TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 359, Number 6, June 2007, Pages 2899–2913
S 0002-9947(07)04113-X
Article electronically published on January 4, 2007

HYPERINVARIANT SUBSPACES FOR SOME SUBNORMAL OPERATORS

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Abstract. In this article we employ a technique originated by Enflo in 1998 and later modified by the authors to study the hyperinvariant subspace problem for subnormal operators. We show that every "normalized" subnormal operator S such that either $\{(S^{*n}S^n)^{1/n}\}$ does not converge in the SOT to the identity operator or $\{(S^nS^{*n})^{1/n}\}$ does not converge in the SOT to zero has a nontrivial hyperinvariant subspace.

1. Introduction

In this article $\mathcal{H}$ will always denote a separable, infinite-dimensional, complex Hilbert space and $\mathcal{L}(\mathcal{H})$ the algebra of all bounded linear operators on $\mathcal{H}$. For T in $\mathcal{L}(\mathcal{H})$ we write $\sigma(T)$ for the spectrum, $\sigma_p(T)$ for the point spectrum, and $\{T\}' = \{X \in \mathcal{L}(\mathcal{H}) : XT = TX\}$ for the commutant of T. Recall that a nontrivial hyperinvariant subspace (n.h.s.) for T is a subspace $\mathcal{M}$ of $\mathcal{H}$ such that $(0) \neq \mathcal{M} \neq \mathcal{H}$ and $X\mathcal{M} \subset \mathcal{M}$ for every $X \in \{T\}'$ and also that an operator S in $\mathcal{L}(\mathcal{H})$ is called *subnormal* if there exists a complex Hilbert space $\mathcal{K} \supset \mathcal{H}$ and a normal operator N in $\mathcal{L}(\mathcal{K})$ such that $N\mathcal{H} \subset \mathcal{H}$ and $N|_{\mathcal{H}} = S$. It is well known that every subnormal operator S in $\mathcal{L}(\mathcal{H})$ has a unique (up to unitary equivalence) *minimal* normal extension in $\mathcal{L}(\mathcal{K})$, which we consistently denote by N_S or N in what follows. Recall that a subnormal operator $S \in \mathcal{L}(\mathcal{H})$ is called *pure* if it has no nonzero reducing subspace $\mathcal{M}$ such that $S|_{\mathcal{M}}$ is a normal operator. The question of whether every subnormal operator in $\mathcal{L}(\mathcal{H})\backslash\mathbb{C}1_{\mathcal{H}}$ (where $\mathbb{C}$, as usual, denotes the complex field) has an n.h.s. remains open, despite the beautiful work of S. Brown ([3]) and J. Thomson ([12], [13]) concerning the existence of invariant subspaces and bounded point evaluations associated with subnormal operators. One of the best known results of a general nature concerning n.h.s. for subnormal operators is the following, which is a result in [12] (with a little help from [5, Theorem 1.4]). As usual, if K is a compact set in $\mathbb{C}$, we write $\mathrm{Rat}(K)$ for the algebra of all rational functions with poles off K.

Theorem 1.1 (Thomson)**.** *If S is either a nonpure subnormal operator in $\mathcal{L}(\mathcal{H})\backslash\mathbb{C}1_{\mathcal{H}}$ or a pure subnormal operator in $\mathcal{L}(\mathcal{H})$ for which there exists a nonzero vector x_0 in $\mathcal{H}$ such that the subspace $\{r(S)x_0 : r \in \mathrm{Rat}(\sigma(S))\}^-$ reduces S, then S has an n.h.s.*

Received by the editors January 18, 2005 and, in revised form, June 24, 2005.
2000 *Mathematics Subject Classification.* Primary 47A15, 47B20.
Key words and phrases. Subnormal operators, hyperinvariant subspaces, spectral measures.

The purpose of this article is to contribute to the solution of the n.h.s. problem for subnormal operators by utilizing a technique initiated by Enflo in [1] and modified by the authors in [8], [9] and [10] to show, after making some preliminary normalizations, that either one of some obvious candidates is an n.h.s. for an arbitrary (normalized) subnormal operator or that operator has some (perhaps surprising) structure (Theorems 4.10 and 5.6).

2. Some preparatory material

In this section we establish some results that will be needed in Sections 4 and 5.

Definition 2.1. If Λ is a compact subset of the complex plane $\mathbb{C}$, we write $\partial_u(\Lambda)$ for the boundary of the unbounded component of $\mathbb{C}\backslash\Lambda$, and we say that a subnormal operator S in $\mathcal{L}(\mathcal{H})$ is in *spectral general position* if

a) S is pure,
b) $\sigma_p(S) = \sigma_p(S^*) = \varnothing$,
c) $\sigma(S)$ is connected,
d) $0 \in \partial_u(\sigma(S))$,
e) $\|S\| = 1$, and
f) $\sigma_p(N_S^* N_S) = \varnothing$.

It follows easily that if a subnormal operator S in $\mathcal{L}(\mathcal{H})$ is in spectral general position, then $0 \in \partial_u(\sigma(N_S))$, the set $\sigma(N_S)$ is connected, and for every $n \in \mathbb{N}$, $\sigma(N_S^{*n} N_S^n) = [0,1]$, $\|N_S^n\| = 1$, and $\sigma_p(N_S^{*n} N_S^n) = \varnothing$.

Proposition 2.2. *If S is a subnormal operator in spectral general position, then S is a completely nonunitary contraction in the class C_{00} (i.e., both sequences $\{S^n\}_{n\in\mathbb{N}}$ and $\{S^{*n}\}_{n\in\mathbb{N}}$ converge to $0_{\mathcal{H}}$ in the strong operator topology* (SOT)*).*

Proof. Since $\|S\| = 1$ and S is pure, obviously S is a completely nonunitary contraction. Note that

$$\|S^{*n}x\|^2 \leq \|S^n x\|^2 = \langle S^{*n} S^n x, x\rangle = \langle (N_S^* N_S)^n x, x\rangle\,, \qquad x \in \mathcal{H}.$$

Since $\sigma(N_S^* N_S) = [0,1]$ and $\sigma_p(N_S^* N_S) = \varnothing$, it is an easy consequence of the spectral theorem that the sequence $\{(N_S^* N_S)^n\}_{n\in\mathbb{N}}$ is monotonically decreasing to $0_{\mathcal{H}}$ in the SOT, and the result follows. □

We begin our program with the following geometric lemma that plays a role in what follows.

Lemma 2.3. *Let Λ be any nonempty perfect set in $\mathbb{C}$, and let μ be a finite Borel measure supported on some compact subset of $\mathbb{C}$ such that μ vanishes on singletons. Then there exists a countable set $\Lambda_\omega \subset \Lambda$ such that for every $\lambda_0 \in \Lambda\backslash\Lambda_\omega$ and for every $r \geq 0$, the circle*

$$C_r(\lambda_0) = \{\zeta \in \mathbb{C} : |\zeta - \lambda_0| = r\}$$

satisfies $\mu(C_r(\lambda_0)) = 0$.

Proof. The basic idea used in the proof is the obvious fact that no such finite measure μ has the property that there is a countably infinite disjoint family of Borel sets in $\mathbb{C}$ each of which has measure greater than some fixed positive number. Now fix an arbitrary $n \in \mathbb{N}$, and let S_n denote the set of those circles

$$C_r(\lambda) = \{\zeta \in \mathbb{C} : |\zeta - \lambda| = r\}, \quad \lambda \in \Lambda, \quad r > 0,$$

such that $\mu(C_r(\lambda)) > 1/n$. We will show that $S_0 = \bigcup_{n\in\mathbb{N}} S_n$ is a countable collection of circles. Since $\mu(\mathbb{C}\backslash \bigcup_n S_n)$ is obviously zero and Λ is uncountable, this will show that if Λ_ω is the collection of centers of the circles in S_0, then $\Lambda_\omega \subset \Lambda$ is countable, and if $\lambda_0 \in \Lambda\backslash\Lambda_\omega$, then $\mu(C_r(\lambda_0)) = 0$ for every $r > 0$. To this end, we now show that each S_n is a finite set. If not, then there is some $n_0 \in \mathbb{N}$ such that $\operatorname{card} S_{n_0} \geq \aleph_0$, and we enumerate some countably infinite subset $\widetilde{S}_{n_0} \subset S_{n_0}$ as

$$\widetilde{S}_{n_0} = \{C_{r_i}(\lambda_i)\}_{i\in\mathbb{N}}.$$

Clearly any two distinct circles in $\widetilde{S}_{n_0}$ can intersect in at most two points, so discarding all of these points of intersection only removes a countable set Ω from $\bigcup_{i\in\mathbb{N}} C_{r_i}(\lambda_i)$, and $\mu(\Omega) = 0$ since μ vanishes on singletons. This means that the collection $\{C_{r_i}(\lambda_i)\backslash\Omega\}_{i\in\mathbb{N}}$ is disjoint and each set therein has measure at least $1/n_0$, contradicting the finiteness of μ. Thus, for each $n \in \mathbb{N}$, the collection S_n of circles is finite, as desired. □

The above lemma is useful here in that it allows us to obtain the following.

Proposition 2.4. *If every subnormal operator S in $\mathcal{L}(\mathcal{H})$ in spectral general position has an n.h.s., then every subnormal operator in $\mathcal{L}(\mathcal{H})\backslash\mathbb{C}1_{\mathcal{H}}$ has an n.h.s. (In other words, when looking for an n.h.s. for a subnormal operator S, no generality is lost by assuming that S is in spectral general position, and, in particular (Proposition 2.2), that S is a c.n.u. contraction in the class C_{00}.)*

Proof. Let S be an arbitrary subnormal operator in $\mathcal{L}(\mathcal{H})\backslash\mathbb{C}1_{\mathcal{H}}$. If S is not pure, then by [5, Theorem 1.4] S has an n.h.s., so, without loss of generality, we may suppose that S is pure. Also we may obviously suppose that $\sigma_p(S) = \sigma_p(S^*) = \varnothing$ and that $\sigma(S)$ is connected. Moreover, since S has an n.h.s. if and only if for some (every) $\lambda \in \mathbb{C}$, $S - \lambda 1_{\mathcal{H}}$ has an n.h.s., we may translate S by any desirable λ_0 that yields d) and f) of Definition 2.1. Next write $\Lambda = \partial_u(\sigma(S)) = \partial_u(\sigma(N_S))$, which is obviously closed, and observe that since $\sigma_p(S) = \varnothing$, Λ has no isolated points and thus is perfect. Write

$$N_S = \int_{\sigma(N_S)} \lambda \, dE,$$

and let μ be a scalar spectral measure for N_S, so μ and E are mutually absolutely continuous. Since the minimality of N_S (together with $\sigma_p(S^*) = \varnothing$) guarantees that $\sigma_p(N_S) = \varnothing$, μ vanishes on singletons. We now apply Lemma 2.3 to the set Λ and the measure μ to obtain a point $\lambda_0 \in \Lambda\backslash\Lambda_\omega$ such that $\mu(C_r(\lambda_0)) = 0 = E(C_r(\lambda_0))$ for all $r \geq 0$. Finally, suppose that there exists y_0 in $\mathcal{K}$ (where $N_S \in \mathcal{L}(\mathcal{K})$) and $r_0 \geq 0$ such that

$$0 = \|((N_S{}^* - \bar{\lambda}_0)(N_S - \lambda_0) - r_0)y_0\|^2 = \int_{\sigma(N_S)} \left||\lambda - \lambda_0|^2 - r_0\right|^2 \, dE_{y_0,y_0},$$

which implies that the measure E_{y_0,y_0} defined, as usual, by

$$E_{y_0,y_0}(\mathcal{B}) = \langle E(\mathcal{B})y_0, y_0\rangle$$

for every Borel set $\mathcal{B} \subset \mathbb{C}$, satisfies $E_{y_0,y_0}(\mathbb{C}\backslash C_{r_0}(\lambda_0)) = 0$ and therefore that

$$E_{y_0,y_0}(C_{r_0}(\lambda_0)) = \|y_0\|^2 .$$

But from above we know that $E(C_{r_0}(\lambda_0))$ is the zero projection, so $y_0 = 0$, and we conclude that the operator $(N_S{}^* - \bar{\lambda}_0 1_{\mathcal{H}})(N_S - \lambda_0 1_{\mathcal{H}})$ has empty point spectrum and that $0 \in \partial_u(S - \lambda_0 1_{\mathcal{H}})$. Upon replacing S by $\gamma(S - \lambda_0 1_{\mathcal{H}})$ with $\gamma = 1/\|S - \lambda_0 1_{\mathcal{H}}\|$, we obtain a pure subnormal operator in spectral general position that has an n.h.s. if and only if S does, as desired. □

3. The main construction

In this section we outline a construction from [10] which (under appropriate supplementary hypotheses) produces nontrivial hyperinvariant subspaces for an operator T in $\mathcal{L}(\mathcal{H})$. The main idea is to employ the sequence of spectral measures associated with the sequences $\{T^n T^{*n}\}_{n\in\mathbb{N}}$ and $\{T^{*n}T^n\}_{n\in\mathbb{N}}$ in a certain fashion. This construction has a certain generality, and we hope that it will be found useful elsewhere (which to some extent has already happened; cf. [6]). As usual, $\mathbb{N}[\mathbb{N}_0]$ will denote the set of positive [nonnegative] integers.

We begin the sketch of our construction by fixing an arbitrary operator $T \in \mathcal{L}(\mathcal{H})\backslash\mathbb{C}1_{\mathcal{H}}$ with dense range. For $n \in \mathbb{N}$, let $E^{(n)}$ be the spectral measure associated with the operator T^nT^{*n}, so

$$T^nT^{*n} = \int_{[0,\|T^n\|^2]} \lambda \; dE^{(n)}(\lambda), \qquad n \in \mathbb{N}.$$

Define

$$E_\lambda^{(n)} = E^{(n)}([0,\lambda]),\; E_{\lambda^-}^{(n)} = E^{(n)}([0,\lambda)), \quad 0 \le \lambda \le \|T^n\|^2, \quad n \in \mathbb{N},$$

and observe that all of the functions $\lambda \to E_\lambda^{(n)}$ and $\lambda \to E_{\lambda^-}^{(n)}$ are monotone increasing. Moreover, since spectral measures are inner and outer regular, it follows easily that all of the functions $\lambda \to \langle E_\lambda^{(n)}x_0, x_0\rangle$ $[\lambda \to \langle E_{\lambda^-}^{(n)}x_0, x_0\rangle]$ are continuous from the right [left]. Next fix an arbitrary $0 < \theta < 1$ and an arbitrary unit vector x_0 in $\mathcal{H}$ (which one may wish to specify later to good advantage) and define

$$\lambda_n = \lambda_n(T,\theta,x_0) = \min\{\lambda \in [0,\|T^n\|^2] : \|E_\lambda^{(n)}x_0\| \ge \theta\}, \quad n \in \mathbb{N}. \tag{1}$$

Then $\lambda_n > 0$ for $n \in \mathbb{N}$ (since $\ker(T^{*n}) = (0)$) and the space

$$\mathcal{M}_n := (1_{\mathcal{H}} - E_{\lambda_n^-}^{(n)})\mathcal{H} = E^{(n)}([\lambda_n, \|T^n\|^2])\mathcal{H}, \quad n \in \mathbb{N},$$

is a reducing (spectral) subspace for T^nT^{*n} such that $T^nT^{*n}|_{\mathcal{M}_n}$ is an invertible operator. Thus there exists a unique $x_n \in \mathcal{M}_n$ such that

$$T^n{T^*}^n x_n = (1_{\mathcal{H}} - E_{\lambda_n^-}^{(n)})x_0, \qquad n \in \mathbb{N}; \tag{2}$$

namely,

$$x_n = \left(\int_{[\lambda_n,\|T^n\|^2]} (1/\lambda)\; dE^{(n)}(\lambda)\right) x_0, \qquad n \in \mathbb{N}. \tag{3}$$

Next we define

$$y_n = {T^*}^n x_n, \qquad z_n = E_{\lambda_n}^{(n)} x_0, \qquad n \in \mathbb{N}, \tag{4}$$

which yields, via (2),

$$T^n y_n = E^{(n)}([\lambda_{n,}\, \|T^n\|^2])x_0, \quad n \in \mathbb{N}.$$

The following lemma, which is an easy consequence of (1)-(4), relates these sequences.

Lemma 3.1. *With the notation as above, for an arbitrary* $X \in \{T\}^{'}$ *and for each* $n \in \mathbb{N}$, *we have*

a) $\|x_0 - T^n y_n\| = \|E^{(n)}_{\lambda_n^-} x_0\| = \langle E^{(n)}_{\lambda_n^-} x_0, x_0\rangle^{1/2} \leq \theta < 1$,

b) $\langle z_n, x_0\rangle = \langle E^{(n)}_{\lambda_n} x_0, x_0\rangle \geq \theta^2$, *and*

c) $|\langle XT^n y_n, z_n\rangle| = |\langle X y_n, T^{*n} z_n\rangle| \leq \|X\|\,\|y_n\|\|T^{*n} z_n\|$.

We next briefly indicate the utility of (a slight variant of) this construction (in which $\|T^{*n} z_n\|$ is replaced by $\|T^{*n+1} z_n\|$) in the quasinilpotent case treated in [10]. Observe first from a) that since x_0 is a unit vector, the sequence $\{T^n y_n\}_{n\in\mathbb{N}}$ is bounded and no subsequence can converge weakly to zero. Moreover b) ensures that the (obviously bounded) sequence $\{z_n\}_{n\in\mathbb{N}}$ has no subsequence weakly convergent to zero. Finally, in case $\sigma(T) = \{0\}$, one knows (cf. [10]) that some subsequence of the (modified) sequence on the right-hand side of c) tends to zero. Thus by dropping down to successive subsequences one can arrange (in the quasinilpotent case) that $\{T^{n_j} y_{n_j}\}_{j\in\mathbb{N}}$ converges weakly to an $s_0 \neq 0$, $\{z_{n_j}\}_{j\in\mathbb{N}}$ converges weakly to $z_0 \neq 0$, and the sequence $\{\langle XT^{n_j} y_{n_j}, z_{n_j}\rangle\}_{j\in\mathbb{N}}$ converges to zero. One then needs some hypothesis or device (usually implying that one of the two weakly convergent sequences above is norm convergent) that ensures that this last sequence converges to $\langle Xs_0, z_0\rangle$. Then, since X is arbitrary in $\{T\}^{'}$, the subspace $(\{T\}^{'} s_0)^-$ is an n.h.s. for T.

To illustrate another (although somewhat banal) context in which the above construction can be used to produce an n.h.s., we now briefly treat the case in which the subnormal operator T under consideration is normal.

Proposition 3.2. *Suppose* N *is a normal operator such that* $\sigma(N^*N) = [0,1]$ *and* $\sigma_p(N^*N) = \varnothing$. *With the definitions and notation as above, let* s_0 *be the (weak) limit of some weakly convergent subsequence of* $\{N^n y_n\}_{n\in\mathbb{N}}$. *Then* $(\{N\}^{'} s_0)^-$ *is an n.h.s. for* N.

Proof. Let $\{n_k\}_{k\in\mathbb{N}} \subset \mathbb{N}$ be a sequence such that $\{N^{n_k} y_{n_k}\}_{k\in\mathbb{N}}$ converges weakly to s_0. It follows immediately from a) of Lemma 3.1 that $s_0 \neq 0$. Moreover, if $X \in \{N\}^{'}$ is arbitrary, then from c) of Lemma 3.1 we obtain

$$
\begin{aligned}
|\langle XN^n y_n, z_n\rangle|^2 &\leq \|X\|^2 \|y_n\|^2 \,\|N^{*n} z_n\|^2 \\
&= \|X\|^2 \langle N^n N^{*n} x_n, x_n\rangle \langle N^n N^{*n} z_n, z_n\rangle \\
&= \|X\|^2 \left(\int_{[\lambda_n,1]} (1/\lambda)\, dE^{(n)}_{x_0,x_0}\right)\left(\int_{[0,\lambda_n]} \lambda\, dE^{(n)}_{x_0,x_0}\right).
\end{aligned} \tag{5}
$$

Moreover, since N is normal, we have $N^n N^{*n} = (NN^*)^n$ for $n \in \mathbb{N}$, and it is an easy consequence of this and the definitions above that

$$E^{(n)}_{\lambda_n} = E^{(1)}_{(\lambda_n)^{1/n}} = E^{(1)}_{\lambda_1},\ n \in \mathbb{N}, \tag{6}$$

so, in particular, $\lambda_n = \lambda_1^n$ and $z_n = z_1$ for all $n \in \mathbb{N}$. Thus, rewriting the right-hand side of (5) by using (6), we obtain

(7)
$$\begin{aligned}|\langle XN^n y_n, z_1\rangle|^2 &\le \|X\|^2 \left(\int_{[\lambda_1,1]} (1/\lambda^n)\ dE^{(1)}_{x_0,x_0}\right)\left(\int_{[0,\lambda_1]} \lambda^n\ dE^{(1)}_{x_0,x_0}\right)\\ &= \|X\|^2 \left(\int_{[\lambda_1,1]} (\lambda_1/\lambda)^n\ dE^{(1)}_{x_0,x_0}\right)\left(\int_{[0,\lambda_1]} (\lambda/\lambda_1)^n\ dE^{(1)}_{x_0,x_0}\right).\end{aligned}$$

By the Lebesgue dominated convergence theorem, the sequence on the right-hand side of (7) converges to $\|X\|^2 E^{(1)}_{x_0,x_0}(\{\lambda_1\})$ which is zero since $\sigma_p(NN^*) = \varnothing$. By Lemma 3.1b) we have $z_1 \neq 0$, and thus we obtain from (7) that $\langle Xs_0, z_1\rangle = 0$ for each $X \in \{N\}'$, which is the desired conclusion. $\square$

4. Hyperinvariant subspaces: Subnormal operators

In this section we employ the construction of Section 3 to obtain some information about hyperinvariant subspaces for subnormal operators. The set-up, which will remain fixed throughout the remainder of the paper, is as follows.

We begin with a subnormal operator $S \in \mathcal{L}(\mathcal{H})$ in spectral general position (so $\sigma_p(S) = \sigma_p(S^*) = \varnothing$, $0 \in \partial_u(\sigma(S))$, $\sigma(S)$ is connected, and $\|S\| = 1$), with minimal normal extension $N = N_S$ in $\mathcal{L}(\mathcal{K})$, so

$$\mathcal{K} = \bigvee_{n\in\mathbb{N}_0} (N^*)^n\, \mathcal{H},$$

and $\sigma_p(N^{*n}N^n) = \varnothing$ for $n \in \mathbb{N}$. (Note also that $\sigma(N) = \sigma(S)$, $\|N\| = 1$ and $\sigma(N^{*n}N^n) = [0,1]$ for $n \in \mathbb{N}$.) We write $P = P^* = P^2$ for the projection in $\mathcal{L}(\mathcal{K})$ whose range is $\mathcal{H}$, and

(8) $$S^{*n}S^n = \int_{[0,1]} \lambda\, dE^{(n)}(\lambda), \qquad n \in \mathbb{N},$$

so $E^{(n)}$ is the spectral measure of $S^{*n}S^n$. Furthermore we write

(9) $$(S^{*n}S^n)^{1/n} = \int_{[0,1]} \nu\, dG^{(n)}(\nu), \qquad n \in \mathbb{N}.$$

Once again there is an obvious relation between the sequences $\{E^{(n)}\}_{n\in\mathbb{N}}$ and $\{G^{(n)}\}_{n\in\mathbb{N}}$, which yields, in particular,

(10) $$G^{(n)}_\lambda = E^{(n)}_{\lambda^n}, \qquad G^{(n)}_{\lambda^-} = E^{(n)}_{(\lambda^n)^-}, \qquad \lambda \in [0,1], \quad n \in \mathbb{N},$$

where these functions are defined and have the same properties as in Section 3. Finally we write

(11) $$(N^*N)^n = N^{*n}N^n = \int_{[0,1]} \mu\, dF^{(n)}(\mu) = \int_{[0,1]} \mu^n\, dF^{(1)}(\mu),$$

and again it follows easily that

(12) $$F^{(n)}_{\mu^n} = F^{(1)}_\mu, \qquad F^{(n)}_{(\mu^n)^-} = F^{(1)}_{\mu^-}, \qquad \mu \in [0,1],\ n \in \mathbb{N}.$$

Note also the relation

(13) $$(PN^{*n}N^n)|_{\mathcal{H}} = S^{*n}S^n, \qquad n \in \mathbb{N},$$

which follows from an elementary calculation and implies that

$$\tag{14} \begin{aligned} \langle (N^*N)^n x, x\rangle &= \int_{[0,1]} \mu^n \, dF^{(1)}_{x,x}(\mu) = \int_{[0,1]} \mu \, dF^{(n)}_{x,x}(\mu) \\ &= \int_{[0,1]} \lambda \, dE^{(n)}_{x,x}(\lambda) = \int_{[0,1]} \nu^n \, dG^{(n)}_{x,x}(\nu), \quad x \in \mathcal{H}. \end{aligned}$$

Next, we define

$$\mathcal{H}_\mu = \mathcal{H} \cap F^{(1)}_\mu \mathcal{K}, \qquad \mu \in (0,1),$$

and we show that each $\mathcal{H}_\mu$ is a (perhaps trivial) hyperinvariant subspace for S. To this end, fix $\mu \in (0,1)$, $x \in \mathcal{H}_\mu$, and let $X \in \{S\}'$. Since $Xx \in \mathcal{H}$, to show that $Xx \in \mathcal{H}_\mu$ it suffices to show that $Xx \in F^{(1)}([0,\mu])\mathcal{K}$, and by virtue of the known characterization of spectral subspaces, this is accomplished by the following computation, which uses (14) and the polar decompositions $N^{2n} = U_n(N^*N)^n$, where $n \in \mathbb{N}$ and each U_n is unitary:

$$\begin{aligned} \|(N^*N)^n Xx\| &= \|N^{2n}Xx\| = \left\|S^{2n}Xx\right\| \\ &= \left\|XN^{2n}x\right\| \le \|X\| \, \|(N^*N)^n x\| \\ &\le \|X\| \, \|x\| \, \mu^n, \quad n \in \mathbb{N}. \end{aligned}$$

This shows that the sequence $\{\|((N^*N)/\mu)^n Xx\|\}_{n\in\mathbb{N}}$ is bounded and thus (cf. [11, p. 66]) that

$$\|((N^*N)/\mu)^n Xx\| \le \|Xx\|, \qquad n \in \mathbb{N},$$

which, in turn, shows that $Xx \in F^{(1)}_\mu \mathcal{K}$. Thus, for every $\mu \in (0,1)$, $\mathcal{H}_\mu$ is either (0), $\mathcal{H}$, or an n.h.s. for S. But if $\mathcal{H}_\mu = \mathcal{H}$ for some $\mu \in (0,1)$, then

$$\|Sx\|^2 = \|Nx\|^2 = \langle N^*Nx, x\rangle \le \mu \, \|x\|^2, \qquad x \in \mathcal{H},$$

which is impossible since $\|S\| = 1$. Thus we have proved the following.

Proposition 4.1. *For each $\mu \in (0,1)$, $\mathcal{H}_\mu = \mathcal{H} \cap F^{(1)}_\mu \mathcal{K}$ is either (0) or an n.h.s. for S.*

This proposition has an interesting corollary.

Corollary 4.2. *If $S \in \mathcal{L}(\mathcal{H})$ is a subnormal operator in spectral general position and S has a nonzero invariant subspace $\widetilde{\mathcal{H}} \subset \mathcal{H}$ such that $\|S|_{\widetilde{\mathcal{H}}}\| < 1$, then S has an n.h.s.*

Proof. By virtue of Proposition 4.1, it suffices to show that if we set $\mu_0^{1/2} = \|S|_{\widetilde{\mathcal{H}}}\|$ (< 1), then $\mathcal{H}_{\mu_0} \ne (0)$. Let $w \in \widetilde{\mathcal{H}}$. Then

$$\|(N^*N)^n w\| = \|N^{2n}w\| = \|S^{2n}w\| = \|(S|_{\widetilde{\mathcal{H}}})^{2n} w\| \le \mu_0^n \|w\|,$$

which shows, as in the proof of Proposition 4.1, that $w \in F^{(1)}_{\mu_0} \cap \mathcal{H} = \mathcal{H}_{\mu_0}$, and hence $\mathcal{H}_{\mu_0} \supset \widetilde{\mathcal{H}} \ne (0)$. □

Now we proceed with the general construction as sketched in Section 3 (with S replacing T^*) and using (8)-(14). In other words, we let x_0 be an arbitrary unit vector in $\mathcal{H}$ and θ an arbitrary real number satisfying $0 < \theta < 1$, and we define

$$\lambda_n = \lambda_n(\theta, x_0) = \min\{\lambda : \langle E^{(n)}_\lambda x_0, x_0\rangle^{1/2} \ge \theta\}, \quad n \in \mathbb{N},$$

$$\mu_n = \mu_n(\theta, x_0) = \min\{\mu : \langle F^{(n)}_\mu x_0, x_0\rangle^{1/2} \ge \theta\}, \quad n \in \mathbb{N},$$

and

$$\nu_n = \nu_n(\theta, x_0) = \min\{\nu : \langle G_\nu^{(n)} x_0, x_0\rangle^{1/2} \geq \theta\}, \qquad n \in \mathbb{N}. \tag{15}$$

From (10) we get that $\lambda_n = (\nu_n)^n$ for every $n \in \mathbb{N}$. Next, proceeding as in Section 3, we define

$$\begin{aligned} x_n &= \left(\int_{[\lambda_n,1]} (1/\lambda)\ dE^{(n)}(\lambda)\right) x_0 \\ &= \left(\int_{[\nu_n,1]} (1/\nu^n)\ dG^{(n)}(\nu)\right) x_0, \qquad n \in \mathbb{N}, \\ y_n &= S^n x_n, \quad z_n = E_{\lambda_n}^{(n)} x_0, \qquad n \in \mathbb{N}, \end{aligned}$$

which yields

$$S^{*n} y_n = S^{*n} S^n x_n = E^{(n)}([\lambda_n, 1]) x_0.$$

Then, by Lemma 3.1 we have that

$$\|x_0 - S^{*n} y_n\| \leq \theta < 1, \quad \langle z_n, x_0\rangle \geq \theta^2, \qquad n \in \mathbb{N}.$$

Moreover, if $X^* \in \{S^*\}'$, then from c) of Lemma 3.1 we obtain, for each $n \in \mathbb{N}$,

$$\begin{aligned} |\langle X^* S^{*n} y_n, z_n\rangle|^2 &\leq \|X\|^2 \|y_n\|^2 \|S^n z_n\|^2 \\ &= \|X\|^2 \langle S^{*n} S^n x_n, x_n\rangle \langle S^{*n} S^n z_n, z_n\rangle \\ &= \|X\|^2 \left(\int_{[\lambda_n,1]} (1/\lambda)\ dE_{x_0,x_0}^{(n)}\right)\left(\int_{[0,\lambda_n]} \lambda\ dE_{x_0,x_0}^{(n)}\right) \\ &= \|X\|^2 \left(\int_{[\nu_n,1]} (\nu_n/\nu)^n\, dG_{x_0,x_0}^{(n)}\right)\left(\int_{[0,\nu_n]} (\nu/\nu_n)^n\, dG_{x_0,x_0}^{(n)}\right). \end{aligned} \tag{16}$$

But note that (16) differs from (7) (after the necessary changes in notation) because in (7) there is only one measure in play, whereas in (16) the sequence of measures $\{G_{x_0,x_0}^{(n)}\}_{n\in\mathbb{N}}$ is involved.

To continue with the general construction, we need now the following computational lemmas.

Lemma 4.3. *Let $x \in \mathcal{H}$, $0 < \nu_0 < 1$, and $r > 1$ be such that $r\nu_0 \leq 1$. Then*

$$\|(1_{\mathcal{K}} - F_{r\nu_0}^{(1)}) G_{\nu_0}^{(n)} x\|^2 \leq (1/r^n)\|G_{\nu_0}^{(n)} x\|^2 \leq (1/r^n)\, \|x\|^2, \qquad n \in \mathbb{N}, \tag{17}$$

and consequently

$$\|(1_{\mathcal{K}} - F_{r\nu_0}^{(1)}) G_{\nu_0}^{(n)}\|^2 \leq 1/r^n, \qquad n \in \mathbb{N}. \tag{18}$$

Proof. Write $y_0 = y_0(n, x, \nu_0) = G^{(n)}_{\nu_0} x$. Then we have

$$
\begin{aligned}
(r\nu_0)^n \langle (1_{\mathcal{K}} - F^{(1)}_{r\nu_0}) G^{(n)}_{\nu_0} x, G^{(n)}_{\nu_0} x\rangle &= \int_{[r\nu_0,1]} (r\nu_0)^n \; dF^{(1)}_{y_0,y_0}(\mu) \\
&\le \int_{[r\nu_0,1]} \mu^n \; dF^{(1)}_{y_0,y_0}(\mu) \\
&\le \langle N^{*n} N^n y_0, y_0\rangle = \langle S^{*n} S^n y_0, y_0\rangle \\
&= \int_{[0,\nu_0]} \nu^n \; dG^{(n)}_{y_0,y_0} \le \nu_0^n \left\|y_0\right\|^2 \\
&= \nu_0^n \|G^{(n)}_{\nu_0} x\|^2 \le \nu_0^n \left\|x\right\|^2, \qquad n \in \mathbb{N},
\end{aligned} \tag{19}
$$

and dividing each side of (19) by $(r\nu_0)^n$ yields (17). Then taking the supremum over $\{x \in \mathcal{H} : \|x\| \le 1\}$ of each side of (17) yields (18). □

Lemma 4.4. *If $\mathcal{H}_\mu = (0)$ (i.e., $F^{(1)}_\mu \mathcal{K} \cap \mathcal{H} = (0)$) for every $\mu \in (0, 1)$, then the sequence $\{\nu_n\}_{n\in\mathbb{N}} = \{\nu_n(x_0, \theta)\}_{n\in\mathbb{N}}$ from* (15) *satisfies $\lim_n \nu_n = 1$ independent of the choices of the (originally given) unit vector x_0 and the θ satisfying $0 < \theta < 1$.*

Proof. Suppose, to the contrary, that for some x_0 in $\mathcal{H}$ with $\|x_0\| = 1$ and θ satisfying $0 < \theta < 1$, we have $\liminf_n \nu_n = \nu_\infty < 1$. Choose a subsequence $\{\nu_{n_j}\}_{j\in\mathbb{N}}$ converging to ν_∞. By dropping down to a further subsequence (but without changing the notation) we may suppose that the sequence $\{G^{(n_j)}_{\nu_{n_j}}\}_{j\in\mathbb{N}}$ of projections is WOT-convergent, say to G_∞. Since

$$\theta^2 \le \langle G^{(n_j)}_{\nu_{n_j}} x_0, x_0\rangle, \qquad j \in \mathbb{N},$$

we have $\theta^2 \le \langle G_\infty x_0, x_0\rangle$, so $G_\infty x_0 \ne 0$. Next let $\widetilde{\nu} \in (\nu_\infty, 1)$. Then there exists $J_0 \in \mathbb{N}$ such that for $j \ge J_0$, $\nu_{n_j} < (\widetilde{\nu} + \nu_\infty)/2$. Setting $\nu_0 = \nu_{n_j}$, $r = \widetilde{\nu}/\nu_{n_j}$ and $n = n_j$ in (18), we obtain

$$\|(1_{\mathcal{K}} - F^{(1)}_{\widetilde{\nu}}) G^{(n_j)}_{\nu_{n_j}}\|^2 \le \left(\frac{\nu_{n_j}}{\widetilde{\nu}}\right)^{n_j} \le \left(\frac{\nu_\infty + \widetilde{\nu}}{2\widetilde{\nu}}\right)^{n_j}, \quad j \ge J_0, \tag{20}$$

and letting $j \to \infty$ in (20) we obtain $G_\infty = F^{(1)}_{\widetilde{\nu}} G_\infty$. But this implies that

$$0 \ne G_\infty x_0 = F^{(1)}_{\widetilde{\nu}} G_\infty x_0 \in \mathcal{H}_{\widetilde{\nu}},$$

contrary to our hypothesis. Thus $\nu_\infty = \lim_n \nu_n = 1$, as desired. □

Corollary 4.5. *Under the hypotheses of Lemma* 4.4*, for every $\nu_0 \in (0, 1)$, the sequence $\{G^{(n)}_{\nu_0}\}_{n\in\mathbb{N}}$ of projections is* SOT*-convergent to* 0.

Proof. By Lemma 4.4, for every $0 < \theta < 1$ and every unit vector x_0 in $\mathcal{H}$, the sequence $\{\nu_n = \nu_n(\theta, x_0)\}_{n\in\mathbb{N}}$ from (15) converges to 1. Thus with $\theta = \varepsilon$ and for n sufficiently large, $\nu_n > \nu_0$ and therefore $\|G^{(n)}_{\nu_0} x_0\| < \varepsilon$, from which the result follows easily. □

We can now prove the following.

Proposition 4.6. *If S is a subnormal operator in spectral general position, and for every $\mu \in (0, 1)$, the hyperinvariant subspace $\mathcal{H}_\mu = \mathcal{H} \cap F^{(1)}_\mu \mathcal{K} = (0)$, then the sequence $\{(S^{*n} S^n)^{1/n}\}_{n\in\mathbb{N}}$ is* SOT*-convergent to $1_{\mathcal{H}}$.*

Proof. Let x_0 be any unit vector in $\mathcal{H}$ and let $\varepsilon > 0$ be given. Choose $0 < \nu_0 < 1$ such that $(1-\nu_0)^2 < \varepsilon$. Then we have from (9) that

$$\begin{aligned}\|(1_{\mathcal{H}} - (S^{*n}S^n)^{1/n})x_0\|^2 &= \int_{[0,1]} (1-\gamma)^2 \; dG^{(n)}_{x_0,x_0}(\gamma) \\ &= \int_{[0,\nu_0]} (1-\gamma)^2 \; dG^{(n)}_{x_0,x_0}(\gamma) + \int_{(\nu_0,1]} (1-\gamma)^2 dG^{(n)}_{x_0,x_0}(\gamma) \\ &\le \|G^{(n)}_{\nu_0} x_0\|^2 + (1-\nu_0)^2 \\ &\le \|G^{(n)}_{\nu_0} x_0\|^2 + \varepsilon.\end{aligned}$$

Upon letting $n \to \infty$ and applying Corollary 4.5 one obtains the result. □

The converse of Proposition 4.6 is also true and admits a very easy proof.

Proposition 4.7. *If S is a subnormal operator in spectral general position, and the sequence $\{(S^{*n}S^n)^{1/n}\}_{n\in\mathbb{N}}$ is* SOT*-convergent to $1_{\mathcal{H}}$, then for each $\mu \in (0,1)$, the space $\mathcal{H}_\mu$ is the zero subspace.*

Proof. Suppose, to the contrary, that for some $\mu_0 \in (0,1)$, $\mathcal{H}_{\mu_0} \neq (0)$, and let x be a unit vector in $\mathcal{H}_{\mu_0}$. Then, by the well-known Hölder-McCarthy inequality, we have

$$\begin{aligned}\langle (S^{*n}S^n)^{1/n}x, x\rangle &\le \langle (S^{*n}S^n)x, x\rangle^{1/n}, \qquad n \in \mathbb{N}, \\ &= \langle (N^{*n}N^n)x, x\rangle^{1/n} \\ &= \left(\int_{[0,\mu_0]} \mu^n \; dF^{(1)}_{x,x}\right)^{1/n} \\ &\le \mu_0 < 1,\end{aligned}$$

which proves that $\{(S^{*n}S^n)^{1/n}\}_{n\in\mathbb{N}}$ cannot converge in the WOT to $1_{\mathcal{H}}$. □

Putting together Propositions 4.6 and 4.7 we immediately obtain one of our main results.

Theorem 4.8. *If S is a subnormal operator in spectral general position, then there exists $\mu_0 \in (0,1)$ such that $\mathcal{H}_{\mu_0} = \mathcal{H} \cap F^{(1)}_{\mu_0}\mathcal{K}$ is an n.h.s. for S if and only if the sequence $\{(S^{*n}S^n)^{1/n}\}_{n\in\mathbb{N}}$ does not converge in the* SOT *to $1_{\mathcal{H}}$.*

A careful perusal of the arguments in this section shows that the hypothesis that S is in spectral general position was never needed. Thus, we have proved the following.

Proposition 4.9. *If S is a subnormal operator in $\mathcal{L}(\mathcal{H})$, then there exists $\mu_0 \in [0, \|S\|^2)$ such that $\mathcal{H}_{\mu_0} = \mathcal{H} \cap F^{(1)}_{\mu_0}\mathcal{K}$ is an n.h.s. for S if and only if the sequence $\{(S^{*n}S^n)^{1/n}\}_{n\in\mathbb{N}}$ does not converge in the* SOT *to $\|S\|^2$.*

As a corollary we immediately obtain the following.

Theorem 4.10. *Let S be any subnormal operator in $\mathcal{L}(\mathcal{H})$ such that S has no n.h.s. Then for every $r \in \mathrm{Rat}(\sigma(S))$, $r(S)$ is subnormal, and the sequence*

$$\{[(r(S)^*)^n(r(S)^n)]^{1/n}\}_{n\in\mathbb{N}}$$

converges in the SOT *to $\|r(S)\|^2$.*

Proof. It is easy to see that no generality is lost by supposing that S is pure. Moreover, if $\sigma(S) \supsetneqq \sigma(N)$, then, as is well known, $\sigma_p(S^*) \neq \varnothing$, contradicting the hypothesis. Thus $\mathrm{Rat}(\sigma(S)) = \mathrm{Rat}(\sigma(N))$, and one knows that for any $r \in \mathrm{Rat}(\sigma(S))$, $r(S)$ is subnormal. Recalling that $\{T\}' \subset \{r(T)\}'$ for $r \in \mathrm{Rat}(\sigma(S))$, we now obtain the result from Proposition 4.9. □

Remark 4.11. It is not difficult to show that the spaces $\mathcal{H}_\mu$ in use above are exactly the spectral maximal spaces for S corresponding to the closed discs centered at 0 with radius $\mu^{1/2}$ (which are well known to be hyperinvariant for S).

5. The sequence $\{(S^n S^{*n})^{1/n}\}_{n\in\mathbb{N}}$

In this section we obtain further information about a subnormal operator S in $\mathcal{L}(\mathcal{H})$ in spectral general position by studying the sequence $\{(S^n S^{*n})^{1/n}\}_{n\in\mathbb{N}}$. We write

$$(S^n S^{*n})^{1/n} = \int_{[0,1]} \alpha \; d\widetilde{G}^{(n)}(\alpha), \qquad n \in \mathbb{N},$$

so $\widetilde{G}^{(n)}$ is the spectral measure of $(S^n S^{*n})^{1/n}$. As before we write

$$\widetilde{G}^{(n)}_\alpha = \widetilde{G}^{(n)}([0,\alpha]), \quad \widetilde{G}^{(n)}_{\alpha^-} = \widetilde{G}^{(n)}([0,\alpha)), \quad \alpha \in [0,1],$$

and note that the function $\alpha \to \widetilde{G}^{(n)}_\alpha$ $[\alpha \to \widetilde{G}^{(n)}_{\alpha^-}]$ is continuous from the right [left]. Since

$$S^{*n}(S^n S^{*n}) = (S^{*n} S^n) S^{*n}, \qquad n \in \mathbb{N},$$

it follows easily from the Weierstrass approximation theorem that

$$S^{*n}\widetilde{G}^{(n)}_\alpha = G^{(n)}_\alpha S^{*n}, \qquad n \in \mathbb{N}, \quad \alpha \in [0,1].$$

We first establish the counterpart of Lemma 4.3.

Lemma 5.1. *Suppose $0 < \mu_0 < 1$ and $r > 1$ is such that $\mu_0 r \le 1$. Then*

$$\|(1_\mathcal{H} - \widetilde{G}^{(n)}_{\mu_0 r}) P F^{(1)}_{\mu_0} z\|^2 \le (1/r^n)\|F^{(1)}_{\mu_0} z\|^2 \le (1/r^n)\|z\|^2, \qquad n \in \mathbb{N}, \quad z \in \mathcal{K},$$

and consequently

$$\|(1_\mathcal{H} - \widetilde{G}^{(n)}_{\mu_0 r}) P F^{(1)}_{\mu_0}\|^2 \le 1/r^n, \qquad n \in \mathbb{N}.$$

Proof. We first observe that for each $n \in \mathbb{N}$ and $z \in \mathcal{K}$,

$$\begin{aligned}\|N^{*n} F^{(1)}_{\mu_0} z\|^2 &= \langle (NN^*)^n F^{(1)}_{\mu_0} z, F^{(1)}_{\mu_0} z\rangle \\ &= \int_{[0,\mu_0]} \mu^n \; dF^{(1)}_{z,z}(\mu) \le \mu_0^n \|F^{(1)}_{\mu_0} z\|^2,\end{aligned}$$

and consequently that

$$\langle S^n S^{*n} P F^{(1)}_{\mu_0} z, P F^{(1)}_{\mu_0} z\rangle = \|S^{*n} P F^{(1)}_{\mu_0} z\|^2 = \|P N^{*n} F^{(1)}_{\mu_0} z\|^2 \le \mu_0^n \|F^{(1)}_{\mu_0} z\|^2. \tag{21}$$

(Here we use the fact that if we regard P as an operator from $\mathcal{K}$ to $\mathcal{H}$, then $S^{*n} P = P N^{*n}$, as an easy calculation shows.) Thus,

$$\begin{aligned}(\mu_0 r)^n \|(1_\mathcal{H} - \widetilde{G}^{(n)}_{\mu_0 r}) P F^{(1)}_{\mu_0} z\|^2 &= (\mu_0 r)^n \langle (1_\mathcal{H} - \widetilde{G}^{(n)}_{\mu_0 r}) P F^{(1)}_{\mu_0} z, P F^{(1)}_{\mu_0} z\rangle \\ &\le \int_{[\mu_0 r, 1]} \alpha^n \; d\widetilde{G}^{(n)}_{P F^{(1)}_{\mu_0} z, P F^{(1)}_{\mu_0} z}(\alpha) \\ &= \langle (S^n S^{*n}) P F^{(1)}_{\mu_0} z, P F^{(1)}_{\mu_0} z\rangle \\ &\le \mu_0^n \|F^{(1)}_{\mu_0} z\|^2 \le \mu_0^n \|z\|^2, \quad n \in \mathbb{N}, \quad z \in \mathcal{K},\end{aligned} \tag{22}$$

where we used (21) in the last step. Thus dividing each side of (22) by $(\mu_0 r)^n$ gives the desired conclusion. $\square$

Lemma 5.2. *For every $\mu_0 \in (0,1)$,*

$$\limsup_n \|(S^nS^{*n})^{1/2n}PF^{(1)}_{\mu_0}z\|^2 \le \mu_0\|PF^{(1)}_{\mu_0}z\|^2, \quad z \in \mathcal{K}.$$

Proof. Fix $r > 1$ such that $r\mu_0 \le 1$. Then, with $z \in \mathcal{K}$ given, we have

$$\begin{aligned}\|(S^nS^{*n})^{1/2n}PF^{(1)}_{\mu_0}z\|^2 &= \langle (S^nS^{*n})^{1/n}PF^{(1)}_{\mu_0}z, PF^{(1)}_{\mu_0}z\rangle \\ &= \int_{[0,1]} \alpha \, d\widetilde{G}^{(n)}_{PF^{(1)}_{\mu_0}z, PF^{(1)}_{\mu_0}z}(\alpha) \\ &\le \int_{[0,\mu_0 r]} \alpha \, d\widetilde{G}^{(n)}_{PF^{(1)}_{\mu_0}z, PF^{(1)}_{\mu_0}z}(\alpha) + \|(1_{\mathcal{H}} - \widetilde{G}^{(n)}_{\mu_0 r})PF^{(1)}_{\mu_0}z\|^2 \\ &\le \mu_0 r\|PF^{(1)}_{\mu_0}z\|^2 + (1/r^n)\|F^{(1)}_{\mu_0}z\|^2,\end{aligned}$$

where in the last inequality we used Lemma 5.1. Thus, for any fixed $z \in \mathcal{K}$,

$$\limsup_n \|(S^nS^{*n})^{1/2n}PF^{(1)}_{\mu_0}z\|^2 \le \mu_0 r\|PF^{(1)}_{\mu_0}z\|^2, \tag{23}$$

and since (23) is valid for all $r > 1$ such that $\mu_0 r \le 1$, the result follows. $\square$

Corollary 5.3. *If for every $\mu \in (0,1)$, we have $(PF^{(1)}_{\mu}\mathcal{K})^- = \mathcal{H}$, then the sequence $\{(S^nS^{*n})^{1/n}\}_{n\in\mathbb{N}}$ converges to 0 in the* SOT.

Proof. Fix $\mu_0 \in (0,1)$. By hypothesis, the set $\{PF^{(1)}_{\mu_0}z : z \in \mathcal{K}\}$ is dense in $\mathcal{H}$, and thus it follows immediately from Lemma 5.2 that $\|(S^nS^{*n})^{1/2n}\| \le \mu_0$. Since this is true for all $\mu_0 \in (0,1)$, we conclude that $\lim_n \|(S^nS^{*n})^{1/2n}\| = 0$, and since the product operation is jointly sequentially continuous in the SOT, the result follows. $\square$

Proposition 5.4. *For each $\mu \in (0,1)$, we have $PF^{(1)}_{\mu}\mathcal{K} \ne (0)$ and $\mathcal{M}_{\mu} := \mathcal{H} \cap (1_{\mathcal{K}} - F^{(1)}_{\mu})\mathcal{K} \ne \mathcal{H}$. Moreover, there exists $\mu_0 \in (0,1)$ such that $(PF^{(1)}_{\mu_0}\mathcal{K})^- \ne \mathcal{H}$ if and only if $\mathcal{M}_{\mu_0} \ne (0)$.*

Proof. Suppose there exists $\mu_1 \in (0,1)$ such that $PF^{(1)}_{\mu_1}\mathcal{K} = (0)$, and note that since $\sigma(N^*N) = [0,1]$, $(0) \ne F^{(1)}_{\mu_1}\mathcal{K} \subset \mathcal{K} \ominus \mathcal{H}$. Then, since N commutes with $F^{(1)}_{\mu_1}$, $N|_{(1_{\mathcal{H}} - F^{(1)}_{\mu_1})\mathcal{K}}$ is a normal extension of S acting on a proper subspace of $\mathcal{K}$, which contradicts the minimality of N, and proves the first half of the first statement of the proposition. Moreover, if there exists $\mu_2 \in (0,1)$ such that $\mathcal{M}_{\mu_2} = \mathcal{H}$, then $(1_{\mathcal{K}} - F^{(1)}_{\mu_2})\mathcal{K} \supset \mathcal{H}$, contradicting the already proved fact that $PF^{(1)}_{\mu_2}\mathcal{K} \ne (0)$, which proves the second half of the first statement above. Finally, observe that there exists $\mu_0 \in (0,1)$ such that $(PF^{(1)}_{\mu_0}\mathcal{K})^- \ne \mathcal{H}$ if and only if there exists a nonzero vector $y_0 \in \mathcal{H} \ominus (PF^{(1)}_{\mu_0}\mathcal{K})^- = \mathcal{H} \cap (\mathcal{F}^{(1)}_{\mu_0}\mathcal{K})^{\perp} = \mathcal{M}_{\mu_0}$. $\square$

We next wish to show that if there exists $\mu_0 \in (0,1)$ such that $(PF^{(1)}_{\mu_0}\mathcal{K})^- \ne \mathcal{H}$, then the nontrivial subspace $\mathcal{M}_{\mu_0}$ defined above is an n.h.s. for S. To establish this, we must first obtain another characterization of $\mathcal{M}_{\mu_0}$.

Proposition 5.5. *If $\mu_0 \in (0,1)$ is such that the space $\mathcal{M}_{\mu_0} = \mathcal{H} \cap (1_{\mathcal{K}} - F^{(1)}_{\mu_0})\mathcal{K}$ from Proposition* 5.4 *is nonzero, then $N|_{\mathcal{M}_{\mu_0}} = S|_{\mathcal{M}_{\mu_0}}$ is invertible and $\mathcal{M}_{\mu_0} = P_{\mu_0}$, where*

(24) $\mathcal{P}_{\mu_0} = \{x \in \mathcal{H} : \exists \{x_n\} \subset \mathcal{H}$ *with* $x = N^n x_n \forall n$ *and* $\sup_n (\mu_0^{n/2} \|x_n\|) < +\infty\}$.

Proof. Note that since $\sigma_p(N^*N) = \varnothing$, we have $F^{(1)}_{\mu^-} = F^{(1)}_{\mu}$ for all $\mu \in (0,1]$, and thus, upon defining $\mathcal{K}_1 := F^{(1)}([\mu_0,1])\mathcal{K}$, we get $\mathcal{M}_{\mu_0} = \mathcal{H} \cap \mathcal{K}_1$. Clearly $\mathcal{K}_1$ reduces N and both $N|_{\mathcal{K}_1}$ and its adjoint have lower bounds $\mu_0^{1/2}$; that is, $N|_{\mathcal{K}_1}$ is invertible. Observe also that $N\mathcal{M}_{\mu_0} \subset \mathcal{M}_{\mu_0}$, and since $\mathcal{M}_{\mu_0} \subset \mathcal{K}_1$, the operator $N|_{\mathcal{M}_{\mu_0}} = S|_{\mathcal{M}_{\mu_0}}$ has lower bound $\delta \geq \mu_0^{1/2}$. Thus $S|_{\mathcal{M}_{\mu_0}}$ is a semi-Fredholm operator with trivial kernel. Furthermore, since $0 \in \partial_u(\sigma(N))$ there exists an open Jordan arc J lying in $\mathbb{C}\backslash\sigma(N)$ with endpoints 0 and $+\infty$. Moreover, for every $\lambda \in J$, $N - \lambda 1_{\mathcal{K}}$ is invertible, so $(N - \lambda 1_{\mathcal{K}})|_{\mathcal{M}_{\mu_0}} = S|_{\mathcal{M}_{\mu_0}} - \lambda 1_{\mathcal{M}_{\mu_0}}$ is bounded below and is thus a semi-Fredholm operator with constant index (independent of λ). Since for $|\lambda| > 1$, $S|_{\mathcal{M}_{\mu_0}} - \lambda 1_{\mathcal{M}_{\mu_0}}$ is invertible, the Fredholm index $i(S|_{\mathcal{M}_{\mu_0}} - \lambda 1_{\mathcal{M}_{\mu_0}}) = 0$ for all $\lambda \in J$ and also for $\lambda = 0$, so $N|_{\mathcal{M}_{\mu_0}}$ is invertible, which proves the first statement of the proposition. Next, for an arbitrary $x_0 \in \mathcal{M}_{\mu_0}$, we define $x_n = (N|_{\mathcal{M}_{\mu_0}})^{-n} x_0$, so $x_n \in \mathcal{M}_{\mu_0} \subset \mathcal{H}$ and

$$\|x_n\| \leq (1/\delta)^n \|x_0\| \leq (1/\mu_0^{1/2})^n \|x_0\|, \qquad n \in \mathbb{N}.$$

Thus $x_0 \in \mathcal{P}_{\mu_0}$ and $\mathcal{M}_{\mu_0} \subset \mathcal{P}_{\mu_0}$. Now suppose that $y_0 \in \mathcal{P}_{\mu_0}(\subset \mathcal{H})$ and let $\{y_n\}_{n\in\mathbb{N}} \subset \mathcal{H}$ be the corresponding sequence from (24). We wish to show that $y_0 \in \mathcal{K}_1$, or, in other words, that $F^{(1)}_{\mu_0} y_0 = 0$. Suppose the contrary. Then, since $F^{(1)}_{\mu_0} y_0 \neq 0$ and the function $\mu \to \|F^{(1)}_{\mu} y_0\|$ is continuous on $[0,1]$ (because $\sigma_p(N^*N) = \varnothing$), there exist $0 < \nu_1 < \nu_2 < \mu_0$ such that

$$0 < \|F^{(1)}_{\nu_1} y_0\| < \|F^{(1)}_{\nu_2} y_0\| < \|F^{(1)}_{\mu_0} y_0\|;$$

in other words, $\|F^{(1)}([\nu_1,\nu_2]) y_0\| \neq 0$. Thus

$$N^n F^{(1)}([\nu_1,\nu_2]) y_n = F^{(1)}([\nu_1,\nu_2]) N^n y_n = F^{(1)}([\nu_1,\nu_2]) y_0 \neq 0$$

and

$$\begin{aligned}\|y_n\| \geq \|F^{(1)}([\nu_1,\nu_2]) y_n\| &= \|(N|_{F^{(1)}([\nu_1,\nu_2])\mathcal{K}})^{-n} F^{(1)}([\nu_1,\nu_2]) y_0\| \\ &\geq (1/\nu_2)^{n/2} \|F^{(1)}([\nu_1,\nu_2]) y_0\|.\end{aligned}$$

Since $\nu_2 < \mu_0$, this contradicts the fact that the sequence $\{\mu_0^{n/2}\|y_n\|\}_{n\in\mathbb{N}}$ is bounded. Thus $y_0 \in \mathcal{M}_{\mu_0}$, which proves the result. □

Finally we can obtain the main result of the present section.

Theorem 5.6. *Let S be a subnormal operator in $\mathcal{L}(\mathcal{H})$ in spectral general position. Then there exists $\mu_0 \in (0,1)$ such that $\mathcal{M}_{\mu_0} = \mathcal{H} \cap (1_{\mathcal{K}} - F^{(1)}_{\mu_0})\mathcal{K}$ is an n.h.s. for S if and only if the sequence $\{(S^n S^{*n})^{1/n}\}_{n\in\mathbb{N}}$ does not converge to zero in the* SOT.

Proof. Suppose that the sequence $\{(S^n S^{*n})^{1/n}\}$ does not converge to zero in the SOT. Then, by Corollary 5.3 there exists $\mu_0 \in (0,1)$ such that $(PF^{(1)}_{\mu_0}\mathcal{K})^- \neq \mathcal{H}$, and by Proposition 5.4, $(0) \neq \mathcal{M}_{\mu_0} \neq \mathcal{H}$. Next, let x_0 be arbitrary in $\mathcal{M}_{\mu_0}$ and let

X be arbitrary in $\{S\}'$. Then there exists a sequence $\{x_n\}_{n\in\mathbb{N}} \subset \mathcal{H}$ as in (24), so $Xx = N^n(Xx_n)$ and

$$\sup_n(\mu_0^{n/2}\,\|Xx_n\|) \leq \|X\|\sup(\mu_0^{n/2}\,\|x_n\|) < +\infty.$$

By Proposition 5.5, $Xx \in \mathcal{M}_{\mu_0}$, and thus $\mathcal{M}_{\mu_0}$ is a nontrivial hyperinvariant subspace for S.

To go in the other direction, suppose now that there exists $\mu_0 \in (0,1)$ such that $(0) \neq \mathcal{M}_{\mu_0} \neq \mathcal{H}$. Then, by Proposition 5.5, $S|_{\mathcal{M}_{\mu_0}} = N|_{\mathcal{M}_{\mu_0}}$ is invertible with lower bound $\delta \geq \mu_0^{1/2}$, and thus for $n \in \mathbb{N}$, $((N|_{\mathcal{M}_{\mu_0}})^n(N|_{\mathcal{M}_{\mu_0}})^{*n})^{1/n}$ has lower bound no smaller than δ^2. Upon writing S as a 2×2 operator matrix relative to the decomposition $\mathcal{H} = \mathcal{M}_{\mu_0} \oplus (\mathcal{H} \ominus \mathcal{M}_{\mu_0})$, a short calculation shows that for every $n \in \mathbb{N}$, the difference $S^nS^{*n} - ((N|_{\mathcal{M}_{\mu_0}})^n(N|_{\mathcal{M}_{\mu_0}})^{*n} \oplus 0_{\mathcal{H}\ominus\mathcal{M}_{\mu_0}})$ is positive semidefinite, and thus by the Löwner-Heinz inequality,

$$(S^nS^{*n})^{1/n} \geq \big((N|_{\mathcal{M}_{\mu_0}})^n(N|_{\mathcal{M}_{\mu_0}})^{*n}\big)^{1/n} \oplus 0_{\mathcal{H}\ominus\mathcal{M}_{\mu_0}}.$$

Thus for every unit vector $x_0 \in \mathcal{M}_{\mu_0}$, $\big\langle (S^nS^{*n})^{1/n}x_0, x_0\big\rangle \geq \delta^2$, and hence the sequence $\{(S^nS^{*n})^{1/n}\}_{n\in\mathbb{N}}$ does not converge to zero even in the weak operator topology. □

As an immediate corollary of Proposition 2.2 and Theorems 4.8 and 5.6, we also have the following rather intriguing result.

Theorem 5.7. *If S is a subnormal operator in spectral general position, then the sequences $\{S^{*n}S^n\}_{n\in\mathbb{N}}$ and $\{S^nS^{*n}\}_{n\in\mathbb{N}}$ converge in the* SOT *to $0_{\mathcal{H}}$. Moreover, either S has an n.h.s. or the sequences $\{(S^{*n}S^n)^{1/n}\}_{n\in\mathbb{N}}$ and $\{(S^nS^{*n})^{1/n}\}_{n\in\mathbb{N}}$ converge in the* SOT *to $1_{\mathcal{H}}$ and $0_{\mathcal{H}}$, respectively.*

Remark 5.8. We believe that the general approach presented in Sections 3-5 may eventually provide additional structure theorems for the class of subnormal operators.

Remark 5.9. Most of the results in this paper can be extended to the class of subdecomposable operators. These results will be presented in a subsequent paper.

Remark 5.10. We take this opportunity to express our thanks to the referee for several excellent remarks that allowed us to correct some minor errors and to improve the exposition herein.

Acknowledgement

This research was supported by a grant from the Korea Research Foundation (R14-2003-006-01000-0).

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