

THE ČECH FILTRATION AND MONODROMY IN LOG CRYSTALLINE COHOMOLOGY

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ABSTRACT. For a strictly semistable log scheme Y over a perfect field k of characteristic p we investigate the canonical Čech spectral sequence $(C)_T$ abutting the Hyodo-Kato (log crystalline) cohomology $H_{crys}^*(Y/T)_{\mathbb{Q}}$ of Y and beginning with the log convergent cohomology of its various component intersections Y^i . We compare the filtration on $H_{crys}^*(Y/T)_{\mathbb{Q}}$ arising from $(C)_T$ with the monodromy operator N on $H_{crys}^*(Y/T)_{\mathbb{Q}}$. We also express N through residue maps and study relations with singular cohomology. If Y lifts to a proper strictly semistable (formal) scheme X over a finite totally ramified extension of $W(k)$, with generic fibre X_K , we obtain results on how the simplicial structure of X_K^{an} (as analytic space) is reflected in $H_{dR}^*(X_K) = H_{dR}^*(X_K^{an})$.

INTRODUCTION

Let A be a complete discrete valuation ring of mixed characteristic $(0, p)$, with perfect residue field k and fraction field K , and let K_0 be the fraction field of the ring of Witt vectors $W(k)$ of k . Let X be a proper strictly semistable A -scheme. Besides its Hodge filtration the de Rham cohomology $H_{dR}^*(X_K)$ of the generic fibre X_K of X comes with a K_0 -lattice with a Frobenius operator F and a nilpotent operator N : these are obtained, via the Hyodo-Kato isomorphism (which depends on the choice of a uniformizer π in A)

$$(*) \quad H_{dR}^*(X_K) \cong H_{crys}^*(Y/T)_{\mathbb{Q}} \otimes_{K_0} K$$

from the Frobenius operator F and the nilpotent operator N on the Hyodo-Kato (log crystalline) cohomology $H_{crys}^*(Y/T)_{\mathbb{Q}}$ of the special fibre Y of X . It follows from the theorem of Tsuji and (independently) Faltings that $H_{dR}^*(X_K)$, together with its Hodge filtration and the operators F and N on its K_0 -lattice defined by $(*)$, allows the reconstruction of the p -adic étale cohomology group $H_{et}^*(X_{\bar{K}}, \mathbb{Q}_p)$ together with its $\text{Gal}(\bar{K}/K)$ -action. We have $N = 0$ if X has good reduction. It is a general and important problem to reconstruct as much as possible of the N -structure on $H_{dR}^*(X_K) = H_{dR}^*(X_K^{an})$ (where X_K^{an} is the rigid analytic space

Received by the editors January 5, 2005 and, in revised form, July 11, 2005.

2000 *Mathematics Subject Classification*. Primary 14F30.

Key words and phrases. Logarithmic crystalline cohomology, monodromy operator, weight filtration, Steenbrink complex, analytic spaces.

Most of this work was done during my visit at the University of California, Berkeley. I wish to thank Robert Coleman (and Bishop) for welcoming me there so warmly. Thanks are also due to Ehud de Shalit, Yukiyoishi Nakkajima and Arthur Ogus for useful related discussions. I thank the referee for his careful reading of the manuscript and his suggestions for improving the exposition. I am grateful to the Deutsche Forschungsgemeinschaft for supporting my stay at Berkeley.

associated with X_K) solely from X_K or X_K^{an} . For this purpose the obvious idea is to look at the following spectral sequence $(C)_S$. Let $\{Y_j\}_{j \in R}$ be the set of irreducible components of Y . Each Y_j is (classically) smooth over k . For $i \geq 1$ let $Y^i = \coprod_{|I|=i} (\bigcap_{j \in I} Y_j)_{j \in I}$, where I runs through the set of subsets of R with precisely i elements. We assume that all connected components of Y are of the same dimension d . For direct sums $E = \coprod E_s$ of subschemes $E_s \subset Y$ let $]E[_X = \coprod]E_s[_X$ be the direct sum of the preimages of the E_s under the specialization map $X_K^{an} \rightarrow Y$: these $]E_s[_X$ are admissible open subspaces of X_K^{an} . The admissible open covering $X_K^{an} = \bigcup_{j \in R}]Y_j[_X$ is a covering by *contractible* spaces in the sense of Berkovich. It gives rise to the spectral sequence

$$(C)_S \quad E_1^{pq} = H_{dR}^q(]Y^{p+1}[_X) \implies H_{dR}^{p+q}(X_K).$$

In applications, for example when X_K is a Shimura variety, $(C)_S$ often has arithmetical meaning. In the case $d = 1$, Coleman and Iovita [5] gave a description of N on $H_{dR}^1(X_K)$ in terms of $(C)_S$; namely, they proved that it is the composite

$$H_{dR}^1(X_K) \xrightarrow{Res} H_{dR}^0(]Y^2[_X) \xrightarrow{\delta} H_{dR}^1(X_K),$$

where Res is an explicit residue map and δ is the connecting homomorphism in $(C)_S$. For any d , Alon and de Shalit [1] gave a tentative definition of N for varieties X_K uniformized by Drinfel'd's symmetric spaces; their central concept of harmonic cochains on the Bruhat Tits building is intimately related to $(C)_S$. The observation from [9] that for such varieties the filtration F_C^\bullet on $H_{dR}^d(X_K)$ defined by $(C)_S$ is the weight filtration for the Frobenius action plays a role in de Shalit's recent proof of the monodromy weight conjecture for such varieties [6] (another proof was given by Ito [13]). Besides these example we are not aware of other investigations of $(C)_S$. The general reference for monodromy operators arising in semistable families (considered there in the ℓ -adic and in the complex analytic setting) is Illusie's article [12]. Specifically, the Čech complex from [12] 2.1.5, 3.2 does *not* correspond to $(C)_S$ but rather to the canonical Čech spectral sequence

$$(C)_{rig} \quad E_1^{pq} = H_{rig}^q(Y^{p+1}) \implies H_{rig}^{p+q}(Y)$$

with H_{rig}^* denoting non-logarithmic rigid cohomology.

In this paper we begin the study of how N interacts with $(C)_S$ for general X by using log convergent cohomology. Since Y is a normal crossings divisor on the regular scheme X it gives rise to a natural log structure on X , and by pull back to natural log structures on subschemes of X . Thus, since Y is a log scheme over the log point $T_1 = (\text{Spec}(k), (\mathbb{N} \rightarrow k, 1 \mapsto 0))$ over k , all subschemes of Y become T_1 -log schemes. As such we ask for their log convergent cohomology $H_{conv}^*(\cdot/T)$ relative to $T = (\text{Spf}(W(k)), (\mathbb{N} \rightarrow W(k), 1 \mapsto 0))$ which takes its values in K_0 -vector spaces. For example, $H_{conv}^*(Y/T) = H^*(Y, \mathcal{C}\omega_Y^\bullet)$ for a certain logarithmic de Rham complex $\mathcal{C}\omega_Y^\bullet$ on analytic tubular neighbourhoods of Y in local T -log smooth liftings. Since Y is log smooth over T_1 the Hyodo-Kato cohomology $H_{crys}^*(Y/T)_\mathbb{Q}$ is isomorphic to $H_{conv}^*(Y/T)$. There is a natural Čech spectral sequence

$$(C)_T \quad E_1^{pq} = H_{conv}^q(Y^{p+1}/T) \implies H_{conv}^{p+q}(Y/T) = H_{crys}^{p+q}(Y/T)_\mathbb{Q}.$$

By [9], 3.8 and 3.9, the isomorphism $(*)$ extends to an isomorphism of spectral sequences $(C)_S \cong (C)_T \otimes_{K_0} K$. The descending filtration $(F_C^r H_{crys}^*(Y/T)_\mathbb{Q})_{r \geq 0}$ on $H_{crys}^*(Y/T)_\mathbb{Q}$ induced by $(C)_T$, in particular, redefines the descending filtration $(F_C^r H_{dR}^*(X_K))_{r \geq 0}$ on $H_{dR}^*(X_K)$ induced by $(C)_S$ (we call them the *canonical*

Čech filtrations). As our main tool to analyse $(C)_T$ we introduce the *Čech-double complex* $B^{\bullet\bullet}$ on Y (or rather on a simplicial scheme U_\bullet associated with a suitable Zariski open covering of Y) whose total complex B^\bullet computes $H_{conv}^*(Y/T)$ and whose $(k-1)$ -st column $B^{\bullet k-1}$ computes $H_{conv}^q(Y^k/T)$. (Since Y^k/T_1 is not log smooth it is doubtful if its log *crystalline* cohomology (relative to T) is a useful object.) This means that $(C)_T$ is the spectral sequence associated with the stupid vertical filtration F_C^\bullet of $B^{\bullet\bullet}$. On the other hand we consider a double complex $A^{\bullet\bullet}$ with associated total complex A^\bullet which is the straightforward analog (in the log convergent setting, as opposed to the log crystalline setting) of the Hyodo-Steenbrink-complex $W_\bullet A^\bullet$ on Y constructed by Mokrane [19]. This A^\bullet computes $H_{crys}^*(Y/T)_\mathbb{Q} = H_{conv}^*(Y/T)_\mathbb{Q}$, too. Moreover A^\bullet is endowed with an endomorphism ν which induces N on $H_{conv}^*(Y/T)_\mathbb{Q}$. The complexes A^\bullet and B^\bullet can be related in two ways: First, there is a natural quasiisomorphism $\psi : A^\bullet \rightarrow B^\bullet$. Second, there is a third complex C^\bullet endowed with a filtration F_C^\bullet and an endomorphism ν such that: there is a natural quasiisomorphism $A^\bullet \rightarrow C^\bullet$ respecting the respective endomorphisms ν , and there is a natural quasiisomorphism $B^\bullet \rightarrow C^\bullet$, filtered with respect to the respective filtrations F_C^\bullet . In particular, ν on C^\bullet induces an operator N on $(C)_T$. We prove (cf. 3.6, 3.8, 4.2):

Theorem. $N = 0$ on $H^*(Y, B^{\bullet k-1})$ and $N(F_C^{k-1}H_{crys}^*(Y/T)_\mathbb{Q}) \subset F_C^k H_{crys}^*(Y/T)_\mathbb{Q}$ for any $k \geq 0$. For $i \geq 0$ the i -fold iterated monodromy operator N^i on $H_{crys}^*(Y/T)_\mathbb{Q} = H^*(Y, C\omega_Y^\bullet) = H^*(Y, B^\bullet)$ is induced by a composite map of sheaf complexes

$$C\omega_Y^\bullet \xrightarrow{\rho_i} F_C^i B^\bullet \subset B^\bullet$$

with an explicit residue map ρ_i .

Retransposing to our lifted situation we derive (cf. 7.4):

Theorem. N on $H_{dR}^*(X_K)$ naturally extends to an operator N on $(C)_S$. However, $N = 0$ on $H_{dR}^*(Y^k|_X)$, and $N(F_C^{k-1}H_{dR}^*(X_K)) \subset F_C^k H_{dR}^*(X_K)$ for any $k \geq 1$. For $i \geq 0$ the i -fold iterated monodromy operator N^i on $H_{dR}^*(X_K)$ has the form

$$H_{dR}^*(X_K) \xrightarrow{Res} F_C^i H_{dR}^*(X_K) \subset H_{dR}^*(X_K)$$

with a residue map Res .

We view this as the generalization, to any d , of the description of N given by Iovita and Coleman in the case $d = 1$. In particular we get an upper bound for the vanishing order of N described in terms of X_K^{an} . We do not know if in general the residue map $H_{dR}^*(X_K) \xrightarrow{Res} F_C^i H_{dR}^*(X_K)$ can be made explicit *without* involving the log basis T . However, for $i = * = d$ there is a natural candidate, generalizing the residue map used by Iovita and Coleman in the case $i = * = d = 1$.

The inclusion $\text{Im}(N^k) \subset F_C^k H_{dR}^*(X_K)$ is an equality if for example the canonical map $H_{rig}^*(Y) \rightarrow H_{crys}^*(Y/T)_\mathbb{Q}$ is *strict* with respect to canonical Čech filtrations, which on $H_{rig}^*(Y)$ is defined through the above spectral sequence $(C)_{rig}$ (see Proposition 5.2). In general, however, it is *not* an equality, even if Y is projective, k is finite and the monodromy weight conjecture holds; see section 5 for a counter example.

Yet there is more structure on B^\bullet : if k is finite and the monodromy weight conjecture holds, the monodromy filtration on $H_{crys}^*(Y/T)_\mathbb{Q}$ is induced from a filtration

on B^\bullet (Theorem 4.8). Moreover B^\bullet has a product structure inducing Poincaré duality (in contrast to A^\bullet , it seems); see section 2.

We also ask how N is related to the singular cohomology $H^*(Y_{Zar}, \mathbb{Q})$. Note that $H^*(Y_{Zar}, \mathbb{Q}) = H^*(X_K^{an}, \mathbb{Q})$; see [2]. We show (Theorem 6.3):

Theorem. *Suppose Y satisfies the monodromy weight conjecture, is of weak Lefschetz type (see section 6), and for each $i \geq 1$, each component of Y^i is geometrically connected. Suppose $H^i(Y_{Zar}, \mathbb{Q}) = 0$ for all $d > i > 0$. Then $N = 0$ on $H_{crys}^s(Y/T)_{\mathbb{Q}}$ for all $s \neq d = \dim(Y)$.*

We hope that our techniques are useful to further elucidate the constraints which the homotopy type of X^{an} (as a Berkovich space) imposes on N , and to return to this question in the future. In this connection we mention the work of Le Stum [18] dealing with the case of curves. However, in the present paper we work with an abstract strictly semistable log scheme Y over k , and only in the final section 7 we consider a lifting X of Y as above.

Notations. For log algebraic geometry we refer to K. Kato [15]. For a (formal) log scheme $(X, \mathcal{N}_X \rightarrow \mathcal{O}_X)$ we will often just write X if it is clear from the context as to which log structure on X we refer. In this text, all log schemes and morphisms of log schemes have charts for the *Zariski* topology. For elements $\{f_j\}_j$ in the structure sheaf of a (formal) scheme we denote by $\mathbb{V}(\{f_j\}_j)$ the closed (formal) subscheme defined by dividing out $\{f_j\}_j$ from the structure sheaf. We let k be a perfect field of characteristic $p > 0$ and $W(k)$ its ring of Witt vectors with Frobenius endomorphism σ . For (sheaves of) $W(k)$ -modules M endowed with a σ -linear endomorphism F , and $a \in \mathbb{N}$, we denote by $M(-a)$ the same $W(k)$ -module, but now endowed with the endomorphism $p^a \cdot F$. We let $K_0 = \text{Quot}(W(k))$ and denote by $W(k)\{t\}$ the p -adic completion of $W(k)[t]$. We need the formal log scheme

$$T = (\text{Spf}(W(k)), (\mathbb{N} \longrightarrow W(k), 1 \mapsto 0)).$$

We denote by T_1 its reduction modulo p : the log point. For a p -adic formal $W(k)$ -scheme \mathcal{F} topologically of finite type we denote by $\mathcal{F}_{\mathbb{Q}}$ its generic fibre as a K_0 -rigid space. For *any* such \mathcal{F} we write sp for the specialization map $\mathcal{F}_{\mathbb{Q}} \rightarrow \mathcal{F}$. For a subscheme F of \mathcal{F} we denote by $]F[_{\mathcal{F}}$ the tube of F in $\mathcal{F}_{\mathbb{Q}}$, i.e. the preimage of F under sp ; thus $]F[_{\mathcal{F}}$ is an admissible open subspace of $\mathcal{F}_{\mathbb{Q}}$. We use repeatedly and without further comment Kiehl’s acyclicity theorem [17] which implies that for a coherent sheaf \mathcal{M} on $]F[_{\mathcal{F}}$ and $i > 0$ the push forward sheaves $R^i sp_* \mathcal{M}$ on F vanish.

1. THE TUBE COHOMOLOGY OF THE STRATA OF Y

1.1. Our basic object of study in this paper is a strictly semistable log scheme Y over k . By definition, Y is a fine T_1 -log scheme (Y, \mathcal{N}_Y) which allows a *Zariski* open covering by open subschemes $Y' \subset Y$ with the following property: there exist integers $m \geq 1$ and charts $\mathbb{N}^m \rightarrow \mathcal{N}_Y|_{Y'}$ for $\mathcal{N}_Y|_{Y'}$ such that

- (i) if on the log scheme T_1 we use the chart $\mathbb{N} \rightarrow k, 1 \mapsto 0$, the diagonal morphism $\mathbb{N} \xrightarrow{\delta} \mathbb{N}^m$ is a chart for the structure morphism of log schemes $Y' \rightarrow T_1$, and
- (ii) the induced morphism of schemes

$$Y' \longrightarrow \text{Spec}(k) \times_{\text{Spec}(k[t])} \text{Spec}(k[t_1, \dots, t_m])$$

is smooth in the classical sense. If not said otherwise we endow subschemes of Y with the pull back structure of the T -log scheme induced by that of Y . By $\{Y_s\}_{s \in R}$ we denote the set of irreducible components of Y . We fix an ordering of R . The existence of charts as above for the Zariski-topology implies that all Y_j are classically smooth. We assume that all connected components of Y are of the same dimension d . For $i \in \mathbb{N}$ let S_i be the set of subsets of R with precisely i distinct elements. We identify S_1 with R . For $\sigma \in S_i$ let

$$Y_\sigma^i = \bigcap_{s \in \sigma} Y_s, \quad Y^i = \prod_{\sigma \in S_i} Y_\sigma^i.$$

(In Y_σ^i the upper index i is redundant, but it reminds us of the cardinality of σ .)

1.2. Define the formal log scheme $V = (\mathrm{Spf}(W(k)\{t\}), (\mathbb{N} \rightarrow W(k)\{t\}, 1 \mapsto t))$. An *admissible lift* of the semistable k -log scheme (Y, \mathcal{N}_Y) is a formal V -log scheme $(\mathcal{Z}, \mathcal{N}_\mathcal{Z})$ together with an isomorphism of T_1 -log schemes

$$(Y, \mathcal{N}_Y) \cong (\mathcal{Z}, \mathcal{N}_\mathcal{Z}) \times_V T_1$$

(where $T_1 \rightarrow V$ is given by $t \mapsto 0$) satisfying the following conditions: on underlying formal schemes \mathcal{Z} is smooth over $\mathrm{Spf}(W(k))$, flat over $\mathrm{Spf}(W(k)\{t\})$ and its reduction mod (p) is generically smooth over $\mathrm{Spec}(k[t])$, the fibre $\mathcal{Y} = \mathbb{V}(t)$ above $t = 0$ is a divisor with normal crossings on \mathcal{Z} , and $\mathcal{N}_\mathcal{Z}$ is the log structure defined by this divisor. We usually denote an admissible lift by $(\mathcal{Z}, \mathcal{Y})$. Locally on Y , admissible lifts exist. Indeed, by [14], 11.3 we locally find embeddings of Y as a normal crossings divisor into smooth k -schemes Z_1 . Assuming Z_1 is affine we can lift Z_1 to a formally smooth affine formal $W(k)$ -scheme \mathcal{Z} . Then we lift equations of Y in \mathcal{O}_{Z_1} (which form part of a local system of coordinates on Z_1) to equations in $\mathcal{O}_\mathcal{Z}$: these define \mathcal{Y} .

1.3. The following construction of diagonal embeddings in the logarithmic context is classical; see for example [10], [19]. Choose an open covering $Y = \bigcup_{h \in H} U_h$ of Y , together with admissible lifts $(\mathcal{Z}_h, \mathcal{Y}_h)$ of the U_h (so U_h is the reduction of \mathcal{Y}_h). For a subset $G \subset H$ let $U_G = \bigcap_{h \in G} U_h$. For $h \in G$ and $s \in R$ we let $\mathcal{Y}_{h,s}$ be the unique $W(k)$ -flat irreducible component of \mathcal{Y}_h which lifts $Y_s \cap U_h$; if $Y_s \cap U_h$ is empty we also let $\mathcal{Y}_{h,s}$ be the empty formal scheme. Let \mathcal{K}'_G be the blowing up of $\times_W (\mathcal{Z}_h)_{h \in G}$ along $\sum_{s \in R} (\times_W (\mathcal{Y}_{h,s})_{h \in G})$, let \mathcal{K}''_G be the complement of the strict transforms in \mathcal{K}'_G of all $\mathcal{Y}_{h_0,s} \times (\times (\mathcal{Z}_h)_{h \in G - \{h_0\}})$ (i.e. all $h_0 \in G$, all $s \in R$), and let \mathcal{Y}'_G be the exceptional divisor in \mathcal{K}'_G .

Let V'_G be the blowing up of the G -indexed self-product $\times_W V_{h \in G}$ of V along $\times_W T_{h \in G}$; the diagonal embedding $V \rightarrow \times_W V_{h \in G}$ lifts to an embedding $V \rightarrow V'_G$. There is a natural morphism of formal log schemes $\mathcal{Y}'_G \rightarrow V'_G$, and $\mathcal{K}_G = \mathcal{K}'_G \times_{V'_G} V$ is a smooth formal W -scheme and V -log formal scheme with relative normal crossings divisor $\mathcal{Y}_G = \mathcal{Y}'_G \times_{V'_G} T$. Given $s \in R$ and any $h \in G$, the closed subscheme $\mathcal{Y}_{G,s} = \mathcal{Y}_{h,s} \times_{\mathcal{Y}_h} \mathcal{Y}_G$ of \mathcal{Y}_G is independent of h . The non-empty $\mathcal{Y}_{G,s}$ form the set of $W(k)$ -flat irreducible components of \mathcal{Y}_G . By construction, the diagonal embedding $U_G \rightarrow \times_W (\mathcal{Y}_h)_{h \in G}$ lifts canonically to an embedding $U_G \rightarrow \mathcal{K}_G$, which in turn factors over the closed formal subscheme \mathcal{Y}_G of \mathcal{K}_G . Similarly, if for $i \geq 1$ and $\sigma \in S_i$ we let $\mathcal{Y}_{G,\sigma}^i = \bigcap_{s \in \sigma} \mathcal{Y}_{G,s}$; then $Y_{G,\sigma}^i$ maps to $\mathcal{Y}_{G,\sigma}^i$.

Lemma 1.4. *For any $\emptyset \neq G_1 \subset G$ the maps*

$$\begin{aligned} &]U_G[\mathcal{Y}_G \longrightarrow]U_G[\mathcal{Y}_{G_1}, \\ &]U_G \cap Y_\sigma^i[\mathcal{Y}_{G,\sigma}^i \longrightarrow]U_G \cap Y_\sigma^i[\mathcal{Y}_{G_1,\sigma}^i \end{aligned}$$

are relative open polydisks.

Proof. We give a description in local coordinates. Since the statement is local we may assume that there are an m with $1 \leq m \leq i$ and for all $h \in G$ étale maps

$$\mathcal{Z}_h \longrightarrow \mathrm{Spf}(W(k)\{t_{h,1}, \dots, t_{h,d}\})$$

such that $\mathcal{Y}_h = \mathbb{V}(t_h)$ with $t_h = \prod_{j=1}^m t_{h,j}$ and such that they induce étale maps

$$U_G \longrightarrow \mathrm{Spec}(k) \times_{\mathrm{Spec}(k[t_h])} \mathrm{Spec}(k[t_{h,1}, \dots, t_{h,d}])$$

which are *the same* for all h if (for each fixed j) we identify the free variables $t_{h,j}$ for all h . We then obtain an étale map

$$\mathcal{K}_G \longrightarrow \mathrm{Spf}\left(\frac{W(k)\{t_{h,1}, \dots, t_{h,d}\}_{h \in G}}{(t_h - t_{h'})_{h,h' \in G}}\right).$$

Here the relations $t_h = t_{h'}$ are due to the base change $V \rightarrow V'_G$ in the definition of \mathcal{K}_G . Now since in the definition of \mathcal{K}'_G we removed the strict transforms of all $\mathcal{Y}_{h_0,s} \times (\times (\mathcal{Z}_h)_{h \in G - \{h_0\}})$ we may speak of the global sections $t_{h,j} t_{h',j}^{-1}$ in $\mathcal{O}_{\mathcal{K}_G}$ for all h, h' . Thus, fixing an element $h_0 \in G_1$ we may speak of $v_{h,j} = t_{h,j} t_{h_0,j}^{-1}$ (all h). We then have the étale map

$$\mathcal{K}_G \longrightarrow \mathrm{Spf}(W(k)\{t_{h_0,1}, \dots, t_{h_0,d}, v_{h,2}^\pm, \dots, v_{h,d}^\pm\}_{h \in G - \{h_0\}})$$

(we sold the relations $t_h = t_{h'}$ for the price of omitting the terms $v_{h,1}^\pm$). Now \mathcal{Y}_G is defined inside \mathcal{K}_G through t_{h_0} , and we may arrange the situation in such a way that $\mathcal{Y}_{G,\sigma}^i$ is defined through $t_{h_0,1}, \dots, t_{h_0,i}$. We get the étale maps

- (1) $\mathcal{Y}_G \longrightarrow \mathrm{Spf}(W(k)\{t_{h_0,1}, \dots, t_{h_0,d}, v_{h,2}^\pm, \dots, v_{h,d}^\pm\}_{h \in G - \{h_0\}} / (t_{h_0}))$,
- (2) $\mathcal{Y}_{G,\sigma}^i \longrightarrow \mathrm{Spf}(W(k)\{t_{h_0,1}, \dots, t_{h_0,d}, v_{h,2}^\pm, \dots, v_{h,d}^\pm\}_{h \in G - \{h_0\}} / (t_{h_0,1}, \dots, t_{h_0,i}))$.

The closed immersions $U_G \rightarrow \mathcal{Y}_G$ and $U_G \cap Y_\sigma^i \rightarrow \mathcal{Y}_{G,\sigma}^i$ are defined by p and all $v_{h,j} - 1$ (all $h \in G - \{h_0\}$, all $2 \leq j \leq d$), hence the map (1) (resp. (2)) induces an isomorphism from $]U_G[\mathcal{Y}_G$ (resp. from $]U_G \cap Y_\sigma^i[\mathcal{Y}_{G,\sigma}^i$) to the tube over the closed subscheme of the right-hand side in (1) (resp. (2)) defined by all $v_{j,h} - 1$. To compute this tube in the right-hand side we look at the completion along the ideal defined by all $v_{j,h} - 1$: it is a formal power series ring over $W(k)\{t_{h_0,1}, \dots, t_{h_0,d}\} / (t_{h_0})$ (resp. over $W(k)\{t_{h_0,i+1}, \dots, t_{h_0,d}\} / (t_{h_0,1}, \dots, t_{h_0,i})$). Repeating all this with the subset G_1 of G simply means omitting the free variables $v_{j,h} - 1$ for $h \in G - (G_1 \cup \{h_0\})$: in both cases the difference is a relative formal powers series ring in the free variables $v_{j,h} - 1$ for $h \in G - (G_1 \cup \{h_0\})$; but formal power series rings correspond to open polydisks.

1.5. Endow \mathcal{K}_G with the log structure defined by \mathcal{Y}_G , and endow \mathcal{Y}_G with the pull back log structure. Then \mathcal{K}_G (resp. \mathcal{Y}_G) is log smooth over V (resp. T), and $U_G \rightarrow \mathcal{Y}_G$ is an exact closed embedding of (formal) T -log schemes. Denote by $\tilde{\omega}_{\mathcal{K}_G}^\bullet$ the logarithmic de Rham complex of $\mathcal{K}_G \rightarrow \mathrm{Spf}(W(k))$ with the trivial log structure on $\mathrm{Spf}(W(k))$. Let $P_\bullet \tilde{\omega}_{\mathcal{K}_G}^\bullet$ be the weight filtration on $\tilde{\omega}_{\mathcal{K}_G}^\bullet$:

$$P_j \tilde{\omega}_{\mathcal{K}_G}^k = \mathrm{Im}(\tilde{\omega}_{\mathcal{K}_G}^j \otimes \Omega_{\mathcal{K}_G}^{k-j} \longrightarrow \tilde{\omega}_{\mathcal{K}_G}^k),$$

where $\Omega_{\mathcal{K}_G}^\bullet$ is the usual de Rham complex of the morphism of schemes underlying $\mathcal{K}_G \rightarrow \mathrm{Spf}(W(k))$. For $G_1 \subset G_2$ we have natural transition maps $\mathcal{K}_{G_2} \rightarrow \mathcal{K}_{G_1}$, hence a simplicial formal scheme $\mathcal{K}_\bullet = \{\mathcal{K}_G\}_{G \subset H}$ with sheaf complexes $P_j \tilde{\omega}_{\mathcal{K}_\bullet}^\bullet$ on it. Also we have the closed simplicial formal sub schemes $\mathcal{Y}_\bullet = \{\mathcal{Y}_G\}_{G \subset H}$ and $\mathcal{Y}_{\bullet, \sigma}^i = \{\mathcal{Y}_{G, \sigma}^i\}_{G \subset H}$ for $i \geq 1$ and $\sigma \in S_i$. Denote by $\mathcal{J}_{\mathcal{Y}_\bullet}$ (resp. $\mathcal{J}_{\mathcal{Y}_{\bullet, \sigma}^i}$) the ideal of \mathcal{Y}_\bullet (resp. of $\mathcal{Y}_{\bullet, \sigma}^i$) in $\mathcal{O}_{\mathcal{K}_\bullet}$. Write $\theta = \mathrm{dlog}(t)$. Using the structure sheaf of the simplicial rigid space $]U_\bullet[_{\mathcal{Y}_\bullet}$ we give analytic analogs of the crystalline definitions from [10], [19]:

$$\begin{aligned} \mathbf{C}\tilde{\omega}_{\mathcal{Y}_\bullet}^\bullet &= \frac{\tilde{\omega}_{\mathcal{K}_\bullet}^\bullet}{\mathcal{J}_{\mathcal{Y}_\bullet} \otimes \tilde{\omega}_{\mathcal{K}_\bullet}^\bullet} \otimes_{\mathcal{O}_{\mathcal{Y}_\bullet}} sp_* \mathcal{O}_{]U_\bullet[_{\mathcal{Y}_\bullet}}, \\ P_j \mathbf{C}\tilde{\omega}_{\mathcal{Y}_\bullet}^\bullet &= \frac{P_j \tilde{\omega}_{\mathcal{K}_\bullet}^\bullet}{\mathcal{J}_{\mathcal{Y}_\bullet} \otimes \tilde{\omega}_{\mathcal{K}_\bullet}^\bullet} \otimes_{\mathcal{O}_{\mathcal{Y}_\bullet}} sp_* \mathcal{O}_{]U_\bullet[_{\mathcal{Y}_\bullet}}, \\ \mathbf{C}\omega_{\mathcal{Y}_\bullet}^\bullet &= \frac{\mathbf{C}\tilde{\omega}_{\mathcal{Y}_\bullet}^\bullet}{\mathbf{C}\tilde{\omega}_{\mathcal{Y}_\bullet}^{\bullet-1} \wedge \theta}. \end{aligned}$$

The following definitions which use the structure sheaves of the simplicial rigid spaces $]U_\bullet \cap Y_\sigma^i[_{\mathcal{Y}_{\bullet, \sigma}^i}$ have no analog in [10], [19]:

$$\begin{aligned} \mathbf{C}\tilde{\omega}_{\mathcal{Y}_{\bullet, \sigma}^i}^\bullet &= \frac{\tilde{\omega}_{\mathcal{K}_\bullet}^\bullet}{\mathcal{J}_{\mathcal{Y}_{\bullet, \sigma}^i} \otimes \tilde{\omega}_{\mathcal{K}_\bullet}^\bullet} \otimes_{\mathcal{O}_{\mathcal{Y}_{\bullet, \sigma}^i}} sp_* \mathcal{O}_{]U_\bullet \cap Y_\sigma^i[_{\mathcal{Y}_{\bullet, \sigma}^i}}, \\ P_j \mathbf{C}\tilde{\omega}_{\mathcal{Y}_{\bullet, \sigma}^i}^\bullet &= \frac{P_j \tilde{\omega}_{\mathcal{K}_\bullet}^\bullet + \mathcal{J}_{\mathcal{Y}_{\bullet, \sigma}^i} \otimes \tilde{\omega}_{\mathcal{K}_\bullet}^\bullet}{\mathcal{J}_{\mathcal{Y}_{\bullet, \sigma}^i} \otimes \tilde{\omega}_{\mathcal{K}_\bullet}^\bullet} \otimes_{\mathcal{O}_{\mathcal{Y}_{\bullet, \sigma}^i}} sp_* \mathcal{O}_{]U_\bullet \cap Y_\sigma^i[_{\mathcal{Y}_{\bullet, \sigma}^i}}, \\ &= \frac{P_j \tilde{\omega}_{\mathcal{K}_\bullet}^\bullet}{\mathcal{J}_{\mathcal{Y}_{\bullet, \sigma}^i} \otimes \tilde{\omega}_{\mathcal{K}_\bullet}^\bullet \cap P_j \tilde{\omega}_{\mathcal{K}_\bullet}^\bullet} \otimes_{\mathcal{O}_{\mathcal{Y}_{\bullet, \sigma}^i}} sp_* \mathcal{O}_{]U_\bullet \cap Y_\sigma^i[_{\mathcal{Y}_{\bullet, \sigma}^i}}, \\ \mathbf{C}\omega_{\mathcal{Y}_{\bullet, \sigma}^i}^\bullet &= \frac{\mathbf{C}\tilde{\omega}_{\mathcal{Y}_{\bullet, \sigma}^i}^\bullet}{\mathbf{C}\tilde{\omega}_{\mathcal{Y}_{\bullet, \sigma}^i}^{\bullet-1} \wedge \theta}, \\ \mathbf{C}\Omega_{\mathcal{Y}_{\bullet, \sigma}^i}^\bullet &= P_0 \mathbf{C}\tilde{\omega}_{\mathcal{Y}_{\bullet, \sigma}^i}^\bullet. \end{aligned}$$

We drop the σ in these notations when we sum over all $\sigma \in S_i$:

$$\begin{aligned} \mathbf{C}\tilde{\omega}_{\mathcal{Y}_\bullet}^\bullet &= \bigoplus_{\sigma \in S_i} \mathbf{C}\tilde{\omega}_{\mathcal{Y}_{\bullet, \sigma}^i}^\bullet, & P_j \mathbf{C}\tilde{\omega}_{\mathcal{Y}_\bullet}^\bullet &= \bigoplus_{\sigma \in S_i} P_j \mathbf{C}\tilde{\omega}_{\mathcal{Y}_{\bullet, \sigma}^i}^\bullet, \\ \mathbf{C}\omega_{\mathcal{Y}_\bullet}^\bullet &= \bigoplus_{\sigma \in S_i} \mathbf{C}\omega_{\mathcal{Y}_{\bullet, \sigma}^i}^\bullet, & \mathbf{C}\Omega_{\mathcal{Y}_\bullet}^\bullet &= \bigoplus_{\sigma \in S_i} \mathbf{C}\Omega_{\mathcal{Y}_{\bullet, \sigma}^i}^\bullet. \end{aligned}$$

We view all these sheaf complexes as living on $U_\bullet = \{U_G\}_G$. In our notation we will frequently drop the subscript bullet below Y (which holds the place for the varying G), thus we understand

$$\begin{aligned} \mathbf{C}\tilde{\omega}_Y^\bullet &= \mathbf{C}\tilde{\omega}_{\mathcal{Y}_\bullet}^\bullet, & \mathbf{C}\omega_Y^\bullet &= \mathbf{C}\omega_{\mathcal{Y}_\bullet}^\bullet, \\ \mathbf{C}\tilde{\omega}_{Y_\sigma^i}^\bullet &= \mathbf{C}\tilde{\omega}_{\mathcal{Y}_{\bullet, \sigma}^i}^\bullet, & \mathbf{C}\omega_{Y_\sigma^i}^\bullet &= \mathbf{C}\omega_{\mathcal{Y}_{\bullet, \sigma}^i}^\bullet, \\ \mathbf{C}\tilde{\omega}_{Y^i}^\bullet &= \mathbf{C}\tilde{\omega}_{\mathcal{Y}_\bullet}^\bullet, & \mathbf{C}\omega_{Y^i}^\bullet &= \mathbf{C}\omega_{\mathcal{Y}_\bullet}^\bullet. \end{aligned}$$

Moreover, to be consistent with the introduction we keep the names of complexes on U_\bullet also for their derived push forward on Y (via the morphism of simplicial schemes $U_\bullet \rightarrow Y$). Note that $\mathbf{C}\Omega_{Y_\sigma^i}^\bullet$ computes the non-logarithmic convergent (or equivalently: crystalline) cohomology of the classically smooth k -scheme Y_σ^i .

Recall from [11], 3.1 that on the formal log scheme T we have a Frobenius action: the unique endomorphism which equals σ on $W(k)$ and multiplication by p on the standard chart $\mathbb{N} \xrightarrow{0} W(k)$. We may assume that for each $h \in H$ there is an endomorphism of $(\mathcal{Z}_h, \mathcal{Y}_h)$ which lifts the Frobenius endomorphism (i.e. the p -power map on the structure sheaf) of its reduction modulo (p) and which sends equations for the divisors $\mathcal{Y}_{h,s}$ on \mathcal{Z}_h to their p -th power. Then we also get such Frobenius endomorphisms F on $\mathcal{K}_\bullet, \mathcal{Y}_\bullet$ and $\mathcal{Y}_{\bullet,\sigma}^i$ and on the various logarithmic de Rham complexes defined above. By abuse of notation we will frequently write $\mathbb{R}\Gamma(Y, K^\bullet)$ instead of $\mathbb{R}\Gamma(U_\bullet, K^\bullet)$ for sheaf complexes K^\bullet on U_\bullet .

Proposition 1.6. *The objects $\mathbb{R}\Gamma(Y, \mathbf{C}\omega_{\mathcal{Y}_\bullet}^\bullet)$, $\mathbb{R}\Gamma(Y, P_j \mathbf{C}\tilde{\omega}_{\mathcal{Y}_\bullet}^\bullet)$, $\mathbb{R}\Gamma(Y, \mathbf{C}\omega_{\mathcal{Y}_{\bullet,\sigma}^i}^\bullet)$ and $\mathbb{R}\Gamma(Y, P_j \mathbf{C}\tilde{\omega}_{\mathcal{Y}_{\bullet,\sigma}^i}^\bullet)$ are independent of the chosen system $\{(\mathcal{Z}_h, \mathcal{Y}_h)\}_{h \in H}$.*

Proof. Given another system $\{(\mathcal{Z}'_{h'}, \mathcal{Y}'_{h'})\}_{h' \in H'}$ one also performs the constructions from section 1.3 for $\{(\mathcal{Z}'_{h'}, \mathcal{Y}'_{h'})\}_{h' \in H'}$ and for the union of the systems $\{(\mathcal{Z}_h, \mathcal{Y}_h)\}_{h \in H}$ and $\{(\mathcal{Z}'_{h'}, \mathcal{Y}'_{h'})\}_{h' \in H'}$. We get canonical maps from the cohomology object formed with respect to this union to those formed with respect to $\{(\mathcal{Z}_h, \mathcal{Y}_h)\}_{h \in H}$ and $\{(\mathcal{Z}'_{h'}, \mathcal{Y}'_{h'})\}_{h' \in H'}$. That these are isomorphisms is a local statement and follows from 1.4 and the Poincaré lemma for relative open polydisks.

1.7. The logarithmic de Rham complex $\omega_{\mathcal{Y}_\bullet/T}^\bullet$ of $\mathcal{Y}_\bullet \rightarrow T$ is

$$\omega_{\mathcal{Y}_\bullet/T}^\bullet = \frac{\tilde{\omega}_{\mathcal{Y}_\bullet/T}^\bullet}{\tilde{\omega}_{\mathcal{Y}_\bullet/T}^{\bullet-1} \wedge \theta} \quad \text{with} \quad \tilde{\omega}_{\mathcal{Y}_\bullet/T}^\bullet = \tilde{\omega}_{\mathcal{K}_\bullet}^\bullet \otimes_{\mathcal{O}_{\mathcal{K}_\bullet}} \mathcal{O}_{\mathcal{Y}_\bullet}.$$

The logarithmic convergent cohomology in our context (see Ogus [22] and Shiho [24] for more general definitions) is given by the objects

$$\begin{aligned} \mathbb{R}\Gamma_{conv}(Y/T) &= \mathbb{R}\Gamma(Y, \omega_{\mathcal{Y}_\bullet/T}^\bullet \otimes_{\mathcal{O}_{\mathcal{Y}_\bullet}} sp_* \mathcal{O}_{U_\bullet[\mathcal{Y}_\bullet]}), \\ \mathbb{R}\Gamma_{conv}(Y_\sigma^i/T) &= \mathbb{R}\Gamma(Y, \omega_{\mathcal{Y}_\bullet/T}^\bullet \otimes_{\mathcal{O}_{\mathcal{Y}_\bullet}} sp_* \mathcal{O}_{U_\bullet \cap Y_\sigma^i[\mathcal{Y}_\bullet]}). \end{aligned}$$

Proposition 1.8. (i) *We have a canonical isomorphism*

$$\mathbb{R}\Gamma_{conv}(Y/T) \cong \mathbb{R}\Gamma_{crys}(Y/T)_{\mathbb{Q}}.$$

(ii) *There are canonical isomorphisms*

$$\mathbb{R}\Gamma_{conv}(Y_\sigma^i/T) \cong \mathbb{R}\Gamma(Y, \mathbf{C}\omega_{\mathcal{Y}_{\bullet,\sigma}^i}^\bullet).$$

Proof. (i) is due to a general comparison isomorphism between log convergent and log crystalline cohomology of a log smooth morphism; see e.g. [24]. In (ii) the canonical map is induced by the inclusion of simplicial rigid spaces $]U_\bullet \cap Y_\sigma^i[\mathcal{Y}_{\bullet,\sigma}^i \rightarrow]U_\bullet \cap Y_\sigma^i[\mathcal{Y}_\bullet$. That it is an isomorphism can be checked locally, so we may assume that there exists an affine admissible lift $(\mathcal{Z}, \mathcal{Y})$ of Y . For $s \in R$ let \mathcal{Y}_s be the $W(k)$ -flat irreducible component of \mathcal{Y} lifting Y_s . For $\tau \subset R$ let $\mathcal{Y}_\tau = \bigcap_{s \in \tau} \mathcal{Y}_s$ if $\emptyset \neq \tau$ and $\mathcal{Y}_\emptyset = \mathcal{Y}$. Then let

$$\mathcal{O}_\tau = \Gamma(]Y_{\tau \cup \sigma}^{\tau \cup \sigma}[_{\mathcal{Y}_\tau}, \mathcal{O}_{\mathcal{Y}_\tau}).$$

Let $\omega_{\mathcal{Y}}^\bullet$ be the logarithmic de Rham complex of \mathcal{Y}/T and write $L^\bullet = \Gamma(Y, \omega_{\mathcal{Y}}^\bullet \otimes \mathbb{Q})$. Then

$$\begin{aligned} \mathbb{R}^* \Gamma_{conv}(Y_\sigma^i/T) &= h^*(L^\bullet \otimes \mathcal{O}_\emptyset), \\ \mathbb{R}\Gamma(Y, \mathbf{C}\omega_{\mathcal{Y}_{\bullet,\sigma}^i}^\bullet) &= h^*(L^\bullet \otimes \mathcal{O}_\sigma). \end{aligned}$$

Thus we need to show that $L^\bullet \otimes \mathcal{O}_\emptyset \rightarrow L^\bullet \otimes \mathcal{O}_\sigma$ is a quasiisomorphism. The following purely formal reduction to Lemma 1.9 below is literally the same as in [9], Proposition 4.2 or [8], Theorem 3.14. For subsets $\mu \subset R$ we may form the closed formal subscheme $\mathcal{Y}^\mu = \bigcup_{s \in \mu} \mathcal{Y}_s$ of \mathcal{Y} (not to be confused with our previous notation \mathcal{Y}^j for $j \in \mathbb{N}$) and write

$$\mathcal{O}^\mu = \Gamma(\bigcup_{\sigma} Y_\sigma^i \cap \mathcal{Y}^\mu |_{\mathcal{Y}^\mu}, \mathcal{O}_{\mathcal{Y}^\mu}).$$

Similarly, for two subsets μ_1, μ_2 of R we write

$$\mathcal{O}^{\mu_1, \mu_2} = \Gamma(\bigcup_{\sigma} Y_\sigma^i \cap \mathcal{Y}^{\mu_1} \cap \mathcal{Y}^{\mu_2} |_{\mathcal{Y}^{\mu_1} \cap \mathcal{Y}^{\mu_2}}, \mathcal{O}_{\mathcal{Y}^{\mu_1} \cap \mathcal{Y}^{\mu_2}})$$

and

$$\mathcal{O}^{\mu_1}_{\mu_2} = \Gamma(\bigcup_{\sigma} Y_\sigma^{|\sigma \cup \mu_2|} \cap \mathcal{Y}^{\mu_1} |_{\mathcal{Y}^{\mu_1} \cap \mathcal{Y}^{\mu_2}}, \mathcal{O}_{\mathcal{Y}^{\mu_1} \cap \mathcal{Y}^{\mu_2}}).$$

We will show that in

$$L^\bullet \otimes \mathcal{O}_\emptyset \xrightarrow{\alpha} L^\bullet \otimes \mathcal{O}^\sigma \xrightarrow{\beta} L^\bullet \otimes \mathcal{O}_\sigma$$

both α and β are quasiisomorphisms. The exact sequences

$$\begin{aligned} 0 \longrightarrow \mathcal{O}_\emptyset \longrightarrow \mathcal{O}^\sigma \oplus \mathcal{O}^{R-\sigma} \longrightarrow \mathcal{O}^{\sigma, R-\sigma} \longrightarrow 0, \\ 0 \longrightarrow \mathcal{O}^\sigma \longrightarrow \mathcal{O}^\sigma \oplus \mathcal{O}^{\sigma, R-\sigma} \longrightarrow \mathcal{O}^{\sigma, R-\sigma} \longrightarrow 0 \end{aligned}$$

show that, to prove that α is a quasiisomorphism, it is enough to prove that $L^\bullet \otimes \mathcal{O}^{R-\sigma} \rightarrow L^\bullet \otimes \mathcal{O}^{\sigma, R-\sigma}$ is a quasiisomorphism. To see this, it is enough to show that both $L^\bullet \otimes \mathcal{O}^{R-\sigma} \xrightarrow{\gamma} L^\bullet \otimes \mathcal{O}_\sigma^{R-\sigma}$ and $L^\bullet \otimes \mathcal{O}^{\sigma, R-\sigma} \xrightarrow{\delta} L^\bullet \otimes \mathcal{O}_\sigma^{R-\sigma}$ are quasiisomorphisms. Consider the exact sequence

$$(*) \quad 0 \longrightarrow \mathcal{O}^{R-\sigma} \longrightarrow \bigoplus_{s \in R-\sigma} \mathcal{O}_s \longrightarrow \bigoplus_{\substack{\rho \subset R-\sigma \\ |\rho|=2}} \mathcal{O}_\rho \longrightarrow \dots \longrightarrow \mathcal{O}_{R-\sigma} \longrightarrow 0.$$

Comparison of the exact sequences $(*) \otimes L^\bullet$ and $(*) \otimes \mathcal{O}_\sigma^{R-\sigma} \otimes L^\bullet$ shows that to prove that γ is a quasiisomorphism, it is enough to show this for $L^\bullet \otimes \mathcal{O}_\rho \rightarrow L^\bullet \otimes \mathcal{O}_{\rho \cup \sigma}$ for all $\emptyset \neq \rho \subset R - \sigma$; but this is Lemma 1.9. Comparison of $(*) \otimes L^\bullet \otimes \mathcal{O}^{\sigma, R-\sigma}$ and $(*) \otimes L^\bullet \otimes \mathcal{O}_\sigma^{R-\sigma}$ shows that to prove that δ is a quasiisomorphism, it is enough to show this for $L^\bullet \otimes \mathcal{O}_\rho^\sigma \xrightarrow{\epsilon_\rho} L^\bullet \otimes \mathcal{O}_{\rho \cup \sigma}$ for all $\emptyset \neq \rho \subset R - \sigma$. Consider the exact sequence

$$(**) \quad 0 \longrightarrow \mathcal{O}^\sigma \longrightarrow \bigoplus_{s \in \sigma} \mathcal{O}_s \longrightarrow \bigoplus_{\substack{\gamma \subset \sigma \\ |\gamma|=2}} \mathcal{O}_\gamma \longrightarrow \dots \longrightarrow \mathcal{O}_\sigma \longrightarrow 0.$$

The exact sequence $(**) \otimes \mathcal{O}_\rho^\sigma \otimes L^\bullet$ shows that to prove that ϵ_ρ is a quasiisomorphism, it is enough to show this for $L^\bullet \otimes \mathcal{O}_{\rho \cup \gamma} \rightarrow L^\bullet \otimes \mathcal{O}_{\rho \cup \sigma}$ for all $\emptyset \neq \gamma \subset \sigma$; but this is Lemma 1.9. The exact sequence $(**) \otimes L^\bullet$ shows that to prove that β is a quasiisomorphism, it is enough to show this for $L^\bullet \otimes \mathcal{O}_\gamma \rightarrow L^\bullet \otimes \mathcal{O}_\sigma$ for all $\emptyset \neq \gamma \subset \sigma$; but this is Lemma 1.9.

Lemma 1.9. *For any inclusion $\emptyset \neq \tau_1 \subset \tau_2 \subset R$ with $\tau_2 - \tau_1 \subset \sigma$ the projection,*

$$\mu : L^\bullet \otimes \mathcal{O}_{\tau_1} \longrightarrow L^\bullet \otimes \mathcal{O}_{\tau_2}$$

is a quasiisomorphism.

Proof. Also this is as in [9], Proposition 4.2. By induction we may suppose $\tau_2 = \tau_1 \cup \{s_0\}$ for some $s_0 \in \sigma$ with $s_0 \notin \tau_2$. We may assume that there is a smooth morphism

$$\mathcal{Y} \longrightarrow \mathrm{Spf}(W(k)) \times_{\mathrm{Spf}(W(k)\{t\})} \mathrm{Spf}(W(k)\{t_1, \dots, t_m\})$$

lifting the situation described in section 1.1, and furthermore that \mathcal{Y}_{τ_2} is the closed formal subscheme of \mathcal{Y}_{τ_1} defined by $U := t_1$ (both are closed formal subschemes of \mathcal{Y}). Then

$$\mathcal{O}_{\tau_1} = \left\{ \sum_{m \geq 0} a_m U^m; a_m \in \mathcal{O}_{\tau_2}, \mathrm{ord}_p(a_m) \longrightarrow \infty \right\}.$$

Localizing further we may assume that there exist $s_1, \dots, s_n \in \mathcal{O}_{\mathcal{Y}}(\mathcal{Y})$ (with $n + m - 1 = \dim(Y)$) such that $\{ds_1, \dots, ds_n, \mathrm{dlog}(t_1), \dots, \mathrm{dlog}(t_{m-1})\}$ is an $\mathcal{O}_{\mathcal{Y}}(\mathcal{Y})$ -basis of L^1 . The images of these elements (denoted by the same names) in $L^1 \otimes \mathcal{O}_{\tau_i}$ then form an \mathcal{O}_{τ_i} -basis of $L^1 \otimes \mathcal{O}_{\tau_i}$ ($i = 1, 2$). Let L_c^1 , resp. $\langle \mathrm{dlog}(U) \rangle$, be the \mathcal{O}_{τ_2} -submodule of $L^1 \otimes \mathcal{O}_{\tau_2}$ generated by the set $\{ds_1, \dots, ds_n, \mathrm{dlog}(t_2), \dots, \mathrm{dlog}(t_{m-1})\}$, resp. the single element $\mathrm{dlog}(U) = \mathrm{dlog}(t_1)$. Let L_c^\bullet , resp. $\langle \mathrm{dlog}(U) \rangle^\bullet$, be the sub- \mathcal{O}_{τ_2} -algebra of $L^\bullet \otimes \mathcal{O}_{\tau_2}$ generated by L_c^1 , resp. by $\langle \mathrm{dlog}(U) \rangle$. These are in fact subcomplexes and we have the decomposition

$$L^\bullet \otimes \mathcal{O}_{\tau_2} = L_c^\bullet \otimes \langle \mathrm{dlog}(U) \rangle^\bullet.$$

By the above description of \mathcal{O}_{τ_1} it follows that the map μ in question has a natural “zero”-section ν , and it suffices to show that ν induces surjective maps in cohomology. Let $\omega \in L^k \otimes \mathcal{O}_{\tau_1}$. It can be written as

$$\omega = \sum_{m \geq 0} a_m U^m \mathrm{dlog}(U) + \sum_{m \geq 0} b_m U^m$$

with $a_m \in L_c^{k-1}$ and $b_m \in L_c^k$. Subtracting $d(\sum_{m > 0} m^{-1} a_m U^m)$ and renaming the coefficients we may write ω modulo exact forms as

$$\omega = a_0 \mathrm{dlog}(U) + \sum_{m \geq 0} b_m U^m.$$

If $d\omega = 0$ we get $\omega = a_0 \mathrm{dlog}(U) + b_0$ which lies in the image of ν .

1.10. *Remarks.* (1) An idealized log scheme is a log scheme together with an ideal in its log structure which maps to the zero element of the structure sheaf; see Ogus [21]. There is a notion of ideally log smooth morphisms between idealized log schemes, defined as usual by a lifting condition over exact closed nilimmersions. For $\sigma \in S_i$ let $\mathcal{F}_{i,\sigma} \subset \mathcal{N}_Y$ be the preimage of $\mathrm{Ker}(\mathcal{O}_Y \rightarrow \mathcal{O}_{Y_\sigma^i})$ in the log structure \mathcal{N}_Y of Y . On Y_σ^i the pair $(\mathcal{N}_Y, \mathcal{F}_{i,\sigma})$ induces the structure of an idealized log scheme. If we view T_1 as an idealized log scheme by simply taking the zero ideal in its log structure, the morphism $Y_\sigma^i \rightarrow T_1$ becomes ideally log smooth. One may define the idealized log crystalline site of Y_σ^i/T and will find that the cohomology (tensoring with \mathbb{Q}) of its structure sheaf is isomorphic to our $\mathbb{R}\Gamma(Y, \mathbb{C}\omega_{Y_\sigma^i}^\bullet)$. There are also Cartier isomorphisms valid in this context; one may therefore define corresponding de Rham-Witt complexes, and everything we develop here could be done with them as well.

(2) We could similarly define the idealized log convergent cohomology of Y_σ^i/T : it is isomorphic to the non-idealized version, as follows from 1.8. (We do not claim

coincidence of the idealized and the non-idealized log *crystalline* cohomology of Y_σ^i/T ; we suspect that the latter is not a useful theory.)

Proposition 1.11. *For $k \geq r \geq 1$ let $\tau \in S_k$, $\rho \in S_r$ such that $\rho \subset \tau$. Taking the residue along $\mathcal{Y}_{\bullet, \rho}^r$ defines a map $Res_\rho : \tilde{\omega}_{\mathcal{K}_\bullet}^q \rightarrow \tilde{\omega}_{\mathcal{K}_\bullet}^{q-r} \otimes \mathcal{O}_{\mathcal{Y}_{\bullet, \rho}^r}$. For $j \geq r$ it extends to a map*

$$Res_\rho : P_j \mathbf{C} \tilde{\omega}_{Y_{\bullet, \tau}^k}^q \longrightarrow P_{j-r} \mathbf{C} \tilde{\omega}_{Y_{\bullet, \tau}^k}^{q-r}.$$

Proof. Taking residues involves the choice of local coordinates; the problem is to show the independence of this choice. We must work on each \mathcal{K}_G (for $G \subset H$) separately. Let us write $\rho = \{s_1, \dots, s_r\}$ with $s_1 < \dots < s_r$ in our fixed ordering of R . We assume that $\mathcal{Y}_{G, \rho}^r \neq \emptyset$. Since we work locally we may suppose that there exist $t_1, \dots, t_r \in \mathcal{O}_{\mathcal{K}_G}$ such that $\mathcal{Y}_{G, s_i} = \mathbb{V}(t_i)$ for $1 \leq i \leq r$. Let

$$d\log(t_\rho) = d\log(t_r) \wedge \dots \wedge d\log(t_1).$$

For $\mu \subset R$ denote by $\tilde{\omega}_{\mathcal{K}_G, \mu}^\bullet$ the logarithmic differential module on \mathcal{K}_G with logarithmic poles along $\bigcup_{s \in \mu} \mathcal{Y}_{G, s}$. Let

$$\mathcal{P}^q = \text{Im}[\tilde{\omega}_{\mathcal{K}_G}^{r-1} \otimes \tilde{\omega}_{\mathcal{K}_G, R-\rho}^{q+1-r} \longrightarrow \tilde{\omega}_{\mathcal{K}_G}^q].$$

Now let $\omega \in \tilde{\omega}_{\mathcal{K}_G}^q$. It can be written as

$$\omega = \omega_0 + \eta \wedge d\log(t_\rho)$$

with $\omega_0 \in \mathcal{P}^q$ and $\eta \in \tilde{\omega}_{\mathcal{K}_G, R-\rho}^{q-r}$. We set

$$Res_\rho(\omega) = \eta \otimes 1 \in \tilde{\omega}_{\mathcal{K}_G}^{q-r} \otimes \mathcal{O}_{\mathcal{Y}_{G, \rho}^r}.$$

To see that this is well defined consider first another sum decomposition $\omega = \omega'_0 + \eta' \wedge d\log(t_\rho)$ with $\omega'_0 \in \mathcal{P}^q$ and $\eta' \in \tilde{\omega}_{\mathcal{K}_G, R-\rho}^{q-r}$. Then

$$(\eta - \eta') \wedge d\log(t_\rho) = \omega'_0 - \omega_0 \in \mathcal{P}^q$$

which implies that at least one of t_1, \dots, t_r divides $\eta - \eta'$, hence $\eta \otimes 1 = \eta' \otimes 1$ in $\tilde{\omega}_{\mathcal{K}_G}^{q-r} \otimes \mathcal{O}_{\mathcal{Y}_{G, \rho}^r}$.

To see independence of the chosen system t_1, \dots, t_r consider another system $t'_1, \dots, t'_r \in \mathcal{O}_{\mathcal{K}_G}$ such that $\mathcal{Y}_{G, s_i} = \mathbb{V}(t'_i)$ for $1 \leq i \leq r$. Any such system t'_1, \dots, t'_r arises from t_1, \dots, t_r by finitely many operations of the following type: multiply t_i for a single $1 \leq i \leq r$ by a unit in $\mathcal{O}_{\mathcal{K}_G}$. Thus we may assume $t'_{i_0} = \epsilon \cdot t_{i_0}$ for some $1 \leq i_0 \leq r$ and $\epsilon \in \mathcal{O}_{\mathcal{K}_G}^\times$, and $t'_i = t_i$ for all $i \neq i_0$. We want to show $Res'_\rho(\omega) = Res_\rho(\omega)$ for the residue map Res'_ρ defined with respect to $\{t'_i\}_{1 \leq i \leq r}$. Write

$$\omega = \omega_0 + \eta \wedge d\log(t'_\rho) + \eta \wedge (d\log(t_\rho) - d\log(t'_\rho)).$$

Clearly $Res'_\rho(\omega_0 + \eta \wedge d\log(t'_\rho)) = \eta$. Therefore we need to show $Res'_\rho(\mu) = 0$ for $\mu = \eta \wedge (d\log(t_\rho) - d\log(t'_\rho))$. Now

$$-\mu = \eta \wedge d\log(t_r) \wedge \dots \wedge d\log(t_{i_0+1}) \wedge d\log(\epsilon) \wedge d\log(t_{i_0-1}) \wedge \dots \wedge d\log(t_1)$$

and $d\log(\epsilon) \in P_0 \tilde{\omega}_{\mathcal{K}_G}^1$, hence $\mu \in \mathcal{P}^q$ and $Res'_\rho(\mu) = 0$.

Proposition 1.12. *For $i, j \geq 1$, $\sigma \in S_i$, there is a canonical isomorphism*

$$\text{Gr}_j \mathbf{C} \tilde{\omega}_{Y_{\bullet, \sigma}^i}^\bullet \cong \bigoplus_{\tau \in S_j} \mathbf{C} \Omega_{Y_{\bullet, \sigma \cup \tau}^i}^\bullet-j.$$

Proof. (Given $\sigma \in S_i$ and $\tau \in S_j$ we may form the union $\sigma \cup \tau \subset R$, an element of $S_{|\sigma \cup \tau|}$ with $|\sigma \cup \tau| \leq |\sigma| + |\tau| = i + j$.) Recall that \mathcal{Y}_\bullet is a relative normal crossings divisor in the smooth formal simplicial $W(k)$ -scheme \mathcal{K}_\bullet and that $\{\mathcal{Y}_{\bullet, \tau}^j\}_{\tau \in S_j}$ is the set of its j -codimensional intersection strata. In such a situation it is a classical fact that taking Poincaré residues (our 1.11 in the extreme case $j = r$) induces an isomorphism

$$\mathrm{Gr}_j \tilde{\omega}_{\mathcal{K}_\bullet}^\bullet \cong \bigoplus_{\tau \in S_j} \Omega_{\mathcal{Y}_{\bullet, \tau}^j}^\bullet-j$$

where $(\Omega_{\mathcal{Y}_{\bullet, \tau}^j}^\bullet, d)$ denotes the classical de Rham complex on the smooth formal simplicial $W(k)$ -scheme $\mathcal{Y}_{\bullet, \tau}^j$. We claim that this isomorphism restricts to an isomorphism between the $\mathcal{O}_{\mathcal{Y}_\bullet}$ -submodule generated by $\mathcal{J}_{\mathcal{Y}_{\bullet, \sigma}^i} \otimes \tilde{\omega}_{\mathcal{K}_\bullet}^\bullet \cap P_j \tilde{\omega}_{\mathcal{K}_\bullet}^\bullet$ and the $\mathcal{O}_{\mathcal{Y}_\bullet}$ -submodule generated by $(\Omega_{\mathcal{Y}_{\bullet, \tau}^j}^{\bullet-1} \wedge d(\mathcal{J}_{\mathcal{Y}_{\bullet, \sigma}^i}) + \mathcal{J}_{\mathcal{Y}_{\bullet, \sigma}^i} \cdot \Omega_{\mathcal{Y}_{\bullet, \tau}^j}^\bullet)[-j]$. Indeed, we may work locally around $\mathcal{Y}_{\bullet, \sigma \cup \tau}^{|\sigma \cup \tau|}$ for $\tau \in S_j$ and assume $\mathcal{Y}_{\bullet, \sigma}^i = \mathbb{V}(t_s)_{s \in \sigma}$ and $\mathcal{Y}_{\bullet, \tau}^j = \mathbb{V}(t_s)_{s \in \tau}$ for suitable $\{t_s\}_{s \in \sigma \cup \tau} \in \mathcal{O}_{\mathcal{K}_\bullet}$. Then the $\mathcal{O}_{\mathcal{Y}_\bullet}$ -submodule of the above left-hand side generated by $\mathcal{J}_{\mathcal{Y}_{\bullet, \sigma}^i} \otimes \tilde{\omega}_{\mathcal{K}_\bullet}^\bullet \cap P_j \tilde{\omega}_{\mathcal{K}_\bullet}^\bullet$ is in fact generated by elements of the form

$$\omega = t_{s_0} \omega' \wedge \bigwedge_{s \in \tau} \mathrm{dlog}(t_s)$$

for some $s_0 \in \sigma$ and $\omega' \in \tilde{\omega}_{\mathcal{K}_\bullet}^{\bullet-j}$ such that $t_{s_0} \omega' \in P_0 \tilde{\omega}_{\mathcal{K}_\bullet}^{\bullet-j}$. If $\mathrm{dlog}(t_{s_0})$ divides ω' (in the graded algebra $\tilde{\omega}_{\mathcal{K}_\bullet}^\bullet$), then dt_{s_0} divides $t_{s_0} \omega'$ and $\mathrm{Res}_\tau(\omega) \in \Omega_{\mathcal{Y}_{\bullet, \tau}^j}^{\bullet-1} \wedge d(\mathcal{J}_{\mathcal{Y}_{\bullet, \sigma}^i})$. If $\mathrm{dlog}(t_{s_0})$ does not divide ω' , then $\omega' \in P_0 \tilde{\omega}_{\mathcal{K}_\bullet}^{\bullet-j}$ and $\mathrm{Res}_\tau(\omega) \in \mathcal{J}_{\mathcal{Y}_{\bullet, \sigma}^i} \cdot \Omega_{\mathcal{Y}_{\bullet, \tau}^j}^\bullet$. Conversely, these local considerations show how to fabricate preimages of elements of $(\Omega_{\mathcal{Y}_{\bullet, \tau}^j}^{\bullet-1} \wedge d(\mathcal{J}_{\mathcal{Y}_{\bullet, \sigma}^i}) + \mathcal{J}_{\mathcal{Y}_{\bullet, \sigma}^i} \cdot \Omega_{\mathcal{Y}_{\bullet, \tau}^j}^\bullet)[-j]$ under the above isomorphism. The claim is established.

Dividing out these submodules we get the isomorphism

$$\frac{P_j \tilde{\omega}_{\mathcal{K}_\bullet}^\bullet}{P_{j-1} \tilde{\omega}_{\mathcal{K}_\bullet}^\bullet + (\mathcal{J}_{\mathcal{Y}_{\bullet, \sigma}^i} \otimes \tilde{\omega}_{\mathcal{K}_\bullet}^\bullet \cap P_j \tilde{\omega}_{\mathcal{K}_\bullet}^\bullet)} \cong \bigoplus_{\tau \in S_j} \Omega_{\mathcal{Y}_{\bullet, \sigma \cup \tau}^{|\sigma \cup \tau|}}^\bullet-j.$$

Tensoring over $\mathcal{O}_{\mathcal{Y}_\bullet}$ with $sp_* \mathcal{O}_{U_\bullet[y_\bullet]}$ we get the wanted isomorphism.

2. THE ČECH DOUBLE COMPLEX $B^{\bullet\bullet}$ AND THE STEENBRINK DOUBLE COMPLEX $A^{\bullet\bullet}$

2.1. For $e \geq 1$ and $\tau = \{\tau_1, \tau_2, \dots, \tau_e\} \in S_e$ with $\tau_1 < \tau_2 < \dots < \tau_{k+1}$ and $\gamma \in \tau$ define $\mathrm{pos}(\gamma \in \tau) \in \mathbb{N}$ by $\tau_{\mathrm{pos}(\gamma \in \tau)} = \gamma$. We continue to work with a fixed system $\{(\mathcal{Z}_h, \mathcal{Y}_h)\}_{h \in H}$ as before. Let $\mathbf{C}\omega_{Y_i}^q = 0$ if $q < 0$ or $i \leq 0$. For $s, t \in \mathbb{Z}$ we define the bidegree (s, t) -term of the Čech double complex $B^{\bullet\bullet}$ by

$$B^{st} = \mathbf{C}\omega_{Y^{t+1}}^s = \bigoplus_{\tau \in S_{t+1}} \mathbf{C}\omega_{Y_\tau}^s = \bigoplus_{\tau \in S_{t+1}} \frac{\mathbf{C}\tilde{\omega}_{Y_\tau}^{s-t+1}}{\mathbf{C}\tilde{\omega}_{Y_\tau}^{s-1} \wedge \theta}$$

The vertical differentials

$$B^{q(i-1)} = \mathbf{C}\omega_{Y^i}^q \xrightarrow{d} B^{(q+1)(i-1)} = \mathbf{C}\omega_{Y^i}^{q+1}$$

are those of $\mathbf{C}\omega_{Y^i}^\bullet$; the horizontal differentials are

$$B^{q(i-1)} = \mathbf{C}\omega_{Y^i}^q = \bigoplus_{\sigma \in S_i} \mathbf{C}\omega_{Y_\sigma^i}^q \xrightarrow{\epsilon} B^{qi} = \mathbf{C}\omega_{Y^{i+1}}^q = \bigoplus_{\tau \in S_{i+1}} \mathbf{C}\omega_{Y_\tau^{i+1}}^q$$

$$(\eta_\sigma)_{\sigma \in S_i} \mapsto \left(\sum_{j \in \tau} (-1)^{\text{pos}(j \in \tau) + q + 1} \eta_{(\tau-j)} \right)_{\tau \in S_{i+1}}.$$

Here $(\tau - j)$ means the element $\tau - \{j\}$ of S_i , and by $\eta_{(\tau-j)}$ we actually mean the image of $\eta_{(\tau-j)}$ under the obvious restriction map $\mathbf{C}\omega_{Y_{\tau-j}^i}^q \rightarrow \mathbf{C}\omega_{Y_\tau^{i+1}}^q$. We define the augmentation

$$\mathbf{C}\omega_Y^\bullet \xrightarrow{\epsilon_0} \mathbf{C}\omega_{Y^1}^\bullet = B^{\bullet 0}$$

as the sum of the canonical restriction maps (similar to the above differentials ϵ , but without alternating signs). It induces a morphism of complexes $\mathbf{C}\omega_Y^\bullet \rightarrow B^\bullet$ with B^\bullet the total complex of $B^{\bullet\bullet}$.

2.2. Copying [19] we define the *Steenbrink double complex* $(A^{ij})_{i,j}$ by

$$A^{ij} = \frac{\mathbf{C}\tilde{\omega}_Y^{i+j+1}}{P_j \mathbf{C}\tilde{\omega}_Y^{i+j+1}}$$

if $i \geq 0, j \geq 0$, and $= 0$ else. The vertical differentials $A^{ij} \rightarrow A^{i+1,j}$ are induced by $(-1)^j d : \mathbf{C}\tilde{\omega}_Y^{i+j+1} \rightarrow \mathbf{C}\tilde{\omega}_Y^{i+j+2}$; the horizontal differentials $A^{ij} \rightarrow A^{i,j+1}$ are induced by the assignment $\omega \mapsto \omega \wedge \theta$. The augmentation

$$\mathbf{C}\omega_Y^\bullet \longrightarrow A^{\bullet 0}, \quad \omega \mapsto \omega \wedge \theta$$

defines a morphism of complexes $\mathbf{C}\omega_Y^\bullet \rightarrow A^\bullet$ with A^\bullet the total complex of $A^{\bullet\bullet}$.

Proposition 2.3. (1) $\mathbf{C}\omega_Y^\bullet \rightarrow B^\bullet$ is a quasiisomorphism.

(2) $\mathbf{C}\omega_Y^\bullet \rightarrow A^\bullet$ is a quasiisomorphism.

(3) There is a morphism of double complexes $\psi : A^{\bullet\bullet} \rightarrow B^{\bullet\bullet}$ compatible with the respective augmentations by $\mathbf{C}\omega_Y^\bullet$. It induces a quasiisomorphism

$$\psi : A^\bullet \longrightarrow B^\bullet.$$

In particular we may identify

$$H_{\text{crys}}^*(Y/T)_{\mathbb{Q}} = H^*(Y, \mathbf{C}\omega_Y^\bullet) = H^*(Y, B^\bullet) = H^*(Y, A^\bullet).$$

Proof. For sheaf complexes K^\bullet on $U_\bullet = \{U_G\}_{\emptyset \neq G \subset H}$ and $\emptyset \neq G \subset H$ let $(K^\bullet)_G$ be the component of K^\bullet on U_G . For each G the complex $(\mathbf{C}\omega_Y^\bullet)_G$, resp. $(B^\bullet)_G$, resp. $(A^\bullet)_G$, is quasiisomorphic with the corresponding complex formed with respect to a single admissible lift of U_G : this follows from Lemma 1.4 and the Poincaré lemma for open polydisks (applied in the case of $(B^\bullet)_G$, resp. $(A^\bullet)_G$, to the subquotient complexes $(B^{\bullet,j})_G$, resp. $(A^{\bullet,j})_G$, for all j). Therefore we may assume for (1) and (2) that we are working with one global admissible lift $(\mathcal{Z}, \mathcal{Y})$ and drop G from our notation. To show (1) it is enough to show that

$$(*) \quad 0 \longrightarrow \mathbf{C}\omega_Y^q \longrightarrow \mathbf{C}\omega_{Y^1}^q \longrightarrow \mathbf{C}\omega_{Y^2}^q \longrightarrow \dots$$

is exact for each q . Since $\mathbf{C}\omega_{Y^i}^q = \mathbf{C}\omega_Y^q \otimes_{\mathcal{O}_Y} \mathcal{O}_{Y^i}$ and $\mathbf{C}\omega_Y^q$ is locally free over \mathcal{O}_Y , this follows from the exactness (Chinese remainder theorem) of

$$0 \longrightarrow \mathcal{O}_Y \longrightarrow \mathcal{O}_{Y^1} \longrightarrow \mathcal{O}_{Y^2} \longrightarrow \dots$$

Also statement (2) is reduced to the exactness of (*), literally as [19], 3.16 is reduced to [19], 3.15.1. Indeed, the proof in [19], 3.16, although written for logarithmic de

Rham Witt complexes, is in fact a completely general argument valid for logarithmic de Rham complexes for any (relative) normal crossings divisor on a (relative) smooth (formal) scheme (note that since we are working with one global admissible lift we are not taking rigid analytic tubes of proper subschemes; just as in [19], 3.16 one does not need to deal with divided power envelopes of proper subschemes).

We turn to (3). The assertion on quasiisomorphy follows from (1) and (2) once ψ with the stated $\mathbf{C}\omega_Y^\bullet$ -compatibility is defined. To do this first note that for $\sigma \in S_k$ the residue map $Res_\sigma : \tilde{\omega}_{\mathcal{K}^\bullet}^{q+k} \rightarrow \tilde{\omega}_{\mathcal{K}^\bullet}^q \otimes \mathcal{O}_{\mathcal{Y}_{\sigma}^k}$ described in 1.11 induces a residue map $Res_\rho : \mathbf{C}\tilde{\omega}_Y^{q+k} \rightarrow \mathbf{C}\tilde{\omega}_{Y_\sigma}^q$, and by construction the latter vanishes on $P_{k-1}\mathbf{C}\tilde{\omega}_Y^{q+k}$. In particular we may define

$$A^{q(k-1)} = \frac{\mathbf{C}\tilde{\omega}_Y^{q+k}}{P_{k-1}\mathbf{C}\tilde{\omega}_Y^{q+k}} \xrightarrow{\psi} B^{q(k-1)} = \bigoplus_{\sigma \in S_k} \frac{\mathbf{C}\tilde{\omega}_{Y_\sigma}^q}{\mathbf{C}\tilde{\omega}_{Y_\sigma}^{q-1} \wedge \theta},$$

$$\eta \mapsto (\alpha_{q,k} \cdot Res_\sigma(\eta))_{\sigma \in S_k}$$

for $k \geq 1$, where we set $\alpha_{q,k} = -1$ if $(q-1, k) \in 2\mathbb{Z} \times 2\mathbb{Z}$, and $\alpha_{q,k} = 1$ otherwise. We first verify that for the horizontal differentials $\wedge\theta$ of $A^{\bullet\bullet}$ and ϵ of $B^{\bullet\bullet}$ we have $\psi \circ (\wedge\theta) = \epsilon \circ \psi$. This can be done locally, in particular we may forget about distant components. So we may assume that there are $\{t_i\}_{1 \leq i \leq l} \in \mathcal{O}_{\mathcal{K}_G}$, units $\{t_i\}_{l+1 \leq i \leq g} \in \mathcal{O}_{\mathcal{K}_G}^\times$ (with $G \subset H$ fixed) and an order preserving identification between R and $\{1, \dots, l\}$ such that $\mathcal{Y}_{G,i} = \mathbb{V}(t_i)$ for all $i \in R = \{1, \dots, l\}$ and such that $\text{dlog}(t_1), \dots, \text{dlog}(t_g)$ is an $\mathcal{O}_{\mathcal{K}_G}$ -basis of $\tilde{\omega}_{\mathcal{K}_G}^1$. We may more specifically assume (after multiplying one of t_1, \dots, t_l by an appropriate unit) that

$$t = \prod_{1 \leq i \leq l} t_i \in \mathcal{O}_{\mathcal{K}_G},$$

the image of the distinguished element $t \in \mathcal{O}_V$. We identify $R = \{1, \dots, l\}$ with the set S_1 of subsets of $R = \{1, \dots, l\}$ with precisely one element. Then $\theta = \sum_{1 \leq i \leq l} \text{dlog}(t_i) = \sum_{i \in S_1} \text{dlog}(t_i)$. For $e \leq g$ we denote by \tilde{S}_e the set of subsets of $\{1, \dots, g\}$ with precisely e elements. We write

$$\text{dlog}(t_\nu) = \text{dlog}(t_{\nu_e}) \wedge \dots \wedge \text{dlog}(t_{\nu_1})$$

for $\nu \in \tilde{S}_e$ with elements $\nu_e > \dots > \nu_1$, and for $\gamma \in \nu$ we define $pos(\gamma \in \nu) \in \mathbb{N}$ by $\nu_{pos(\gamma \in \nu)} = \gamma$. By definition, for $\eta \in A^{q(k-1)}$ we have

$$\psi(\eta \wedge \theta) = (\alpha_{q,k+1} \sum_{i \in S_1} Res_\tau(\eta \wedge \text{dlog}(t_i)))_{\tau \in S_{k+1}},$$

$$\epsilon(\psi(\eta)) = (\alpha_{q,k} \sum_{i \in \tau} (-1)^{pos(i \in \tau) + q + 1} Res_{\tau-i}(\eta))_{\tau \in S_{k+1}}$$

in

$$B^{qk} = \mathbf{C}\omega_{Y^{k+1}}^q = \bigoplus_{\tau \in S_{k+1}} \frac{\mathbf{C}\tilde{\omega}_{Y_\tau}^q}{\mathbf{C}\tilde{\omega}_{Y_\tau}^{q-1} \wedge \theta}.$$

Now η can be written as a sum of elements of the type $\beta \wedge \text{dlog}(t_\alpha)$ with $\alpha \in \tilde{S}_{q+k}$ and $\beta \in \mathbf{C}\tilde{\omega}_Y^0$. Therefore we may assume $\eta = \beta \wedge \text{dlog}(t_\alpha)$ for some $\alpha \in \tilde{S}_{q+k}$ and $\beta \in \mathbf{C}\tilde{\omega}_Y^0$. We check equality of the τ -components of the above expressions for fixed $\tau \in S_{k+1}$. In the case $|\tau - (\alpha \cap \tau)| \geq 2$ both of them vanish. Now consider the case

$\tau = (\alpha \cap \tau) \cup i_0$ for some $i_0 \notin \alpha$. We have $\eta = \beta' \wedge \text{dlog}(t_{\alpha - (\alpha \cap \tau)}) \wedge \text{dlog}(t_{\alpha \cap \tau})$ with $\beta' = \beta$ or $\beta' = -\beta$. Then

$$\begin{aligned} \alpha_{q,k+1} \sum_{i \in S_1} \text{Res}_\tau(\eta \wedge \text{dlog}(t_i)) &= \alpha_{q,k+1} \text{Res}_\tau(\eta \wedge \text{dlog}(t_{i_0})) \\ &= \alpha_{q,k+1} (-1)^{\text{pos}(i_0 \in \tau) + 1} \text{Res}_\tau(\beta' \wedge \text{dlog}(t_{\alpha - (\alpha \cap \tau)}) \wedge \text{dlog}(t_\tau)) \\ &= \alpha_{q,k+1} (-1)^{\text{pos}(i_0 \in \tau) + 1} \beta' \wedge \text{dlog}(t_{\alpha - (\alpha \cap \tau)}) \end{aligned}$$

and

$$\alpha_{q,k} \sum_{i \in \tau} (-1)^{\text{pos}(i \in \tau) + q + 1} \text{Res}_{\tau-i} \eta = \alpha_{q,k} (-1)^{\text{pos}(i_0 \in \tau) + q + 1} \beta' \wedge \text{dlog}(t_{\alpha - (\alpha \cap \tau)})$$

and we see equality. Finally consider the case $\tau \subset \alpha$. We have $\eta = \beta' \wedge \text{dlog}(t_{\alpha - \tau}) \wedge \text{dlog}(t_\tau)$ with $\beta' = \beta$ or $\beta' = -\beta$. Then

$$\begin{aligned} \alpha_{q,k+1} \sum_{i \in S_1} \text{Res}_\tau(\eta \wedge \text{dlog}(t_i)) &= \alpha_{q,k+1} \sum_{i \in S_1 - \alpha} \text{Res}_\tau(\eta \wedge \text{dlog}(t_i)) \\ &= \alpha_{q,k+1} \sum_{i \in S_1 - \alpha} (-1)^{k+1 + \text{pos}(i \in i \cup (\alpha - \tau)) + 1} \text{Res}_\tau(\beta' \wedge \text{dlog}(t_{i \cup (\alpha - \tau)}) \wedge \text{dlog}(t_\tau)) \\ &= \alpha_{q,k+1} \sum_{i \in S_1 - \alpha} (-1)^{k + \text{pos}(i \in i \cup (\alpha - \tau))} \beta' \wedge \text{dlog}(t_{i \cup (\alpha - \tau)}) \end{aligned}$$

and

$$\begin{aligned} \alpha_{q,k} \sum_{i \in \tau} (-1)^{\text{pos}(i \in \tau) + q + 1} \text{Res}_{\tau-i} \eta &= \alpha_{q,k} \sum_{i \in \tau} (-1)^{k+1 + q + 1} \text{Res}_{\tau-i}(\beta' \wedge \text{dlog}(t_{\alpha - \tau}) \wedge \text{dlog}(t_i) \wedge \text{dlog}(t_{\tau-i})) \\ &= \alpha_{q,k} \sum_{i \in \tau} (-1)^{k + q + \text{pos}(i \in i \cup (\alpha - \tau)) + 1} \beta' \wedge \text{dlog}(t_{i \cup (\alpha - \tau)}). \end{aligned}$$

But in $\mathbf{C}\tilde{\omega}_{Y_\tau^{k+1}}^{q+1} / (\mathbf{C}\tilde{\omega}_{Y_\tau^{k+1}}^q \wedge \theta)$ the element

$$\begin{aligned} \beta' \wedge \text{dlog}(t_{\alpha - \tau}) \wedge \theta &= \sum_{i \in S_1} \beta' \wedge \text{dlog}(t_{\alpha - \tau}) \wedge \text{dlog}(t_i) \\ (3) \quad &= \sum_{i \in S_1 - (\alpha - \tau)} (-1)^{\text{pos}(i \in i \cup (\alpha - \tau))} \beta' \wedge \text{dlog}(t_{i \cup (\alpha - \tau)}) \end{aligned}$$

vanishes, thus again we obtain the desired equality and the proof of $\psi \circ (\wedge \theta) = \epsilon \circ \psi$ is finished. The compatibility of ψ with the vertical differentials d is easy to check. Now we check $\epsilon_0 = \psi \circ (\wedge \theta)$ for the augmentation maps $\epsilon_0 : \mathbf{C}\omega_Y^q \rightarrow B^{q0}$ and $(\wedge \theta) : \mathbf{C}\omega_Y^q \rightarrow A^{q0}$. Let $\eta \in \mathbf{C}\omega_Y^q$. We work with a system of coordinates $\{t_i\}$ as above, and again we may assume $\eta = \beta \wedge \text{dlog}(t_\alpha)$ for some $\alpha \in \tilde{S}_q$ and $\beta \in \mathbf{C}\omega_Y^0$. We find

$$\psi(\eta \wedge \theta) = (\text{Res}_j \sum_{i \in S_1} \eta \wedge \text{dlog}(t_i))_{j \in S_1} = (\text{Res}_j \sum_{i \in S_1 - \alpha} \eta \wedge \text{dlog}(t_i))_{j \in S_1},$$

and we see that for $j \in S_1 - \alpha$ the j -component is η , i.e. the j -component of $\epsilon_0(\eta)$. Now consider the respective j -components for $j \in \alpha$. We have $\eta = \beta' \wedge \text{dlog}(t_\sigma) \wedge$

$\mathrm{dlog}(t_j)$ with $\sigma = \alpha - j$ and $\beta' = \beta$ or $\beta' = -\beta$. Then the j -component of $\psi(\eta \wedge \theta)$ is

$$\begin{aligned} \mathrm{Res}_j \sum_{i \in S_1 - \alpha} (-1)^{\mathrm{pos}(i \in (\sigma \cup i))} \beta' \wedge \mathrm{dlog}(t_{\sigma \cup i}) \wedge \mathrm{dlog}(t_j) \\ = \sum_{i \in S_1 - \alpha} (-1)^{\mathrm{pos}(i \in (\sigma \cup i))} \beta' \wedge \mathrm{dlog}(t_{\sigma \cup i}). \end{aligned}$$

Since

$$\beta' \wedge \mathrm{dlog}(t_\sigma) \wedge \theta = \sum_{i \in S_1 - \alpha} (-1)^{\mathrm{pos}(i \in (\sigma \cup i)) + 1} \beta' \wedge \mathrm{dlog}(t_{\sigma \cup i})$$

vanishes in $\mathbf{C}\omega_{Y^1}^q$, we find that also in this case the j -component is η , i.e. the j -component of $\epsilon_0(\eta)$.

2.4. Both B^\bullet and A^\bullet are designed to compute $H_{\mathrm{crys}}^*(Y/T) \otimes \mathbb{Q}$ in terms of cohomology groups of the components of Y . A first advantage of B^\bullet against A^\bullet is the following. There does not seem to exist an obvious pairing on A^\bullet , compatible with the differential in the usual sense, extending the cup product on $\mathbf{C}\omega_Y^\bullet$ which induces the Poincaré-duality in cohomology from [10]. By contrast, such a pairing does exist on B^\bullet . Namely, define

$$\mathbf{C}\omega_{Y^{j+1}}^i \otimes \mathbf{C}\omega_{Y^{s+1}}^k \longrightarrow \mathbf{C}\omega_{Y^{j+s+1}}^{i+k}$$

by composing the restriction maps

$$\mathbf{C}\omega_{Y^{j+1}}^i \longrightarrow \mathbf{C}\omega_{Y^{j+s+1}}^i \quad \text{and} \quad \mathbf{C}\omega_{Y^{s+1}}^k \longrightarrow \mathbf{C}\omega_{Y^{j+s+1}}^k$$

(for these we do *not* use alternating signs as we did in the definition of the differential ϵ of B^\bullet) with the cup product

$$\mathbf{C}\omega_{Y^{j+s+1}}^i \otimes \mathbf{C}\omega_{Y^{j+s+1}}^k \longrightarrow \mathbf{C}\omega_{Y^{j+s+1}}^{i+k}.$$

2.5. On the double complex $B^{\bullet\bullet}$ define the Čech filtration $F_C^\bullet B^{\bullet\bullet}$ by setting $F_C^r B^{ik} = B^{ik}$ if $r < k + 1$, and $F_C^r B^{ik} = 0$ if $r \geq k + 1$. It gives rise to a spectral sequence

$$(C)_T \quad E_1^{pq} = H^q(Y, B^{\bullet p}) \implies H^{p+q}(Y, B^\bullet) = H_{\mathrm{crys}}^{p+q}(Y/T)_\mathbb{Q}.$$

We denote by $F_C^\bullet H_{\mathrm{crys}}^*(Y/T)_\mathbb{Q}$ the induced filtration on $H_{\mathrm{crys}}^*(Y/T)_\mathbb{Q}$.

2.6. Let ν be the bihomogeneous endomorphism of bidegree $(-1, 1)$ of $A^{\bullet\bullet}$ such that $(-1)^{j+1}\nu$ is the natural projection $A^{i,j} \rightarrow A^{i-1,j+1}$. By [19], 3.18 it induces the usual monodromy operator N on $H_{\mathrm{crys}}^*(Y/T)_\mathbb{Q} = H^*(Y, A^\bullet)$. (In fact our definition of ν differs from the one written in [19], 3.13 by the sign $(-1)^i$. It seems to us that to ensure that [19], 3.18 holds, our definition is the correct one). ν anticommutes with the differential of $A^{\bullet\bullet}$. Note that the filtration $\nu^\bullet A^{\bullet\bullet}$ of $A^{\bullet\bullet}$ by the images of the iterated applications of ν is just the stupid horizontal filtration. Since the morphism ψ from 2.3 sends $\nu^r A^{\bullet\bullet}$ to $F_C^r B^{\bullet\bullet}$ we see:

Corollary 2.7. *For all $r \geq 0$,*

$$\mathrm{Im}(H^*(Y, \nu^r A^\bullet) \longrightarrow H_{\mathrm{crys}}^*(Y/T)_\mathbb{Q}) \subset F_C^r H_{\mathrm{crys}}^*(Y/T)_\mathbb{Q}$$

in $H_{\mathrm{crys}}^(Y/T)_\mathbb{Q}$. In particular $\mathrm{Im}N^r \subset F_C^r H_{\mathrm{crys}}^*(Y/T)_\mathbb{Q}$ inside $H_{\mathrm{crys}}^*(Y/T)_\mathbb{Q}$.*

3. THE ČECH-STEENBRINK TRICOMPLEX

3.1. We now develop another tool to compare B^\bullet with A^\bullet : the Čech-Steenbrink tricomplex $C^{\bullet\bullet\bullet}$. For $k \geq 0, i \geq 0, j \geq 0$ its tridegree (ijk) -term is

$$C^{ijk} = \frac{\mathbf{C}\tilde{\omega}_{Y^{k+1}}^{i+j+1}}{P_j \mathbf{C}\tilde{\omega}_{Y^{k+1}}^{i+j+1}}.$$

For other triples (i, j, k) we let $C^{ijk} = 0$. The differentials $C^{ijk} \rightarrow C^{(i+1)jk}$ are those induced from $(-1)^j d : \mathbf{C}\tilde{\omega}_{Y^{k+1}}^{i+j+1} \rightarrow \mathbf{C}\tilde{\omega}_{Y^{k+1}}^{i+j+2}$. The differentials $C^{ijk} \rightarrow C^{i(j+1)k}$ are $\omega \mapsto \omega \wedge \theta$. The differentials $C^{ij(k-1)} \rightarrow C^{ijk}$ are

$$C^{ij(k-1)} = \bigoplus_{\sigma \in S_k} \frac{\mathbf{C}\tilde{\omega}_{Y^\sigma}^{i+j+1}}{P_j \mathbf{C}\tilde{\omega}_{Y^\sigma}^{i+j+1}} \xrightarrow{\epsilon} C^{ijk} = \bigoplus_{\tau \in S_{k+1}} \frac{\mathbf{C}\tilde{\omega}_{Y^\tau}^{i+j+1}}{P_j \mathbf{C}\tilde{\omega}_{Y^\tau}^{i+j+1}},$$

$$(\eta_\sigma)_{\sigma \in S_k} \mapsto \left(\sum_{\gamma \in \tau} (-1)^{\text{pos}(\gamma \in \tau) + i + 1} \eta_{(\tau - \gamma)} \right)_{\tau \in S_{k+1}}$$

(here $(\tau - \gamma)$ means the element $\tau - \{\gamma\}$ of S_k , and by $\eta_{(\tau - \gamma)}$ we actually mean the image of $\eta_{(\tau - \gamma)}$ under the restriction map).

Fix $k \geq 1$. The differentials of the tricomplex $C^{\bullet\bullet\bullet}$ induce on its subsheaf $C^{\bullet\bullet(k-1)}$ a structure of double complex; we denote its associated total complex by $C^{\bullet(k-1)}$, with the convention that $C^{ij(k-1)}$ is placed in degree $i + j$. Similarly we can form the complex $B^{\bullet(k-1)}$ with the convention that $B^{i(k-1)}$, when regarded as a position in $B^{\bullet(k-1)}$, is placed in degree i . The augmentation $B^{\bullet(k-1)} \rightarrow C^{\bullet 0(k-1)}$ is defined by $\omega \mapsto \omega \wedge \theta$.

Proposition 3.2. *This induces a quasiisomorphism $B^{\bullet(k-1)} \rightarrow C^{\bullet(k-1)}$.*

Proof. It is enough to show that for all $q \geq 0$ the sequence

$$0 \longrightarrow \mathbf{C}\omega_{Y^k}^q \longrightarrow C^{q,0,(k-1)} \longrightarrow C^{q,1,(k-1)} \longrightarrow C^{q,2,(k-1)} \longrightarrow \dots$$

is exact, in other words that

$$\mathbf{C}\tilde{\omega}_{Y^k}^{q-1} \xrightarrow{\wedge\theta} \mathbf{C}\tilde{\omega}_{Y^k}^q \xrightarrow{\wedge\theta} \frac{\mathbf{C}\tilde{\omega}_{Y^k}^{q+1}}{P_0 \mathbf{C}\tilde{\omega}_{Y^k}^{q+1}} \xrightarrow{\wedge\theta} \frac{\mathbf{C}\tilde{\omega}_{Y^k}^{q+2}}{P_1 \mathbf{C}\tilde{\omega}_{Y^k}^{q+2}} \xrightarrow{\wedge\theta} \dots$$

is exact. As in [19], 3.15 this is reduced to proving that

$$0 \longrightarrow \text{Gr}_0 \mathbf{C}\tilde{\omega}_{Y^k}^\bullet \xrightarrow{\wedge\theta} \text{Gr}_1 \mathbf{C}\tilde{\omega}_{Y^k}^\bullet[1] \xrightarrow{\wedge\theta} \text{Gr}_2 \mathbf{C}\tilde{\omega}_{Y^k}^\bullet[2] \xrightarrow{\wedge\theta} \dots$$

is an exact sequence of complexes. In view of 1.12 it is enough to show that for each component Y_s^k of Y^k the complex

$$0 \longrightarrow \mathbf{C}\Omega_{Y_s^k}^\bullet \longrightarrow \bigoplus_{t \in S_1} \mathbf{C}\Omega_{Y_t^1 \cap Y_s^k}^\bullet \longrightarrow \bigoplus_{t \in S_2} \mathbf{C}\Omega_{Y_t^2 \cap Y_s^k}^\bullet \longrightarrow \dots$$

is exact. The differentials in this complex are the sums of all restriction maps (with alternating signs) between the respective summands which make sense. It is then enough to show that for arbitrary q the complex

$$M^\bullet = [\mathbf{C}\Omega_{Y_s^k}^q \longrightarrow \bigoplus_{t \in S_1} \mathbf{C}\Omega_{Y_t^1 \cap Y_s^k}^q \longrightarrow \bigoplus_{t \in S_2} \mathbf{C}\Omega_{Y_t^2 \cap Y_s^k}^q \longrightarrow \dots]$$

is exact. This is the total complex of the following double complex $M^{\bullet\bullet}$. Let $J = \{u \in R \mid Y_s^k \subset Y_u\}$ and $I = R - J$. Let

$$M^{ij} = \bigoplus_{\substack{t \in S_{i+j} \\ |t \cap I| = j}} \mathbf{C}\Omega_{Y_t^{i+j} \cap Y_s^k}^q$$

for $i \geq 0$ and $j \geq 0$, and $M^{ij} = 0$ otherwise. The differentials $M^{ij} \rightarrow M^{i+1,j}$ and $M^{ij} \rightarrow M^{i,j+1}$ are the sums of all restriction maps (with alternating signs) between the respective summands which make sense. Now it is enough to show that for each fixed j the complex $M^{\bullet j}$ is exact. We have a direct sum decomposition of complexes

$$M^{\bullet j} = \bigoplus_{\substack{r \in S_j \\ r \subset \bar{I}}} M_r^\bullet,$$

$$M_r^i = \bigoplus_{\substack{t \in S_i \\ t \subset J}} \mathbf{C}\Omega_{Y_r^j \cap Y_s^k}^q$$

where the differentials $M_r^i \rightarrow M_r^{i+1}$ are the sums of identity maps (with appropriate alternating signs) between the respective summands. Since the complexes M_r^\bullet are exact, so is $M^{\bullet j}$.

We remark that as in 2.3 we could also have reduced to the case where we are dealing with a single global admissible lift. At least in that case the complexes M^{\bullet} are exact in positive degrees (with M^{i0} in degree 0). We do not need this.

3.3. Let C^\bullet be the total complex of the tricomplex $C^{\bullet\bullet\bullet}$. The augmentation

$$A^{\bullet\bullet} \longrightarrow C^{\bullet\bullet 0}$$

is defined as the sum of the canonical restriction maps (similar to the differentials $C^{ij(k-1)} \rightarrow C^{ij k}$, but without alternating signs). By 2.3 and 3.2 we have:

Corollary 3.4. *The augmentations $A^{\bullet\bullet} \rightarrow C^{\bullet\bullet 0}$ and $B^{\bullet\bullet} \rightarrow C^{\bullet 0}$ induce quasiisomorphisms $A^\bullet \rightarrow C^\bullet$ and $B^\bullet \rightarrow C^\bullet$. In particular we may identify $H_{crys}^*(Y/T)_{\mathbb{Q}} = H^*(Y, \mathbf{C}\omega_Y^\bullet) = H^*(Y, A^\bullet) = H^*(Y, B^\bullet) = H^*(Y, C^\bullet)$.*

3.5. Let ν be the trihomogeneous endomorphism of tridegree $(-1, 1, 0)$ of $C^{\bullet\bullet\bullet}$ such that $(-1)^{j+1}\nu$ is the natural projection $C^{ij(k-1)} \rightarrow C^{(i-1)(j+1)(k-1)}$. It is clear that the resolution $A^\bullet \rightarrow C^\bullet$ is compatible with the endomorphisms ν on source and target, thus ν on $C^{\bullet\bullet\bullet}$ induces the same endomorphism N in cohomology. For fixed $k \geq 1$ we also denote by N the operator on $H^*(Y, \mathbf{C}\omega_{Y^k}^\bullet)$ induced by the endomorphism $\nu : C^{\bullet\bullet(k-1)} \rightarrow C^{\bullet\bullet(k-1)}$ via the canonical isomorphism $H^*(Y, C^{\bullet(k-1)}) = H^*(Y, B^{\bullet(k-1)}) \cong H^*(Y, \mathbf{C}\omega_{Y^k}^\bullet)$.

Theorem 3.6. *For any $k \geq 1$ we have $N = 0$ on $H^*(Y, \mathbf{C}\omega_{Y^k}^\bullet)$.*

Proof. $C^{\bullet\bullet(k-1)}$ is the direct sum, over all $\sigma \in S_k$, of the double complexes

$$C_\sigma^{\bullet\bullet(k-1)} = \bigoplus_{i,j} C_\sigma^{ij(k-1)} \quad \text{with} \quad C_\sigma^{ij(k-1)} = \left(\frac{\mathbf{C}\tilde{\omega}_{Y_\sigma^k}^{i+j+1}}{P_j \mathbf{C}\tilde{\omega}_{Y_\sigma^k}^{i+j+1}} \right)_{i,j},$$

and ν is given by endomorphisms ν on each of these summands. We will show that on each summand ν is homotopic to zero. Fix $\sigma \in S_k$, choose an auxiliary $\gamma \in \sigma$

and define

$$C_\sigma^{ij(k-1)} \xrightarrow{h^{ij}} C_\sigma^{(i-1)j(k-1)},$$

$$\omega \mapsto (-1)^{j+1} Res_\gamma \omega$$

(this depends on γ). We claim

$$(1) \quad \nu(\omega) = h^{ij}(\omega) \wedge \theta - h^{i(j+1)}(\omega \wedge \theta)$$

for $\omega \in C_\sigma^{ij(k-1)}$. This can be verified locally, in particular we may forget about distant components. So we may assume that there are $\{t_i\}_{1 \leq i \leq l} \in \mathcal{O}_{\mathcal{K}_G}$, units $\{t_i\}_{l+1 \leq i \leq g} \in \mathcal{O}_{\mathcal{K}_G}^\times$ (with $G \subset H$ fixed) and an order preserving identification between R and $\{1, \dots, l\}$ such that $\mathcal{Y}_{G,i} = \mathbb{V}(t_i)$ for all $i \in R = \{1, \dots, l\}$ and such that $dlog(t_1), \dots, dlog(t_g)$ is an $\mathcal{O}_{\mathcal{K}_G}$ -basis of $\tilde{\omega}_{\mathcal{K}_G}^1$. We may more specifically assume (after multiplying one of t_1, \dots, t_l by an appropriate unit) that

$$t = \prod_{1 \leq i \leq l} t_i \in \mathcal{O}_{\mathcal{K}_G},$$

the image of the distinguished element $t \in \mathcal{O}_V$. We identify $R = \{1, \dots, l\}$ with the set S_1 of subsets of $R = \{1, \dots, l\}$ with precisely one element. Then $\theta = \sum_{1 \leq i \leq l} dlog(t_i) = \sum_{i \in S_1} dlog(t_i)$. For $e \leq g$ we denote by \tilde{S}_e the set of subsets of $\{1, \dots, g\}$ with precisely e elements. We write

$$dlog(t_\nu) = dlog(t_{\nu_e}) \wedge \dots \wedge dlog(t_{\nu_1})$$

for $\nu \in \tilde{S}_e$ with elements $\nu_e > \dots > \nu_1$, and for $\gamma \in \nu$ we define $pos(\gamma \in \nu) \in \mathbb{N}$ by $\nu_{pos(\gamma \in \nu)} = \gamma$. We may assume that ω is represented by $\beta dlog(t_\rho)$ for some $\beta \in \mathbb{C}\omega_{\mathcal{Y}_k}^0$ and $\rho \in \tilde{S}_{i+j+1}$. In the case $\gamma \in \rho$ we find

$$h^{ij}(\omega) = (-1)^{pos(\gamma \in \rho)+j} \beta dlog(t_{\rho-\gamma}),$$

hence

$$(i) \quad h^{ij}(\omega) \wedge \theta = \sum_{\alpha \in S_1 - (\rho - \gamma)} (-1)^{pos(\gamma \in \rho)+j} \beta dlog(t_{\rho-\gamma}) \wedge dlog(t_\alpha).$$

On the other hand $\omega \wedge \theta = \sum_{\alpha \in S_1 - \rho} \beta dlog(t_\rho) \wedge dlog(t_\alpha)$ and hence

$$(ii) \quad h^{i(j+1)}(\omega \wedge \theta) = \sum_{\alpha \in S_1 - \rho} (-1)^{pos(\gamma \in \rho)+j} \beta dlog(t_{\rho-\gamma}) \wedge dlog(t_\alpha).$$

The difference between (i) – (ii) is $(-1)^{j+1} \beta dlog(t_\rho) = \nu(\omega)$. In the case $\gamma \notin \rho$ we see $h^{ij}(\omega) = 0$, while as before $\omega \wedge \theta = \sum_{\alpha \in S_1 - \rho} \beta dlog(t_\rho) \wedge dlog(t_\alpha)$ and thus $h^{i(j+1)}(\omega \wedge \theta) = (-1)^{j+1} \beta dlog(t_\rho) = \nu(\omega)$, so (1) is proved. One also has

$$(2) \quad d(h^{ij}(\omega)) - h^{(i+1)j}(d\omega) = 0$$

where d denotes the differential $C_\sigma^{\bullet \bullet (k-1)} \rightarrow C_\sigma^{(\bullet+1) \bullet (k-1)}$. Now (1) and (2) together imply that on the complex $C_\sigma^{\bullet \bullet (k-1)}$, if \tilde{d} denotes its differential, we have $\nu = \tilde{d} \circ h - h \circ \tilde{d}$. This tells us that the endomorphisms ν of the graded algebra $C^{\bullet \bullet (k-1)}$, which anticommutes with \tilde{d} , induces the zero map in cohomology.

3.7. On $C^{\bullet\bullet\bullet}$ define the Čech filtration $F_C^\bullet C^{\bullet\bullet\bullet}$ by setting $F_C^r C^{ijk} = C^{ijk}$ if $r < k + 1$, and $F_C^r C^{ijk} = 0$ if $r \geq k + 1$. Then 3.2 says that $B^\bullet \rightarrow C^\bullet$ is a filtered quasiisomorphism with respect to the respective Čech filtrations. In general, for filtrations (on various complexes) denoted P_\bullet , resp. F_C^\bullet , we will use the notation Gr_\bullet , resp. Gr_C^\bullet , for the associated graded object.

Corollary 3.8. $N(F_C^k H_{\text{crys}}^*(Y/T)_\mathbb{Q}) \subset F_C^{k+1} H_{\text{crys}}^*(Y/T)_\mathbb{Q}$ for $k \geq 0$ (sharpening 2.7).

Proof. It is enough to show that for $k \geq 0$, the image of

$$H^*(Y, F_C^k B^\bullet) \xrightarrow{H^*(\nu)} H^*(Y, F_C^k B^\bullet)$$

is contained in the image of the natural map $H^*(Y, F_C^{k+1} B^\bullet) \xrightarrow{\iota} H^*(Y, F_C^k B^\bullet)$. In view of the exact sequence

$$H^*(Y, F_C^{k+1} B^\bullet) \xrightarrow{\iota} H^*(Y, F_C^k B^\bullet) \xrightarrow{pr} H^*(Y, \text{Gr}_C^k B^\bullet)$$

this is equivalent with showing that the image of $H^*(\nu)$ is contained in $\text{Ker}(pr)$. This follows from 3.6 using the canonical quasiisomorphism $B^{\bullet k}[k] \cong \text{Gr}_C^k B^\bullet$.

4. MONODROMY AND WEIGHT FILTRATION VIA THE ČECH COMPLEX B^\bullet

4.1. The endomorphisms $\Phi := F$ on the complexes $\mathbf{C}\omega_{Y^i}^\bullet$ induce a Frobenius endomorphism Φ on B^\bullet . The Frobenius endomorphism Φ on A^\bullet is defined by $\Phi := p^i F$ on A^{ij} . On A^\bullet the weight filtration $P_\bullet A^\bullet$ is defined by

$$P_k A^\bullet = \bigoplus_{i \geq 0, j \geq 0} P_k A^{ij}, \quad P_k A^{ij} = \frac{P_{2j+k+1} \mathbf{C}\tilde{\omega}_Y^{i+j+1}}{P_j \mathbf{C}\tilde{\omega}_Y^{i+j+1}}.$$

Denote by $\text{Gr}_\bullet A^\bullet$ the associated graded with respect to $P_\bullet A^\bullet$. We have

$$\text{Gr}_k A^\bullet \cong \bigoplus_{\substack{j \geq 0 \\ j \geq -k}} \mathbf{C}\Omega_{Y^{2j+k+1}}^\bullet[-2j-k](-j-k)$$

via residue maps (see [19], 3.22). Passing to the limit $n \rightarrow \infty$ and tensoring with \mathbb{Q} we get the weight spectral sequence

$$E_1^{-k, i+k} = \bigoplus_{\substack{j \geq 0 \\ j \geq -k}} H^{i-2j-k}(Y, \mathbf{C}\Omega_{Y^{2j+k+1}}^\bullet)(-j-k) \implies H_{\text{crys}}^i(Y/T)_\mathbb{Q}.$$

It gives rise to the *weight filtration* on $H_{\text{crys}}^*(Y/T)_\mathbb{Q}$. On the other hand, the operator N on $H_{\text{crys}}^*(Y/T)_\mathbb{Q}$ gives rise to the *monodromy filtration* on $H_{\text{crys}}^*(Y/T)_\mathbb{Q}$, defined in [19], 3.26. Consider the statement:

(MW) The weight spectral sequence degenerates in E_2 , and the weight filtration coincides with the monodromy filtration on $H_{\text{crys}}^*(Y/T)_\mathbb{Q}$.

The standard conjecture ([19], 3.24, 3.27) in this context states that (MW) holds whenever Y is the reduction of a projective semistable A -scheme as in section 7. Nakajima [20] proved the degeneration in E_2 unconditionally.

We see that N and the weight filtration on $H_{\text{crys}}^*(Y/T)_\mathbb{Q}$ can be obtained from corresponding structures on A^\bullet . The following Theorems 4.2 and 4.8 tell us that we can also recover N and the weight filtration from structures on B^\bullet .

Theorem 4.2. For $i \geq 0$ the i -fold iterated monodromy operator N^i on

$$H_{crys}^*(Y/T)_{\mathbb{Q}} = H^*(Y, \mathbf{C}\omega_Y^\bullet) = H^*(Y, B^\bullet)$$

is induced by a composite of complex morphisms

$$\mathbf{C}\omega_Y^\bullet \xrightarrow{\rho_i} F_C^i B^\bullet \xrightarrow{\iota} B^\bullet,$$

where ρ_i is given by explicit residue maps and ι is the inclusion.

Proof. Consider the quasiisomorphisms

$$\mathbf{C}\omega_Y^\bullet \xrightarrow{\wedge\theta} A^\bullet \quad \text{and} \quad A^\bullet \xrightarrow{\psi} B^\bullet.$$

The operator N^i on $H_{crys}^*(Y/T)_{\mathbb{Q}}$ is the map in cohomology induced by the composite $\psi \circ \nu^i \circ (\wedge\theta)$. This is the following map of complexes $\rho_i : \mathbf{C}\omega_Y^\bullet \rightarrow F_C^i B^\bullet$: An element $\eta \in \mathbf{C}\omega_Y^q$ is sent to

$$\epsilon(\text{Res}_\sigma(\eta \wedge \theta))_{\sigma \in S_{i+1}} \in \bigoplus_{\sigma \in S_{i+1}} \mathbf{C}\omega_{Y_\sigma^{i+1}}^{q-i} = B^{(q-i)i}$$

with the sign $\epsilon = \alpha_{(q-i), (i+1)} \prod_{j=0}^{i-1} (-1)^{j+1} \in \{\pm 1\}$ (note $\eta \wedge \theta \in \mathbf{C}\tilde{\omega}_Y^{q+1}$).

4.3. On $B^{\bullet\bullet}$ define the *canonical filtration* by setting

$$P_l^{can} B^{i(k-1)} = \frac{P_{l-1} \mathbf{C}\tilde{\omega}_{Y^k}^i}{(\mathbf{C}\tilde{\omega}_{Y^k}^{i-1} \wedge \theta) \cap P_{l-1} \mathbf{C}\tilde{\omega}_{Y^k}^i}$$

for $l \geq 0$. It induces a filtration $P_\bullet^{can} B^\bullet$ on B^\bullet . On $A^{\bullet\bullet}$ define the *canonical filtration* as the kernel filtration for ν , i.e.

$$P_l^{can} A^{\bullet\bullet} = \text{Ker}(\nu^l : A^{\bullet\bullet} \rightarrow A^{\bullet\bullet}),$$

or equivalently $P_l^{can} A^{ij} = P_{l-j-1} A^{ij}$ for $l \geq 0$. For any filtration denoted P_\bullet^{can} (here and on various other complexes below) we write Gr_\bullet^{can} for the associated graded object.

Lemma 4.4. The map $\psi : A^\bullet \rightarrow B^\bullet$ from 2.3 is a filtered quasiisomorphism with respect to canonical filtrations.

Proof. On $\mathbf{C}\omega_Y^\bullet$ define the canonical filtration by setting

$$P_l^{can} \mathbf{C}\omega_Y^i = \frac{P_{l-1} \mathbf{C}\tilde{\omega}_Y^i}{(\mathbf{C}\tilde{\omega}_Y^{i-1} \wedge \theta) \cap P_{l-1} \mathbf{C}\tilde{\omega}_Y^i} = \frac{P_{l-1} \mathbf{C}\tilde{\omega}_Y^i}{P_{l-2} \mathbf{C}\tilde{\omega}_Y^{i-1} \wedge \theta}$$

for $l \geq 0$. The second equality in this definition can be justified by induction on l , noting that the natural surjection

$$\frac{P_{l-1} \mathbf{C}\tilde{\omega}_Y^i}{(P_{l-2} \mathbf{C}\tilde{\omega}_Y^{i-1} \wedge \theta) + (P_{l-2} \mathbf{C}\tilde{\omega}_Y^i)} \longrightarrow \frac{P_{l-1} \mathbf{C}\tilde{\omega}_Y^i}{((\mathbf{C}\tilde{\omega}_Y^{i-1} \wedge \theta) \cap P_{l-1} \mathbf{C}\tilde{\omega}_Y^i) + (P_{l-2} \mathbf{C}\tilde{\omega}_Y^i)}$$

is also injective because its composition with the map

$$\frac{P_{l-1} \mathbf{C}\tilde{\omega}_Y^i}{((\mathbf{C}\tilde{\omega}_Y^{i-1} \wedge \theta) \cap P_{l-1} \mathbf{C}\tilde{\omega}_Y^i) + (P_{l-2} \mathbf{C}\tilde{\omega}_Y^i)} \xrightarrow{\wedge\theta} \frac{P_l \mathbf{C}\tilde{\omega}_Y^{i+1}}{P_{l-1} \mathbf{C}\tilde{\omega}_Y^{i+1}}$$

is injective by [19], 3.15. Using the second defining expression for $P_l^{can} \mathbf{C}\omega_Y^\bullet$ we get from [19], 3.15 that $\wedge\theta : \mathbf{C}\omega_Y^\bullet \rightarrow A^\bullet$ is a filtered resolution w.r.t. canonical filtrations. Using the first one we get that the same exists for $\epsilon_0 : \mathbf{C}\omega_Y^\bullet \rightarrow B^\bullet$. The lemma follows.

For the rest of this section we assume that k is finite and Y is proper and satisfies (MW). We fix a cohomology degree $*$ and simply write H^* for $H^*_{crys}(Y/T)_{\mathbb{Q}}$.

Proposition 4.5 (Chiarellotto, [3]). (1) Let $r \in \mathbb{N}$. The sequence

$$H^*(Y, P_r^{can} A^\bullet) \longrightarrow H^* \xrightarrow{N^r} H^* \longrightarrow H^*(Y, \text{Coker}(\nu^r))$$

is exact. In particular we have

$$\text{Im}(H^*(Y, \nu^r A^\bullet) \longrightarrow H^*) = \text{Im}(H^* \xrightarrow{N^r} H^*).$$

(2) We have

$$H_{rig}^*(Y) = H_{rig}^*(Y/K_0) = H^*(Y, P_1^{can} A^\bullet).$$

4.6. Define the Čech filtration on $H_{rig}^*(Y)$ by setting

$$F_C^s H_{rig}^*(Y) = \text{Im}(H^*(Y, P_1^{can} F_C^s A^\bullet) \longrightarrow H_{rig}^*(Y))$$

for $s \geq 0$, where we write $F_C^s A^\bullet = \nu^s A^\bullet$. It indeed deserves its name: it is the filtration arising from the spectral sequence

$$(C)_{rig} \quad E_1^{pq} = H_{rig}^q(Y^{p+1}) \implies H_{rig}^{p+q}(Y)$$

(we do not need this fact). Let $H_{rig}^*(Y) \xrightarrow{\iota} H^*$ be the canonical map. By Chiarellotto's result [3], since we assume (MW), we have $\text{Im}(\iota) = \text{Ker}(N)$.

Proposition 4.7. For all $r \geq 0$ we have

$$\iota(F_C^r H_{rig}^*(Y)) = \text{Ker}(N) \cap \text{Im}(N^r).$$

Proof. This follows formally from the fact that the filtrations $F_C^\bullet H_{rig}^*(Y)$ on $H_{rig}^*(Y)$ and $\text{Ker}(N) \cap \text{Im}(N^\bullet)$ on $\text{Im}(\iota) = \text{Ker}(N)$ are in fact weight filtrations for the Frobenius actions, with the same strictly monotone increasing sequences of weights on the graded pieces. Indeed, we have

$$P_1^{can} \text{Gr}_C^r A^\bullet \cong \text{Gr}_{-r} A^{\bullet r}[-r] \cong \text{Gr}_{r+1} \mathbf{C}\tilde{\omega}_Y^\bullet[1](r+1) \cong \mathbf{C}\Omega_{Y^{r+1}}^{\bullet-r}$$

(as in [19], 3.22(2)). Therefore $H^*(Y, P_1^{can} \text{Gr}_C^r A^\bullet)$ is pure of weight $* - r$ in view of [4], hence also its subquotient $\text{Gr}_C^r H_{rig}^*(Y)$. On the other hand, by (MW) the subquotient $(\text{Ker}(N) \cap \text{Im}(N^r))/(\text{Ker}(N) \cap \text{Im}(N^{r+1}))$ of H^* is pure of weight $* - r$.

Theorem 4.8. Denote by G_\bullet the convolution of the filtrations P_\bullet^{can} and F_C^\bullet on B^\bullet , i.e. the filtration defined by

$$G_k = \sum_{i \geq 0} P_{i+1}^{can} F_C^{i-k} B^\bullet.$$

Then G_\bullet induces the monodromy filtration on $H^*_{crys}(Y/T)_{\mathbb{Q}} = H^*(Y, B^\bullet)$.

Proof. We will show that $H^*(Y, \text{Gr}_k^{can} \text{Gr}_C^r B^\bullet)$ is pure of weight $* - r + k - 1$. This then implies that the filtration induced by G_\bullet is the weight filtration for Frobenius on $H^*_{crys}(Y/T)_{\mathbb{Q}}$, i.e. the uniquely determined Frobenius stable filtration whose subquotients are pure and of mutually different weights; but this is the monodromy filtration, by (MW).

Define the double complex

$$\tilde{B}^{\bullet\bullet} = (\mathbf{C}\tilde{\omega}_{Y^{i+1}}^q)_{q,i}$$

with differentials analogous to those in section 2.1. Let $\tilde{B}^\bullet, \tilde{B}^\bullet$ be the associated total complexes. As in section 2.5 define the Čech filtration $F_C^\bullet \tilde{B}^\bullet$ by setting

$F_C^r \tilde{B}^{i(k-1)} = \tilde{B}^{i(k-1)}$ if $r < k$, and $F_C^r \tilde{B}^{i(k-1)} = 0$ if $r \geq k$. Define the canonical filtration by setting $P_l^{can} \tilde{B}^{i(k-1)} = P_{l-1} \mathbf{C}\tilde{\omega}_{Y^k}^i$ for $l \geq 0$.

Claim: $H^*(Y, \mathrm{Gr}_k^{can} \mathrm{Gr}_C^r \tilde{B}^\bullet)$ is pure of weight $* - r + k - 1$, for all $k \geq 1$. For all $k \geq 2$ $H^*(Y, \mathrm{Gr}_k^{can} F_C^r \tilde{B}^\bullet)$ is also pure of weight $* - r + k - 1$.

The assertion in case $k = 1$ follows from the proof of 4.7, thanks to the isomorphism $\psi : P_1^{can} A^\bullet \cong P_1^{can} B^\bullet = P_1^{can} \tilde{B}^\bullet$. Now let $k \geq 2$. We have

$$\mathrm{Gr}_k^{can} \mathrm{Gr}_C^s \tilde{B}^\bullet = \mathrm{Gr}_{k-1} \mathbf{C}\tilde{\omega}_{Y^{s+1}}^\bullet[-s] \cong \mathbf{C}\Omega_{Y^{s+1} \cap Y^{k-1}}^\bullet[-s - k + 1](-k + 1).$$

So $H^*(Y, \mathrm{Gr}_k^{can} \mathrm{Gr}_C^r \tilde{B}^\bullet)$ is pure of weight $* - s - k + 1 - 2(-k + 1) = * - r + k - 1$. It follows that $H^*(Y, \mathrm{Gr}_k^{can} F_C^r \tilde{B}^\bullet)$ is mixed with weights at most $* - r + k - 1$, and $H^*(Y, \mathrm{Gr}_k^{can}(\tilde{B}^\bullet / F_C^r \tilde{B}^\bullet))$ is mixed with weights at least $* - r + k$. On the other hand, $\mathrm{Gr}_k^{can} \tilde{B}^\bullet$ is quasiisomorphic with $\mathrm{Gr}_{k+1} \mathbf{C}\tilde{\omega}_Y^\bullet$ via the canonical augmentation $\mathbf{C}\tilde{\omega}_Y^\bullet \rightarrow \tilde{B}^{\bullet 0}$. But $\mathrm{Gr}_{k+1} \mathbf{C}\tilde{\omega}_Y^\bullet \cong \mathbf{C}\Omega_{Y^j}^\bullet-j$, thus $H^*(Y, \mathrm{Gr}_k^{can} \mathbf{C}\tilde{\omega}_Y^\bullet)$ is pure of weight $* + k - 1$. Now the long cohomology sequence associated to

$$0 \longrightarrow \mathrm{Gr}_k^{can} F_C^r \tilde{B}^\bullet \longrightarrow \mathrm{Gr}_k^{can} \tilde{B}^\bullet \longrightarrow \mathrm{Gr}_k^{can}(\tilde{B}^\bullet / F_C^r \tilde{B}^\bullet) \longrightarrow 0$$

gives the second statement of the claim.

To prove that $H^*(Y, \mathrm{Gr}_k^{can} \mathrm{Gr}_C^r B^\bullet)$ is pure of weight $* - r + k - 1$ it is now enough to show that for all $k \geq 1$ the natural map

$$H^*(Y, \mathrm{Gr}_k^{can} \mathrm{Gr}_C^r \tilde{B}^\bullet) \longrightarrow H^*(Y, \mathrm{Gr}_k^{can} \mathrm{Gr}_C^r B^\bullet)$$

is surjective. For this we need to show that the connecting maps in cohomology associated with the short exact sequence

$$0 \longrightarrow \mathrm{Gr}_{k-1}^{can} \mathrm{Gr}_C^r B^{\bullet-1} \xrightarrow{\wedge^\theta} \mathrm{Gr}_k^{can} \mathrm{Gr}_C^r \tilde{B}^\bullet \longrightarrow \mathrm{Gr}_k^{can} \mathrm{Gr}_C^r B^\bullet \longrightarrow 0$$

are zero. This follows from 3.6, because as in [19], 3.18 one sees that these connecting maps are induced by the endomorphism ν of $\mathrm{Gr}_C^r B^\bullet$.

5. ČECH FILTRATION VERSUS $\mathrm{Im}(N^\bullet)$ FILTRATION

We give a sufficient criterion for the equality $\mathrm{Im}(N^r) = F_C^r H^*$, but we also give an example with $\mathrm{Im}(N^r) \neq F_C^r H^*$, although (MW) holds true.

Lemma 5.1. *Let V be an abelian group, $m, n \in \mathbb{N}$, $n \leq m$, let $(0) = F^m \subset \dots \subset F^1 \subset F^0 = V$ be a descending filtration, and let $N \in \mathrm{Hom}(V, V)$ such that $N(F^{i-1}) \subset F^i$ and $\mathrm{Ker}(N) \cap N^i V = \mathrm{Ker}(N) \cap F^i$ for all $i \geq n$. Then $N^i V = F^i$ for all $i \geq n$.*

Proof. Descending induction on i : let $x \in F^{i-1}$. Then $Nx = N^i y$ for some y by induction hypothesis, thus $x - N^{i-1} y \in \mathrm{Ker}(N) \cap F^{i-1}$. By assumption this means $N^{i-1} z = x - N^{i-1} y$ for some z , i.e. $x = N^{i-1}(z - y)$.

Proposition 5.2. *Suppose k is finite, Y is proper and satisfies (MW). Suppose in addition that the natural map $\iota : H_{rig}^*(Y) \rightarrow H^*$ is strict with respect to the canonical Čech filtrations, i.e. $\iota(F_C^r H_{rig}^*(Y)) = \iota(H_{rig}^*(Y)) \cap F_C^r H^*$ for all r . Then we have $\mathrm{Im}(N^r) = F_C^r H^*$ for all $r \geq 0$.*

Proof. By 5.1 we only need to show $\mathrm{Ker}(N) \cap F_C^r H^* = \mathrm{Ker}(N) \cap \mathrm{Im}(N^r)$ for all r , but this is obviously implied by strictness of ι together with 4.7.

5.3. (1) The condition on ι in 5.2 is formulated in terms of cohomology over the log base T (using the double complex B^\bullet). But note that, given a lift of Y to a semistable scheme X over a finite totally ramified extension \mathcal{O}_K of $W(k)$, the filtration $F_C^r H^*$ becomes the canonical Čech filtration on $H_{dR}^*(X_K)$ induced from $(C)_S$ (see section 7), and the condition on ι becomes an entirely analytic condition not involving log structures.

(2) Assuming (MW), the condition on ι holds if convoluting the filtrations P_{\bullet}^{can} and F_C^\bullet on B^\bullet commutes with passing to cohomology; by 4.8 this is equivalent to: the convolution of the filtrations $\text{Ker}(N^\bullet)$ and $F_C^\bullet H^*$ on H^* is a weight filtration on H^* .

5.4. **Examples.** (1) For curves we always have $\text{Im}(N) = F_C^1 H^1$; see [5].

(2) Let $\Omega_K^{(d+1)}$ be Drinfel'd's p -adic symmetric space of dimension d over a finite totally ramified extension K of K_0 , let X_Γ be the quotient of $\Omega_K^{(d+1)}$ by a cocompact discrete torsionfree subgroup $\Gamma < PGL_{d+1}(K)$ and let Y be its strictly semistable reduction. It has interesting cohomology only in degree $* = d$ where it satisfies (MW), as was recently proven by de Shalit [6] and (independently) by Ito [13]. By [9], if d is odd, we have $\text{Im}(N^r) = F_C^r H^*$ for all r . But if d is even we do *not always* have $\text{Im}(N^r) = F_C^r H^*$. For example, let $d = 2$. On $H_{dR}^2(X_\Gamma)$ we have a covering filtration $(F_\Gamma^r)_{r \geq 0}$ and in [9] it is shown that it coincides with the Čech filtration $(F_C^r)_{r \geq 0}$ on $H_{dR}^2(X_\Gamma)$ (see section 7), and moreover that

$$F_\Gamma^j = \sum_i \text{Ker}(N^{i+1}) \cap \text{Im}(N^{i-2+2j}) = \sum_i \text{Ker}(N^{i+1}) \cap \text{Im}(N^{i-3+2j})$$

for the monodromy operator N on $H_{dR}^2(X_\Gamma)$ induced from $H_{crys}^2(Y/T)$ by means of $H_{dR}^2(X_\Gamma) \cong H_{crys}^2(Y/T) \otimes_{W(k)} K = H^2 \otimes_{K_0} K$. Observing $N^3 = 0$ we get

$$F_\Gamma^1 = \text{Ker}(N) + \text{Im}(N),$$

$$F_\Gamma^2 = \text{Ker}(N) \cap \text{Im}(N) + \text{Im}(N^2) = \text{Im}(N^2).$$

Now *assume* $\text{Im}(N) = F_C^1 H^2$ in $H^2 = H_{crys}^2(Y/T)_\mathbb{Q}$. Then $\text{Im}(N) = F_\Gamma^1$ in $H_{dR}^2(X_\Gamma)$. Combining this with the above identities easily leads to $\text{Ker}(N) = \text{Im}(N^2)$. Therefore N induces an isomorphism $\text{Im}(N)/\text{Im}(N^2) \cong \text{Im}(N^2)$. On the other hand, by the computations in [23], p. 93 we have

$$\dim_K(\text{Im}(N^2)) = \dim_K(F_\Gamma^2) = \dim_K(F_\Gamma^1/F_\Gamma^2) - 1 = \dim_K(\text{Im}(N)/\text{Im}(N^2)) - 1.$$

Together this is a contradiction, disproving our assumption.

6. N AND SINGULAR COHOMOLOGY

6.1. In this section, k is finite and Y is proper. For $j \geq 1$, irreducible components M of Y^j and irreducible components N of Y^{j+1} with $N \subset M$, denote by

$$c_{M,N}^s : H_{crys}^s(M)_\mathbb{Q} \longrightarrow H_{crys}^s(N)_\mathbb{Q}$$

the natural restriction maps. We say Y is of weak Lefschetz type if for every $j \geq 1$, with every pair (N, M) as above, the maps $c_{M,N}^s$ are isomorphisms if $s < (\dim N) = d - j$, and also if the map $c_{M,N}^{d-j}$ is injective, with $\text{Im}(c_{M,N}^{d-j}) = h_N^{d-j} \subset H_{crys}^{d-j}(N)_\mathbb{Q}$ independent of M . For example, by [16], Y is of weak Lefschetz type if it is projective and all embeddings $N \subset M$ as above are ample divisors on M .

Lemma 6.2. *Suppose Y is of weak Lefschetz type, and that for each $i \geq 1$, each component of Y^i is geometrically connected. Let $s \leq d - 1$ and suppose $H^i(Y_{Zar}, \mathbb{Q}) = 0$ for all $d - s > i > 0$. Then the following sequence is exact:*

$$(*) \quad H_{crys}^s(Y^1)_{\mathbb{Q}} \longrightarrow H_{crys}^s(Y^2)_{\mathbb{Q}} \longrightarrow \dots \longrightarrow H_{crys}^s(Y^{d-s+1})_{\mathbb{Q}}.$$

Proof. The geometrical connectedness of each component L of Y^i for each i implies $H_{crys}^0(L)_{\mathbb{Q}} = K_0$ for each such L . Therefore the complex (with $H_{crys}^0(Y^1)_{\mathbb{Q}}$ in degree 0)

$$(**) \quad H_{crys}^0(Y^1)_{\mathbb{Q}} \longrightarrow H_{crys}^0(Y^2)_{\mathbb{Q}} \longrightarrow \dots \longrightarrow H_{crys}^0(Y^{d+1})_{\mathbb{Q}} \longrightarrow 0$$

computes $H^*(Y_{Zar}, K_0) = H^*(Y_{Zar}, \mathbb{Q}) \otimes K_0$, thus is exact by our hypothesis when truncated after the degree $(d - s)$ term. Now note that the weak Lefschetz assumption allows us to naturally identify into one single object H^s all the following K_0 -vector spaces: the cohomology groups $H_{crys}^s(L)_{\mathbb{Q}}$ for all components L of Y^j for all $1 \leq j \leq d - s$, and the subspaces $h_L^s \subset H_{crys}^s(L)_{\mathbb{Q}}$ for all components L of Y^{d-s+1} . We therefore obtain the sequence $(*)$ by tensoring $(**)$ over K_0 with H^s , truncating after the degree $(d - s)$ term and embedding the degree $(d - s)$ term into $H_{crys}^s(Y^{d-s+1})_{\mathbb{Q}}$.

Theorem 6.3. *Suppose Y satisfies (MW), is of weak Lefschetz type, and that for each $i \geq 1$, each component of Y^i is geometrically connected. Suppose $H^i(Y_{Zar}, \mathbb{Q}) = 0$ for all $d > i > 0$. Then $N = 0$ on $H_{crys}^s(Y/T)_{\mathbb{Q}}$ for all $s \neq d = \dim(Y)$.*

Proof. First note that we may assume $s < d$ since the assertion for $s > d$ is reduced to that for $2d - s$ using Poincaré-duality (which commutes with Frobenius, hence with weight filtrations, hence — assuming (MW) — with monodromy filtrations). By 5.1 we need to show $\text{Ker}(N) \cap \text{Im}(N) = (0)$ in $H_{crys}^s(Y/T)_{\mathbb{Q}}$. By 4.7 we can do this by proving $F_C^1 H_{rig}^s(Y) = 0$ (since we assume (MW)). We prove $F_C^t H_{rig}^s(Y) = 0$ for $t \geq 1$ by descending induction on t . Proving $F_C^t H_{rig}^s(Y) = 0$ means proving that

$$\lambda_t : H^s(Y, P_1^{can} F_C^t A^\bullet) \longrightarrow H^s(Y, P_1^{can} A^\bullet) = H_{rig}^s(Y)$$

is the zero map. In view of the exact sequence

$$H^s(Y, P_1^{can} F_C^{t+1} A^\bullet) \longrightarrow H^s(Y, P_1^{can} F_C^t A^\bullet) \xrightarrow{\kappa} H^s(Y, P_1^{can} \text{Gr}_C^t A^\bullet)$$

and the vanishing of λ_{t+1} by induction hypothesis, it is enough to show that for all $x \in H^s(Y, P_1^{can} F_C^t A^\bullet)$ there exists an $x' \in H^s(Y, P_1^{can} F_C^t A^\bullet)$ with $\kappa(x) = \kappa(x')$ and $\lambda_t(x') = 0$. Consider the sequence

$$(S) \quad H^{s-1}(Y, P_1^{can} \text{Gr}_C^{t-1} A^\bullet) \xrightarrow{\alpha} H^s(Y, P_1^{can} \text{Gr}_C^t A^\bullet) \xrightarrow{\beta} H^{s+1}(Y, P_1^{can} \text{Gr}_C^{t+1} A^\bullet)$$

where α and β are the connecting maps in the obvious long exact cohomology sequences. Note $\text{Gr}_C^j A^\bullet = A^{\bullet j}[-j]$ and $P_1^{can} A^{\bullet j} = \text{Gr}_{j+1} \mathbf{C}\tilde{\omega}_Y^\bullet[j + 1] \cong \mathbf{C}\Omega_{Y^{j+1}}^\bullet$; together $P_1^{can} \text{Gr}_C^j A^\bullet \cong \mathbf{C}\Omega_{Y^{j+1}}^\bullet[-j]$. Therefore (S) becomes

$$H_{crys}^{s-t}(Y^t)_{\mathbb{Q}} \longrightarrow H_{crys}^{s-t}(Y^{t+1})_{\mathbb{Q}} \longrightarrow H_{crys}^{s-t}(Y^{t+2})_{\mathbb{Q}},$$

hence is exact by 6.2. Since $\kappa(x) \in \text{Ker}(\beta)$, we therefore find

$$y \in H^{s-1}(Y, P_1^{can} \text{Gr}_C^{t-1} A^\bullet)$$

with $\alpha(y) = \kappa(x)$. The image x' of y under the composite

$$H^{s-1}(Y, P_1^{can} \text{Gr}_C^{t-1} A^\bullet) \longrightarrow H^{s-1}(Y, P_1^{can}(A^\bullet/F_C^t A^\bullet)) \longrightarrow H^s(Y, P_1^{can} F_C^t A^\bullet)$$

has the desired properties.

6.4. Even if $H^i(Y_{Zar}, \mathbb{Q}) = 0$ for all $i > 0$, or even if Y is the reduction of a semistable scheme X over a finite totally ramified extension \mathcal{O}_K of $W(k)$ whose generic fibre is contractible in the sense of Berkovich spaces (this hypothesis guarantees $H^i(Y_{Zar}, \mathbb{Q}) = 0$ for all $d > i > 0$ as required in 6.3), we cannot expect the vanishing of N on the middle degree cohomology $H_{crys}^d(Y/T)_{\mathbb{Q}}$. However, if we assume in addition that for every $j \geq 1$, every irreducible component M of Y^j and every irreducible component N of Y^{j+1} with $N \subset M$, the restriction map

$$H_{crys}^s(M)_{\mathbb{Q}} \longrightarrow H_{crys}^s(N)_{\mathbb{Q}}$$

is an isomorphism if $s = d - j$, and is injective if $s = d - j + 1$, with image independent on M , then we can argue as above to prove $N = 0$ even on $H_{crys}^d(Y/T)_{\mathbb{Q}}$. For example we get $N = 0$ on $H_{crys}^1(Y/T)_{\mathbb{Q}}$ in the case $d = 1$ (for this and a converse of it see also [19], 5.6).

6.5. If Y_σ^i is geometrically connected for all $i \geq 1$, all $\sigma \in S_i$, then $H^s(Y_{Zar}, K_0) = F_C^s H_{crys}^s(Y/T)_{\mathbb{Q}}$. Indeed, $F_C^s H_{crys}^s(Y/T)_{\mathbb{Q}}$ is the $E_\infty^{s,0}$ -term of the spectral sequence

$$(C)_T \quad E_1^{pq} = H^q(Y, B^{\bullet p}) \implies H^{p+q}(Y, B^\bullet) = H_{crys}^{p+q}(Y/T)_{\mathbb{Q}}$$

from section 2.5. But $E_\infty^{s,0} = E_2^{s,0}$ is the s -th cohomology group of the complex

$$H_{conv}^0(Y^1/T) \longrightarrow H_{conv}^0(Y^2/T) \longrightarrow H_{conv}^0(Y^3/T) \longrightarrow \dots$$

(with $H_{conv}^0(Y^1/T)$ in degree 0). Our assumption implies $H_{conv}^0(Y_\sigma^i/T) = K_0$ for all $i \geq 1$, all $\sigma \in S_i$, and therefore $E_\infty^{s,0} = H^s(Y_{Zar}, K_0)$.

Now suppose in addition Y is of weak Lefschetz type and satisfies (MW). Suppose that we have $\text{Im}(N^r) = F_C^r H^*$ (cf. section 5) and $N^r = 0$ on $H_{crys}^r(Y/T)_{\mathbb{Q}}$ for all $0 < r < d$. Then necessarily even $N = 0$ on $H_{crys}^r(Y/T)_{\mathbb{Q}}$ for all $0 < r < d$, as follows from 6.3.

7. LIFTINGS TO MIXED CHARACTERISTIC

7.1. Let A be a complete discrete valuation ring which is a totally ramified finite extension of $W(k)$. Let $K = \text{Quot}(A)$ and fix a uniformizer π in A . Let X be a proper π -adic formal $\text{Spf}(A)$ -scheme with strictly semistable reduction, i.e. Zariski locally it admits étale maps to $\text{Spf}A\langle X_1, \dots, X_{d+1} \rangle / (X_1 \dots X_a - \pi)$ for some $1 \leq a \leq d + 1$. It is naturally a log smooth formal log-scheme over

$$S = (\text{Spf}(A), (\mathbb{N} \longrightarrow A, 1 \mapsto \pi)).$$

Taking reduction modulo π we get a semistable k -log scheme Y as in section 1.1. Let X_K be the generic fibre of X as a rigid analytic space. For $i \geq 0$ let $]Y^i[_X = \coprod_{r \in S_i}]Y_r^i[_X$, the direct sum of the tubes of the Y_r^i in X . The covering $X_K = \bigcup_{j \in S_1}]Y_j^1[_X$ is an admissible open covering, it therefore gives rise to the Čech spectral sequence

$$(C)_S \quad E_1^{pq} = H_{dR}^q(]Y^{p+1}[_X) \implies H_{dR}^{p+q}(X_K).$$

On the other hand we have from section 2.5 the spectral sequence

$$(C)_T \quad E_1^{pq} = H^q(Y, B^{\bullet p}) \implies H^{p+q}(Y, B^{\bullet}) = H_{crys}^{p+q}(Y/T)_{\mathbb{Q}}.$$

Proposition 7.2. *Depending on the choice of π there is an isomorphism of spectral sequences*

$$(C)_S \cong (C)_T \otimes_{K_0} K.$$

Proof. Note $H^q(Y, B^{\bullet p}) \cong H_{conv}^q(Y^{p+1}/T)$ by 1.8 and $H_{crys}^q(Y/T)_{\mathbb{Q}} \cong H_{conv}^q(Y/T)$. Thus we get the isomorphism from [9].

7.3. By transport of structure the monodromy operator N on $H_{crys}^*(Y/T)_{\mathbb{Q}}$ induces a monodromy operator N on $H_{dR}^*(X_K)$ which does not depend on our choice of π ; see [11], sect. 5. Denote by $F_C^r H_{dR}^*(X_K)$ the filtration on $H_{dR}^*(X_K)$ induced by the spectral sequence $(C)_S$. Via $(C)_S \cong (C)_T \otimes_{K_0} K$ this is the filtration obtained by scalar extension from the filtration $F_C^{\bullet} H_{crys}^*(Y/T)_{\mathbb{Q}}$ on $H_{crys}^*(Y/T)_{\mathbb{Q}}$. Therefore we get from 3.6 and 3.8 the following theorem, which in particular gives an upper bound for the vanishing order of N in terms of the rigid space X_K :

Theorem 7.4. *There is a natural operator N acting on $(C)_S$ inducing the monodromy operator N on $H_{dR}^*(X_K)$. However, we have $N = 0$ on all E_1 -terms. In particular, $\text{Im}(N^r) \subset N(F_C^{r-1} H_{dR}^*(X_K)) \subset F_C^r H_{dR}^*(X_K)$ in $H_{dR}^*(X_K)$.*

7.5. We do not know if in general the residue map $H_{dR}^*(X_K) \xrightarrow{Res} F_C^i H_{dR}^*(X_K)$ can be made explicit *without* involving the log basis T . However, for $i = * = d$ this map should be the following (generalizing that of [5]): restrict a class in $H_{dR}^d(X_K)$ to $H_{dR}^d(\mathbb{A}^1[X])$; there choose a representing d -form, take its residue and view it as an element in $H_{dR}^0(\mathbb{A}^1[X])$.

We also mention that the tentative definition of N given in [1] for varieties X_K uniformized by Drinfel'd's symmetric spaces is based on residue maps, and we expect that our description of N can be used for a comparison with the N from [1].

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