\beta}-\frac{1}{p\beta}\right)}, & p > \frac{2(\beta-1)}{\beta-2}. \end{cases}$$

Proof. Now let $\xi = (\xi', \xi_n) = (\rho\omega' \sin\theta, \rho\cos\theta)$, with $\omega' \in \mathbb{S}^{n-2}$ and $0 \leq \theta \leq \pi$. When $\varepsilon < \theta < \pi - \varepsilon$ we have the uniform estimate

$$|\widehat{\mu_\beta}(\rho\omega'\sin\theta, \rho\cos\theta)| \lesssim \rho^{-\frac{n-1}{2}}.$$

Hence, when $p > \frac{2(\beta-1)}{\beta-2}$,

$$\begin{aligned}
&\int_0^\pi \int_{\mathbb{S}^{n-2}} |\widehat{\mu_\beta}(\rho\omega'\sin\theta, \rho\cos\theta)|^p \sin^{n-2}\theta d\omega' d\theta \\
&\quad\lesssim \rho^{-\frac{n-1}{2}p} \\
&\quad+ \int_0^\varepsilon \left(\min\left(\rho^{-\frac{n-1}{\beta}}, \rho^{-\frac{n-1}{2}} (\sin\theta)^{-\frac{(n-1)(\beta-2)}{2(\beta-1)}} (\cos\theta)^{-\frac{(n-1)}{2(\beta-1)}}\right)\right)^p (\sin\theta)^{n-2} d\theta \\
&\quad\lesssim \rho^{-\frac{n-1}{2}p} + \int_0^\varepsilon \left(\min\left(\rho^{-\frac{n-1}{\beta}}, \rho^{-\frac{n-1}{2}} \theta^{-\frac{(n-1)(\beta-2)}{2(\beta-1)}}\right)\right)^p \theta^{n-2} d\theta \\
&\quad\lesssim \rho^{-\frac{n-1}{2}p} + \int_0^{\rho^{-1+1/\beta}} \rho^{-\frac{n-1}{\beta}p} \theta^{n-2} d\theta \\
&\quad+ \int_{\rho^{-1+1/\beta}}^\varepsilon \rho^{-\frac{n-1}{2}p} \theta^{-\frac{(n-1)(\beta-2)p}{2(\beta-1)}} \theta^{n-2} d\theta \\
&\quad\lesssim \rho^{-\frac{n-1}{2}p} + \rho^{-(n-1)\frac{p+\beta-1}{\beta}} + \rho^{-(n-1)\frac{p+\beta-1}{\beta}} \lesssim \rho^{-(n-1)\frac{p+\beta-1}{\beta}}.
\end{aligned}$$

The computations when $p = \frac{2(\beta-1)}{\beta-2}$ or $p < \frac{2(\beta-1)}{\beta-2}$ are similar. □

Incorporating the average Fourier transform decay estimate of Proposition 2 into the proof of [RT] as described in §2 then yields Theorem 6.

§4. Translates of transformations of a fixed surface

We now turn to the proof of Theorem 4. If the number of parameters m is greater than $n-1$, then under the rank assumption of Theorem 4, we may select $n-1$ variables $s_{i_1}, \dots, s_{i_{n-1}}$ such that the corresponding square submatrix of (1.4) is nonsingular. The estimate (1.5) then holds, with respect to $ds_{i_1} \cdots ds_{i_{n-1}}$, uniformly in the remaining s variables. Since $K \subset\subset \mathbb{R}^m$, we may integrate in all the variables and see that (1.5) holds. Hence it sufffices to assume that $m = n-1$ and (1.4) is nonsingular.

Starting with the two-dimensional case, let

$$T_s : \mathbb{R}^2 \to \mathbb{R}^2$$

be a smooth family of transformations of the plane with $s \in [a, b]$. Let $\gamma^0 : [-\varepsilon, \varepsilon] \to \mathbb{R}^2$ be a convex curve in $\mathbb{R}^2$ and let $\gamma_s(t) = T_s\left(\gamma^0(t)\right)$. We are interested in the operator

$$Rf(x, s) = \int f(x - \gamma_s(t))\, dt = (f * \mu_s)(x),$$

where μ_s is the measure defined by

$$\int f(x)\, d\mu_s = \int f\left(T_s(\gamma^0(t))\right) dt.$$

Splitting the curve into a finite number of segments if necessary, we may assume the existence of two constants φ_1 and φ_2 such that for any t the left and the right tangent lines at $\gamma(t)$ have slopes between φ_1 and φ_2, and $\varphi_2 - \varphi_1$ is small.

Proposition 3. *Let $\Phi^t = (\cos\varphi, \sin\varphi)$ with $\varphi_1 \le \varphi \le \varphi_2$. We assume that for every $\varphi_1 \le \varphi \le \varphi_2$ the matrix*

$$C = \begin{bmatrix} J_{T_s} \Phi^t \\ J_{\frac{dT_s}{ds}} \Phi^t \end{bmatrix} \tag{4.1}$$

is nonsingular. Then

$$\left(\int \left| \widehat{\mu_s}(\xi) \right|^2 ds \right)^{\frac{1}{2}} \lesssim |\xi|^{-\frac{1}{2}}. \tag{4.2}$$

Proof. For any $\xi \neq 0$ let

$$\delta = \max_s \left(\frac{\left|\xi \cdot J_{T_s} \Phi\right|}{|\xi|}, \frac{\left|\xi \cdot J_{\frac{dT_s}{d\theta}} \Phi\right|}{|\xi|} \right).$$

The nonsingularity of C ensures that $\delta > c > 0$. Let $\eta_1(s), \eta_2(s) \in C^\infty$ be cut-off functions such that $\eta_1(s) + \eta_2(s) \equiv 1$, $\frac{\left|\xi \cdot J_{\frac{dT_s}{ds}} \Phi\right|}{|\xi|} > c$ on the support of η_1 and $\frac{\left|\xi \cdot J_{T_s} \Phi\right|}{|\xi|} > c$ on the support of η_2. Then

$$\int \left|\widehat{\mu_s}(\xi)\right|^2 ds = \int \left|\widehat{\mu_s}(\xi)\right|^2 \eta_1(s)\, ds + \int \left|\widehat{\mu_s}(\xi)\right|^2 \eta_2(s)\, ds = I + II.$$

To estimate I observe that

$$I = \int \left|\widehat{\mu_s}(\xi)\right|^2 \eta_1(s)\, ds = \int \int \int e^{-2\pi i \xi \cdot \left(\gamma_s(t) - \gamma_s(t')\right)} \eta_1(s)\, ds dt' dt.$$

Since

$$\begin{aligned} \left| \frac{d}{ds} \xi \cdot (\gamma_s(t) - \gamma_s(t')) \right| &= \left| \xi \cdot \frac{dT_s}{ds}(\gamma(t)) - \xi \cdot \frac{dT_s}{ds}(\gamma(t')) \right| \\ &= \left| \xi \cdot J_{\frac{dT_s}{ds}}(c)\, (\gamma(t) - \gamma(t')) \right| \geqslant c\, |\xi|\, |\gamma(t) - \gamma(t')| \end{aligned}$$

(observe that the direction of $\gamma(t) - \gamma(t')$ is between φ_1 and φ_2 by the convexity of γ)), integrating by parts in s yields

$$I \lesssim \int \int \frac{1}{1 + |\xi|\, |\gamma(t) - \gamma(t')|} dt' dt \lesssim \frac{1}{|\xi|}.$$

We now consider II. When s belongs to the support of η_2 we have, a.e. in t,

$$\left|\frac{d}{dt}\xi\cdot T_s\left(\gamma\left(t\right)\right)\right| = \left|\xi\cdot J_{T_s}\left(\gamma\left(t\right)\right)\gamma'\left(t\right)\right| \geqslant c\left|\xi\right|$$

and therefore, integrating by parts, we get

$$\left|\widehat{\mu_s}\left(\xi\right)\right| = \left|\int e^{-2\pi i\xi\cdot T_s(\gamma(t))}dt\right| \lesssim \frac{1}{|\xi|},$$

finishing the proof of Proposition 3. □

It is also possible to apply the geometric combinatorics technique of Christ [C2] (see also [TaW]) to obtain all but the sharp $L^{\frac{3}{2}} \longrightarrow L^3$ result, with a restricted weak-type estimate at the endpoints, under the same geometric condition (4.1). Let $\gamma_{t,s}(x) = x - T_s(\gamma^0(t))$, thought of as a family of diffeomorphisms of $\mathbb{R}^2$, indexed by t, s. Define, for $y_0 \in \mathbb{R}^2$ fixed, a map $\Psi : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ by

$$\Psi(t', t, s) = \gamma_{t',s}(\gamma_{t,s}^{-1}(y_0)).$$

Then the crucial things one needs for the argument of [C2] to work are:

(i) $$||D\Psi||_{2\times 2} \geq c|t' - t|,$$

where $||A||_{2\times 2}$ is the maximal 2×2 minor of a 2×3 matrix A, and

(ii) $$|\Psi^{-1}(x)|_1 \leq C,$$

an upper bound on the lengths of the preimages of points under Ψ. In this translation-invariant situation,

$$\Psi(t', t, s) = y_0 + T_s(\gamma(t)) - T_s(\gamma(t'))$$

so that

$$\begin{aligned} D\Psi =& \Big[-DT_s(\dot\gamma(t')), \quad DT_s(\dot\gamma(t)), \quad T_s'(\gamma(t)) - T_s'(\gamma(t'))\Big] \\ \simeq& \Big[-DT_s(\dot\gamma(s)), \quad (t'-t)DT_s(\ddot\gamma(t')), \quad (t'-t)DT_s'(\dot\gamma(t'))\Big], \end{aligned}$$

from which one sees that (i) will follow if

$$rank\Big[J_{T_s}\dot\gamma, \quad J_{T_s}\ddot\gamma, \quad J_{\frac{dT_s}{ds}}\dot\gamma\Big] = 2.$$

Since this approach does not yield the endpoint result, but only restricted weak type, we shall not describe it in more detail.

In dimensions $n \geq 3$, we need to impose a regularity condition on the surface in addition to convexity, so we now assume that γ is a C^2 convex parametrized hypersurface in $\mathbb{R}^n$. By a partition of unity on the surface, we may assume that $\gamma\left(t\right)$ is in a small neighborhood of a fixed point $x_o = \gamma\left(t_o\right)$. We can also assume that the image of the Gauss map is a small compact subset Ω of $\mathbb{S}^{n-1}$, and denote by $\Omega^{\perp}$ the set of directions that are orthogonal to a direction in Ω.

Proposition 4. *Under the assumption that the matrix in* (1.4) *has rank* n,

(4.3) $$\left\{\int \left|\widehat{\mu_s}\left(\xi\right)\right|^2 \Psi\left(s\right)ds\right\}^{\frac{1}{2}} \lesssim |\xi|^{-\frac{n-1}{2}},$$

where $\Psi \in C_0^{\infty}\left(K\right)$.

Remark. Let $\omega = \Xi^t J_{T_s}(x_0)$. By the rank assumption in Theorem 4, we have that the vectors $\omega, \frac{\partial\omega}{\partial s_1}, \dots, \frac{\partial\omega}{\partial s_{n-1}}$ are linearly independent. Hence, writing $s = (s', s'') \in \mathbb{R}^{n-1} \times \mathbb{R}^{m-n+1}$, it follows that for all $s_0'' \in \mathbb{R}^{m-n+1}$, the map $s' \to \omega(s', s_0'')$ defines a hypersurface in $\mathbb{R}^n$ whose tangent hyperplane does not contain the origin.

Proof. Let $\xi = \rho\Xi$ where $\rho = |\xi|$ and let

$$\delta(s) = \inf_{\substack{\Phi\in\Omega^\perp \\ |\Phi|=1}} \max_{k=1,\dots,n-1} \left|\Xi^t J_{\frac{\partial T}{\partial s_k}}(x_o)\,\Phi\right|. \tag{4.4}$$

By a smooth partition of unity we can assume that Ψ is supported in a small neighborhood of a fixed point s_0 and that on the support of Ψ either $\delta(s) \geq \varepsilon$ or $\delta(s) < \varepsilon$ holds for a suitable ε to be chosen later.

Assume we have $\delta(s) \geq \varepsilon$. Let us consider

$$I = \int |\widehat{\mu_s}(\xi)|^2\,\Psi(t')\,ds = \int\int\int e^{-i\xi\cdot(\gamma_s(t)-\gamma_s(t'))}\,\Psi(t')\,dsdt'dt.$$

We have

$$\begin{aligned}\left|\frac{\partial}{\partial s_k}\xi\cdot(\gamma_s(t)-\gamma_s(t'))\right| &= \left|\xi\cdot\frac{\partial T_s}{\partial s_k}(\gamma(t)) - \xi\cdot\frac{\partial T_s}{\partial s_k}(\gamma(t'))\right| \\ &= \rho\left|\Xi^t J_{\frac{\partial T_s}{\partial s_k}}(\gamma(c))\,\gamma'(c)\,(t-t')\right| \geqslant \varepsilon\rho\,|\gamma'(c)\,(t-t')| \geq c\varepsilon\rho\,|t-t'|\,.\end{aligned}$$

We used the fact that $\gamma'(c)(t-t')$ is in the tangent hyperplane at $\gamma(c)$ and therefore is in $\Omega^\perp$, and that γ' has maximal rank. Integrating by parts $n-1$ times in s_k we get

$$I \lesssim \int\int \frac{1}{[1+\rho\,|t-t'|]^{n-1}}\,dsdt \lesssim \frac{1}{\rho^{n-1}}.$$

We now consider the case

$$\delta(s) < \varepsilon. \tag{4.5}$$

Since the map $T_s(x)$ is smooth we can write $T_s(x) = T_s(x_0) + J_{T_s}(x-x_0) + E_s(x, x_0)$. Then

$$\begin{aligned}|\widehat{\mu_s}(\xi)| &= \left|\int e^{i\rho\Xi\cdot T_s(\gamma(t))}dt\right| \\ &= \left|\int e^{i\rho\Xi\cdot[J_{T_s}(\gamma(t)-\gamma(t_0))+E_s(\gamma(t),\gamma(t_o))]}dt\right| \\ &= \left|\int e^{i\rho\Xi\cdot J_{T_s}\gamma(t)+i\rho\Xi\cdot E_s(\gamma(t),\gamma(t_o))}dt\right|.\end{aligned}$$

Setting $\omega = J^t_{T_s}\Xi$, one has

$$|\widehat{\mu_s}(\xi)| = \left|\int e^{i\rho\omega\cdot\gamma(t)+i\rho\Xi\cdot E_s(\gamma(t),\gamma(t_o))}dt\right|. \tag{4.6}$$

By (4.4) and (4.5) there exists $\Phi \in \Omega^\perp$ so that for every k,

$$\left|\Xi^t J_{\frac{\partial T}{\partial s_k}}(x_o)\,\Phi\right| < \varepsilon, \tag{4.7}$$

i.e.,

$$\left|\frac{\partial\omega}{\partial s_k}\cdot\Phi\right| < \varepsilon. \tag{4.8}$$

Without loss of generality we can assume that $\gamma(t) = (t, \Gamma(t))$ with $\gamma'(t_0) = 0$ and that $\Phi = (1, 0, \ldots, 0)^t$. We claim that the Jacobian associated to the change of variables

$$\begin{aligned} \omega_2 &= \omega_2(s) \\ &\vdots \\ \omega_N &= \omega_N(s) \end{aligned} \tag{4.9}$$

is nonsingular. Indeed, let $\omega' = (\omega_2, \ldots, \omega_N)^t$ and assume that the vectors

$$\frac{\partial \omega'}{\partial s_1}, \ldots, \frac{\partial \omega'}{\partial s_{n-1}}$$

are linearly dependent. Then for suitable $(c_1, \ldots, c_{n-1}) \neq 0$ we have

$$c_1 \frac{\partial \omega'}{\partial s_1} + \cdots + c_{n-1} \frac{\partial \omega'}{\partial s_{n-1}} = (0, \ldots 0)^t.$$

Let

$$c_1 \frac{\partial \omega_1}{\partial s_1} + \cdots + c_{n-1} \frac{\partial \omega_1}{\partial s_{n-1}} = \alpha;$$

then

$$c_1 \frac{\partial \omega}{\partial s_1} + \cdots + c_{n-1} \frac{\partial \omega}{\partial s_{n-1}} = (\alpha, 0, \ldots, 0)^t.$$

Since the vectors $\frac{\partial \omega}{\partial s_1}, \ldots, \frac{\partial \omega}{\partial s_{n-1}}$ are linearly independent by the assumption (1.4), we have $\alpha \neq 0$ and therefore $\Phi = (1, 0, \ldots, 0)^t$ is linearly dependent of $\frac{\partial \omega}{\partial s_1}, \ldots, \frac{\partial \omega}{\partial s_{n-1}}$. This contradicts (4.8).

Also observe that, by (4.7) and (1.4),

$$\omega_1 = \omega \cdot \Phi = \Xi^t J_{T_s} \Phi$$

is bounded away from zero.

Let us consider the integral in (4.6). In order to integrate by parts in the t_1 variable, we observe that

$$\begin{aligned} &\frac{\partial}{\partial t_1} \left([\omega' t + \omega_n \Gamma(t)] + \Xi \cdot E_s(\gamma(t), \gamma(t_o)) \right) \\ &= \omega_1 + \omega_n \frac{\partial \Gamma}{\partial t_1} + \frac{\partial}{\partial t_1} \Xi \cdot E_s(\gamma(t), \gamma(t_o)). \end{aligned}$$

Since $\gamma'(t_0) = 0$, taking t in a sufficiently small neighborhood of t_0 we get that $\omega_n \frac{\partial \Gamma}{\partial t_1}$ is small. Moreover the same can be shown for the last term since γ is C^1 and T_s is smooth. This ensures that the above derivative is bounded away from zero. Hence,

$$\begin{aligned} &\int e^{i\rho\left[\omega' \cdot t + \omega_n \Gamma(t)\right] + i\rho \Xi \cdot E_s(\gamma(t), \gamma(t_o))} dt \\ &= \frac{1}{\rho} \int e^{i\rho\left[\omega' \cdot t + \omega_n \Gamma(t)\right] + i\rho \Xi \cdot E_s(\gamma(t), \gamma(t_o))} H(\omega', t)\, dt, \end{aligned}$$

where H is a bounded function smooth in the ω' variable. It follows that

$$\begin{aligned} I^2 &= \int \left|\widehat{\mu_s}(\xi)\right|^2 \Psi(s)\, ds \\ &= \frac{1}{\rho^2} \int \left| \int e^{i\rho\left[\omega'\cdot t + \omega_n \Gamma(t)\right] + i\rho \Xi \cdot E_s(\gamma(t),\gamma(t_o))} H(\omega', t)\, dt \right|^2 \Psi(s)\, ds. \end{aligned}$$

Performing the change of variables (4.9), we obtain

$$I^2 = \frac{1}{\rho^2} \int \left| \int e^{i\rho\left[\omega'\cdot t + \omega_n \Gamma(t)\right] + i\rho \Xi \cdot E_s(\gamma(t),\gamma(t_o))} H(\omega', t)\, dt \right|^2 \widetilde{\Psi}(\omega')\, d\omega'$$

and thus

$$I^2 = \frac{1}{\rho^2} \int \left| \int \int e^{iR\left[\omega'\cdot t + \omega_n \Gamma(t)\right] + iR \Xi \cdot E_s(\gamma(t),\gamma(t_o))} H(\omega', t)\, dt' dt_1 \right|^2 \widetilde{\Psi}(\omega')\, d\omega',$$

where $t' = (t_2, \ldots, t_{n-1})$. By the Minkowski integral inequality we can bound I by

$$\frac{1}{\rho} \int \left\{ \int \left| \int e^{i\rho\left[\omega''\cdot t' + \omega_n \Gamma(t)\right] + i\rho \Xi \cdot E_s(\gamma(t),\gamma(t_o))} H(\omega', t)\, dt' \right|^2 \widetilde{\Psi}(\omega')\, d\omega' \right\}^{1/2} dt_1.$$

Expanding and rewriting the term inside the brackets, we turn it into

$$\begin{aligned} &\int\int\int e^{i\rho\left[\omega''\cdot\left(t'-u'\right) + \omega_n\left(\Gamma\left(t_1,t'\right) - \Gamma\left(t_1,u'\right)\right)\right] + i\rho\Xi\cdot E_s(\gamma(t),\gamma(t_o)) - i\rho\Xi\cdot E_s\left(\gamma\left(t_1,u'\right),\gamma(t_o)\right)} \\ &\qquad \times H(\omega', t_1, t')\, H(\omega', t_1, u')\, \widetilde{\Psi}(\omega')\, d\omega' dt' du'. \end{aligned}$$

For fixed values of t_1, t', u', let k be such that

$$|t_k - u_k| > c\,|t-u|\,.$$

Then the derivative of the phase in ω_k is controlled by

$$\begin{aligned} &\left| t_k - u_k + \frac{\partial \omega_n}{\partial \omega_k} \nabla\Gamma \cdot (0, t'-u') + O(t-u) \max\left(|t-t_0|\,, |u-t_0|\right) \right| \\ &\geq \frac{1}{2}\,|t_k - u_k|\,. \end{aligned}$$

Therefore we can integrate by parts $n-2$ times in the above integral, and we get a term controlled by

$$\int\int \frac{1}{\left(1+\rho\,|t'-u'|\right)^{n-2}} dt' du' \lesssim \rho^{-(n-2)}.$$

Hence,

$$I \lesssim \rho^{-\frac{n}{2}},$$

which is better than we need. This finishes the proof of Proposition 4. □

Using Propositions 3 and 4 in the proof of [RT] yields Theorem 4.

§5. Nondegenerate generalized Radon transforms

Let X and Y be smooth manifolds, with cotangent bundles T^*X and T^*Y having zero sections 0, and let $C \subset (T^*X\backslash 0) \times (T^*Y\backslash 0)$ be a canonical relation. Then $I^r(X, Y; C)$ is the class of Fourier integral operators $F : \mathcal{E}'(Y) \longrightarrow \mathcal{D}'(X)$ of order $r \in \mathbb{R}$ associated with C. (We refer to [H1, H2] for the background material on Fourier integral operators.) L^2 estimates for Fourier integral operators associated with a canonical relation C depend on the structure of the projections $\pi_R : C \longrightarrow T^*Y$ and $\pi_L : C \longrightarrow T^*X$. Assume $dim X = dim Y + m$. The optimal L^2 estimates, with an operator $F \in I^{r-\frac{m}{4}}(X, Y; C)$ mapping $L^2_{\alpha,comp}(Y) \longrightarrow L^2_{\alpha-r,loc}(X)$, hold under the assumption that π_R is a submersion (which guarantees that π_L is an immersion), together with the weak additional requirement that the spatial projections $\pi_X : C \longrightarrow X$ and $\pi_Y : C \longrightarrow Y$ are submersions [H1, H2]; such canonical relations C are called *nondegenerate*. If we consider the special case of a generalized Radon transform R given by (1.6), we have $Y = \mathbb{R}^n$ and $X = \mathbb{R}^{n+m}$. One can embed R in an analytic family of operators by inserting the factor $|\theta|^{-z}$ into the oscillatory representation (1.7); then

$$R^z \in I^{-Re(z)-\frac{k}{2}-\frac{m}{4}}(\mathbb{R}^{n+m}, \mathbb{R}^n; C)$$

with $C = N^*Z'$, where $Z \subset \mathbb{R}^{n+m} \times \mathbb{R}^n$ is the incidence relation for R, as explained in the Introduction. Under the assumption that C is nondegenerate, we have

$$R^z : L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^{n+m}), \quad Re(z) = -\frac{k}{2}. \tag{5.1}$$

On the other hand, the Schwartz kernel of R^z is in L^∞ for $Re(z) = n - k$, so that

$$R^z : L^1(\mathbb{R}^n) \longrightarrow L^\infty(\mathbb{R}^{n+m}), \quad Re(z) = n - k. \tag{5.2}$$

The bounds in both (5.1) and (5.2) grow at most exponentially in $|Im(z)|$, and hence Theorem 5 follows by analytic interpolation.

Next, we make the connection between Theorems 4 and 5 by showing that, if the hypersurface γ in Theorem 4 is C^∞, then condition (1.4) implies that the associated canonical relation C is nondegenerate, so that in this case Theorem 4 becomes a special case of Theorem 5. Note that, by a simple calculation, (1.4) holds for the family $\{T_s\}$ iff it holds for the family $\{T_s^{-1}\}$, and for convenience we will work with the latter. The support of the Schwartz kernel of R is then

$$Z = \{(x, s, y) : x - y \in T_s^{-1}(\gamma)\}.$$

Letting $\nu(t)$ be a unit normal at $t \in \gamma$, we have

$$\begin{aligned} C =& N^*Z' \\ =& \{(x, s, \theta \cdot J^*_{T_s}\nu(t), \theta \cdot (\frac{\partial T_s}{\partial s})^*\nu(t); x - t, \theta \cdot J^*_{T_s}\nu(t)) \\ & \quad : x \in \mathbb{R}^n, s \in \mathbb{R}^m, t \in \gamma, \theta \in \mathbb{R}\backslash 0\} \\ =& \{(*, *; y, \theta \cdot J^*_{T_s}\nu(t)) : y \in \mathbb{R}^n, s \in \mathbb{R}^m, t \in \gamma, \theta \in \mathbb{R}\backslash 0\}, \end{aligned}$$

from which we see that $rank(D\pi_R) = n + rank(\frac{D(\eta)}{D(s,t,\theta)})$. Condition (1.4) then implies that $rank(\frac{D(\eta)}{D(\theta, s_1, \ldots, s_{n-1})}) = n$.

We also point out that Seeger [S] has obtained $L^p - L^q$ estimates for generalized Radon transforms, almost sharp in the finite-type setting in two dimensions. If C

is nondegenerate, one can see that the Z is of type (1,1) in the terminology of [S], and thus $R : L^p_{comp}(Y) \longrightarrow L^q_{loc}(X)$ for

$$(\frac{1}{p}, \frac{1}{q}) \in int\{hull((0,0),(1,1),(\frac{2}{3},\frac{1}{3}))\}.$$

However, the commutator approach of [S] is insensitive to the presence of more variables and therefore, in the particular context of Theorem 5, does not yield estimates outside of this set, regardless of the dimension.

It is possible to formulate geometric criteria under which the canonical relation C is nondegenerate. First consider the case of curves, $k = 1$. Write $\gamma_t(\cdot, s) = \gamma(\cdot, s; t)$, so that $\{\gamma_t\}$ is a one-parameter family of diffeomorphisms of $\mathbb{R}^n$ (parametrized by $s \in \mathbb{R}^m$); by a change of variables in $\mathbb{R}^n$, we may assume $\gamma_0 = Id$. As described in [GS, Eqn. 6.5] (see also [CNSW, §9.3]), we can parametrize C as

$$C = \{(\gamma_t^{-1}(y;s), s, (D_x\gamma)^*(\eta), (D_s\gamma)^*(\eta); y, \eta) : (s,y,t) \in \mathbb{R}^{m+n+1}, \eta \perp \Gamma(y,s;t)\},$$

where Γ is the (right) pullback of $\dot{\gamma}$ by the family of diffeomorphisms $\{\gamma_t\}$, namely

$$\Gamma(y,s;t) = \frac{d}{dt'}\big(\gamma_{t'+t} \circ \gamma_t^{-1}(y)\big)\big|_{t'=0}. \tag{5.3}$$

For each y, s, $\Gamma(y,s;\cdot) : \mathbb{R} \longrightarrow T_y\mathbb{R}^n$.

Example 1. If $\gamma(x,s;t) = x + \gamma^0(s;t)$ is a translation-invariant family, then $\Gamma(y,s;t) = \dot{\gamma}^0(s;t)$ is just the velocity vector of the curve at time t.

Example 2. If, as in [CNSW, §9.1], we prescribe a variable family of curves via a Taylor expansion in t and the exponential map (and allow s-dependence),

$$\gamma(x,s;t) = exp_x(tX(s) + t^2Y(s) + \cdots),$$

where $X(s), Y(s), \ldots$ are vector fields on $\mathbb{R}^n$ depending on $s \in \mathbb{R}^k$, then, as calculated in [GS, §6.4],

$$\Gamma(y,s;t) = X(s) + 2tY(s) + \cdots, \tag{5.4}$$

which is enough to determine whether C is nondegenerate.

If we work locally in x, s and t, so that the first component $\dot{\gamma}_1 \neq 0$, then $\Gamma_1 \neq 0$ as well (for $|t|$ small). Writing $\eta = (\eta_1, \eta')$, etc., we may then solve $\eta \perp \Gamma(y,s;t)$ for η_1 in terms of $\eta' \in \mathbb{R}^{n-1}\backslash 0$ and write the projection $\pi_R : C \longrightarrow T^*\mathbb{R}^n$ as

$$\pi_R(s,y,t,\eta') = (y, -\frac{\Gamma'(y,s;t)\cdot\eta'}{\Gamma_1(y,s;t)}, \eta'). \tag{5.5}$$

The canonical relation C is nondegenerate if $\pi_R : C \longrightarrow T^*\mathbb{R}^n$ is a submersion, which also implies that $\pi_L : C \longrightarrow T^*(\mathbb{R}^n \times \mathbb{R}^k)$ is an immersion. Thus, we have

Theorem 7. *If $\gamma(x,s;t)$ is a C^∞ family of curves in $\mathbb{R}^n$ such that*

$$\mathbb{R}^{k+1} \ni (s,t) \longrightarrow \frac{\Gamma'(y,s;t)\cdot\eta'}{\Gamma_1(y,s;t)} \text{ has no critical points } \forall \eta' \in \mathbb{R}^{n-1}\backslash 0, \tag{5.6}$$

then

$$R : L^{\frac{2n-1}{n}}(\mathbb{R}^n) \longrightarrow L^{\frac{2n-1}{n-1}}(\mathbb{R}^n \times \mathbb{R}^m). \tag{5.7}$$

Condition (5.6) can be restated as a maximal rank condition on an $(m+1)\times(n-1)$ matrix:

$$rank\begin{bmatrix}\Gamma_1 \cdot D_s\Gamma' - D_s\Gamma_1 \otimes \Gamma' \\ \Gamma_1 \cdot \partial_t\Gamma' - \partial_t\Gamma_1 \otimes \Gamma'\end{bmatrix} = n-1.$$

Thus, a necessary condition for C to be nondegenerate is that $m \geq n-2$.

For the translation-invariant Example 1 above, we may write $\gamma^0(s;t)=(t,g(s;t))$, where $g:\mathbb{R}^{m+1}\longrightarrow\mathbb{R}^{n-1}$; then $\Gamma(y,s;t)=\dot{\gamma}^0=(1,\dot{g}(s;t))$, so that (5.6) becomes

$$rank\begin{bmatrix}D_s\dot{g} \\ \ddot{g}\end{bmatrix} = n-1. \tag{5.8}$$

For $n=2, m=0$ (i.e., no s parameter) we need $\ddot{g}\neq 0$ as in the result of Littman [L] and Strichartz [Str], while for $n=2, m=1$, (5.8) becomes $\ddot{g}\neq 0$ or $\partial_s\dot{g}\neq 0$, which includes the result of [RT] in the smooth setting. In $\mathbb{R}^3$, we need at least $m=1$, and then (5.8) becomes $\ddot{g}\wedge\partial_s\dot{g}\neq 0$, i.e., $\{\dot{g},\ddot{g},\partial_s\dot{g}\}$ linearly independent. If the family $\gamma^0(s;\cdot)$ arises from rotation of an initial curve $\gamma_0(\cdot)$ about an axis $\mathbb{R}\cdot v$, then we need $\dot{\gamma}_0\wedge v\neq 0, \ddot{\gamma}_0\cdot v\neq 0$. For example, convolution with the rotations of $\gamma_0(t)=(t,t^2,0)$ about the x_2 axis in $\mathbb{R}^3$ already maps $L^{\frac{5}{3}}(\mathbb{R}^3)\longrightarrow L^{\frac{5}{2}}(\mathbb{R}^3\times S^1)$.

Theorem 3 for curves ($k=1$) follows from Theorem 7, since we may take $s\in\mathbb{R}^{\frac{n(n-1)}{2}}$ to be local coordinates on $SO(n)$ and (5.8) holds; essentially this says that $SO(n)$ acts transitively on the sphere.

If one wants to formulate the results in terms of averages over m-dimensional families of k-surfaces in $\mathbb{R}^n$, then only a few changes are necessary. Starting with a C^∞ map

$$\gamma:\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^k\longrightarrow\mathbb{R}^n,\quad \gamma(x,s;0)=x,\quad D_t\gamma \text{ an injection},$$

the resulting generalized Radon transform belongs to $I^{-\frac{k}{2}-\frac{m}{4}}(\mathbb{R}^{n+m},\mathbb{R}^n;C)$. To describe the canonical relation C, we use the pullback

$$\Gamma(y,s;t)=D_{t'}(\gamma_{t+t'}\circ\gamma_t^{-1})|_{t'=0},$$

which is a map $\Gamma:\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^k\longrightarrow\mathbb{R}^{k^*}\otimes T_y\mathbb{R}^n$. We can assume that, with $x=(x',x'')\in\mathbb{R}^k\times\mathbb{R}^{n-k}$, etc., we have that $D_{x'}\gamma'$ is nonsingular, and thus Γ' is nonsingular for $|t|$ small. Condition (5.6) is then replaced by

$$rank(D_{s,t}(({\Gamma'}^*)^{-1}(\Gamma'')(\eta''))) = k,\quad \forall\eta''\in\mathbb{R}^{n-k}\backslash 0. \tag{5.9}$$

Under this assumption, C is nondegenerate. Again, specializing to the translation-invariant case and letting $s\in\mathbb{R}^{\frac{n(n-1)}{2}}$ be local coordinates on $SO(n)$, it is not hard to see that (5.9) is satisfied for any smooth initial k-surface, and thus Theorem 3 follows.

References

[BHI] L. Brandolini, S. Hoffman and A. Iosevich, *Sharp rate of average decay of the Fourier transform of a bounded set*, Geom. Func. Anal. **13** (2003), 671–680. MR2006553 (2004g:42015)

[BRT] L. Brandolini, M. Rigoli and G. Travaglini, *Average decay of Fourier transforms and geometry of convex sets*, Rev. Mat. Iberoamericana **14** (1998), 519–560. MR1681584 (2000a:42017)

[Bu] M. Burak-Erdoğan, *Mixed norm estimates for a restricted X-ray transform in $\mathbb{R}^4$ and $\mathbb{R}^5$*, Internat. Math. Res. Notices (2001), 575–600. MR1836731 (2002i:44003)

[BuC] M. Burak-Erdoğan and M. Christ, *Mixed norm estimates for a restricted X-ray transform*, J. Anal. Math. **87** (2002), 187–198. MR1945281 (2003k:42030)

[C1] M. Christ, *Estimates for the k-plane transform*, Indiana Univ. Math. Jour. **33** (1984), 891–910. MR0763948 (86k:44004)

[C2] ———, *Convolution, curvature and combinatorics: a case study*, Internat. Math. Res. Notices (19) (1998), 1033–1048. MR1654767 (2000a:42026)

[CNSW] M. Christ,A. Nagel, E. Stein and S. Wainger, *Singular and maximal Radon transforms: analysis and geometry.*, Ann. of Math. **150** (1999), 489-577. MR1726701 (2000j:42023)

[D] S. Dury, *Generalizations of Riesz potentials and L^p estimates for certain k-plane transforms*, Illinois Jour. Math. **28** (1984), 495–512. MR0748958 (85h:44004)

[GS] A. Greenleaf and A. Seeger, *Oscillatory and Fourier Integral operators with degenerate canonical relations*, Publicacions Matemátiques, Proc. of the 6th International Conf. on Harmonic Analysis and PDE (2002), 93-141. MR1964817 (2004b:42027)

[GSW] A. Greenleaf, A. Seeger and S. Wainger, *On X-ray transforms for rigid line complexes and integrals over curves in $\mathbb{R}^4$*, Proc. Amer. Math. Soc. **127** (1999), 3533-3545. MR1670367 (2001a:44002)

[Gu] V. Guillemin, *On some results of Gelfand in integral geometry*, Proc. Symp. Pure Math. **43** (1985), 149–155. MR0812288 (87d:58137)

[GuSt] V. Guillemin and S. Sternberg, *Geometric asymptotics*, AMS, Providence, 1977. MR0516965 (58:24404)

[H1] L. Hörmander, *Fourier integral operators I*, Acta Math. **127** (1971), 79–183. MR0388463 (52:9299)

[H2] ———, *The analysis of linear partial differential operators, IV*, Springer-Verlag, New York, Berlin, 1985. MR0781537 (87d:35002b)

[L] W. Littman, *$L^p - L^q$ estimates for singular integral operators arising from hyperbolic equations*, Proc. Symp. Pure Math. **23** (1973), 479-481. MR0358443 (50:10909)

[M] B. Marshall, *Decay rates of Fourier transforms of curves*, Trans. Amer. Math. Soc. **310** (1988), 115–126. MR0948194 (89m:42012)

[O] D. Oberlin, *Convolution estimates for some measures on curves*, Proc. Amer. Math. Soc. **99** (1987), 56–60. MR0866429 (88f:42033)

[P] A.N. Podkorytov, *The asymptotic of a Fourier transform of a convex curve*, Vest. Leningr. Univ. Mat. **24** (1991), 57–65. MR1166380 (93h:42019)

[PhS] D. H. Phong and E.M. Stein, *Models of degenerate Fourier integral operators and Radon transforms*, Ann. Math. **140** (1994), 703–722. MR1307901 (96c:35206)

[RT] F. Ricci and G. Travaglini, *Convex curves, Radon transforms and convolution operators defined by singular measures*, Proc. A.M.S. **129** (2001), 1739–1744. MR1814105 (2002i:42010)

[S] A. Seeger, *Radon transforms and finite type conditions*, J. Amer. Math. Soc. **11** (1998), 869–897. MR1623430 (99f:58202)

[Str] R. Strichartz, *Convolutions with kernels having singularities on a sphere*, Trans. Amer. Math. Soc. **148** (1970), 461–471. MR0256219 (41:876)

[TaW] T. Tao and J. Wright, *L^p improving bounds for averages along curves*, Jour. Amer. Math. Soc. **16** (2003), 605–638. MR1969206 (2004j:42005)

[V] A. Varchenko, *Number of lattice points in families of homothetic domains in $\mathbb{R}^n$*, Funk. An. App. **17** (1983), 1–6. MR0705041 (85e:11077)

Dipartimento di Ingegneria Gestionale e dell'Informazione, Università degli Studi di Bergamo, V.le G Marconi 5, 24044 Dalmine, Italy
E-mail address: brandolini@unibg.it

Department of Mathematics, University of Rochester, Rochester, New York 14627
E-mail address: allan@math.rochester.edu

Dipartimento di Statistica, Università di Milano-Bicocca, Edificio U7, Via Bicocca degli Arcimboldi 8, 20126 Milano, Italy
E-mail address: giancarlo.travaglini@unimib.it