

LINEAR RELATIONS IN THE CALKIN ALGEBRA FOR COMPOSITION OPERATORS

THOMAS KRIETE AND JENNIFER MOORHOUSE

ABSTRACT. We consider this and related questions: When is a finite linear combination of composition operators, acting on the Hardy space or the standard weighted Bergman spaces on the unit disk, a compact operator?

1. INTRODUCTION

For $\beta \geq 1$, let \mathcal{D}_β denote the reproducing kernel Hilbert space of functions analytic in the unit disk $D = \{z : |z| < 1\}$ and having the kernel functions $k_w(z) = (1 - \bar{w}z)^{-\beta}$. Thus, $f(w) = \langle f, k_w \rangle$ for w in D and f in \mathcal{D}_β . The Hardy space H^2 is exactly \mathcal{D}_1 , and when $\beta > 1$, \mathcal{D}_β is the standard weighted Bergman space A_α^2 with $\alpha + 2 = \beta$; see Section 2.1. We consider composition operators $C_\varphi : f \rightarrow f \circ \varphi$ acting on \mathcal{D}_β , where φ is an analytic self-map of D . When $\beta \geq 1$, every C_φ lies in $\mathcal{B}(\mathcal{D}_\beta)$, the algebra of bounded operators on \mathcal{D}_β . Unlike the classes of Toeplitz and Hankel operators which act on \mathcal{D}_β , the set of composition operators in $\mathcal{B}(\mathcal{D}_\beta)$ has no obvious additive or linear structure. However, in the Bergman space case $\beta > 1$, the second author observed additive structure modulo the ideal \mathcal{K} of compact operators and characterized those pairs φ and ψ for which $C_\varphi - C_\psi$ is compact [16]. Our purpose here is twofold: to present some analogous results for the H^2 case $\beta = 1$, and to pass from additive to linear structure in the Calkin algebra $\mathcal{B}(\mathcal{D}_\beta)/\mathcal{K}$.

Composition operators which are themselves compact were characterized in the A_α^2 case by MacCluer and Shapiro [15] and on H^2 by Shapiro [25]; in [22] Sarason found a different condition, sufficient for H^2 and necessary and sufficient for L^1 , later shown by Shapiro and Sundberg [27] to be necessary in the H^2 case as well. The problem of compact difference was raised in an explicit form by Shapiro and Sundberg [26] and MacCluer [13]; these authors found several criteria, some necessary and some sufficient. More recently, MacCluer, Ohno, and Zhao [14] have shown that compactness of $C_\varphi - C_\psi$ acting on H^∞ , the space of bounded analytic functions on D , is characterized in terms of the quantity $\rho(z) = \left| \frac{\varphi(z) - \psi(z)}{1 - \overline{\varphi(z)}\psi(z)} \right|$, the pseudo-hyperbolic distance between values of φ and ψ . The second author of the present article subsequently found that ρ plays a related role on the Bergman spaces, as follows.

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Theorem A ([16]). *Suppose that φ and ψ are analytic self-maps of D and $\alpha > -1$. Then $C_\varphi - C_\psi$ is a compact operator on A_α^2 if and only if*

$$\lim_{|z| \rightarrow 1} \rho(z) \left\{ \frac{1 - |z|^2}{1 - |\varphi(z)|^2} + \frac{1 - |z|^2}{1 - |\psi(z)|^2} \right\} = 0.$$

Let $F(\varphi)$ denote the set of points in the unit circle ∂D at which φ has a finite angular derivative in the sense of Caratheodory; see Section 2.2. Also, we will use the notation $A \equiv B \pmod{\mathcal{K}}$ to indicate that two bounded operators A and B have a compact difference. There is a “sum theorem,” as follows:

Theorem B ([16]). *Let $\varphi, \varphi_1, \dots, \varphi_n$ be analytic self-maps of D for which the sets $F(\varphi_i)$, $i = 1, \dots, n$, are pairwise disjoint and with $F(\varphi) = F(\varphi_1) \cup \dots \cup F(\varphi_n)$. Consider $C_\varphi, C_{\varphi_1}, \dots, C_{\varphi_n}$ as acting on A_α^2 where $\alpha > -1$, and let*

$$\rho_i = \left| \frac{\varphi - \varphi_i}{1 - \overline{\varphi}\varphi_i} \right|, \quad i = 1, \dots, n.$$

Then the following are equivalent:

(i) *For each $i = 1, \dots, n$ and each ζ in $F(\varphi_i)$,*

$$\lim_{z \rightarrow \zeta} \rho_i(z) \left\{ \frac{1 - |z|^2}{1 - |\varphi(z)|^2} + \frac{1 - |z|^2}{1 - |\varphi_i(z)|^2} \right\} = 0.$$

(ii) $C_\varphi \equiv C_{\varphi_1} + \dots + C_{\varphi_n} \pmod{\mathcal{K}}$.

The following useful modification of Theorem A, which localizes the notion of compact difference, is implicit in [16].

Theorem C. *Suppose that φ and ψ are analytic self-maps of D , $\alpha > -1$ and G is a measurable subset of D . If*

$$\lim_{|z| \rightarrow 1} \chi_G(z) \rho(z) \left\{ \frac{1 - |z|^2}{1 - |\varphi(z)|^2} + \frac{1 - |z|^2}{1 - |\psi(z)|^2} \right\} = 0,$$

then $M_{\chi_G}(C_\varphi - C_\psi)$ is a compact operator from A_α^2 into its containing L^2 space. Here M_{χ_G} denotes the operator of multiplication by the characteristic function χ_G .

Other recent related work includes the following: Bourdon, Levi, Narayan, and J. H. Shapiro [4] show that a composition operator associated with an “almost linear fractional” map is, in fact, a compact perturbation of a linear fractional composition operator; Bourdon [3] treats the question of compact difference vs. topological connectedness for linear fractional maps (see Section 6 below); J. E. Shapiro [24] shows that if $C_\varphi - C_\psi$ is compact on H^2 , the singular parts of the Clark measures of φ and ψ coincide (see Section 2.3). Most recently, Nieminen and Saksman [19] have shown that the just-quoted condition of J. E. Shapiro on singular parts, plus uniform integrability of the differences of the absolutely continuous densities of the respective Clark measures of φ and ψ , are together equivalent to compactness of $C_\varphi - C_\psi$ on L^1 (or, on the space of complex Borel measures on $\partial\mathbb{D}$) and are thus sufficient on H^2 . In a different direction, Gorkin and Mortini [10] have characterized compactness for finite linear combinations of composition operators acting on uniform algebras.

Our analogues of Theorems A, B, and C for H^2 require some different methods. For us the key is an application of Clark measures, following ideas of Sarason [22], Cima and Matheson [5] and J. E. Shapiro [24]. In Section 3 we obtain essential

norm estimates for weighted composition operators on H^2 analogous to the Cima-Matheson essential norm formula for (unweighted) composition operators [5]. We combine these results in Section 4 with a general principle of Moorhouse and Toews [17] and Carleson measure estimates as in [16] to obtain our H^2 results.

Section 5 is devoted to the question of when a given finite linear combination of composition operators is compact. We look at lower bounds, given in terms of first- and higher-order boundary data, for the essential norm of a linear combination; these results further develop ideas of MacCluer [13]. We introduce the class \mathcal{S} of analytic self-maps φ of D having “sufficient data” at every point in ∂D where φ makes significant contact with the boundary. For φ and ψ in \mathcal{S} we make clear the obstructions to the essential norm $\|C_\varphi - C_\psi\|_e$ being small, the conditions under which it must be small, and when it is zero, that is, when $C_\varphi - C_\psi$ is compact. For $\varphi_1, \dots, \varphi_n$ in \mathcal{S} , we characterize, via a finite system of linear equations involving boundary data of these maps, those coefficients for which $c_1 C_{\varphi_1} + \dots + c_n C_{\varphi_n}$ is compact. An application is a simple algorithm for determining the dimension of the vector space in $\mathcal{B}(\mathcal{D}_\beta)/\mathcal{K}$ spanned by the cosets $[C_{\varphi_1}], \dots, [C_{\varphi_n}]$.

The final Section 6 concerns a problem first studied by Berkson [2], subsequently considered by MacCluer [13], Shapiro and Sundberg [26], and most recently by Toews and the second author [17] and Bourdon [3]: to characterize those pairs φ and ψ for which C_φ and C_ψ lie in the same connected component of the topological space of composition operators, equipped with the norm topology on $\mathcal{B}(\mathcal{D}_\beta)$. We observe that a general sufficient condition of the second author for the Bergman space case [16] extends to H^2 and apply this result to those C_φ with φ lying in a certain subclass \mathcal{S}_0 of \mathcal{S} to determine when C_φ and C_ψ lie in the same component of $\{C_\varphi : \varphi \in \mathcal{S}_0\}$.

A variation on ideas of Berkson [2], Shapiro and Sundberg [26], and MacCluer [13] (see Exercise 9.3.2 in [8]) states that if $\varphi_1, \dots, \varphi_n$ are distinct analytic self-maps of D , and if $J(\varphi)$ denotes the set of points $e^{i\theta}$ in ∂D with $|\varphi(e^{i\theta})| = 1$, then

$$\left\| \sum_{j=1}^n c_j C_{\varphi_j} \right\|_e^2 \geq \frac{1}{2\pi} \sum_{j=1}^n |c_j|^2 |J(\varphi_j)|,$$

where $|J(\varphi)|$ is the arclength measure of $J(\varphi)$. Accordingly, to study the questions discussed above, we assume throughout that our analytic self-maps φ of D satisfy $|\varphi(e^{i\theta})| < 1$ a.e.

2. PRELIMINARIES

Here we collect some preliminary facts used in the sequel.

2.1. The Hardy and Bergman spaces. The Hardy space $H^2 = \mathcal{D}_1$ is the set of all functions $f(z) = \sum_{n=0}^\infty a_n z^n$ analytic in D with

$$\|f\|^2 \equiv \sum_{n=0}^\infty |a_n|^2 < \infty.$$

Given f in H^2 , the non-tangential limit $f(e^{i\theta}) = \lim_{\angle z \rightarrow e^{i\theta}} f(z)$ exists for $d\theta$ -almost every $e^{i\theta}$ in ∂D . Moreover, the correspondence $f(z) \rightarrow f(e^{i\theta})$ allows one to think of H^2 as the closed subspace of $L^2 = L^2(\partial D, \frac{d\theta}{2\pi})$ with orthonormal basis $\{e^{in\theta}\}_{n=0}^\infty$.

For $\alpha > -1$, the Bergman space A_α^2 is the set of functions f analytic in D with

$$\|f\|^2 = \frac{\alpha + 1}{\pi} \int_D |f(z)|^2 (1 - |z|^2)^\alpha dA(z) < \infty,$$

where dA is a Lebesgue area measure on D . As mentioned above, $\mathcal{D}_\beta = A_{\beta-2}^2$ for $\beta > 1$, with equality of norms.

For information about H^2 and A_α^2 , see [9] and [8].

2.2. Angular derivatives. Let φ be an analytic self-map of D . Then φ has a (finite) angular derivative at ζ in ∂D provided $\varphi(\zeta)$, the non-tangential limit of φ at ζ , exists and has modulus one, and

$$\varphi'(\zeta) \equiv \lim_{\zeta z \rightarrow \zeta} \frac{\varphi(z) - \varphi(\zeta)}{z - \zeta}$$

exists as a finite complex number. If the angular derivative $\varphi'(\zeta)$ fails to exist, we write $|\varphi'(\zeta)| = \infty$. In either case the Julia-Caratheodory Theorem [8] asserts in part that

$$\liminf_{z \rightarrow \zeta} \frac{1 - |\varphi(z)|}{1 - |z|} = |\varphi'(\zeta)|,$$

where the limit inferior is taken unrestrictedly in D ; moreover $|\varphi'(\zeta)| > 0$. Throughout we write $F(\varphi)$ for the set of all points in ∂D where φ has a finite angular derivative. For ζ in $F(\varphi)$ we have the relation $\varphi'(\zeta) = \bar{\zeta}\varphi(\zeta)|\varphi'(\zeta)|$. A condition necessary for the composition operator C_φ to act compactly on H^2 is that $F(\varphi)$ be empty [28]. On the Bergman space A_α^2 , this condition is both necessary and sufficient [15].

2.3. Clark measures. Let φ be an analytic self-map of D . If $|\alpha| = 1$, there exists a finite positive Borel measure μ_α on ∂D such that

$$(2.1) \quad \frac{1 - |\varphi(z)|^2}{|\alpha - \varphi(z)|^2} = \operatorname{Re} \left(\frac{\alpha + \varphi(z)}{\alpha - \varphi(z)} \right) = \int_{\partial D} P_z(e^{it}) d\mu_\alpha(t)$$

for z in D , where

$$P_z(e^{it}) = \frac{1 - |z|^2}{|e^{it} - z|^2}$$

is the Poisson kernel at z . The existence of μ_α follows since the left side of equation (2.1) is a positive harmonic function. The measures μ_α (the *Clark measures* of φ) were introduced as an operator-theoretic tool by D. N. Clark [6], and have been further analyzed by Aleksandrov [1], Poltoratski [20], and Sarason [23].

On decomposing $\mu_\alpha = \mu_\alpha^{ac} + \mu_\alpha^s$, where μ_α^{ac} and μ_α^s are, respectively, the absolutely continuous and singular parts with respect to Lebesgue measure, one finds by Fatou's theorem [9] that

$$\mu_\alpha^{ac} = \frac{1 - |\varphi(e^{i\theta})|^2}{|\alpha - \varphi(e^{i\theta})|^2} \frac{d\theta}{2\pi}.$$

The singular part μ_α^s is carried by $\varphi^{-1}(\{\alpha\})$, the set of those ζ in ∂D where $\varphi(\zeta)$ exists and equals α , and is itself the sum of the pure point measure

$$(2.2) \quad \mu_\alpha^{pp} = \sum_{\varphi(\zeta)=\alpha} \frac{1}{|\varphi'(\zeta)|} \delta_\zeta$$

(here δ_ζ is the unit point mass at ζ) and a continuous singular measure μ_α^{cs} , either of which can vanish.

Let us write

$$E(\varphi) = \overline{\bigcup_{|\alpha|=1} \text{spt}(\mu_\alpha^s)},$$

where $\text{spt}(\mu)$ denotes the closed support of a measure μ . It is clear from Eqn. (2.2) that $F(\varphi)$ is a subset of $E(\varphi)$.

2.4. Essential norms. Let H and L be separable Hilbert spaces and write $\mathcal{B}(H, L)$ for the space of bounded operators from H to L . Let \mathcal{K} denote the subspace of compact operators in $\mathcal{B}(H, L)$. The essential norm $\|T\|_e$ of an operator T in $\mathcal{B}(H, L)$ is the operator-norm distance from T to \mathcal{K} . We will find this alternate description useful:

$$(2.3) \quad \|T\|_e = \sup_{\{f_n\} \in \mathcal{U}} \left(\limsup_{n \rightarrow \infty} \|Tf_n\| \right),$$

where \mathcal{U} is the collection of all sequences $\{f_n\}$ of unit vectors in H which tend to zero weakly.

2.5. Carleson measures. For a point ζ on the unit circle and $\delta > 0$, let $S(\zeta, \delta) = \{z \in D : |\zeta - z| < \delta\}$. If μ is a finite positive Borel measure on D and $\beta \geq 1$, we consider the quantities

$$\Delta_\beta(\mu) = \sup_{|\zeta|=1, \delta>0} \frac{\mu(S(\zeta, \delta))}{\delta^\beta}, \quad \Delta_\beta^*(\mu) = \limsup_{\delta \rightarrow 0} \left\{ \sup_{|\zeta|=1} \frac{\mu(S(\zeta, \delta))}{\delta^\beta} \right\}.$$

One often says μ is a β -Carleson measure if $\Delta_\beta(\mu) < \infty$, and a *vanishing* β -Carleson measure if $\Delta_\beta^*(\mu) = 0$. The reader might consult [8] for the history of the following well-known result. The statement about $\|J\|_e$ can be deduced from ideas in the proof of Theorem 3.12 in [8].

Theorem 2.1. *Let μ be a finite positive Borel measure on D and assume $\beta \geq 1$. Then:*

(i) *The space \mathcal{D}_β (considered as a space of analytic functions on D) is contained in $L^2(\mu)$ if and only if μ is a β -Carleson measure. In this case the inclusion map $J : \mathcal{D}_\beta \rightarrow L^2(\mu)$ is bounded with norm comparable to $\sqrt{\Delta_\beta(\mu)}$.*

(ii) *If \mathcal{D}_β is contained in $L^2(\mu)$, then $\|J\|_e$, the essential norm of the inclusion map, is comparable to $\sqrt{\Delta_\beta^*(\mu)}$. In particular, J is compact if and only if μ is a vanishing β -Carleson measure.*

2.6. A general scheme for compact difference and arc-connectedness. For a bounded analytic function w on ∂D , one can form the associated multiplication operator $M_w : f \rightarrow wf$. If φ is an analytic self-map of D , then we have the weighted composition operator $M_w C_\varphi$. Given two analytic self-maps of D , φ , and ψ , consider the self-maps $\varphi_t = t\varphi + (1-t)\psi$, $0 \leq t \leq 1$. Based on the formal operator identity

$$(2.4) \quad C_{\varphi_s} - C_{\varphi_r} = M_{\varphi-\psi} \left[\int_r^s C_{\varphi_t} dt \right] X,$$

$0 \leq r < s \leq 1$, and the fact that the differentiation operator $X = \frac{d}{dz}$ is a topological isomorphism of H_0^2 (the subspace of H^2 consisting of all functions that vanish at

the origin) and the Bergman space A_1^2 (see [8]), the second author and C. Toews proved the following:

Theorem 2.2 ([17]). *Let φ, ψ , and $\varphi_t, 0 \leq t \leq 1$, be as above.*

(i) *Suppose the weighted composition operators $M_{\varphi-\psi}C_{\varphi_t}$ act boundedly from A_1^2 to H^2 , with uniformly bounded norms, $0 \leq t \leq 1$. Then, there is a constant $B > 0$ such that, as operators on H^2 ,*

$$\|C_{\varphi_s} - C_{\varphi_r}\| \leq B|s - r|, \quad 0 \leq r < s < 1.$$

(ii) *Suppose that φ, ψ , and φ_t satisfy the hypotheses of part (i) above, and in addition, that for each $t, 0 \leq t \leq 1, M_{\varphi-\psi}C_{\varphi_t}$ is a compact operator from A_1^2 to H^2 . Then $C_\varphi - C_\psi$ is a compact operator on H^2 .*

The above result remains true if one replaces H^2 and A_1^2 by A_α^2 and $A_{\alpha+2}^2$, respectively, where $\alpha > -1$; see [17].

3. WEIGHTED COMPOSITION OPERATORS ON H^2 AND L^2

For the analytic self-maps of D considered here (those with $|\varphi(e^{i\theta})| < 1$ a.e. on ∂D), Sarason [22] found a convenient representation of C_φ as an integral operator on H^2 and even on the larger space $L^2 = L^2(\partial D, \frac{d\theta}{2\pi})$. For f in L^2 , extend f to a harmonic function in D via the Poisson integral: $f(z) = \int_{\partial D} P_z(e^{it})f(e^{it})\frac{dt}{2\pi}$. Putting $(C_\varphi f)(e^{i\theta}) = f(\varphi(e^{i\theta}))$ using the extended f (since $|\varphi(e^{i\theta})| < 1$ a.e.), one has

$$(3.1) \quad (C_\varphi f)(e^{i\theta}) = \int_{\partial D} \frac{1 - |\varphi(e^{i\theta})|^2}{|e^{it} - \varphi(e^{i\theta})|^2} f(e^{it})\frac{dt}{2\pi}.$$

Using the Schur test for boundedness of integral operators (stated below), Sarason showed that C_φ is compact on H^2 if the Clark measures μ_α of φ are absolutely continuous for all α in ∂D ; J. H. Shapiro and C. Sundberg [27] established the converse via function-theoretic methods. Subsequently Cima and Matheson [5] discovered the following expression for the essential norm of an arbitrary C_φ acting on H^2 :

$$(3.2) \quad \|C_\varphi\|_e^2 = \sup_{|\alpha|=1} \mu_\alpha^s(\partial D),$$

a formula foreshadowed by C. Cowen’s inequalities for smooth φ [7, p. 84].

Here we adapt the integral operator approach to investigate essential norms of *weighted* composition operators

$$M_w C_\varphi : f \rightarrow w \cdot (f \circ \varphi).$$

We allow w in L^∞ and consider $M_w C_\varphi$ as mapping L^2 to L^2, H^2 to L^2 or (in the event that w is in H^∞), H^2 to H^2 . In all cases we have the following:

Theorem 3.1. *Let φ be an analytic self-map of D with $|\varphi(e^{i\theta})| < 1$ a.e. and having Clark measures $\mu_\alpha, |\alpha| = 1$. Suppose w is a bounded measurable function on ∂D such that $|w|$ is continuous at every point of $E(\varphi)$. Then*

$$\sup_{|\alpha|=1} \int_{\partial D} |w|^2 d\mu_\alpha^s \leq \|M_w C_\varphi\|_e^2 \leq 4 \sup_{|\alpha|=1} \int_{\partial D} |w|^2 d\mu_\alpha^s.$$

In particular, $M_w C_\varphi$ is compact if and only if $w \equiv 0$ on $E(\varphi)$.

For the proof, our essential tool is the following.

The Schur test [18, p. 282]. Consider two measure spaces (X, μ) and (Y, ν) , and let N be a measurable function on the product space $Y \times X$. Suppose there exist positive measurable functions p on X and q on Y and constants $A, B > 0$ satisfying

$$\int_X |N(y, x)|p(x)d\mu(x) \leq Aq(y), \quad y \text{ in } Y,$$

$$\int_Y |N(y, x)|q(y)d\nu(y) \leq Bp(x), \quad x \text{ in } X.$$

Then the formula

$$(Tf)(y) = \int_X N(y, x)f(x)d\mu(x)$$

defines a bounded operator T from $L^2(\mu)$ to $L^2(\nu)$ with $\|T\| \leq \sqrt{AB}$.

We will also need several lemmas. The first, and the final conclusion in the second, are due to J. E. Shapiro [24].

Lemma 3.2 ([24]). Let φ be an analytic self-map of D with Clark measures μ_α , $|\alpha| = 1$. If f is continuous on ∂D , then

$$\lim_{r \nearrow 1} \int_{\partial D} f \frac{1 - r^2}{|\alpha - r\varphi|^2} \frac{d\theta}{2\pi} = \int_{\partial D} f d\mu_\alpha^s,$$

for $|\alpha| = 1$.

Lemma 3.3. If f is continuous on ∂D , then

$$\lim_{r \nearrow 1} \int_{\partial D} f \frac{1 - |r\varphi|^2}{|\alpha - r\varphi|^2} \frac{d\theta}{2\pi} = \int_{\partial D} f d\mu_\alpha,$$

uniformly in α , $|\alpha| = 1$. Thus $\int_{\partial D} f d\mu_\alpha$ is a continuous function of α .

Proof. For $0 < r < 1$, the function $\frac{1 - |r\varphi|^2}{|\alpha - r\varphi|^2}$ is bounded and harmonic on D , and thus is the Poisson integral of its boundary function. It follows from this, and the definition of μ_α , that the conclusion holds for $f = P_z$, the Poisson kernel at any z in D . Thus the conclusion holds when f is a finite linear combination of Poisson kernels. Such finite linear combinations are uniformly dense in the continuous functions on ∂D , and the lemma follows. \square

The third lemma is a variant of exercise (7) in §26 of Halmos' treatise [11]; for the proof the interested reader can easily adapt the hint given there.

Lemma 3.4. Let f and f_n , for $n = 1, 2, 3, \dots$, be non-negative integrable functions on a measure space (X, μ) and suppose $\lambda \geq 0$. If $f_n \rightarrow f$, μ -a.e. as $n \rightarrow \infty$, and

$$\limsup_{n \rightarrow \infty} \left\{ \int_X f_n d\mu - \int_X f d\mu \right\} \leq \lambda,$$

then

$$\limsup_{n \rightarrow \infty} \int_X |f_n - f| d\mu \leq \lambda.$$

Proof of Theorem 3.1. We can write $w = |w|v$ where v is measurable and unimodular on ∂D . Since $M_w = M_v M_{|w|}$ and M_v is unitary, the theorem is unaltered by assuming that $w = |w| \geq 0$. For now we also assume that w is continuous on ∂D .

Consider the normalized kernel function

$$K_a(z) = \frac{\sqrt{1-|a|^2}}{1-\bar{a}z}, \quad |a| < 1.$$

If $|a| \rightarrow 1$, then $K_a \rightarrow 0$ weakly in H^2 . Now take $a = r\alpha$ where $0 \leq r < 1$ and α is fixed in ∂D . Then $|C_\varphi K_{r\alpha}|^2 = \frac{1-r^2}{|\alpha-r\varphi|^2}$ and

$$\int |w|^2 d\mu_\alpha^s = \lim_{r \nearrow 1} \int |w|^2 \frac{1-r^2}{|\alpha-r\varphi|^2} \frac{d\theta}{2\pi} = \limsup_{r \nearrow 1} \|M_w C_\varphi K_{r\alpha}\|^2$$

where Lemma 3.2 gives the first equality; the lower bound for $\|M_w C_\varphi\|_e^2$ follows from Eqn. (2.3). (Throughout the proof, all integrals are taken over ∂D .)

For the upper bound we consider $M_w C_\varphi$ as an integral operator from $L^2 = L^2(\partial D, \frac{d\theta}{2\pi})$ to $L^2(G, \frac{d\theta}{2\pi})$, where $G = \{e^{i\theta} : w(e^{i\theta}) > 0\}$. By Eqn. (3.1) the kernel of this operator is

$$K(e^{i\theta}, e^{it}) = w(e^{i\theta}) \frac{1-|\varphi(e^{i\theta})|^2}{|e^{it}-\varphi(e^{i\theta})|^2}.$$

Similarly, if $0 < r < 1$, the kernel of the integral operator $M_w C_{r\varphi}$ is

$$K_r(e^{i\theta}, e^{it}) = w(e^{i\theta}) \frac{1-|r\varphi(e^{i\theta})|^2}{|e^{it}-r\varphi(e^{i\theta})|^2}.$$

Since $\|r\varphi\|_\infty \leq r < 1$, $C_{r\varphi}$, and thus $M_w C_{r\varphi}$, are compact. We apply the Schur test to the integral operator $M_w C_{r\varphi} - M_w C_\varphi$, which has integral kernel $N = K_r - K$. We take $\mu = \frac{d\theta}{2\pi}$, ν to be the restriction of $\frac{d\theta}{2\pi}$ to G , $p(e^{i\theta}) = 1$ and $q(e^{i\theta}) = w(e^{i\theta})$. Then

$$\begin{aligned} & \int |K_r(e^{i\theta}, e^{it}) - K(e^{i\theta}, e^{it})| p(e^{it}) \frac{dt}{2\pi} \\ & \leq w(e^{i\theta}) \int \left(\frac{1-|r\varphi(e^{i\theta})|^2}{|e^{it}-r\varphi(e^{i\theta})|^2} + \frac{1-|\varphi(e^{i\theta})|^2}{|e^{it}-\varphi(e^{i\theta})|^2} \right) \frac{dt}{2\pi} \\ & = 2w(e^{i\theta}) = 2q(e^{i\theta}), \end{aligned}$$

for all $e^{i\theta}$ in the circle. This is the first Schur hypothesis.

For the second Schur hypothesis, we write

$$\lambda = \sup_{|\alpha|=1} \int |w|^2 d\mu_\alpha^s,$$

and consider sequences $r_n \nearrow 1$ and $\{\alpha_n\}$ in ∂D . It is enough to show that

$$(3.3) \quad \limsup_{n \rightarrow \infty} \int |K_{r_n}(e^{i\theta}, \alpha_n) - K(e^{i\theta}, \alpha_n)| w(e^{i\theta}) \frac{d\theta}{2\pi} \leq 2\lambda,$$

for then, since $M_w C_{r_n\varphi}$ is compact, the Schur test with $A = 2$ and $B = 2\lambda$ will imply that

$$\|M_w C_\varphi\|_e^2 \leq \limsup_{n \rightarrow \infty} \|M_w C_{r_n\varphi} - M_w C_\varphi\|^2 \leq 4\lambda,$$

as desired.

We may assume that α_n tends to some α in ∂D as $n \rightarrow \infty$. First we use Lemma 3.4 with

$$f_n = |w|^2 \frac{1-|\varphi|^2}{|\alpha_n-\varphi|^2}, \quad f = |w|^2 \frac{1-|\varphi|^2}{|\alpha-\varphi|^2}.$$

Note that

$$\int f \frac{d\theta}{2\pi} = \int |w|^2 d\mu_\alpha - \int |w|^2 d\mu_\alpha^s,$$

and similarly for f_n, μ_{α_n} and $\mu_{\alpha_n}^s$, so that

$$\int (f_n - f) \frac{d\theta}{2\pi} = \int |w|^2 d\mu_{\alpha_n} - \int |w|^2 d\mu_\alpha + \int |w|^2 d\mu_\alpha^s - \int |w|^2 d\mu_{\alpha_n}^s.$$

By the final conclusion in Lemma 3.3, the difference of the first two terms tends to zero as $n \rightarrow \infty$, so that

$$\limsup_{n \rightarrow \infty} \int (f_n - f) \frac{d\theta}{2\pi} \leq \lambda.$$

Since $f_n \rightarrow f$ a.e., Lemma 3.4 implies that

$$(3.4) \quad \limsup_{n \rightarrow \infty} \int |K(e^{i\theta}, \alpha_n) - K(e^{i\theta}, \alpha)| w(e^{i\theta}) \frac{d\theta}{2\pi} \leq \lambda.$$

Now use Lemma 3.4 again, this time with

$$f_n = |w|^2 \frac{1 - |r_n \varphi|^2}{|\alpha_n - r_n \varphi|^2}, \quad f = |w|^2 \frac{1 - |\varphi|^2}{|\alpha - \varphi|^2}.$$

We have

$$\begin{aligned} \int (f_n - f) \frac{d\theta}{2\pi} &\leq \left| \int |w|^2 \frac{1 - |r_n \varphi|^2}{|\alpha_n - r_n \varphi|^2} \frac{d\theta}{2\pi} - \int |w|^2 d\mu_{\alpha_n} \right| \\ &\quad + \left| \int |w|^2 d\mu_{\alpha_n} - \int |w|^2 d\mu_\alpha \right| \\ &\quad + \left| \int |w|^2 d\mu_\alpha - \int |w|^2 \frac{1 - |\varphi|^2}{|\alpha - \varphi|^2} \frac{d\theta}{2\pi} \right|. \end{aligned}$$

As $n \rightarrow \infty$, the first two terms on the right tend to zero by Lemma 3.3, while the last term is exactly $\int |w|^2 d\mu_\alpha^s$, which does not exceed λ . Thus, according to Lemma 3.4,

$$\limsup_{n \rightarrow \infty} \int |K_{r_n}(e^{i\theta}, \alpha_n) - K(e^{i\theta}, \alpha)| w(e^{i\theta}) \frac{d\theta}{2\pi} \leq \lambda.$$

Combining this with Eqn. (3.4) yields Eqn. (3.3) and thus the desired upper bound for $\|M_w C_\varphi\|_e^2$.

It remains to consider the case where $w = |w|$ is continuous at each point of $E(\varphi)$, but not necessarily on all of ∂D . Since $E(\varphi)$ is compact, there exists a function g continuous on ∂D with $g = w$ on $E(\varphi)$. Then $w - g$ is continuous at and vanishes at every point of $E(\varphi)$. Given $\epsilon > 0$ one can construct a function h continuous on ∂D with $h = \epsilon$ on $E(\varphi)$ and $|w - g| \leq h$ on ∂D . Since $\|M_{w-g} C_\varphi f\| \leq \|M_h C_\varphi f\|$ for all f in L^2 , $\|M_{w-g} C_\varphi\|_e \leq \|M_h C_\varphi\|_e$ by Eqn. (2.3). An application of this inequality and the triangle inequality yields

$$\|M_g C_\varphi\|_e - \|M_h C_\varphi\|_e \leq \|M_w C_\varphi\|_e \leq \|M_g C_\varphi\|_e + \|M_h C_\varphi\|_e.$$

Because g and h are both continuous on ∂D , we can apply our earlier argument to estimate $\|M_g C_\varphi\|_e$ and $\|M_h C_\varphi\|_e$; in particular, $\|M_h C_\varphi\|_e = O(\epsilon)$ as $\epsilon \rightarrow 0$. Since ϵ is arbitrary and $w = g$ on $E(\varphi)$, the theorem follows. \square

4. LOCAL COMPACT DIFFERENCE AND A SUM THEOREM FOR H^2

Our plan for studying linear combinations in $\mathcal{B}(H^2)/\mathcal{K}$ is to decompose, mod \mathcal{K} , a composition operator into pieces associated to subsets of $E(\varphi)$ in a manner analogous to the decomposition in Theorem B above. This depends on an H^2 analogue of Theorem C above using Theorem 3.1 and H^2 versions of ideas from [16]. Throughout, φ and ψ are analytic self-maps of D with $|\varphi| < 1$ a.e. and $|\psi| < 1$ a.e. on ∂D , $\rho = \left| \frac{\varphi - \psi}{1 - \overline{\varphi}\psi} \right|$, and $\varphi_t = t\varphi + (1 - t)\psi$, $0 \leq t \leq 1$. We require three lemmas.

Lemma 4.1. *Let $a < 1$ and suppose G is a measurable subset of ∂D with $\rho \leq a$ on G . Assume that $0 \leq t \leq 1$. Then $M_{\chi_G} M_{\varphi - \psi} C_{\varphi_t}$ acts boundedly from A_1^2 to L^2 , and*

$$\|M_{\chi_G} M_{\varphi - \psi} C_{\varphi_t}\|_{\mathcal{B}(A_1^2, L^2)} \leq \frac{c}{1 - a} \|M_{\chi_G} M_{\rho} C_{\varphi_t}\|_{\mathcal{B}(H^2, L^2)},$$

where c is an absolute constant. Moreover, the same inequality (with a different c) holds if both norms are replaced by the corresponding essential norms. In particular, if the operator $M_{\chi_G} M_{\rho} C_{\varphi_t} : H^2 \rightarrow L^2$ is compact, so is $M_{\chi_G} M_{\varphi - \psi} C_{\varphi_t} : A_1^2 \rightarrow L^2$.

Proof. We consider the measures

$$\nu_t = \left(\chi_G |\varphi - \psi|^2 \frac{d\theta}{2\pi} \right) \circ \varphi_t^{-1}, \quad \beta_t = \left(\chi_G \rho^2 \frac{d\theta}{2\pi} \right) \circ \varphi_t^{-1}$$

on the disk D . As in [16] we have

$$\begin{aligned} \frac{1 - |\varphi_t|^2}{|1 - \overline{\varphi}\psi|} &= \left| 1 + \overline{\varphi} \frac{(\psi - \varphi_t)}{1 - \overline{\varphi}\psi} + \varphi_t \frac{(\overline{\varphi} - \overline{\varphi}_t)}{1 - \overline{\varphi}\psi} \right| \\ &= \left| 1 - t\overline{\varphi} \frac{(\varphi - \psi)}{1 - \overline{\varphi}\psi} + (1 - t)\varphi_t \frac{(\overline{\varphi} - \overline{\psi})}{1 - \overline{\varphi}\psi} \right| \\ &\geq 1 - \rho. \end{aligned}$$

Thus, if ζ lies in ∂D , $\delta > 0$ and $S(\zeta, \delta) = \{z \in D : |z - \zeta| < \delta\}$, on the set $G \cap \varphi_t^{-1}(S(\zeta, \delta))$ we have

$$|\varphi - \psi|^2 = \rho^2 |1 - \overline{\varphi}\psi|^2 \leq \rho^2 \left(\frac{1 - |\varphi_t|^2}{1 - a} \right)^2 \leq \frac{4\delta^2}{(1 - a)^2} \rho^2.$$

It follows that

$$\frac{\nu_t(S(\zeta, \delta))}{\delta^3} \leq \frac{4}{(1 - a)^2} \frac{\beta_t(S(\zeta, \delta))}{\delta},$$

so in the terminology of Section 2.5,

$$\Delta_3(\nu_t) \leq \frac{4}{(1 - a)^2} \Delta_1(\beta_t) \text{ and } \Delta_3^*(\nu_t) \leq \frac{4}{(1 - a)^2} \Delta_1^*(\beta_t).$$

Since for all bounded analytic functions h we have

$$\int_D |h|^2 d\nu_t = \|M_{\chi_G} M_{\varphi - \psi} C_{\varphi_t} h\|_{L^2}^2 \text{ and } \int_D |h|^2 d\beta_t = \|M_{\chi_G} M_{\rho} C_{\varphi_t} h\|_{L^2}^2,$$

the desired conclusion follows from Theorem 2.1 and the formula (2.3). □

Lemma 4.2. *Let G be an open set in ∂D on which ρ is essentially bounded away from one, and assume that $E(\varphi) \cap E(\psi) \cap G$ is closed. Then $E(\varphi_t) \cap G$ is contained in $E(\varphi) \cap E(\psi) \cap G$ for $0 < t < 1$.*

Proof. For z in D we have

$$\frac{1 - |\varphi_t(z)|}{1 - |z|} \geq t \frac{1 - |\varphi(z)|}{1 - |z|} + (1 - t) \frac{1 - |\psi(z)|}{1 - |z|}.$$

On letting $z \rightarrow \zeta$ in ∂D , we see that if φ_t has a finite angular derivative at ζ , so do φ and ψ , that is, $F(\varphi_t) \subset F(\varphi) \cap F(\psi)$. The opposite containment holds by linearity of the angular derivative and so $F(\varphi_t) = F(\varphi) \cap F(\psi)$.

Suppose that I is an open arc whose closure lies in G and does not intersect both $E(\varphi)$ and $E(\psi)$. Let $\mu_{\alpha,t}$ be a Clark measure for φ_t . The point masses of $\mu_{\alpha,t}$, if any, are carried by $F(\varphi_t) \subset E(\varphi) \cap E(\psi)$, so that $\mu_{\alpha,t}$ puts no mass at the endpoints of I . A theorem of J. E. Shapiro [24] then states that

$$(4.1) \quad \mu_{\alpha,t}^s(I) = \lim_{r \nearrow 1} \int_I \frac{1 - r^2}{|\alpha - r\varphi_t|^2} \frac{d\theta}{2\pi},$$

where $\mu_{\alpha,t}^s$ is the singular part of $\mu_{\alpha,t}$. It is enough to show that this quantity is zero, for then $spt(\mu_{\alpha,t}^s)$ cannot intersect I , and consequently neither can $E(\varphi_t)$.

For some $a < 1$ we have $\rho \leq a$ on G . Thus for $e^{i\theta}$ in G , $\psi(e^{i\theta})$ lies in the closed pseudo-hyperbolic disk with pseudo-hyperbolic radius a and pseudo-hyperbolic center $\varphi(e^{i\theta})$. On noting the Euclidean center and radius of this (also Euclidean) disk [8, p. 44], one can verify that there is a positive constant c , depending only on $a < 1$, such that

$$|\alpha - r\varphi_t(e^{i\theta})| \geq c \max\{|\alpha - r\varphi(e^{i\theta})|, |\alpha - r\psi(e^{i\theta})|\}$$

for all α in ∂D , $e^{i\theta}$ in G and $0 < r < 1$.

Given any ζ in the closure \bar{I} of the above arc I , there exists an open arc $A(\zeta)$ in $\partial \mathbb{D}$ containing ζ with either $\overline{A(\zeta)} \cap E(\varphi)$ or $\overline{A(\zeta)} \cap E(\psi)$ empty. The open cover $\{A(\zeta) : \zeta \in \bar{I}\}$ for \bar{I} has a finite subcover $\{A_1, \dots, A_n, B_1, \dots, B_m\}$ such that all of the sets $\overline{A_i} \cap E(\varphi)$, $i = 1, \dots, n$, and $\overline{B_j} \cap E(\psi)$, $j = 1, \dots, m$, are empty; of course, this subcover could consist only of A_i 's, or only of B_j 's. By Eqn. (4.1) and the previous paragraph,

$$\begin{aligned} \mu_{\alpha,t}^s(I) &= \lim_{r \nearrow 1} \int_I \frac{1 - r^2}{|\alpha - r\varphi_t|^2} \frac{d\theta}{2\pi} \\ &\leq \lim_{r \nearrow 1} \frac{1}{c} \left[\sum_{i=1}^n \int_{A_i} \frac{1 - r^2}{|\alpha - r\varphi|^2} \frac{d\theta}{2\pi} + \sum_{j=1}^m \int_{B_j} \frac{1 - r^2}{|\alpha - r\psi|^2} \frac{d\theta}{2\pi} \right] \\ &= \frac{1}{c} \left[\sum_{i=1}^n \mu_{\alpha,0}^s(A_i) + \sum_{j=1}^m \mu_{\alpha,1}^s(B_j) \right] \\ &= 0, \end{aligned}$$

as desired. □

Our third lemma localizes Theorem 2.2. The proof is as in [17], but now uses the operator Eqn. (2.5) left-multiplied by M_{χ_G} .

Lemma 4.3. *Let G be a measurable subset of ∂D . Suppose that the weighted composition operators $M_{\chi_G} M_{\varphi-\psi} C_{\varphi_t}$ act boundedly from A_1^2 to L^2 with norms uniformly bounded for $0 \leq t \leq 1$. If $M_{\chi_G} M_{\varphi-\psi} C_{\varphi_t}$ is compact for $0 \leq t \leq 1$, then $M_{\chi_G}(C_\varphi - C_\psi)$ is compact from H^2 to L^2 .*

We can now state our local compact difference theorem.

Theorem 4.4. *Let U be an open subset of ∂D whose boundary intersects neither $E(\varphi)$ nor $E(\psi)$, and with $E(\varphi) \cap U = E(\psi) \cap U$. Suppose that ρ can be re-defined on a set of measure zero (if necessary), so that $\lim_{e^{i\theta} \rightarrow \zeta} \rho(e^{i\theta}) = \rho(\zeta) = 0$ for every ζ in $E(\varphi) \cap E(\psi) \cap U$. Then $M_{\chi_U}(C_\varphi - C_\psi)$ is a compact operator from H^2 to L^2 .*

Proof. Since the intersection of $E(\varphi) \cap E(\psi)$ with U must be compact, there is an open subset G of U , containing this intersection and such that $\rho \leq \frac{1}{2}$ on G . We have

$$M_{\chi_U}(C_\varphi - C_\psi) = M_{\chi_G}(C_\varphi - C_\psi) + M_{\chi_{U \setminus G}} C_\varphi - M_{\chi_{U \setminus G}} C_\psi.$$

The last two operators on the right are compact by Theorem 3.1. According to Lemma 4.3, compactness of $M_{\chi_G}(C_\varphi - C_\psi)$ will follow if we can show that the operators $M_{\chi_G} M_{\varphi-\psi} C_{\varphi_t}$ map A_1^2 into L^2 , are uniformly bounded in norm and are each compact. Lemma 4.1 gives uniform boundedness since

$$(4.2) \quad \|C_{\varphi_t}\|_{\mathcal{B}(H^2)}^2 \leq \frac{2}{1 - |\varphi_t(0)|} \leq 2 \max \left\{ \frac{1}{1 - |\varphi(0)|}, \frac{1}{1 - |\psi(0)|} \right\}$$

(see [8]), while compactness follows from Lemmas 4.1, 4.2, and Theorem 3.1. \square

We close this section with a sum theorem for H^2 . Here $\varphi, \varphi_1, \varphi_2, \dots, \varphi_n$ are analytic self-maps of D with $|\varphi| < 1$ a.e. and $|\varphi_i| < 1$ a.e. on ∂D , $i = 1, 2, \dots, n$.

Theorem 4.5. *Let $\varphi, \varphi_1, \dots, \varphi_n$ be as above. Suppose that*

- (a) *The sets $E(\varphi_i)$, $i = 1, \dots, n$, are pairwise disjoint, and $E(\varphi)$ coincides with $E(\varphi_1) \cup \dots \cup E(\varphi_n)$, and*
- (b) *the functions $\rho_i = \left| \frac{\varphi - \varphi_i}{1 - \overline{\varphi} \varphi_i} \right|$ can be altered on a set of measure zero in ∂D (if necessary) to guarantee that $\lim_{e^{i\theta} \rightarrow \zeta} \rho_i(e^{i\theta}) = \rho_i(\zeta) = 0$ for all ζ in $E(\varphi_i)$, $i = 1, \dots, n$.*

Then, as operators on H^2 , $C_\varphi \equiv C_{\varphi_1} + \dots + C_{\varphi_n} \pmod{\mathcal{K}}$.

Proof. Clearly, we can find pairwise disjoint open sets U_1, \dots, U_n in ∂D such that U_i contains $E(\varphi_i)$ and $\rho_i \leq \frac{1}{2}$ on U_i , $i = 1, \dots, n$. Let $G = \partial D \setminus \bigcup_{i=1}^n U_i$, so that

$\partial D = G \cup U_1 \cup \dots \cup U_n$. For a measurable subset B of ∂D , let us write M_B for M_{χ_B} . Then

$$\begin{aligned} & C_\varphi - C_{\varphi_1} - C_{\varphi_2} - \dots - C_{\varphi_n} \\ &= (M_G + M_{U_1} + \dots + M_{U_n})(C_\varphi - C_{\varphi_1} - \dots - C_{\varphi_n}) \\ &= M_G C_\varphi - \sum_{i=1}^n M_G C_{\varphi_i} \\ &+ M_{U_1}(C_\varphi - C_{\varphi_1}) - \sum_{i \neq 1} M_{U_1} C_{\varphi_i} \\ &+ M_{U_2}(C_\varphi - C_{\varphi_2}) - \sum_{i \neq 2} M_{U_2} C_{\varphi_i} \\ &+ \vdots \\ &+ M_{U_n}(C_\varphi - C_{\varphi_n}) - \sum_{i \neq n} M_{U_n} C_{\varphi_i}. \end{aligned}$$

Theorem 3.1 tells us that every individual term on the right is compact except possibly $M_{U_i}(C_\varphi - C_{\varphi_i})$, $i = 1, \dots, n$. These, however, must be compact by Theorem 4.4. □

5. LINEAR COMBINATIONS MOD \mathcal{K}

In this section we consider a class \mathcal{S} of analytic self-maps of D for which definitive computations can be done. For $\varphi_1, \dots, \varphi_n$ in \mathcal{S} , we determine which linear combinations $c_1 C_{\varphi_1} + \dots + c_n C_{\varphi_n}$ are compact. We begin with some results which hold for arbitrary analytic self-maps of D .

5.1. A first-order lower bound for $\|c_1 C_{\varphi_1} + \dots + c_n C_{\varphi_n}\|_e$. Two analytic self-maps of D have the *same first-order data* at ζ in ∂D provided ζ lies in both $F(\varphi)$ and $F(\psi)$, $\varphi(\zeta) = \psi(\zeta)$ and $\varphi'(\zeta) = \psi'(\zeta)$. A special case of a theorem of MacCluer [13] states that if $C_{\varphi_1}, \dots, C_{\varphi_n}$ act on \mathcal{D}_β , ζ is in ∂D , and no two of the maps $\varphi_1, \dots, \varphi_n$ have the same first-order data at ζ , then

$$\|c_1 C_{\varphi_1} + \dots + c_n C_{\varphi_n}\|_e^2 \geq \sum_{k=1}^n |c_k|^2 \frac{1}{|\varphi'_k(\zeta)|^\beta}.$$

Here it is understood that if ζ is not in $F(\varphi_k)$, then $|\varphi'_k(\zeta)| = \infty$. We will need a minor but useful extension of this result. For ζ in ∂D and $M > 0$, let $\Gamma_{\zeta, M}^1$ denote the curve in D given by $\frac{|z-\zeta|}{1-|z|^2} = M$, the boundary of a non-tangential approach region with vertex at ζ . At the point ζ the sides of this region make angle θ with the radius to ζ , where $2 \cos \theta = \frac{1}{M}$. We will use the notation “ $\lim_{\Gamma_{\zeta, M}^1}$ ” to indicate a limit taken as $z \rightarrow \zeta$ along the starboard leg of $\Gamma_{\zeta, M}^1$ (that is, *counterclockwise*).

Lemma 5.1. *Suppose φ and ψ are analytic self-maps of D . Then*

$$\lim_{\Gamma_{\zeta, M}^1} \frac{1 - |z|^2}{1 - \overline{\varphi(z)}\psi(z)} = \begin{cases} \frac{2}{(1 + i \tan \theta)|\varphi'(\zeta)| + (1 - i \tan \theta)|\psi'(\zeta)|} & \text{if } \zeta \in F(\varphi) \cap F(\psi) \\ & \text{and } \varphi(\zeta) = \psi(\zeta), \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The Schwarz inequality applied to the H^2 kernel functions gives

$$(5.1) \quad \frac{1 - |z|^2}{|1 - \overline{\varphi(z)}\psi(z)|} \leq \left(\frac{1 - |z|^2}{1 - |\varphi(z)|^2} \right)^{\frac{1}{2}} \left(\frac{1 - |z|^2}{1 - |\psi(z)|^2} \right)^{\frac{1}{2}}.$$

Both factors on the right are bounded in D , and if ζ is not in $F(\varphi)$, say, the first factor tends to zero as $z \rightarrow \zeta$. Thus we may assume that ζ lies in $F(\varphi) \cap F(\psi)$. It is also clear that if $\varphi(\zeta) \neq \psi(\zeta)$, the left side of (5.1) tends to zero as $z \rightarrow \zeta$ along $\Gamma_{\zeta, M}^1$.

In the remaining case, ζ lies in $F(\varphi) \cap F(\psi)$ and $\varphi(\zeta) = \psi(\zeta)$. For z in $\Gamma_{\zeta, M}^1$, we have

$$\frac{1 - \overline{\varphi(z)}\psi(z)}{1 - |z|^2} = \frac{1 - |\varphi(z)|^2}{1 - |z|^2} + \overline{\varphi(z)} M \frac{\varphi(z) - \psi(z)}{|\zeta - z|}.$$

Since $\Gamma_{\zeta, M}^1$ is nontangential at ζ , the first term on the right tends to $|\varphi'(\zeta)|$ as $z \rightarrow \zeta$ along $\Gamma_{\zeta, M}^1$. Moreover on the counterclockwise leg of $\Gamma_{\zeta, M}^1$, $\zeta - z \sim \zeta e^{i\theta}|\zeta - z|$ as $z \rightarrow \zeta$. Recall that $\zeta \overline{\varphi(\zeta)}\varphi'(\zeta) = |\varphi'(\zeta)|$ and similarly for ψ . Since $Me^{i\theta} = \frac{1}{2}(1 + i \tan \theta)$ and $\varphi(\zeta) = \psi(\zeta)$, we have

$$\begin{aligned} \lim_{\Gamma_{\zeta, M}^1} \frac{1 - \overline{\varphi(z)}\psi(z)}{1 - |z|^2} &= |\varphi'(\zeta)| + M \zeta \overline{\varphi(\zeta)} e^{i\theta} (\psi'(\zeta) - \varphi'(\zeta)) \\ &= \frac{1}{2} \left[(1 + i \tan \theta)|\psi'(\zeta)| + \overline{(1 + i \tan \theta)}|\varphi'(\zeta)| \right], \end{aligned}$$

as desired. □

For ζ in $F(\varphi)$ we call the vector $D_1(\varphi, \zeta) = (\varphi(\zeta), \varphi'(\zeta))$ the first-order data of φ at ζ . Suppose we fix analytic self-maps of D , $\varphi_1, \dots, \varphi_n$. For ζ in ∂D , we denote by $\mathcal{D}_1(\zeta)$ the set of first-order data vectors at ζ associated to these self-maps:

$$\mathcal{D}_1(\zeta) = \{D_1(\varphi_j, \zeta) : 1 \leq j \leq n \text{ and } \zeta \in F(\varphi_j)\}.$$

Let us put $F = F(\varphi_1) \cup \dots \cup F(\varphi_n)$.

Theorem 5.2. *With the above notation, let $C_{\varphi_1}, \dots, C_{\varphi_n}$ act on \mathcal{D}_β . Then for any complex numbers c_1, \dots, c_n and ζ in ∂D ,*

$$\|c_1 C_{\varphi_1} + \dots + c_n C_{\varphi_n}\|_e^2 \geq \sum_{\mathbf{d} \in \mathcal{D}_1(\zeta)} \left| \sum_{\substack{\zeta \in F(\varphi_j) \\ D_1(\varphi_j, \zeta) = \mathbf{d}}} c_j \right|^2 \frac{1}{|d_1|^\beta},$$

where $\mathbf{d} = (d_0, d_1)$.

Proof. Referring to Lemma 5.1, we see that if φ and ψ are analytic self-maps of D , then M tending to infinity means that $\theta \rightarrow \frac{\pi}{2}$, so that

$$\lim_{M \rightarrow \infty} \lim_{\Gamma_{\zeta, M}^1} \frac{1 - |z|^2}{1 - \overline{\varphi(z)}\psi(z)} = \begin{cases} \frac{1}{|\varphi'(\zeta)|} & \text{if } \zeta \in F(\varphi) \cap F(\psi) \text{ and} \\ & D_1(\varphi, \zeta) = D_1(\psi, \zeta), \\ 0 & \text{otherwise.} \end{cases}$$

As $|z| \rightarrow 1$ the normalized kernel functions $\frac{k_z}{\|k_z\|}$ tend weakly to zero in \mathcal{D}_β . Since $C_\varphi^* k_z = k_{\varphi(z)}$, we see from Eqn. (2.3) that

$$\begin{aligned} & \|c_1 C_{\varphi_1} + \cdots + c_n C_{\varphi_n}\|_e^2 \\ & \geq \lim_{M \rightarrow \infty} \lim_{\Gamma_{\zeta, M}^1} \left\| (\bar{c}_1 C_{\varphi_1}^* + \cdots + \bar{c}_n C_{\varphi_n}^*) \frac{k_z}{\|k_z\|} \right\|_e^2 \\ & = \sum_{j, \ell=1}^n \bar{c}_j c_\ell \lim_{M \rightarrow \infty} \lim_{\Gamma_{\zeta, M}^1} \left(\frac{1 - |z|^2}{1 - \overline{\varphi_j(z)}\varphi_\ell(z)} \right)^\beta \\ & = \sum_{\substack{\zeta \in F(\varphi_j) \cap F(\varphi_\ell) \\ D_1(\varphi_j, \zeta) = D_1(\varphi_\ell, \zeta)}} c_j \bar{c}_\ell \frac{1}{|\varphi_j(\zeta)|^\beta}, \end{aligned}$$

which is a restatement of the desired conclusion. □

Corollary 5.3. *If $c_1 C_{\varphi_1} + \cdots + c_n C_{\varphi_n}$ is compact on \mathcal{D}_β , then for every ζ in F and every \mathbf{d} in $\mathcal{D}_1(\zeta)$,*

$$\sum_{\substack{\zeta \in F(\varphi_j) \\ D_1(\varphi_j, \zeta) = \mathbf{d}}} c_j = 0.$$

5.2. A remark on Theorem B. Corollary 5.3 shows that the hypothesis on angular derivative sets in Theorem B is actually implied by condition (ii) of the theorem.

Corollary 5.4. *Suppose that $C_\varphi, C_{\varphi_1}, \dots, C_{\varphi_n}$ act on \mathcal{D}_β and*

$$C_\varphi \equiv C_{\varphi_1} + \cdots + C_{\varphi_n} \pmod{\mathcal{K}}.$$

Then $F(\varphi_1), \dots, F(\varphi_n)$ are pairwise disjoint and $F(\varphi_1) \cup \cdots \cup F(\varphi_n)$ coincides with $F(\varphi)$.

Proof. Let us write $\varphi = \varphi_0$, $c_0 = 1$, and $c_j = -1$ for $j = 1, \dots, n$, so that

$$c_0 C_{\varphi_0} + c_1 C_{\varphi_1} + \cdots + c_n C_{\varphi_n} \in \mathcal{K}.$$

By Corollary 5.3, each of the sets $\{j : \zeta \in F(\varphi_j) \text{ and } D_1(\varphi_j, \zeta) = \mathbf{d}\}$ is either empty or contains exactly two elements, namely zero and an integer from $\{1, \dots, n\}$. The conclusion follows. □

5.3. Lower bounds from higher-order data. We have considered the first-order data $D_1(\varphi, \zeta) = (\varphi(\zeta), \varphi'(\zeta))$ for ζ in $F(\varphi)$. In what follows, we look at higher-order data vectors

$$D_k(\varphi, \zeta) = (\varphi(\zeta), \varphi'(\zeta), \varphi''(\zeta), \dots, \varphi^{(k)}(\zeta))$$

at points where the corresponding derivatives make sense. Specifically, we say φ has k^{th} -order data at ζ in ∂D if there exist complex numbers b_0, b_1, \dots, b_k with $|b_0| = 1$ so that

$$\varphi(z) = b_0 + b_1(z - \zeta) + \dots + b_k(z - \zeta)^k + o(|z - \zeta|^k)$$

as $z \rightarrow \zeta$ unrestrictedly in D . In this case $\lim_{z \rightarrow \zeta} \varphi^{(j)}(z)$ exists and equals $j!b_j$ for $j = 1, \dots, k$ (see, for example, the argument on p. 47 in [23]); we refer to this limit as $\varphi^{(j)}(\zeta)$. Since $|b_0| = 1$, ζ is in $F(\varphi)$ and b_1 is the angular derivative $\varphi'(\zeta)$.

The model for this definition is of course a map which continues analytically across ∂D near ζ . Aside from the partial Taylor expansion we want our φ to inherit another property of analyticity: order of contact. We say an analytic self-map φ of D has *order of contact* $c > 0$ at ζ if $|\varphi(\zeta)| = 1$ and

$$\frac{1 - |\varphi(e^{i\theta})|^2}{|\varphi(\zeta) - \varphi(e^{i\theta})|^c}$$

is essentially bounded above and away from zero as $e^{i\theta} \rightarrow \zeta$. To clarify this and subsequent calculations, we map to the upper half-plane $\Omega = \{w : \text{Im } w > 0\}$. For any α in ∂D , consider the conformal map $\tau_\alpha(z) = i \frac{\alpha - z}{\alpha + z}$, which takes D onto Ω and α to 0. If φ has a finite angular derivative at ζ , then $u = \tau_{\varphi(\zeta)} \circ \varphi \circ \tau_\zeta^{-1}$ is an analytic self-map of Ω having a non-tangential limit $u(0) = 0$ and a finite angular derivative $u'(0) = |\varphi'(\zeta)|$. Suppose for the moment that φ has an analytic continuation to a neighborhood of ζ , so the same is true of u at the origin. For w near zero, $u(w) = \sum_{n=1}^\infty a_n w^n$ with $a_1 = u'(0)$. If we assume $|\varphi| < 1$ a.e. on ∂D , then

$$\text{Im } u(x) = \sum_{n=1}^\infty (\text{Im } a_n) x^n$$

is positive for real x near (but not equal to) zero. The smallest natural number n with a_n non-real must be even, say $n = 2m$, and so $\text{Im } u(x) \sim (\text{Im } a_{2m})x^{2m}$ as $x \rightarrow 0$; moreover, $\text{Im } a_{2m} > 0$. It follows that near zero the image of \mathbb{R} under u is approximated by the curve $y = cx^{2m}$ for appropriate $c > 0$. Further, if $\tau_\zeta(e^{i\theta}) = x$, then

$$\text{Im } u(x) = \frac{1 - |\varphi(e^{i\theta})|^2}{|\varphi(\zeta) - \varphi(e^{i\theta})|^2},$$

and we find that $\frac{1 - |\varphi(e^{i\theta})|^2}{|\varphi(\zeta) - \varphi(e^{i\theta})|^{2m}}$ tends to a positive number as $e^{i\theta} \rightarrow \zeta$, so that φ has order of contact $2m$ at ζ .

Definition 5.5. We say an analytic self-map φ of D has *sufficient data* at ζ in ∂D if

- (i) φ has a finite angular derivative at ζ ;
- (ii) φ has an order of contact $2m$ at ζ for some natural number m ;
- (iii) φ has a $2m^{\text{th}}$ -order data at ζ .

If φ has sufficient data at ζ with an order of contact $2m$ and $u = \tau_{\varphi(\zeta)} \circ \varphi \circ \tau_{\zeta}^{-1}$ is as above, then u has an analogous expansion at the origin,

$$u(w) = \sum_{j=0}^{2m} \frac{u^{(j)}(0)}{j!} w^j + o(|w|^{2m});$$

here the derivatives can be realized as the non-tangential limits $\lim_{\angle w \rightarrow 0} u^{(j)}(w)$, $j = 0, 1, \dots, 2m$. Moreover, if $1 \leq k \leq 2m$, the k^{th} order data vector $D_k(\varphi, \zeta)$ determines and is determined by the corresponding data $u'(0), \dots, u^{(k)}(0)$ of u at zero.

Given ζ in ∂D , a natural number $k \geq 2$ and $M > 0$, let $\Gamma_{\zeta, M}^k$ denote the locus of the equation $\frac{|\zeta - z|^k}{1 - |z|^2} = M$ in D , a curve having “order of contact k ” with ∂D at ζ . We write “ $\lim_{\Gamma_{\zeta, M}^k}$ ” to indicate a limit taken as z tends to ζ along $\Gamma_{\zeta, M}^k$.

Lemma 5.6. *Suppose analytic self-maps of D , φ and ψ , both have sufficient data at ζ in ∂D . Let $u = \tau_{\varphi(\zeta)} \circ \varphi \circ \tau_{\zeta}^{-1}$ and $v = \tau_{\psi(\zeta)} \circ \psi \circ \tau_{\zeta}^{-1}$, so that for w near zero in Ω ,*

$$u(w) = \sum_{j=1}^{2m} \frac{u^{(j)}(0)}{j!} w^j + o(|w|^{2m}), \quad v(w) = \sum_{j=1}^{2n} \frac{v^{(j)}(0)}{j!} w^j + o(|w|^{2n}),$$

where $2m$ and $2n$ are the respective orders of contact of φ and ψ at ζ . Then, if $k \geq 2$ and $M > 0$,

$$\lim_{\Gamma_{\zeta, M}^k} \frac{1 - |z|^2}{1 - \overline{\varphi(z)}\psi(z)} = \left[\left(\frac{u'(0)}{2} - i \frac{M}{2^{k-1}k!} u^{(k)}(0) \right) + \left(\frac{v'(0)}{2} - i \frac{M}{2^{k-1}k!} v^{(k)}(0) \right) \right]^{-1}$$

provided $k \leq 2m$, $k \leq 2n$, and $D_{k-1}(\varphi, \zeta) = D_{k-1}(\psi, \zeta)$, while the limit is zero otherwise.

Proof. First we assume that $2m \geq k$, $2n \geq k$ and $D_{k-1}(\varphi, \zeta) = D_{k-1}(\psi, \zeta)$. We put $\alpha = \varphi(\zeta) = \psi(\zeta)$ and observe from direct calculation that with $w = \tau_{\zeta}(z)$,

$$\frac{v(w) - \overline{u(w)}}{2i \operatorname{Im} w} = \frac{1 - \overline{\varphi(z)}\psi(z)}{1 - |z|^2} \cdot \frac{|\zeta + z|^2}{(\overline{\alpha} + \overline{\varphi(z)})(\alpha + \psi(z))}.$$

If we let $z \rightarrow \zeta$ along the curve $\Gamma_{\zeta, M}^k$, $w = \tau_{\zeta}(z)$ tends to zero along its image in Ω , which is a slight enough perturbation of the curve $\tilde{\Gamma}_k$ defined by the equation $\frac{|w|^k}{\operatorname{Im} w} = \frac{M}{2^{k-2}}$ that the latter can be used to compute our limit. That is,

$$\lim_{\Gamma_{\zeta, M}^k} \frac{1 - \overline{\varphi(z)}\psi(z)}{1 - |z|^2} = \lim_{\tilde{\Gamma}_k} \frac{v(w) - \overline{u(w)}}{2i \operatorname{Im} w}.$$

Moreover,

$$(5.2) \quad \frac{v(w) - \overline{u(w)}}{2i \operatorname{Im} w} = \frac{\operatorname{Im} u(w)}{\operatorname{Im} w} + \frac{v(w) - u(w)}{2i \operatorname{Im} w}.$$

We write

$$u(w) = \sum_{j=1}^{2m} a_j w^j + o(|w|^{2m}), \quad v(w) = \sum_{j=1}^{2n} b_j w^j + o(|w|^{2n}),$$

and put $w = re^{i\theta}$. Consider the first term on the right in Eqn. (5.2). Since $a_1, a_2, \dots, a_{2m-1}$ are real and $\text{Im } a_{2m} > 0$, we have

$$\begin{aligned} \text{Im } u(w) &= a_1 r \sin \theta + a_2 r^2 \sin 2\theta + \dots + a_{2m-1} r^{2m-1} \sin(2m-1)\theta \\ &\quad + |a_{2m}| r^{2m} \sin(t_{2m} + 2m\theta) + o(r^{2m}), \end{aligned}$$

where $a_{2m} = |a_{2m}| e^{it_{2m}}$ with $0 < t_{2m} < \pi$. We take w in $\tilde{\Gamma}^k$ and divide by $\text{Im } w = r \sin \theta = \frac{2^{k-2}}{M} r^k$ to see that

$$(5.3) \quad \lim_{\tilde{\Gamma}^k} \frac{\text{Im } u(w)}{\text{Im } w} = a_1 + (\text{Im } a_k) \frac{M}{2^{k-2}}, \quad 2 \leq k \leq 2m,$$

where we use the fact that a_k is real if $k < 2m$.

Now consider the second term in Eqn. (5.2). Since $D_{k-1}(\varphi, \zeta) = D_{k-1}(\psi, \zeta)$, we have $a_j = b_j$ for $j < k$. Thus for w in $\tilde{\Gamma}^k$,

$$\frac{v(w) - u(w)}{2i \text{Im } w} = -i \frac{M}{2^{k-1}} (b_k - a_k) e^{ik\theta} + o(1).$$

Since $\theta \rightarrow 0$ as $w \rightarrow 0$ along $\tilde{\Gamma}^k$,

$$\lim_{\tilde{\Gamma}^k} \frac{v(w) - u(w)}{2i \text{Im } w} = -i \frac{M}{2^{k-1}} (b_k - a_k).$$

This equation, Eqn. (5.3) and Eqn. (5.2) give the desired result.

Suppose now that $2 \leq k \leq \min\{2m, 2n\}$ and $D_{k-1}(\varphi, \zeta) \neq D_{k-1}(\psi, \zeta)$. Let p be the smallest integer with $p \leq k-1$ and $a_p \neq b_p$. Then for $w = re^{i\theta}$ in $\tilde{\Gamma}^k$,

$$\frac{v(w) - u(w)}{2i \text{Im } w} = -i \frac{M}{2^{k-1}} \left(\frac{b_p - a_p}{r^{k-p}} \right) e^{ip\theta} + o\left(\frac{1}{r^{k-p}} \right),$$

a quantity whose modulus tends to infinity as $w \rightarrow 0$ along $\tilde{\Gamma}^k$. Thus

$$(5.4) \quad \lim_{\Gamma_{\zeta, M}^k} \frac{1 - |z|^2}{1 - \overline{\varphi(z)}\psi(z)} = 0.$$

Finally, if $k > 2m$, Eqn. (5.4) follows from Eqn. (5.1), the definition of ‘‘order of contact’’ and simple estimates applied to the Clark measure inequality

$$\frac{1 - |\varphi(z)|^2}{|\varphi(\zeta) - \varphi(z)|^2} \geq \int_{\partial D} P_z \frac{1 - |\varphi|^2}{|\varphi(\zeta) - \varphi|^2} \frac{d\theta}{2\pi}.$$

The case $k > 2n$ is similar. □

Now fix analytic self-maps $\varphi_1, \dots, \varphi_n$ of D and ζ in F . Assume that any φ_j having finite angular derivative at ζ (that is, ζ is in $F(\varphi_j)$) in fact has sufficient data at ζ . Given an integer $k \geq 2$, we write $\mathbb{M}_k(\zeta)$ for the set of those integers j , $1 \leq j \leq n$, for which $F(\varphi_j)$ contains ζ and the order of contact of φ_j at ζ is at least k . Let us write $\mathcal{D}_k(\zeta) = \{D_k(\varphi_j, \zeta) : j \in \mathbb{M}_k(\zeta)\}$. We have a higher-order analogue of Theorem 5.2.

Theorem 5.7. *Assume that $\varphi_1, \dots, \varphi_n$ are analytic self-maps of D as described above, and let ζ be in F . If c_1, \dots, c_n are complex, $k \geq 3$ and notation is as above,*

$$\|c_1 C_{\varphi_1} + \dots + c_n C_{\varphi_n}\|_e^2 \geq \sum_{\mathbf{d} \in \mathcal{D}_{k-1}(\zeta)} \left| \sum_{\substack{j \in \mathbb{M}_k(\zeta) \\ D_{k-1}(\varphi_j, \zeta) = \mathbf{d}}} c_j \right|^2 \frac{1}{|d_1|^\beta},$$

where $\mathbf{d} = (d_0, d_1, \dots, d_{k-1})$ and each C_{φ_j} acts on \mathcal{D}_β .

Proof. First consider analytic self-maps φ and ψ of D , each of which has sufficient data, with respective orders of contact $2m$ and $2n$, at a given ζ in ∂D . Let u and v be related to φ and ψ as in Lemma 5.6. Since $u^{(j)}(0)$ is real for $1 \leq j < 2m$ and $\text{Im } u^{(2m)}(0) > 0$ (and similarly for $v^{(j)}(0)$), we see from Lemma 5.6 that

$$\lim_{M \rightarrow \infty} \lim_{\Gamma_{\zeta, M}^{k-1}} \frac{1 - |z|^2}{1 - \overline{\varphi(z)}\psi(z)} = \begin{cases} \frac{1}{|\varphi'(\zeta)|} & \text{if } k \leq 2m, k \leq 2n, \text{ and} \\ & D_{k-1}(\varphi, \zeta) = D_{k-1}(\psi, \zeta), \\ 0 & \text{otherwise.} \end{cases}$$

Proceeding as in the proof of Theorem 5.2, we find

$$\begin{aligned} \|c_1 C_{\varphi_1} + \dots + c_n C_{\varphi_n}\|_e^2 &\geq \sum_{j, \ell=1}^n \overline{c_j} c_\ell \lim_{M \rightarrow \infty} \lim_{\Gamma_{\zeta, M}^{k-1}} \left(\frac{1 - |z|^2}{1 - \overline{\varphi_j(z)}\varphi_\ell(z)} \right)^\beta \\ &= \sum_{\substack{j, \ell \in \mathbb{M}_k(\zeta) \\ D_{k-1}(\varphi_j, \zeta) = D_{k-1}(\varphi_\ell, \zeta)}} \overline{c_j} c_\ell \frac{1}{|\varphi'_j(\zeta)|^\beta}, \end{aligned}$$

which is the desired conclusion. □

The above theorem yields a higher-order version of MacCluer’s lower bound for $\|C_\varphi - C_\psi\|_e$ in [13].

Corollary 5.8. *Fix ζ in ∂D and analytic self-maps φ and ψ of D , both of which have sufficient data at ζ , with respective orders of contact $2m$ and $2n$.*

- (i) *If $n < m$, then $\|C_\varphi - C_\psi\|_e^2 \geq \frac{1}{|\varphi'(\zeta)|^\beta}$.*
- (ii) *If $n = m$ and $D_{2m-1}(\varphi, \zeta) \neq D_{2m-1}(\psi, \zeta)$, then*

$$\|C_\varphi - C_\psi\|_e^2 \geq \frac{1}{|\varphi'(\zeta)|^\beta} + \frac{1}{|\psi'(\zeta)|^\beta}.$$

Proof. Apply Theorem 5.7 with $\varphi_1 = \varphi$, $\varphi_2 = \psi$, $c_1 = 1$, and $c_2 = -1$. □

We need a more delicate version of Theorem 5.7, which is conveniently expressed in terms of the following formalism. For $\beta \geq 1$, \mathcal{D}_β^+ will denote the reproducing kernel Hilbert space of functions on the right half-plane $\Omega^+ = \{z : \text{Re } z > 0\}$ having the kernel functions $k_w^+(z) = (z + \overline{w})^{-\beta}$, w in Ω^+ . These spaces appear in the literature with various defining normalizations. When $\beta = 1$, \mathcal{D}_β^+ is the Hardy space on Ω^+ ; see [12]. For $\beta > 1$, \mathcal{D}_β^+ is the weighted Bergman space of all functions f analytic on Ω^+ for which

$$\|f\|_{\mathcal{D}_\beta^+}^2 \equiv \frac{2^{\beta-2}(\beta-1)}{\pi} \int_{\Omega^+} |f(x + iy)|^2 x^{\beta-2} dx dy$$

is finite; see [21, p. 74]. For us a key fact (used in Section 5.5) is this: $\{k_w^+ : w \in \Omega_+\}$ is a linearly independent set in \mathcal{D}_β^+ .

Now let $\mathbb{M}_k(\zeta)$ and $\mathcal{D}_k(\zeta)$ be as defined prior to Theorem 5.7.

Lemma 5.9. *Let $\varphi_1, \dots, \varphi_n$ be analytic self-maps of D . Fix ζ in F and suppose that if $F(\varphi_j)$ contains ζ , then φ_j has sufficient data at ζ , $j = 1, \dots, n$. Let u_j be related to φ_j as u is related to φ in Lemma 5.6. Then if $A > 0$ and k is an even natural number,*

$$\|c_1 C_{\varphi_1} + \dots + c_n C_{\varphi_n}\|_e^2 \geq \sum_{\mathbf{d} \in \mathcal{D}_{k-1}(\zeta)} \left\| \sum_{\substack{j \in \mathbb{M}_k(\zeta) \\ D_{k-1}(\varphi_j, \zeta) = \mathbf{d}}} \overline{c_j} k_{\frac{u_j'(0)}{2} - iAu_j^{(k)}(0)}^+ \right\|_{\mathcal{D}_\beta^+}^2.$$

Proof. We use Lemma 5.6 with $M = A2^{k-1}k!$. Then we have

$$\begin{aligned} & \|c_1 C_{\varphi_1} + \dots + c_n C_{\varphi_n}\|_e^2 \\ & \geq \lim_{\Gamma_{\zeta, M}^k} \left\| (\overline{c_1} C_{\varphi_1}^* + \dots + \overline{c_n} C_{\varphi_n}^*) \frac{k_z}{\|k_z\|} \right\|_{\mathcal{D}_\beta}^2 \\ & \geq \lim_{\Gamma_{\zeta, M}^k} \sum \overline{c_j} c_\ell \left(\frac{1 - |z|^2}{1 - \overline{\varphi_j(z)} \varphi_\ell(z)} \right)^\beta \\ & = \sum_{\substack{j, \ell \in \mathbb{M}_k(\zeta) \\ D_{k-1}(\varphi_j, \zeta) = D_{k-1}(\varphi_\ell, \zeta)}} \overline{c_j} c_\ell \left[\overline{\left(\frac{u_j'(0)}{2} - iAu_j^{(k)}(0) \right)} + \left(\frac{u_\ell'(0)}{2} - iAu_\ell^{(k)}(0) \right) \right]^{-\beta} \\ & = \sum_{\mathbf{d} \in \mathcal{D}_{k-1}(\zeta)} \left\| \sum_{\substack{j \in \mathbb{M}_k(\zeta) \\ D_{k-1}(\varphi_j, \zeta) = \mathbf{d}}} \overline{c_j} k_{\frac{u_j'(0)}{2} - iAu_j^{(k)}(0)}^+ \right\|_{\mathcal{D}_\beta^+}^2, \end{aligned}$$

as desired. Note that since $u_j'(0) > 0$, $A > 0$ and $\text{Im } u_j^{(k)}(0) \geq 0$, the complex number $\frac{u_j'(0)}{2} - iAu_j^{(k)}(0)$ lies in Ω^+ . □

5.4. The class \mathcal{S} and making $\|C_\varphi - C_\psi\|_e$ small. For maps φ and ψ with sufficient data at a given ζ in ∂D , Corollary 5.8 describes two obstructions to $\|C_\varphi - C_\psi\|_e$ being small:

- (a) unequal orders of contact at ζ ;
- (b) equal order of contact $2m$ but $D_{2m-1}(\varphi, \zeta) \neq D_{2m-1}(\psi, \zeta)$.

In this section we estimate $\|C_\varphi - C_\psi\|_e$ in the absence of these obstructions and characterize when it is zero. We work within the class \mathcal{S} of analytic self-maps φ of D for which $E(\varphi)$ is a finite set (so that $E(\varphi) = F(\varphi)$) and such that φ has sufficient data at each point of $F(\varphi)$. For simplicity we restrict our attention to composition operators on $\mathcal{D}_1 = H^2$. We write Ω for the upper half-plane $\{z : \text{Im } z > 0\}$; the

pseudo-hyperbolic metric Λ on Ω is given by

$$\Lambda(z, w) = \left| \frac{z - w}{z - \bar{w}} \right|.$$

Note that $0 \leq \Lambda < 1$ on $\Omega \times \Omega$. Recall from Section 5.3 that if φ has sufficient data at ζ in $F(\varphi)$ with order of contact $2m$, and $u = \tau_{\varphi(\zeta)} \circ \varphi \circ \tau_{\zeta}^{-1}$, then $u^{(2m)}(0)$ lies in Ω .

Proposition 5.10. *Fix ζ in ∂D and suppose that φ and ψ have sufficient data with respective orders of contact $2m$ and $2n$ at ζ , and moreover that $\varphi(\zeta) = \psi(\zeta)$.*

Write $\rho = \left| \frac{\varphi - \psi}{1 - \bar{\varphi}\psi} \right|$, $u = \tau_{\varphi(\zeta)} \circ \varphi \circ \tau_{\zeta}^{-1}$ and $v = \tau_{\psi(\zeta)} \circ \psi \circ \tau_{\zeta}^{-1}$.

- (a) *If $2m \neq 2n$, or $2m = 2n$ and $D_{2m-1}(\varphi, \zeta) \neq D_{2m-1}(\psi, \zeta)$, then $\rho(e^{i\theta}) \rightarrow 1$ as $e^{i\theta} \rightarrow \zeta$.*
- (b) *If $2m = 2n$ and $D_{2m-1}(\psi, \zeta) = D_{2m-1}(\varphi, \zeta)$, then*

$$\lim_{e^{i\theta} \rightarrow \zeta} \rho(e^{i\theta}) = \Lambda \left(u^{(2m)}(0), v^{(2m)}(0) \right) < 1.$$

- (c) *$\rho(e^{i\theta}) \rightarrow 0$ as $e^{i\theta} \rightarrow \zeta$ if and only if $2m = 2n$ and $D_{2m}(\varphi, \zeta) = D_{2m}(\psi, \zeta)$.*

Proof. First assume that the hypotheses of (b) hold. Since $\psi(\zeta) = \varphi(\zeta)$, direct computation shows that if $\tau_{\zeta}(e^{i\theta}) = x$,

$$\rho(e^{i\theta}) = \left| \frac{u(x) - v(x)}{u(x) - \overline{v(x)}} \right|.$$

For $i = 1, 2, \dots, 2m - 1$, the MacLaurin coefficient a_i for u is real and equal to the corresponding coefficient b_i for v . On the other hand, $a_{2m} = u^{(2m)}(0)/(2m)!$ and $b_{2m} = v^{(2m)}(0)/(2m)!$ lie in Ω . Clearly,

$$\lim_{x \rightarrow 0} \left| \frac{u(x) - v(x)}{u(x) - \overline{v(x)}} \right| = \Lambda(a_{2m}, b_{2m}),$$

which is the desired conclusion for (b).

Clearly (b) implies (c); the interested reader can easily verify (a). □

For φ , ψ , and ζ as in Proposition 5.10, let us use the convention that $\rho(\zeta) = \lim_{e^{i\theta} \rightarrow \zeta} \rho(e^{i\theta})$. The limit exists by Proposition 5.10 and for φ, ψ in \mathcal{S} this convention, at worst, redefines ρ on a finite set, leaving the multiplication operator M_ρ unaltered. Note, however, that $\rho(\zeta)$ as just defined is in general not the same as the non-tangential limit $\lim_{\angle z \rightarrow \zeta} \rho(z)$.

Theorem 5.11. *Suppose that φ and ψ are in \mathcal{S} with $F(\varphi) = F(\psi) = F$, and that C_φ and C_ψ act on H^2 . Suppose that at each ζ in F , φ and ψ have common order of contact $2m(\zeta)$ with $D_{2m(\zeta)-1}(\varphi, \zeta) = D_{2m(\zeta)-1}(\psi, \zeta)$ and moreover that $\rho(\zeta) \leq \frac{1}{2}$. Then*

$$\frac{1}{4} \max_{\zeta \in F} \frac{1}{|\varphi'(\zeta)|} \rho(\zeta)^2 \leq \|C_\varphi - C_\psi\|_e^2 \leq B \max_{\alpha \in \varphi(F)} \sum_{\varphi(\zeta) = \alpha} \frac{1}{|\varphi'(\zeta)|} \rho(\zeta)^2,$$

where B is an absolute constant and $\varphi(F) = \{\varphi(\zeta) : \zeta \in F\}$. Moreover, the lower bound is valid without the assumption that $\rho(\zeta) \leq \frac{1}{2}$, $\zeta \in F$.

Note. Suppose that φ and ψ satisfy the hypotheses of the theorem, and that μ_α and ν_α are their respective Clark measures. Then μ_α^s and ν_α^s are pure point, and since φ and ψ have the same first-order data,

$$\nu_\alpha^s = \mu_\alpha^s = \sum_{\substack{\zeta \in F \\ \varphi(\zeta) = \alpha}} \frac{1}{|\varphi'(\zeta)|} \delta_\zeta.$$

The upper bound for $\|C_\varphi - C_\psi\|_e^2$ in the theorem then takes the form

$$B \sup_{|\alpha|=1} \int \rho^2 d\mu_\alpha^s.$$

We expect that there is a lower bound of the same type.

Proof of Theorem 5.11. For ζ in F write $u_\zeta = \tau_{\varphi(\zeta)} \circ \varphi \circ \tau_\zeta^{-1}$ and $v_\zeta = \tau_{\psi(\zeta)} \circ \psi \circ \tau_\zeta^{-1}$. According to Lemma 5.9, if ζ is in F and $A > 0$,

$$\|C_\varphi - C_\psi\|_e^2 \geq \left\| \left\| k_{\frac{u'_\zeta(0)}{2} - iAu_\zeta^{(2m(\zeta))}(0)}^+ - k_{\frac{v'_\zeta(0)}{2} - iAv_\zeta^{(2m(\zeta))}(0)}^+ \right\|_{H^2_+} \right\|^2.$$

A calculation analogous to [8, Lemma 9.11] shows that for z, w in Ω_+ ,

$$\|k_z^+ - k_w^+\|_{H^2_+}^2 = \left| \frac{z - w}{z + \bar{w}} \right|^2 \left(\frac{1}{z + \bar{z}} + \frac{1}{w + \bar{w}} \right).$$

We put

$$A = \frac{u'_\zeta(0)}{\left| u_\zeta^{(2m(\zeta))}(0) - v_\zeta^{(2m(\zeta))}(0) \right|},$$

$$z = \frac{u'_\zeta(0)}{2} - iAu_\zeta^{(2m(\zeta))}(0), \quad w = \frac{v'_\zeta(0)}{2} - iAv_\zeta^{(2m(\zeta))}(0),$$

and note that $|z + \bar{w}| \leq 2u'_\zeta(0)$ while

$$|z - w| = u'_\zeta(0) \Lambda \left(u_\zeta^{(2m(\zeta))}(0), v_\zeta^{(2m(\zeta))}(0) \right).$$

It follows that

$$\|k_z^+ - k_w^+\|_{H^2_+}^2 \geq \frac{1}{4u'_\zeta(0)} \Lambda \left(u_\zeta^{(2m(\zeta))}(0), v_\zeta^{(2m(\zeta))}(0) \right)^2,$$

which, by Proposition 5.10 and since $u'_\zeta(0) = |\varphi'(\zeta)|$, gives the lower bound for the essential norm of $C_\varphi - C_\psi$.

For the upper bound select a finite union G of pairwise disjoint open arcs which contains F and is such that $\rho \leq \frac{2}{3}$ on G . Now

$$C_\varphi - C_\psi = M_{\chi_G}(C_\varphi - C_\psi) + M_{\chi_{\partial D \setminus G}} C_\varphi - M_{\chi_{\partial D \setminus G}} C_\psi,$$

and the last two terms on the right are compact operators by Theorem 3.1. Therefore $\|C_\varphi - C_\psi\|_e = \|M_{\chi_G}(C_\varphi - C_\psi)\|_e$ which we now estimate. To use Eqn. (2.3) it is permissible to restrict the sequence $\{f_n\}$ to a subspace of finite codimension. With notation as in Section 2.6, we pick a sequence of unit vectors f_n in $H^2_0 = (\ker X)^\perp$ which converges weakly to zero. When applied to an H^2 function, the operator

identity (2.5) holds pointwise in D and thus, since the kernel functions span H^2 , in the weak operator topology. Thus

$$\|M_{\chi_G}(C_\varphi - C_\psi)f_n\|_{L^2} \leq \|X\|_{\mathcal{B}(H_0^2, A_1^2)} \int_0^1 \|M_{\chi_G}M_{\varphi-\psi}C_{\varphi_t}g_n\|_{L^2} dt,$$

where $g_n = Xf_n/\|Xf_n\|$. Since X is bounded below on H_0^2 , $g_n \rightarrow 0$ weakly in A_1^2 . Now $\|M_{\chi_G}M_{\varphi-\psi}C_{\varphi_t}\|_{\mathcal{B}(A_1^2, L^2)}$ is bounded for $0 \leq t \leq 1$ by Lemma 4.1 and Eqn. (4.2), so we can apply Fatou's Lemma to conclude that

$$\begin{aligned} & \limsup_{n \rightarrow \infty} \|M_{\chi_G}(C_\varphi - C_\psi)f_n\|_{L^2} \\ & \leq \|X\|_{\mathcal{B}(H_0^2, A_1^2)} \int_0^1 \limsup_{n \rightarrow \infty} \|M_{\chi_G}M_{\varphi-\psi}C_{\varphi_t}g_n\|_{L^2} dt. \end{aligned}$$

Moreover, by Eqn. (2.3) and Lemma 4.1 we have

$$\begin{aligned} \limsup_{n \rightarrow \infty} \|M_{\chi_G}M_{\varphi-\psi}C_{\varphi_t}g_n\|_{L^2} & \leq \|M_{\chi_G}M_{\varphi-\psi}C_{\varphi_t}\|_{e, \mathcal{B}(A_1^2, L^2)} \\ & \leq b\|M_{\chi_G}M_\rho C_{\varphi_t}\|_{e, \mathcal{B}(H^2, L^2)} \\ & \leq b\|M_h C_{\varphi_t}\|_{e, \mathcal{B}(H^2, L^2)}, \end{aligned}$$

where $b > 0$, h is any continuous function on ∂D with $h \geq \chi_G \rho$ on ∂D and $h(\zeta) = \rho(\zeta)$ for ζ in F . Clearly we can choose such an h while also requiring that $\rho \leq \frac{3}{4}$ on the set $W = \{e^{i\theta} : h(e^{i\theta}) > 0\}$. Then, by Theorem 3.1 and Lemma 3.2, if $\mu_{\alpha,t}$ is a Clark measure for φ_t ,

$$\begin{aligned} \|M_h C_{\varphi_t}\|_e & \leq 2 \sup_{|\alpha|=1} \left\{ \int_W h^2 d\mu_{\alpha,t}^s \right\}^{\frac{1}{2}} \\ & = 2 \sup_{|\alpha|=1} \left\{ \lim_{r \nearrow 1} \int_W h^2 \frac{1-r^2}{|\alpha-r\varphi_t|^2} \frac{d\theta}{2\pi} \right\}^{\frac{1}{2}}. \end{aligned}$$

Since $\rho \leq \frac{3}{4}$ on W we see (as noted in the proof of Lemma 4.2) that there is a constant $c > 0$ such that $|\alpha - r\varphi_t(e^{i\theta})| \geq c|\alpha - r\varphi(e^{i\theta})|$, for α in ∂D , $e^{i\theta}$ in W , $0 < r < 1$ and $0 \leq t < 1$. Thus

$$\begin{aligned} \|M_h C_{\varphi_t}\|_e & \leq \frac{2}{c} \sup_{|\alpha|=1} \left\{ \lim_{r \nearrow 1} \int_{\partial D} h^2 \frac{1-r^2}{|\alpha-r\varphi|^2} \frac{d\theta}{2\pi} \right\}^{\frac{1}{2}} \\ & = \frac{2}{c} \sup_{|\alpha|=1} \int_{\partial D} h^2 d\mu_\alpha^s, \end{aligned}$$

where $\{\mu_\alpha\}$ are the Clark measures for φ . Since each μ_α^s is pure point in the present circumstances,

$$\mu_\alpha^s = \begin{cases} \sum_{\substack{\zeta \in F \\ \varphi(\zeta) = \alpha}} \frac{1}{|\varphi'(\zeta)|} \delta_\zeta & \text{if } \alpha \in \varphi(F), \\ 0 & \text{otherwise.} \end{cases}$$

In view of Eqn. (2.3) and the arbitrariness of $\{f_n\}$ in H_0^2 , this gives the desired upper bound for $\|C_\varphi - C_\psi\|_e$. \square

Corollary 5.12. *Suppose that φ and ψ lie in \mathcal{S} with respective orders of contact $2m(\zeta)$ and $2n(\zeta)$ at each ζ in (respectively) $F(\varphi)$ and $F(\psi)$. Then the following are equivalent:*

- (i) $C_\varphi - C_\psi$ is compact on H^2 .
- (ii) $F(\psi) = F(\varphi)$ (we call this set F) and for all ζ in F , $2n(\zeta) = 2m(\zeta)$ and $D_{2m(\zeta)}(\psi, \zeta) = D_{2m(\zeta)}(\varphi, \zeta)$.

5.5. Linear relations mod \mathcal{K} . We fix $\varphi_1, \dots, \varphi_n$ in \mathcal{S} and again write F for the union $F(\varphi_1) \cup \dots \cup F(\varphi_n)$, a finite set. For ζ in F and $k = 2, 4, 6, \dots$, let

$$\mathbb{N}_k(\zeta) = \{j : F(\varphi_j) \text{ contains } \zeta \text{ and } k \text{ is the order of contact of } \varphi_j \text{ at } \zeta\}.$$

We also write $\mathcal{E}_k(\zeta) = \{D_k(\varphi_j, \zeta) : j \text{ is in } \mathbb{N}_k(\zeta)\}$.

Theorem 5.13. *Let $\varphi_1, \dots, \varphi_n$ be in \mathcal{S} and set notation as above. Given complex numbers c_1, \dots, c_n , the following are equivalent:*

- (i) $c_1 C_{\varphi_1} + \dots + c_n C_{\varphi_n}$ is compact on \mathcal{D}_β ;
- (ii) for every ζ in F , every even $k \geq 2$ and every \mathbf{d} in $\mathcal{E}_k(\zeta)$,

$$\sum_{\substack{j \in \mathbb{N}_k(\zeta) \\ D_k(\varphi_j, \zeta) = \mathbf{d}}} c_j = 0.$$

Proof. First assume that $c_1 C_{\varphi_1} + \dots + c_n C_{\varphi_n}$ is compact. Fix ζ in F and let u_j be related to φ_j as u is related to φ in Lemma 5.6. Let $\mathbb{M}_k(\zeta)$ and $\mathcal{D}_k(\zeta)$ be as defined prior to Theorem 5.7. According to Lemma 5.9 (with $A = 1$), if $k \geq 2$, \mathbf{d} is in $\mathcal{D}_{k-1}(\zeta)$, and $w_j = \frac{u'_j(0)}{2} - iu_j^{(k)}(0)$, then

$$\sum_{\substack{j \in \mathbb{M}_k(\zeta) \\ D_{k-1}(\varphi_j, \zeta) = \mathbf{d}}} \bar{c}_j k_{w_j}^+ = 0.$$

But if $D_{k-1}(\varphi_j, \zeta)$ and $D_{k-1}(\varphi_\ell, \zeta)$ coincide, $w_j = w_\ell$ exactly when $D_k(\varphi_j, \zeta) = D_k(\varphi_\ell, \zeta)$. Using the fact that $\{k_w^+ : \operatorname{Re} w > 0\}$ is linearly independent in \mathcal{D}_β^+ , we see that for every \mathbf{d} in $\mathcal{D}_k(\zeta)$,

$$(5.5) \quad \sum_{\substack{j \in \mathbb{M}_k(\zeta) \\ D_k(\varphi_j, \zeta) = \mathbf{d}}} c_j = 0.$$

Since $\mathbb{M}_k(\zeta)$ is the union of the disjoint sets $\mathbb{N}_k(\zeta)$ and $\mathbb{M}_{k+1}(\zeta)$, we see from Eqn. (5.5) that

$$\sum_{\substack{j \in \mathbb{N}_k(\zeta) \\ D_k(\varphi_j, \zeta) = \mathbf{d}}} c_j + \sum_{\substack{j \in \mathbb{M}_{k+1}(\zeta) \\ D_k(\varphi_j, \zeta) = \mathbf{d}}} c_j = 0.$$

Suppose now $\mathbf{d} = (d_0, d_1, \dots, d_k)$. There are a finite number of elements in $\mathcal{D}_{k+1}(\zeta)$ having the form $(d_0, d_1, \dots, d_k, *)$, call them $\mathbf{d}_i = (d_0, d_1, \dots, d_k, d_{k+1}^i)$, $i = 1, 2, \dots, r$. Then

$$\sum_{\substack{j \in \mathbb{M}_{k+1}(\zeta) \\ D_k(\varphi_j, \zeta) = \mathbf{d}}} c_j = \sum_{i=1}^r \left\{ \sum_{\substack{j \in \mathbb{M}_{k+1}(\zeta) \\ D_{k+1}(\varphi_j, \zeta) = \mathbf{d}_i}} c_j \right\}$$

and each of the r summands on the right vanishes, again by Eqn. (5.5). Therefore (ii) holds.

Now assume that (ii) holds. We consider the case $\beta = 1$. The set F is finite, say $F = \{\zeta_1, \dots, \zeta_s\}$. Let U_1, \dots, U_s be disjoint arcs whose union is ∂D and for which the interior of U_i contains ζ_i for $i = 1, \dots, s$. Clearly,

$$\sum_{j=1}^n c_j C_{\varphi_j} = \sum_{i=1}^s M_{\chi_{U_i}} \left(\sum_{j=1}^n c_j C_{\varphi_j} \right).$$

Taking U to be any U_i and $\zeta = \zeta_i$, it is enough to show that the operator

$$M_{\chi_U} \left(\sum_{j=1}^n c_j C_{\varphi_j} \right),$$

considered as acting from H^2 to L^2 , is compact. This operator can be written as

$$(5.6) \quad \sum_{\zeta \notin F(\varphi_j)} c_j M_{\chi_U} C_{\varphi_j} + \sum_{\zeta \in F(\varphi_j)} c_j M_{\chi_U} C_{\varphi_j}.$$

If $F(\varphi_j)$ does not contain ζ , then χ_U is identically zero in a neighborhood of $F(\varphi_j)$, and we see from Theorem 3.1 that $M_{\chi_U} C_{\varphi_j}$ is compact. Thus the first sum in the expression (5.6) is a compact operator.

Now consider the second sum, which can be rewritten as

$$\sum_{m \geq 1} \left\{ \sum_{j \in \mathbb{N}_{2m}(\zeta)} c_j M_{\chi_U} C_{\varphi_j} \right\};$$

the sum over m is of course finite since $\mathbb{N}_{2m}(\zeta)$ is empty for m large enough. For non-empty $\mathbb{N}_{2m}(\zeta)$,

$$\sum_{j \in \mathbb{N}_{2m}(\zeta)} c_j M_{\chi_U} C_{\varphi_j} = \sum_{\mathbf{d} \in \mathcal{E}_{2m}(\zeta)} \left\{ \sum_{\substack{j \in \mathbb{N}_{2m}(\zeta) \\ D_{2m}(\varphi_j, \zeta) = \mathbf{d}}} c_j M_{\chi_U} C_{\varphi_j} \right\}.$$

If \mathbf{d} is in $\mathcal{E}_{2m}(\zeta)$, j and ℓ are in $\mathbb{N}_{2m}(\zeta)$ and $D_{2m}(\varphi_j, \zeta) = D_{2m}(\varphi_\ell, \zeta) = \mathbf{d}$, Proposition 5.10(c) tells us that

$$\left| \frac{\varphi_j(e^{i\theta}) - \varphi_\ell(e^{i\theta})}{1 - \overline{\varphi_j(e^{i\theta})} \varphi_\ell(e^{i\theta})} \right| \rightarrow 0 \text{ as } e^{i\theta} \rightarrow \zeta.$$

It follows from Theorem 4.4 that $M_{\chi_U}(C_{\varphi_j} - C_{\varphi_\ell})$ is compact. Fix such ℓ and write $T = M_{\chi_U} C_{\varphi_\ell}$. Then for each j in $\mathbb{N}_{2m}(\zeta)$ with $D_{2m}(\varphi_j, \zeta) = \mathbf{d}$, there is a compact operator K_j with $M_{\chi_U} C_{\varphi_j} = T + K_j$. Thus

$$\sum_{\substack{j \in \mathbb{N}_{2m}(\zeta) \\ D_{2m}(\varphi_j, \zeta) = \mathbf{d}}} c_j M_{\chi_U} C_{\varphi_j} = \left(\sum_{\substack{j \in \mathbb{N}_{2m}(\zeta) \\ D_{2m}(\varphi_j, \zeta) = \mathbf{d}}} c_j \right) T + \sum_{\substack{j \in \mathbb{N}_{2m}(\zeta) \\ D_{2m}(\varphi_j, \zeta) = \mathbf{d}}} c_j K_j.$$

The coefficient of T vanishes by the hypothesis (ii), and we are left with a compact operator, verifying (i). The proof for the case $\beta > 1$ is similar, with Theorem C playing the role of Theorem 4.4, the sets U_i taken to lie in D , and Proposition 5.10(c) replaced by the assertion that if $2m = 2n$ and $D_{2m}(\varphi, \zeta) = D_{2m}(\psi, \zeta)$,

then $\rho(z) \rightarrow 0$ as $z \rightarrow \zeta$ unrestrictedly in D , an implication easily established by calculations in the proofs of Lemma 5.6 and Proposition 5.10. \square

Remark 5.14. It is sometimes convenient to rephrase condition (ii) in Theorem 5.13. With notation as in the statement, fix ζ in F , an even natural number k and a vector \mathbf{d} in $\mathcal{E}_k(\zeta)$. We define the vector $\mathbf{x}(\zeta, k, \mathbf{d}) = (a_1, \dots, a_n)$ in \mathbb{C}^n , where $a_j = 1$ if j is in $\mathbb{N}_k(\zeta)$ and $D_k(\varphi_j, \zeta) = \mathbf{d}$, while $a_j = 0$ otherwise. Let $\mathcal{M} = \mathcal{M}(\varphi_1, \dots, \varphi_n)$ denote the linear span in \mathbb{C}^n of all such vectors $\mathbf{x}(\zeta, k, \mathbf{d})$. Clearly, condition (ii) in Theorem 5.13 is equivalent to

$$(ii)' \quad (c_1, \dots, c_n) \in \mathcal{M}^\perp.$$

Put another way, if we define a linear transformation $\mathbb{A} : \mathbb{C}^n \rightarrow \mathcal{B}(\mathcal{D}_\beta)/\mathcal{K}$ by

$$\mathbb{A}(c_1, \dots, c_n) = [c_1 C_{\varphi_1} + \dots + c_n C_{\varphi_n}],$$

where $[B]$ denotes the coset of the operator B , then the content of Theorem 5.13 is that $\ker \mathbb{A} = \mathcal{M}^\perp$. This yields the immediate:

Corollary 5.15. *Fix $\varphi_1, \dots, \varphi_n$ in \mathcal{S} . Then the vector subspace of $\mathcal{B}(\mathcal{D}_\beta)/\mathcal{K}$ spanned by the cosets $[C_{\varphi_1}], \dots, [C_{\varphi_n}]$ has the same dimension as the subspace $\mathcal{M}(\varphi_1, \dots, \varphi_n)$ of \mathbb{C}^n . In particular, these cosets are linearly independent if and only if $\mathcal{M}(\varphi_1, \dots, \varphi_n) = \mathbb{C}^n$.*

For maps φ in \mathcal{S} with order of contact uniformly two, the linear fractional self-maps of D play a special role. Let us write $\mathcal{S}(2)$ for the collection of those φ in \mathcal{S} which have order of contact two at each point of $F(\varphi)$. Further, we denote by \mathcal{L} the collection of linear fractional self-maps φ of D which are not automorphisms but which have $\|\varphi\|_\infty = 1$. We note that any linear fractional map ψ is determined by its second-order data $D_2(\psi, z_0) = (\psi(z_0), \psi'(z_0), \psi''(z_0))$ at any point z_0 of analyticity. Suppose now that φ is in $\mathcal{S}(2)$ and ζ_0 is a point in $F(\varphi)$. Let φ_0 be the unique linear fractional map with $D_2(\varphi_0, \zeta_0) = D_2(\varphi, \zeta_0)$. Since the curvature of the parametric curve $\varphi(e^{i\theta})$ is determined by second-order data, the circle $\varphi_0(\partial D)$ is necessarily the osculating circle of this curve at the point $\varphi(\zeta_0)$. Thus φ_0 maps D to D and lies in \mathcal{L} . The following result was established by the second author on the weighted Bergman spaces [16]; here we extend it to H^2 .

Corollary 5.16. *Let φ be in $\mathcal{S}(2)$ with $F(\varphi) = \{\zeta_1, \dots, \zeta_r\}$. For $i = 1, \dots, r$ let φ_i be the unique linear fractional map with $D_2(\varphi_i, \zeta_i) = D_2(\varphi, \zeta_i)$. Then*

$$C_\varphi \equiv C_{\varphi_1} + \dots + C_{\varphi_r} \pmod{\mathcal{K}},$$

where the operators act on H^2 .

Proof. This is immediate from Theorem 5.13 applied to the linear combination

$$C_\varphi - C_{\varphi_1} - C_{\varphi_2} - \dots - C_{\varphi_r}.$$

\square

Let us write \mathcal{V}_2 for the vector subspace of $\mathcal{B}(\mathcal{D}_\beta)/\mathcal{K}$ spanned by the cosets $[C_\varphi]$ with φ in $\mathcal{S}(2)$. According to Corollary 5.16, $\{[C_\varphi] : \varphi \in \mathcal{L}\}$ is a spanning subset of \mathcal{V}_2 . That it is linearly independent as well follows immediately from Theorem 5.13, which gives the following:

Corollary 5.17. *$\{[C_\varphi] : \varphi \in \mathcal{L}\}$ is a basis for \mathcal{V}_2 .*

In Theorem 5.13, the proof that (ii) implies (i) uses the local compact difference Theorem 4.4 to locally “match up” certain pairs C_{φ_j} and C_{φ_ℓ} by local data. As one changes the locality or local data, the pairs can change. However, the method suggests that with some minimality hypothesis, the coefficients in a linear equation

$$c_1 C_{\varphi_1} + \cdots + c_n C_{\varphi_n} \equiv 0 \pmod{\mathcal{K}}$$

should be integers up to a common scalar factor. The next result makes this precise.

Corollary 5.18. *Suppose that $\varphi, \varphi_1, \dots, \varphi_n$ lie in \mathcal{S} and assume that $[C_{\varphi_1}], \dots, [C_{\varphi_n}]$ are linearly independent in $\mathcal{B}(\mathcal{D}_\beta)/\mathcal{K}$. If*

$$C_\varphi \equiv c_1 C_{\varphi_1} + \cdots + c_n C_{\varphi_n} \pmod{\mathcal{K}},$$

then c_1, \dots, c_n are rational numbers.

Proof. Let us write $\psi_1 = \varphi$ and $\psi_k = \varphi_{k-1}$, $b_1 = -1$ and $b_k = c_{k-1}$ for $k = 2, 3, \dots, n+1$, so that the equation mod \mathcal{K} in the statement becomes

$$b_1 C_{\psi_1} + b_2 C_{\psi_2} + \cdots + b_{n+1} C_{\psi_{n+1}} \equiv 0 \pmod{\mathcal{K}}.$$

Linear independence of $[C_{\psi_2}], \dots, [C_{\psi_{n+1}}]$ implies that $b_1 = -1$ uniquely determines b_2, \dots, b_{n+1} . Thus according to Remark 5.14, $\mathcal{M} = \mathcal{M}(\psi_1, \dots, \psi_n)$ has codimension one in \mathbb{C}^{n+1} . From the vectors $\mathbf{x}(\zeta, k, \mathbf{d})$ which span \mathcal{M} , select a basis $\mathbf{x}_1, \dots, \mathbf{x}_n$ for \mathcal{M} ; the coordinates of each \mathbf{x}_i are zeros and ones. On applying the Gram-Schmidt process to $\{\mathbf{x}_1, \dots, \mathbf{x}_n\}$, we obtain an orthonormal basis for \mathcal{M} of the form $\mathbf{z}_i / \|\mathbf{z}_i\|$, $i = 1, 2, \dots, n$, where each \mathbf{z}_i has rational coefficients. Let P denote the orthogonal projection of \mathbb{C}^{n+1} onto \mathcal{M} and write $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_{n+1}$ for the standard basis vectors in \mathbb{C}^{n+1} . For some j , $(I - P)\mathbf{e}_j$ is non-zero, and clearly

$$(I - P)\mathbf{e}_j = \mathbf{e}_j - \sum_{i=1}^n \frac{\langle \mathbf{e}_j, \mathbf{z}_i \rangle}{\|\mathbf{z}_i\|^2} \mathbf{z}_i$$

is a vector $(r_1, r_2, \dots, r_{n+1})$ with obviously rational coefficients. Since \mathcal{M}^\perp is one-dimensional, $(b_1, \dots, b_{n+1}) = c(r_1, \dots, r_{n+1})$ for some complex number c . Since $cr_1 = b_1 = -1$, c is rational so that b_2, \dots, b_{n+1} are rational as well. \square

Finally, we note that certain weighted composition operators enjoy a decomposition analogous to that in Corollary 5.16, with the sum replaced by a linear combination.

Proposition 5.19. *Let φ be in $\mathcal{S}(2)$ with $F(\varphi) = \{\zeta_1, \dots, \zeta_r\}$ and suppose that φ_i , $i = 1, \dots, r$, are linear fractional maps related to φ as in Corollary 5.16. If w is a bounded measurable function on ∂D which is continuous at each point of $F(\varphi)$, then*

$$M_w C_\varphi \equiv w(\zeta_1) C_{\varphi_1} + \cdots + w(\zeta_r) C_{\varphi_r} \pmod{\mathcal{K}},$$

where the operators are considered as mapping H^2 to L^2 .

Proof. Choose pairwise disjoint arcs I_1, \dots, I_r with union $\partial \mathbb{D}$ and the interior of I_j containing ζ_j for $j = 1, \dots, r$. We write χ_j for χ_{I_j} . We have

$$M_w C_\varphi = \sum_{j=1}^r \left[M_{(w-w(\zeta_j))\chi_j} C_\varphi + w(\zeta_j) M_{\chi_j} C_\varphi \right].$$

The first term in each summand on the right is compact by Theorem 3.1. Moreover the operators

$$M_{\chi_j}(C_\varphi - C_{\varphi_j}) \text{ and } M_{(1-\chi_j)}C_{\varphi_j}$$

are compact, the first by Theorem 4.4 and Proposition 5.10(c) and the second by Theorem 3.1; the proposition follows. \square

Note that left-multiplying all operators in the proposition by the orthogonal projection P of L^2 onto H^2 gives an alternate statement in which the multiplication operator M_w is replaced by the corresponding Toeplitz operator $T_w = PM_w|_{H^2}$, and all operators act from H^2 to H^2 .

6. ARC-CONNECTEDNESS IN $\mathcal{C}(\mathcal{S}_0)$

In [26], Shapiro and Sundberg posed the following interesting question: What is the relationship between the conditions

- (a) $C_\varphi - C_\psi$ is compact;
- (b) C_φ and C_ψ lie in the same component of the topological space of composition operators on \mathcal{D}_β ?

The second author and Toews [17] proposed the scheme discussed in Section 2.6 and used it to give examples of composition operators satisfying (b) but not (a). Bourdon [3] presented analogous examples for linear fractional maps, showing that for φ and ψ in the class \mathcal{L} (see Section 5.5), (a) corresponds to equal second-order data at the (common) point of contact with ∂D whereas equal first-order data characterizes (b). The second author [16] proved the corresponding result on the Bergman spaces for φ and ψ in the class $\mathcal{S}_0 \cap \mathcal{S}(2)$ where \mathcal{S}_0 is as defined below. Here we extend this result to H^2 and to higher orders of contact. First we record a general sufficient condition for (b), proved in [16] for the Bergman spaces.

Proposition 6.1. *Let us write $\varphi_t = t\varphi + (1-t)\psi$, $0 \leq t \leq 1$, where φ and ψ are analytic self-maps of D . If $\rho = \left| \frac{\varphi-\psi}{1-\bar{\varphi}\psi} \right|$ is essentially bounded away from one on ∂D , then there is a constant B such that*

$$\|C_{\varphi_s} - C_{\varphi_r}\|_{\mathcal{B}(H^2)} \leq B|s - r|, \quad 0 \leq r < s \leq 1.$$

The proposition follows immediately from Theorem 2.2 and Lemma 4.1 with G taken to be the appropriate set of full measure.

We write \mathcal{S}_0 for the collection of those φ in \mathcal{S} such that $\|\chi_{\partial D \setminus U}\varphi\|_\infty < 1$ for every open subset U of ∂D containing $F(\varphi)$. Such a φ is allowed to make contact with ∂D only at points of $F(\varphi)$. We also write $\mathcal{C}(\mathcal{S}_0)$ for the set of those C_φ with φ in \mathcal{S}_0 .

Theorem 6.2. *Suppose that φ and ψ lie in \mathcal{S}_0 with respective orders of contact $2m(\zeta)$ and $2n(\zeta)$ at each ζ in (respectively) $F(\varphi)$ and $F(\psi)$. Let $\varphi_t = t\varphi + (1-t)\psi$, $0 \leq t \leq 1$. Then if $\beta \geq 1$, the following are equivalent:*

- (i) C_φ and C_ψ lie in the same component of the space $\mathcal{C}(\mathcal{S}_0)$ equipped with the norm topology of $\mathcal{B}(\mathcal{D}_\beta)$.
- (ii) There is a positive constant B such that

$$\|C_{\varphi_s} - C_{\varphi_r}\| \leq B|s - r|, \quad 0 \leq r < s \leq 1.$$

- (iii) $F(\varphi) = F(\psi)$ (call this set F), and for each ζ in F , $2m(\zeta) = 2n(\zeta)$ and $D_{2m(\zeta)-1}(\varphi, \zeta) = D_{2m(\zeta)-1}(\psi, \zeta)$.

Proof. First consider the H^2 case. Suppose that (iii) holds and write $\rho = \left| \frac{\varphi - \psi}{1 - \overline{\varphi}\psi} \right|$ as usual. If ζ is in F , it is clear from Proposition 5.10 that $\lim_{e^{i\theta} \rightarrow \zeta} \rho(e^{i\theta}) < 1$. Thus there is an open set U in ∂D containing F such that $\|\chi_U \rho\|_\infty < 1$. Since φ and ψ lie in \mathcal{S}_0 , $\|\chi_{\partial D \setminus U} \rho\|_\infty < 1$ as well, and (ii) follows from Proposition 6.1. For the Bergman space version of this implication, replace U by an appropriate subset of D and use the Bergman space analogue of Proposition 6.1 in [16], together with the following replacement for Proposition 5.10(b), a fact easily established by calculations in the proofs of Lemma 5.6 and Proposition 5.10: If φ and ψ have sufficient data and common order of contact $2m$ at ζ in ∂D , and if $D_{2m-1}(\varphi, \zeta) = D_{2m-1}(\psi, \zeta)$, then

$$\limsup_{z \rightarrow \zeta} \rho(z) < 1,$$

the limit superior being taken unrestrictedly in D . Clearly (ii) implies (i) on any \mathcal{D}_β . Finally, we can use the topological argument of Theorem 2.4 in [13], with Corollary 5.8 playing the role of Theorem 2.2 in [13], to show that (i) implies (iii). \square

The last theorem and Corollary 5.12 show that the phenomenon (a) \Rightarrow (b) (and not conversely) persists more broadly. Note, however, that by passing from the class \mathcal{S} to \mathcal{S}_0 , we have eliminated maps φ which touch the unit circle outside of $E(\varphi)$. Such points of contact are immaterial for condition (a); we do not know whether they matter for (b).

REFERENCES

- [1] A. Aleksandrov, Multiplicity of boundary values of inner functions, *Izv. Akad. Nauk Armyan. SSR Ser. Mat.* 22 (1987), no. 5, 490–503, 515. MR0931885 (89e:30058)
- [2] E. Berkson, Composition operators isolated in the uniform operator topology, *Proc. Amer. Math. Soc.* 81 (1981), 230–232. MR0593463 (82f:47039)
- [3] P. S. Bourdon, Components of linear-fractional composition operators, *J. Math. Anal. Appl.* 279 (2003), 228–245. MR1970503 (2004c:47047)
- [4] P. S. Bourdon, D. Levi, S. K. Narayan, and J. H. Shapiro, Which linear fractional composition operators are essentially normal? *J. Math. Anal. Appl.* 280 (2003), 30–53. MR1972190 (2003m:47042)
- [5] J. A. Cima and A. L. Matheson, Essential norms of composition operators and Aleksandrov measures, *Pacific J. Math.* 179 (1997), 59–64. MR1452525 (98e:47047)
- [6] D. N. Clark, One-dimensional perturbations of restricted shifts, *J. D'Analyse Math.* 25 (1972), 169–191. MR0301534 (46:692)
- [7] C. C. Cowen, Composition operators on H^2 , *J. Operator Theory* 9 (1983), 77–106. MR0695941 (84d:47038)
- [8] C. C. Cowen and B. D. MacCluer, *Composition Operators on Spaces of Analytic Functions*, Studies in Advanced Mathematics, CRC Press, Boca Raton, FL, 1995. MR1397026 (97i:47056)
- [9] P. L. Duren, *Theory of H^p Spaces*, Academic Press, New York, 1970. MR0268655 (42:3552)
- [10] P. Gorkin and R. Mortini, Norms and essential norms of linear combinations of endomorphisms, *Trans. Amer. Math. Soc.* 358 (2006), no. 2, 553–571. MR2177030 (2006h:47055)
- [11] P. R. Halmos, *Measure Theory*, Van Nostrand, New York, 1950. MR0033869 (11:504d)
- [12] K. Hoffman, *Banach Spaces of Analytic Functions*, Prentice Hall, Englewood Cliffs, NJ, 1962. MR0133008 (24:A2844)
- [13] B. D. MacCluer, Components in the space of composition operators, *Integral Equations and Operator Theory* 12 (1989), 725–738. MR1009027 (91b:47070)
- [14] B. D. MacCluer, S. Ohno, and R. Zhao, Topological structure of the space of composition operators on H^∞ , *Integral Equations and Operator Theory* 40 (2001), 481–494. MR1839472 (2002d:47039)

- [15] B. D. MacCluer and J. H. Shapiro, Angular derivatives and compact composition operators on the Hardy and Bergman spaces, *Canadian J. Math.* 38 (1986), 878–906. MR0854144 (87h:47048)
- [16] J. Moorhouse, Compact differences of composition operators, *J. Functional Analysis* 219 (2005), 70–92. MR2108359 (2005i:47037)
- [17] J. Moorhouse and C. Toews, Differences of composition operators, Trends in Banach Spaces and Operator Theory (Memphis, TN, 2001), 207–213, *Contemp. Math.* 321, Amer. Math. Soc., Providence, RI, 2003. MR1978818 (2004b:47046)
- [18] N. K. Nikolski, *Operators, Functions, and Systems: An Easy Reading, Volume 1: Hardy, Hankel and Toeplitz*, Mathematical Surveys and Monographs, 92, Amer. Math. Soc., Providence, RI, 2002. MR1864396 (2003i:47001a)
- [19] P. J. Nieminen and E. Saksman, On the compactness of the difference of composition operators, *J. Math. Anal. Appl.* 298 (2004), 501–522. MR2086972 (2005e:30065)
- [20] A. G. Poltoratski, Boundary behavior of pseudocontinuable functions (Russian), *Algebra i Analiz* 5 (1993), 189–210; translation in *St. Petersburg Math. J.* 5 (1994), 389–406. MR1223178 (94k:30090)
- [21] S. Saitoh, *Integral Transforms, Reproducing Kernels and their Applications*, Pitman Research Notes in Mathematics, 369, Longman, Harlow, 1997. MR1478165 (98k:46041)
- [22] D. E. Sarason, Composition operators as integral operators, *Analysis and Partial Differential Equations*, C. Sadosky (ed.), 545–565, Marcel-Dekker, New York, 1990. MR1044808 (92a:47040)
- [23] D. E. Sarason, *Sub-Hardy Hilbert Spaces in the Unit Disk*, University of Arkansas Lecture Notes in the Mathematical Sciences, Volume 10, John Wiley and Sons, New York, 1994. MR1289670 (96k:46039)
- [24] J. E. Shapiro, Aleksandrov measures used in essential norm inequalities for composition operators, *J. Operator Theory* 40 (1998), 133–146. MR1642538 (99i:47062)
- [25] J. H. Shapiro, The essential norm of a composition operator, *Annals Math.* 125 (1987), 375–404. MR0881273 (88c:47058)
- [26] J. H. Shapiro and C. Sundberg, Isolation amongst the composition operators, *Pacific J. Math.* 145 (1990), 117–152. MR1066401 (92g:47041)
- [27] J. H. Shapiro and C. Sundberg, Compact composition operators on L^1 , *Proc. Amer. Math. Soc.* 108 (1990), 443–449. MR0994787 (90d:47035)
- [28] J. H. Shapiro and P. D. Taylor, Compact, nuclear and Hilbert-Schmidt composition operators on H^2 , *Indiana Univ. Math. J.* 23 (1973), 471–496. MR0326472 (48:4816)

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF VIRGINIA, CHARLOTTESVILLE, VIRGINIA 22904
E-mail address: tlk8q@virginia.edu

DEPARTMENT OF MATHEMATICS, COLGATE UNIVERSITY, HAMILTON, NEW YORK 11346
E-mail address: jmoorhouse@mail-colgate.edu