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SHARP SOBOLEV INEQUALITIES IN THE PRESENCE OF A TWIST

STEPHANE COLLION, EMMANUEL HEBEY, AND MICHEL VAUGON

ABSTRACT. Let (M,g) be a smooth compact Riemannian manifold of dimension $n \geq 3$. Let also A be a smooth symmetrical positive $(0,2)$-tensor field in M. By the Sobolev embedding theorem, we can write that there exist $K, B > 0$ such that for any $u \in H_1^2(M)$,

$$\left(\int_M |u|^{2^\star} dv_g\right)^{2/2^\star} \leq K \int_M A_x(\nabla u, \nabla u) dv_g + B \int_M u^2 dv_g$$

where $H_1^2(M)$ is the standard Sobolev space of functions in L^2 with one derivative in L^2. We investigate in this paper the value of the sharp K in the equation above, the validity of the corresponding sharp inequality, and the existence of extremal functions for the saturated version of the sharp inequality.

Let (M,g) be a smooth compact Riemannian manifold of dimension $n \geq 3$. Also let A be a smooth symmetrical $(0,2)$-tensor field in M. In a local chart, $A = (A^{ij})$, $i,j = 1,\dots,n$. We assume that A is positive when acting on 1-forms in the sense that for any $x \in M$, and any η in the cotangent space $T_x(M)^\star$, $A_x = A(x)$ is such that $A_x(\eta,\eta) > 0$ if $\eta \not\equiv 0$. Then, by the Sobolev embedding theorem, we can write that there exist $K, B > 0$ such that for any $u \in H_1^2(M)$,

$$\left(\int_M |u|^{2^\star} dv_g\right)^{2/2^\star} \leq K \int_M A_x(\nabla u, \nabla u) dv_g + B \int_M u^2 dv_g \tag{0.1}$$

where $\nabla u = (\partial_i u)$ is the 1-form consisting (in local charts) of the first derivatives of u, dv_g is the Riemannian volume element of g, and $H_1^2(M)$ is the standard Sobolev space consisting of functions in L^2 with one derivative in L^2. Clearly, the sharp constant B in (0.1) is $V_g^{-2/n}$, where V_g is the volume of (M,g), and the corresponding sharp inequality holds true since it holds true for the classical Sobolev inequality [and $|\nabla u|^2$ is controled by $A_x(\nabla u, \nabla u)$]. On the other hand, as is easily understood by the fact that A charges some parts of the space M more than others, it is expected that A will affect the sharp constant K in (0.1). Note (0.1) is associated to the operator $\Delta_A^g u = -\text{div}_g(A_x \nabla u)$ which appears in several places in mathematical and physics literature.

The questions we ask in this note are: what is the value $K_s = K_s(g)$ of the sharp K in (0.1), does the corresponding sharp inequality hold true, and if yes, does its saturated version (where B is lowered to its minimum value under the constraint $K = K_s$) possess extremal functions? When $A = g^{-1}$, we are back to the classical problem (dealing with the classical Sobolev inequality). Possible references in book

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form for the classical problem are Druet and Hebey [10], and Hebey [14]. When A degenerates, the nature of (0.1) changes and we are led to inequalities studied such as in the very nice Beckner [2] where sharp inequalities involving the degenerate Grushin [12] operator are proved to hold.

When dealing with the general (0.1), in order to answer the above questions, we need to introduce some definitions. We define $A_\sharp$, $A_\sharp = (A_{ij})$ in a local chart to be the smooth symmetrical $(2,0)$-tensor field obtained from A by the g-musical isomorphism, so that $A_{ij} = A^{\alpha\beta} g_{\alpha i} g_{\beta j}$. Then we define the *twist function* K_T of A and g by

$$K_T(x) = \sqrt{\frac{|A_\sharp(x)|}{|g(x)|}} \tag{0.2}$$

where, in a local chart at x, $|A_\sharp(x)|$ stands for the determinant of the matrix $(A_{ij}(x))$, and $|g(x)|$ stands for the determinant of the matrix $(g_{ij}(x))$. Let Ag be the $(1,1)$-tensor field obtained by contracting one index of A with one index of g so that, in a local chart, $(Ag)_i^j = A^{i\alpha} g_{\alpha j}$. For any $x \in M$, $(Ag)_x = Ag(x)$ defines an isomorphism $\Phi(x)$ of $T_x(M)$ by $(\Phi(x).X)^i = (Ag_x)^i_\alpha X^\alpha$. Then, another (more intrinsic) equation for K is that $K_T(x) = \sqrt{|Ag_x|}$, where $|Ag_x|$ is the determinant of $\Phi(x)$. We also define the *twisted metric* $\hat{g}$ by

$$\hat{g} = K_T^{\frac{2}{n-2}} \tilde{g}, \tag{0.3}$$

where $\tilde{g}$ is the Riemannian metric in M such that $\tilde{g}^{-1} = A$. In local coordinates the matrix consisting of the components $\tilde{g}_{ij}$ of $\tilde{g}$ is the inverse of the matrix (A^{ij}) consisting of the components of A, so that $A^{i\alpha}\tilde{g}_{\alpha j} = \delta^i_j$ at any point and for all i, j. We let K_n be the sharp constant for the Euclidean Sobolev inequality $\|u\|_{2^\star} \le K_n \|\nabla u\|_2$. Then, as is well known (see for instance Hebey [14]),

$$K_n = \sqrt{\frac{4}{n(n-2)\omega_n^{2/n}}}, \tag{0.4}$$

where ω_n is the volume of the standard n-dimensional sphere. Our result states as follows.

Theorem 0.1. *Let (M, g) be a smooth compact Riemannian manifold of dimension $n \ge 3$, and let $A = (A^{ij})$ be a smooth positive symmetrical $(0,2)$-tensor field in M. The value $K_s(g)$ of the sharp constant K in (0.1) is $K_s(g) = K_n^2 / \sqrt[n/2]{\min K_T}$, where $\min K_T = \min_{x\in M} K_T(x)$, K_T is the twist function of A and g given by (0.2), and K_n is given by (0.4). Moreover, there exists $B > 0$ such that for any $u \in H_1^2(M)$ the sharp inequality*

$$\left(\int_M |u|^{2^\star} dv_g\right)^{2/2^\star} \le \frac{K_n^2}{\sqrt[n/2]{\min K_T}} \int_M A_x(\nabla u, \nabla u) dv_g + B \int_M u^2 dv_g \tag{0.5}$$

holds true. If $B_0(g)$ stands for the lowest B in (0.5), then $B_0(g) \ge V_g^{-2/n}$ and, if $n \ge 4$, we also have that

$$\frac{4(n-1)\Lambda^{2/(n-2)}}{(n-2)K_s(g)} B_0(g) \ge \max_{x \in MinK_T} \left[S_{\hat{g}}(x) + \frac{n-4}{n-2} \frac{\Delta_{\hat{g}} K_T(x)}{K_T(x)} \right], \tag{0.6}$$

where $MinK_T$ is the subset of M consisting of the x in M which are such that K_T is minimum at x, $\Lambda = 1/\min K_T$, $\hat{g}$ is the twisted metric given by (0.3), $\Delta_{\hat{g}} = -div_{\hat{g}}\nabla$

is the Laplacian associated to $\hat{g}$*, and* $S_{\hat{g}}$ *is the scalar curvature of* $\hat{g}$*. At last, if the inequality in* (0.6) *is strict, then the sharp saturated inequality*

$$\left(\int_M |u|^{2^\star} dv_g\right)^{2/2^\star} \leq \frac{K_n^2}{\sqrt[n]{\min K_T}} \int_M A_x(\nabla u, \nabla u) dv_g + B_0(g) \int_M u^2 dv_g \tag{0.7}$$

possesses extremal functions, namely nontrivial (smooth positive) functions which realize the equality in (0.7)*.*

When $A = g^{-1}$, we are back to the classical Sobolev inequality. The validity of the classical sharp inequality on arbitrary manifolds was proved in Hebey and Vaugon [15]. The existence of extremal functions for the classical sharp inequality (and the above result when $A = g^{-1}$) was studied in Djadli and Druet [6]. Possible references in book form on the sharp classical Sobolev inequality are Druet and Hebey [10], and Hebey [14]. Extensions of the notions of weakly critical and critical functions (introduced in Hebey and Vaugon [16]) to inequalities like (0.1) are studied in Collion [4]. Results for 3-dimensional manifolds, in the spirit of those obtained by Druet [7, 8], are also available in Collion [4]. When $n = 3$, equations like (0.6) have to be replaced by an equation like $M_A(x) \leq 0$ for all $x \in \mathrm{MinK}_T$, where $M_A(x)$ is the mass of a suitably chosen Shrödinger operator $\Delta_{\hat{g}} + h$, and the existence of extremal functions follows from equations like $M_A(x) < 0$ for all $x \in \mathrm{MinK}_T$. Developments on the notions of weakly critical and critical functions may also be found in the papers Humbert and Vaugon [17], and Robert [19].

1. Proof of Theorem 0.1

We prove Theorem 0.1 in this section. As a preliminary remark, let $A = (A^{ij})$ be a positive symmetrical $(0, 2)$-tensor in $\mathbb{R}^n$. If δ stands for the Euclidean metric, and u is smooth, define $\Delta_A u = -\mathrm{div}_\delta(A\nabla u)$ so that $\Delta_A u = -A^{ij}\partial_{ij}u$. Also define $\Phi = 1/\sqrt{A}$ to be a $(1, 1)$-tensor (Φ is not unique) such that $A\Phi^2 = \delta^{-1}$ in the sense that $A^{\alpha\beta}\Phi^i_\alpha\Phi^j_\beta = \delta^{ij}$. We regard Φ as the isomorphism of $\mathbb{R}^n$ given by $(\Phi x)^i = \Phi^i_\alpha x^\alpha$, and if u is a smooth function in $\mathbb{R}^n$, we define u_A by the equation $u_A(x) = u(\Phi x)$. Then u_A is a solution of $\Delta_A u_A = u_A^{2^\star - 1}$ in $\mathbb{R}^n$ if and only if u is a solution of $\Delta u = u^{2^\star - 1}$ in $\mathbb{R}^n$, where Δ is the Euclidean Laplacian. In particular, by the results of Caffarelli-Gidas-Spruck [3], and also Obata [18], u_A is a (positive) solution in $\mathbb{R}^n$ of $\Delta_A u_A = u_A^{2^\star - 1}$ if and only if

$$u_A(x) = \left(\frac{\lambda}{1 + \frac{\lambda^2|\Phi x - a|^2}{n(n-2)}}\right)^{\frac{n-2}{2}} \tag{1.1}$$

for some $\lambda > 0$ and $a \in \mathbb{R}^n$. Let $A\delta$ be the isomorphism of $\mathbb{R}^n$ we get from A by lowering one index with the δ-musical isomorphism. Then, $|\Phi| = 1/\sqrt{|A\delta|}$, where $|A\delta|$ and $|\Phi|$ stand for the determinants of $A\delta$ and Φ, and we can check that the sharp homogeneous Euclidean inequality with respect to A reads as

$$\left(\int_{\mathbb{R}^n} |u|^{2^\star} dx\right)^{2/2^\star} \leq \frac{K_n^2}{\sqrt[n]{|A\delta|}} \int_{\mathbb{R}^n} A(\nabla u, \nabla u) dx \tag{1.2}$$

where K_n, as in (0.4), is the sharp constant for the classical homogeneous Euclidean Sobolev inequality $\|u\|_{2^\star} \leq K_n\|\nabla u\|_2$. Moreover, as for the classical case where $A = \delta^{-1}$, extremal functions for (1.2) and positive solutions of the critical equation

$\Delta_A u = u^{2^\star-1}$ are the same. Following the arguments in Hebey [14] (Proposition 4.2), it follows from (1.2) that for any compact Riemannian manifold (M,g), and any B, constants K in (0.1) are such that $K \ge K_n^2/\sqrt[n/2]{\min K_T}$. A closely related remark is the following: for (M,g) a smooth (compact) Riemannian manifold, and $A=(A^{ij})$ a smooth symmetrical $(0,2)$-tensor field in M, let $\Delta_A^g = -\mathrm{div}_g(A(x)\nabla)$, where div_g is the divergence with respect to g. Then

$$\Delta_A^g u = K_T^{\frac{2}{n-2}}\Delta_{\hat g}u \tag{1.3}$$

for all smooth functions u in M, where K_T is the twist function of A and g given by (0.2), $\hat g$ is the twist metric given by (0.3), and $\Delta_{\hat g} = -\mathrm{div}_{\hat g}\nabla$ is the Laplacian with respect to $\hat g$. Equation (1.3) holds true since

$$\int_M A_x(\nabla u,\nabla u)dv_g = \int_M |\nabla u|_{\hat g}^2 dv_{\hat g}$$

for all $u\in H_1^2(M)$, where $|\cdot|_{\hat g}$ is the norm with respect to $\hat g$. Let f_T be the function given by the equation $f_T^{(n-2)/2}K_T = 1$, and let h be a smooth function in M. Noting that

$$\int_M \left(A_x(\nabla u,\nabla u)+hu^2\right)dv_g = \int_M \left(|\nabla u|_{\hat g}^2+\hat h u^2\right)dv_{\hat g} \tag{1.4}$$

and that

$$\int_M |u|^{2^\star}dv_g = \int_M f_T|u|^{2^\star}dv_{\hat g} \tag{1.5}$$

for all $u\in H_1^2(M)$, where $\hat h = f_T h$, it follows from the result in Hebey and Vaugon [15] that we apply to the $\hat g$-metric and that there exists $B>0$ such that

$$\left(\int_M |u|^{2^\star}dv_g\right)^{2/2^\star} \le \left(\max_M f_T\right)^{2/2^\star}\int_M A_x(\nabla u,\nabla u)dv_g + B\int_M u^2dv_g$$

for all $u\in H_1^2(M)$. In particular, $K_s(g) = K_n^2/\sqrt[n/2]{\min K_T}$ is the sharp constant K in (0.1), and the sharp inequality (0.5) holds true on any compact Riemannian manifold. Equation (0.6) in Theorem 0.1 follows from (1.4), (1.5), and Aubin [1]. Then we are left with the proof that the saturated inequality (0.7) possesses extremal functions if the inequality in (0.6) is strict. By the definition of $B_0(g)$, for any $0<\alpha<B_0(g)$ there exist $u_\alpha\in C^\infty(M)$, $u_\alpha>0$, and $\lambda_\alpha\in\left(0,K_s(g)^{-1}\right)$ such that

$$\Delta_A^g u_\alpha + \frac{\alpha}{K_s(g)}u_\alpha = \lambda_\alpha u_\alpha^{2^\star-1} \tag{1.6}$$

and $\int_M u_\alpha^{2^\star}dv_g = 1$, where $\Delta_A^g = -\mathrm{div}_g(A(x)\nabla)$. The u_α's are bounded in $H_1^2(M)$. Up to a subsequence, $u_\alpha \rightharpoonup u$ weakly in $H_1^2(M)$. If $u\not\equiv 0$, then u is an extremal function for (0.7). By contradiction we assume that $u\equiv 0$ so that, in particular, $\|u_\alpha\|_\infty \to +\infty$ and $\lambda_\alpha \to K_s(g)^{-1}$ as $\alpha\to B_0(g)$. We define an A-bubble by the $\hat g$-extension of equation (1.1) to sequences of functions. Namely we define an A-bubble as a sequence (B_α) of functions given by the equations

$$B_\alpha(x) = \left(\frac{\mu_\alpha}{\mu_\alpha^2+\frac{d_{\hat g}(x_\alpha,x)^2}{n(n-2)}}\right)^{\frac{n-2}{2}}, \tag{1.7}$$

where (x_α) is a convergent sequence of points in M, and (μ_α) is a sequence of positive real numbers such that $\mu_\alpha \to 0$ as $\alpha \to B_0(g)$. In what follows we let the x_α's and μ_α's be given by the equations

$$\begin{aligned} & u_\alpha(x_\alpha) = \|u_\alpha\|_\infty \,, \\ & \mu_\alpha^{-(n-2)/2} = \frac{\sqrt{\Lambda}}{K_s(g)^{(n-2)/4}} \|u_\alpha\|_\infty \,, \end{aligned} \tag{1.8}$$

where $\Lambda = (\min K_T)^{-1}$ is as in Theorem 0.1. Up to a subsequence, the x_α's converge. We let x_0 be their limit. Then we must have that $x_0 \in \mathrm{MinK}_T$. We let also G be the Green's function of the operator $\Delta_A^g + K_s(g)^{-1}B_0(g)$ (or, equivalently, the Green's function of $\Delta_{\hat{g}} + K_s(g)^{-1}B_0(g)f_T$), and we define Φ to be the positive and continuous function in $M \times M$ given by

$$\Phi(x,y) = (n-2)\omega_{n-1} d_{\hat{g}}(x,y)^{n-2} G(x,y)$$

if $x \neq y$, and $\Phi(x,y) = 1$ if $x = y$. Following the arguments developed in Druet, Hebey and Robert [11] (see Chapter 5, where minimum energy is discussed), we can write that

$$\frac{\sqrt{\Lambda} u_\alpha}{K_s(g)^{(n-2)/4}} = \Big(\Phi(x_0, \cdot) + o(1)\Big) B_\alpha, \tag{1.9}$$

where $o(1) \to 0$ in $C^0(M)$ as $\alpha \to B_0(g)$, (B_α) is given by (1.7), the x_α's and μ_α's are given by (1.8), and x_0 and Λ are as above. In particular, it follows from (1.9) that

$$\lim_{\alpha \to B_0(g)} \frac{\int_{B_{x_0}(\delta)} u_\alpha^2 dv_g}{\int_M u_\alpha^2 dv_g} = 1 \tag{1.10}$$

for all $\delta > 0$ when $n \geq 4$, but that (1.10) stops being true when $n = 3$. By the local isoperimetric inequality in Druet [9], and the coarea formula, we can write that for any $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that for any smooth function u with compact support in $B_{x_0}(\delta_\varepsilon)$,

$$\left(\int_M |u|^{2^\star} dv_{\hat{g}}\right)^{2/2^\star} \leq K_n^2 \int_M |\nabla u|_{\hat{g}}^2 dv_{\hat{g}} + B_\varepsilon \int_M u^2 dv_{\hat{g}} \tag{1.11}$$

where $B_\varepsilon = \frac{n-2}{4(n-1)} K_n^2 \left(S_{\hat{g}}(x_0) + \varepsilon\right)$. We fix $\varepsilon > 0$, and let η be a smooth cutoff function such that $\eta = 1$ in $B_{x_0}(\delta_\varepsilon/4)$, $\eta = 0$ in $M \backslash B_{x_0}(\delta_\varepsilon/2)$, and $0 \leq \eta \leq 1$. We plug ηu_α into (1.11). By (1.6), but also (1.3) and (1.10), we get that when $n \geq 4$,

$$\begin{aligned} & \left(B_\varepsilon - \frac{B_0(g)}{K_s(g)} K_n^2 f_T(x_0)\right) \int_M u_\alpha^2 dv_{\hat{g}} + o\left(\int_M u_\alpha^2 dv_{\hat{g}}\right) \\ & \geq \left(\int_M (\eta u_\alpha)^{2^\star} dv_{\hat{g}}\right)^{2/2^\star} - \left(\frac{1}{\max f_T}\right)^{\frac{n-2}{n}} \int_M \eta^2 f_T u_\alpha^{2^\star} dv_{\hat{g}} \,. \end{aligned} \tag{1.12}$$

By Hölder's inequality, writing that $f_T \leq (\max f_T)^{(n-2)/n} f_T^{2/n}$, the right hand side in (1.12) is nonnegative. Choosing $\varepsilon > 0$ sufficiently small, we get a contradiction if the first term in (1.12) is negative. In particular we get a contradiction if $n = 4$ and the inequality in (0.6) is strict, or if $n > 4$, the inequality in (0.6) is strict, and $\Delta_{\hat{g}} K_T(x) = 0$ for all $x \in \mathrm{MinK}_T$. We assume in what follows that $n \geq 5$. We let Λ_α be the right hand side in (1.12). Writing that $f_T = f_T^{(n-2)/n} f_T^{2/n}$, and that $(1+x)^p = 1 + (p + o(1))\, x$, we get by Hölder's inequality that

$$\Lambda_\alpha \geq \left(\frac{2}{2^\star} + o(1)\right)(\max f_T)^{1-\frac{2}{2^\star}} \int_M |h_T|(\eta u_\alpha)^{2^\star} dv_{\hat{g}}, \tag{1.13}$$

where $h_T = \frac{f_T}{\max f_T} - 1$ (so that, in particular, $h_T \leq 0$). By (1.9),

$$\begin{aligned} &\int_M |h_T|(\eta u_\alpha)^{2^\star} dv_{\hat{g}} \\ &= (1+\varepsilon_\delta)\left(\frac{K_s(g)^{(n-2)/4}}{\sqrt{\Lambda}}\right)^{2^\star} \int_{B_{x_\alpha}(\delta)} |h_T| B_\alpha^{2^\star} dv_{\hat{g}} + o\left(\int_M u_\alpha^2 dv_{\hat{g}}\right), \end{aligned} \tag{1.14}$$

where $\varepsilon_\delta \to 0$ as $\delta \to 0$. By the expansion of h_T at x_α in geodesic normal coordinates, by (1.9) and (1.10), and also by Lemma 7 in Demengel and Hebey [5], we can write that

$$\begin{aligned} &\int_{B_{x_\alpha}(\delta)} h_T B_\alpha^{2^\star} dv_{\hat{g}} = h_T(x_\alpha) \int_{B_{x_\alpha}(\delta)} B_\alpha^{2^\star} dv_{\hat{g}} \\ &- \frac{n(n-4)\Lambda}{8(n-1)K_s(g)^{(n-2)/2}} \left(\Delta_{\hat{g}} h_T(x_0)\right) \int_M u_\alpha^2 dv_{\hat{g}} + \varepsilon_\delta^\alpha \int_M u_\alpha^2 dv_{\hat{g}}, \end{aligned} \tag{1.15}$$

where $\lim_{\delta\to 0} \limsup_{\alpha \to B_0(g)} |\varepsilon_\delta^\alpha| = 0$. Plugging (1.13)–(1.15) into (1.12), and recalling Λ_α in (1.13) is the right hand side of (1.12), we get that

$$\begin{aligned} &\left(\frac{4(n-1)\Lambda^{2/(n-2)}}{(n-2)K_s(g)} B_0(g) - S_{\hat{g}}(x_0) - \frac{n-4}{n-2} \frac{\Delta_{\hat{g}} K_T(x_0)}{K_T(x_0)}\right) \int_M u_\alpha^2 dv_{\hat{g}} \\ &\leq C_1\Big(\varepsilon + o(1)\Big) \int_M u_\alpha^2 dv_{\hat{g}} + C_2 h_T(x_\alpha), \end{aligned} \tag{1.16}$$

where $C_1, C_2 > 0$ do not depend on α. In particular, since $h_T(x_\alpha) \leq 0$, if the inequality in (1.12) is strict, then we get a contradiction with (1.16) by choosing $\varepsilon > 0$ sufficiently small. This proves Theorem 0.1.

If we assume that the points in MinK_T are nondegenerate critical points for K_T, then $|h_T(x_\alpha)| \geq C d_{\hat{g}}(x_\alpha, x_0)^2$, where $C > 0$ does not depend on α. In particular, if blow-up occurs, then we get with (1.16) (see also Collion [4] and Hebey [13]) that

$$d_g(x_0, x_\alpha) = o(\mu_\alpha) \tag{1.17}$$

when $n \geq 5$, where, as above, $x_0 \in \mathrm{MinK}_T$ is the limit of the x_α's.

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150 bis rue Legendre, 75017 Paris, France
E-mail address: Stephane.Collion@wanadoo.fr

Département de Mathématiques, Université de Cergy-Pontoise, Site de Saint-Martin, 2 avenue Adolphe Chauvin, 95302 Cergy-Pontoise cedex, France
E-mail address: Emmanuel.Hebey@math.u-cergy.fr

Département de Mathématiques, Université Pierre et Marie Curie, 4 place Jussieu, 75252 Paris cedex 05, France
E-mail address: vaugon@math.jussieu.fr