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HOMOLOGOUS NON-ISOTOPIC SYMPLECTIC SURFACES OF HIGHER GENUS

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Dedicated to Ron Fintushel on the occasion of his sixtieth birthday

Abstract. We construct an infinite family of homologous, non-isotopic, symplectic surfaces of any genus greater than one in a certain class of closed, simply connected, symplectic four-manifolds. Our construction is the first example of this phenomenon for surfaces of genus greater than one.

1. Introduction

Several papers have addressed, in the last few years, the problem of isotopy of symplectic surfaces representing the same homology class $\alpha \in H_2(M;\mathbb{Z})$ of a simply connected, symplectic 4-manifold M. Although a list of the results obtained in those papers is beyond the scope of this Introduction, we just want to mention the existence of both uniqueness and non-uniqueness results, depending on the choice of the pair (M, α). The prototypical example of a non-uniqueness result, further discussed in this paper, is the presence of infinitely many non-isotopic symplectic tori representing any multiple of the fiber class of (some) elliptic surfaces. The idea behind this construction, developed by Fintushel and Stern in [7], is to obtain non-equivalent symplectic tori by braiding parallel copies of a symplectic torus. All known examples of non-uniqueness, so far, have been genus 1 representatives of a homology class of self-intersection 0. On the other hand, the few higher genus cases under control have always led to uniqueness results as in [14]. The interest in obtaining examples of non-isotopic symplectic surfaces of higher genus has been pointed out by many researchers, in particular, in [1], [7] and [16].

There is no special reason to expect that non-isotopy phenomena should be restricted to genus 1, self-intersection 0 surfaces. In fact, if we also take into account 4-manifolds with non-trivial fundamental groups, Smith has shown in [15] the existence of higher (odd) genus examples, and examples of symplectic curves with cusp singularities have been provided (for the projective plane) in [1]. However, all attempts to build higher genus examples in simply connected 4-manifolds by suitably "doubling" the braiding construction have failed. The reason for the failure is the impossibility of detecting non-isotopy (for potential examples) either by using the Seiberg-Witten invariants or by using a more classical topological invariant like the fundamental group of the complement. In the latter approach, the reason is

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that most of the basic building blocks, genus 1 surfaces built through the braiding construction, cannot be distinguished by means of the π_1 either, as first observed in [7].

However, there is a second mechanism, different from the original braiding construction, that allows us to produce non-isotopic symplectic tori in a symplectic 4-manifold (see [4] and [19]). This alternative construction originates from the presence of non-isotopic, nullhomologous Lagrangian tori in a large class of (simply connected) symplectic 4-manifolds (see [9] and [18]). These symplectic tori are obtained by "summing" a preferred symplectic torus with non-isotopic nullhomologous tori. The non-isotopy can be detected, as for the braiding construction, by using Seiberg-Witten theory, but the advantage of this second construction is that, in many cases, π_1 is sufficient to distinguish the tori, as shown in [5]. Also, if we "double" the construction, the fundamental groups of the complements may retain enough information to distinguish the resulting higher genus surfaces.

In fact, following the ideas outlined above, we will be able to prove the following.

Theorem 1. *For any fixed integer $q \geq 1$, there exist simply connected, symplectic 4-manifolds X containing infinitely many homologous, pairwise non-isotopic, symplectic surfaces $\{\Xi_{p,q} \mid \gcd(p,q) = 1\}$ of genus $q+1$. Furthermore, there is no pair homeomorphism between $(X, \Xi_{p,q})$ and $(X, \Xi_{p',q})$ unless $p' = p$.*

We briefly preview the idea of the construction. First, consider the symplectic simply connected 4-manifold $E(2)_K$ obtained by the knot surgery construction of [6] from the elliptic $K3$ surface $E(2)$ and a non-trivial fibered knot K in S^3. $E(2)_K$ contains, for each $q \geq 1$, an infinite family of homologous, symplectic, non-isotopic tori $\{T_{p,q} \mid \gcd(p,q) = 1\}$, together with a symplectic surface Σ_g of genus $g = g(K)+1$ of self-intersection 0. This surface Σ_g intersects each of the tori $T_{p,q}$ at q positive transverse points. By doubling $E(2)_K$ along Σ_g we obtain a symplectic, simply connected 4-manifold $X = D_K$. By summing together two copies of the torus $T_{p,q}$, one copy from each side, we obtain a family of homologous genus $q+1$, self-intersection 0 symplectic surfaces $\Xi_{p,q}$.

By direct computation, we will show that if the fibered knot K used in the construction of D_K is non-trivial, then infinitely many of these surfaces $\Xi_{p,q}$ have non-homeomorphic complements $D_K \setminus \Xi_{p,q}$, distinguished by their fundamental groups. This proves, in particular, that the surfaces are not isotopic.

2. Construction

In this section we will construct an infinite family of non-isotopic, symplectic surfaces of genus greater than or equal to 2 for a class of simply connected, symplectic 4-manifolds. These 4-manifolds are the manifolds denoted by D_K, where K is any fibered knot, introduced in [13].

First we recall the construction of D_K. Let $K \subset S^3$ be a fibered knot and let $g(K)$ denote its genus. Let Σ_K denote the fiber of the fibration of the knot exterior, the minimal genus Seifert surface for the knot K.

For each fibered knot K we can construct a symplectic 4-manifold $E(2)_K$, homeomorphic to the elliptic surface $E(2)$ (the $K3$ surface), obtained by knot surgery on $E(2)$ with the knot K (see [6]). Denote by N_K the fibered 3-manifold obtained by the 0-surgery on S^3 along K. $S^1 \times N_K$ is a symplectic 4-manifold with a framed, self-intersection 0 symplectic torus $S^1 \times \mathbf{m}$, where $\mathbf{m}$ is the core of the surgery solid torus. The manifold $E(2)_K$ can be presented as

$$E(2)_K = \big(E(2) \setminus \nu F\big) \cup \big(S^1 \times (S^3 \setminus \nu K)\big) = E(2) \#_{F=S^1\times \mathbf{m}} S^1 \times N_K\,, \tag{2.1}$$

where the gluing diffeomorphism identifies factorwise the boundary 3-tori $F \times \partial D^2$ and $S^1 \times \mu(K) \times \lambda(K)$ (reversing the orientation on the last factor). The presentation of $E(2)_K$ as a fiber sum shows, by Gompf's theory (see [10]), that it admits a symplectic structure restricting, outside the gluing locus, to the one of the summands.

Inside $E(2)_K$ a disk section S of $(E(2) \setminus \nu F)$ can be glued to a Seifert surface Σ_K to form a closed, symplectic, genus $g(K)$ surface Σ of self-intersection -2. By taking Σ and a regular torus fiber F and resolving their normal intersection, we obtain a symplectic surface Σ_g of self-intersection 0 and genus $g = g(K) + 1$ inside $E(2)_K$. Such a surface can be endowed with a natural framing, inherited from the canonical framings of the fibers of $E(2)$ and $S^1 \times N_K$. We define

$$D_K = \big(E(2)_K \setminus \nu\Sigma_g\big) \cup_\varphi \big(E(2)_K \setminus \nu\Sigma_g\big), \tag{2.2}$$

where the gluing map φ is an orientation-reversing self-diffeomorphism on $\partial(\nu\Sigma_g) \cong \Sigma_g \times \partial D^2$ that is the identity on the Σ_g factor and complex conjugation on the ∂D^2 factor.

The following result is proved in [13].

Proposition 2. *D_K is a closed, simply connected, spin, irreducible, symplectic 4-manifold. The signature of D_K is -32, and its intersection form is given by $4E_8 \oplus (7 + 2g(K)) \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$.*

In order to obtain higher genus symplectic surfaces in D_K, we will start from some symplectic tori in $E(2)_K$. The symplectic 4-manifold $E(2)_K$ contains, for every $q \geq 1$, an infinite family $\{T_{p,q}\}$ of pairwise non-isotopic symplectic tori representing the homology class $q[F] = q[S^1 \times \mathbf{m}] \in H_2(E(2)_K; \mathbb{Z})$ indexed by the integers p coprime to q (see [4] and [19]).

There are two equivalent ways to present these tori. We start with the approach of [19]. First, by looking at the construction of $E(2)_K$ in (2.1), we can recognize a rim torus R given by the image in $E(2)_K$ of $S^1 \times \lambda(K)$. As we can assume, up to isotopy, that a copy of the longitude $\lambda(K)$ lies on the fiber of N_K (given by the Seifert surface Σ_K capped off with a disk of the Dehn filling), the torus R can be assumed to be Lagrangian. Inside $S^1 \times N_K$ the torus $S^1 \times \lambda(K)$ is the boundary of a solid torus (as $\lambda(K)$ bounds a disk of the Dehn filling), but after the knot surgery this is not generally true anymore for R (for non-trivial K). It remains, however, nullhomologous, as it bounds the 3-manifold $S^1 \times \Sigma_K$.

Now take a copy of $\partial\nu K$ pushed slightly inside $S^3 \setminus \nu K$, in such a way that it is transverse to the fibration and intersects each fiber in a curve isotopic to the longitude. The simple closed curves on this torus are the (p, q)-cables of K, where $\gcd(p, q) = 1$ (remember that their linking number with K is q), with $\lambda(K)$ and $\mu(K)$ themselves being the $(1, 0)$- and the $(0, 1)$-cables, respectively. Up to isotopy we can assume that, with the exception of the $(1, 0)$-cable, every (p, q)-cable is a curve transverse to the fibration, and its intersection with a fiber consists of positive points for $q \geq 1$. Denote by $K_{p,q}$ the (p, q)-cable of K, for any $q \geq 1$. We can say that this closed curve is obtained by *circle sum* (in the sense of [9]) of q copies of the meridian and p copies of the longitude of K.

It will be useful in what follows to observe that the cable knot $K_{p,q}$ is contained in an enlarged tubular neighborhood $\mathsf{n}K$ of K (which we can assume to have, say, twice the radius of νK for some metric on S^3). Note that $S^3 \setminus \mathsf{n}K$ also fibers over S^1. Its fiber S_K is strictly contained in Σ_K, with $\mathsf{l}(K) = \partial \mathsf{S}_K$ isotopic to $\lambda(K) = \partial \Sigma_K$ inside Σ_K. The cable $K_{p,q}$ intersects the fiber Σ_K at q points $x_1, \dots, x_q$ contained in the knotted annulus $\Sigma_K \setminus \mathrm{int}(\mathsf{S}_K)$, whose boundary is the union $\mathsf{l}(K) \cup \lambda(K)$. Figure 1 illustrates the relations between these fibers and $K_{p,q}$.

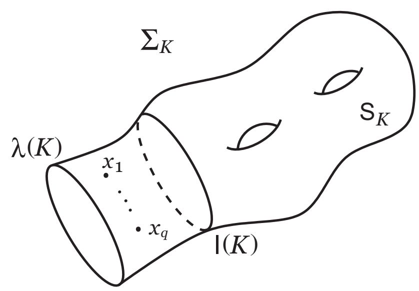

FIGURE 1. $\Sigma_K \cap K_{p,q}$ and fibers of the two knot exteriors

As $K_{p,q}$ is transverse to Σ_K the torus $S^1 \times K_{p,q}$ is symplectic in $S^1 \times N_K$. For a fixed value of q, the tori $S^1 \times K_{p,q}$ are homologous, as the homology class of $K_{p,q}$ in $H_1(S^3 \setminus \nu K; \mathbb{Z})$ is given by $lk(K_{p,q}, K)[\mu(K)] = q[\mu(K)]$ and the Dehn filling does not affect this relation. (If $q = 1$ more is true, namely the tori $K_{p,1}$ are isotopic, the isotopy being supported in the solid torus of the Dehn filling.) After knot surgery, the images $T_{p,q}$ in $E(2)_K$ of these tori remain homologous. In fact, the image of the homology class of the (p,q)-cable of K, through the injective map

$$H_1(S^3 \setminus \nu K; \mathbb{Z}) \longrightarrow H_2(S^1 \times (S^3 \setminus \nu K); \mathbb{Z}) \longrightarrow H_2(E(2)_K; \mathbb{Z}),$$

is $q[F]$. However, infinitely many of the tori $T_{p,q}$ become non-isotopic. Roughly speaking, the torus $T_{p,q}$ is the sum of q copies of F with p copies of the nullhomologous Lagrangian torus R, and can be interpreted in terms of the cabling of F itself.

The family of tori $T_{p,q}$ will be the starting point of our construction, but note that we can extend the previous argument to include, instead of just (p,q)-cables of the knot K, any curve in $S^3 \setminus \nu K$ obtained as the circle sum of a number of copies of the meridian and a closed curve (or a union of closed curves) lying on the surface Σ_K (and thus homologous, but not necessarily isotopic to the longitude). With arguments similar to the ones above, we can achieve transversality for the resulting curves; the symplectic tori obtained this way differ by the addition of nullhomologous Lagrangian tori. Details of this construction are presented in [19].

A second way to construct the family $T_{p,q}$, presented in [4], is the following. Denote by $H = A \cup B$ the Hopf link. In the knot exterior $S^3 \setminus \mathsf{n}K$ we consider a standard pair of meridian and longitude, that we denote, to avoid confusion with $\mu(K)$ and $\lambda(K)$, by $\mathsf{m}(K)$ and $\mathsf{l}(K)$, respectively. We then have the decomposition

$$E(2)_K = \left[E(2) \setminus \nu F\right] \cup \left[S^1 \times (S^3 \setminus \nu H)\right] \cup \left[S^1 \times (S^3 \setminus \mathsf{n}K)\right], \tag{2.3}$$

with the first gluing identifying $F \times \partial D^2$ with $S^1 \times \mu(A) \times \lambda(A)$ and the second one identifying $S^1 \times \lambda(B) \times \bar{\mu}(B)$ with $S^1 \times \mathsf{m}(K) \times \mathsf{l}(K)$ factorwise, reversing the orientation on the last factor. (The bar over a curve refers to the negative orientation.) The equivalence between this construction and the construction of (2.1) (even in the symplectic category, with a suitable presentation as symplectic fiber sum) simply follows by observing that the exterior of the Hopf link is diffeomorphic to $S^1 \times (D^2 \setminus \nu\{0\})$.

The torus $T_{p,q}$ is given by the image of $S^1 \times C_{p,q} \subset \left[S^1 \times (S^3 \setminus \nu H)\right]$, where $C_{p,q}$ is the (p,q)-cable of A, as illustrated in Figure 2 for $q = 1$. The transversality of $C_{p,q}$ to the fibration of $S^3 \setminus \nu H$ with fiber given by a disk spanning A, pierced once by νB, is evident, and this entails that $T_{p,q}$ is symplectic. For fixed q, the curves $C_{p,q}$ are not homologous in $S^3 \setminus \nu H$, but they become so after gluing in $S^3 \setminus \mathsf{n}K$ with the prescription above, as the meridian $\mu(B)$ becomes nullhomologous. Therefore, the tori $T_{p,q}$ are homologous in $E(2)_K$ for fixed q.

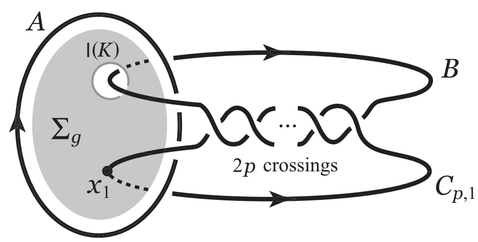

FIGURE 2. 3-component link $A \cup B \cup C_{p,1}$ in S^3

The equivalence between the two constructions of $T_{p,q}$ follows when we observe that the gluing of $S^3 \setminus \mathsf{n}K$ to $S^3 \setminus \nu H$ in (2.3) is the *splicing* of K and H along B, and the result of this operation is to produce a copy of $S^3 \setminus \nu K$, with the image of A giving the knot K and the image of the (p,q)-cable of A (i.e. $C_{p,q}$) giving the (p,q)-cable of K (i.e. $K_{p,q}$). (See [3], Proposition 1.1, for a detailed discussion of this construction.) Stated otherwise, we can think of the curve $C_{p,q}$ in the second construction as spiraling along an inner torus of $S^3 \setminus \nu H = S^1 \times (D^2 \setminus \nu\{0\})$; after the gluing this inner torus gets identified with a copy of $\partial \nu K$ pushed inside $S^3 \setminus \nu K$, and hence $C_{p,q}$ is identified with $K_{p,q}$ in the first construction. Note that, after the splicing, the disk spanning A and pierced by νB (that appears shaded in Figure 2) is identified with $\Sigma_K \setminus \mathrm{int}(\mathsf{S}_K)$, which appears as the neck in Figure 1.

Now that we have defined the family $\{T_{p,q} \mid \gcd(p,q) = 1\}$ we can construct, out of two copies of $T_{p,q}$, the desired genus $q+1$ surface in D_K. Since in $E(2)_K$, $[T_{p,q}] \cdot [\Sigma_g] = q[F] \cdot [\Sigma_g] = q[F] \cdot [S] = q$ for each p, we can internally sum two copies of q-times punctured $T_{p,q}$ from each of $(E(2)_K \setminus \nu\Sigma_g)$ halves in (2.2), and thus obtain a family of homologous, genus $q+1$, self-intersection 0 surfaces

$$\{\, \Xi_{p,q} \;=\; T_{p,q} \# T_{p,q} \;=\; (T_{p,q} \setminus \nu\Sigma_g) \cup (T_{p,q} \setminus \nu\Sigma_g) \,\}$$

inside D_K. (We are going to follow the convention that $U \setminus V := U \setminus (U \cap V)$.) See Figure 3.

Since each $T_{p,q}$ is symplectic in $E(2)_K$ and our ambient 4-manifold D_K is a symplectic sum (cf. [10]) of two copies of $E(2)_K$ along a symplectic surface Σ_g, each $\Xi_{p,q}$ is a symplectic submanifold of D_K. In the next section we discuss how to

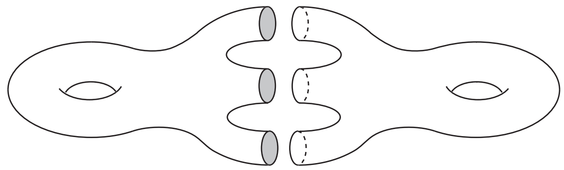

FIGURE 3. Genus 4 surface $\Xi_{p,3}$ as the double of $(T_{p,3} \setminus \nu\Sigma_g)$

distinguish the isotopy classes of submanifolds $\Xi_{p,q}$ by comparing the fundamental groups of their complements in D_K.

3. Fundamental groups

The isotopy class of the torus $T_{p,q}$ inside $E(2)_K$ can be detected via the Seiberg-Witten invariant of the fiber sum manifold $E(1)\#_{F=T_{p,q}}E(2)_K$ (that depends ultimately on the diffeomorphism type of the pair $(E(2)_K, T_{p,q})$). It would be interesting (and probably quicker) to be able to use a similar approach to distinguish the diffeomorphism type of the pair $(D_K, \Xi_{p,q})$. Unfortunately, we currently lack a satisfactory machinery within the Seiberg-Witten theory to carry this out. However, we can use a classical approach to this problem, namely the study of the fundamental group of the exterior of $\Xi_{p,q}$. This allows us to prove an even stronger result, namely the existence of pairs $(D_K, \Xi_{p,q})$ that are not even homeomorphic. We were first led to this approach by the observation that the tori $T_{p,q}$ themselves can be distinguished by means of the fundamental group alone, as illustrated in [5]. The reason underlying that phenomenon seems to be related to the role of the nullhomologous Lagrangian tori in the construction of $T_{p,q}$.

It remains an interesting question to determine whether there are homeomorphic pairs (or at least pairs with the same fundamental group of the exterior) which are not diffeomorphic. The main result of this section is the following theorem.

Theorem 3. *Let $\Xi_{p,q}$ be the genus $q+1$, symplectic surface inside D_K constructed in Section 2. Then we have*

$$\pi_1(D_K \setminus \Xi_{p,q}) \cong \pi_1(S^3_{q/p}(K)), \tag{3.1}$$

where $S^3_{q/p}(K)$ is the rational homology sphere obtained by Dehn (q/p)-surgery on S^3 along the knot K, so that

$$\pi_1(S^3_{q/p}(K)) \cong \frac{\pi_1(S^3 \setminus \nu K)}{\mu(K)^q\lambda(K)^p = 1} \tag{3.2}$$

with $\mu(K)$ and $\lambda(K)$ denoting the meridian and the longitude of K, respectively.

Theorem 1 now follows from Theorem 3 and the fact that when K is a non-trivial knot, for a fixed value of q, infinitely many choices of p give mutually non-isomorphic groups $\pi_1(S^3_{q/p}(K))$. A proof of this latter statement is contained in [5]. For some knot K, we can explicitly determine infinite values of p that lead to pairwise non-homeomorphic pairs. As an example, consider the figure eight knot. This knot is known to be fibered and hyperbolic. Moreover, it is known (see [17]) that, with 6 explicit exceptions all having $|p| < 2$, its (q/p)-Dehn surgery is a hyperbolic manifold. At this point we can invoke the result of [2] which guarantees that, restricting ourselves to positive p, no two surgeries give homeomorphic 3-manifolds. Mostow rigidity then implies that all the resulting manifolds have distinct fundamental groups.

The proof of Theorem 3 will occupy the remainder of this section. We start by observing that we have the following decomposition into halves:

$$D_K \setminus \nu\Xi_{p,q} \cong \big[E(2)_K \setminus \nu(\Sigma_g \cup T_{p,q})\big] \cup \big[E(2)_K \setminus \nu(\Sigma_g \cup T_{p,q})\big]. \tag{3.3}$$

We want to apply Seifert-Van Kampen theorem to this decomposition. Our first step is the computation of the fundamental group of the halves. We claim that

$$\pi_1\big(E(2)_K \setminus \nu(\Sigma_g \cup T_{p,q})\big) \cong \pi_1(S^3_{q/p}(K)). \tag{3.4}$$

Recall from (2.1) and the construction in Section 2 that we have

$$E(2)_K \setminus \nu T_{p,q} = \big[E(2) \setminus \nu F\big] \cup \big[S^1 \times (S^3 \setminus \nu(K \cup K_{p,q}))\big]. \tag{3.5}$$

Our strategy is to use decomposition (3.5) to compute the fundamental group in (3.4). Note that the surface Σ_g intersects both components of decomposition (3.5). We proceed to compute the fundamental group of the complement of $\nu\Sigma_g$ in each piece separately. Denote by X_1 and X_2 the two components in decomposition (3.5). The first result we need is a straightforward modification of Proposition 2.1 of [8], and we present its proof here for completeness. Observe that $X_1 \setminus \nu\Sigma_g$ (i.e. $X_1 \setminus (X_1 \cap \nu\Sigma_g)$), from the definition in (3.5), does not depend on K.

Lemma 4. *Let $X_1 = E(2) \setminus \nu F$. Then $\pi_1(X_1 \setminus \nu\Sigma_g) = 1$.*

Proof. Note that $X_1 \setminus \nu\Sigma_g$ is $[E(2) \setminus \nu F] \setminus \nu\Sigma'$, where Σ' is a genus 1, self-intersection 0 surface with boundary, gotten from the union $F' \cup S$ of another copy of regular fiber and a disk section of $E(2) \setminus \nu F$ by resolving their normal intersection point. It is well known that $[E(2) \setminus \nu F]$ is simply connected (see e.g. [11]). It now follows from the Seifert-Van Kampen theorem that the fundamental group in question is normally generated by the meridian of Σ'. By perturbing the elliptic fibration of $E(2)$ if necessary, we may assume that there is at least one singular cusp fiber which is topologically a sphere. This implies that the meridian of Σ' (which is isotopic to a meridian of S) bounds a topological disk, which is a cusp fiber with a disk removed. □

We now consider the remaining piece in decomposition (3.5). First, observe that $S^3 \setminus \nu(K \cup K_{p,q})$ admits a fibration over S^1, with fiber $\Sigma_{K,q}$ given by the spanning surface Σ_K of K with q disjoint disks $\mathbb{D}_{x_1}, \ldots, \mathbb{D}_{x_q}$ removed, in correspondence with the intersection points $x_1, \ldots, x_q$ of Σ_K with $K_{p,q}$. These q disks all lie in the knotted annulus $\Sigma_K \setminus \mathrm{int}(\mathsf{S}_K)$. The existence of such fibration should be quite apparent from the constructions in Section 2, and comes from gluing, through the splicing, the fibration of $S^3 \setminus \mathsf{n}K$ with fiber S_K and the fibration of $S^3 \setminus \nu(H \cup C_{p,q})$ induced by the spanning disk of A (that gives the punctured knotted annulus above). A section of this fibration is given by a copy of $\mu(K)$.

Denote by a_i, b_i, $i = 1, \ldots, g(K)$, the generators of $\pi_1(\Sigma_K)$ and denote by c_j, $j = 1, \ldots, q$, the homotopy classes of the loops around the disks $\mathbb{D}_{x_j}$. Locating the base point on $\partial\mathsf{S}_K$ we can assume that representatives of the generators a_i and b_i are all contained in $\mathsf{S}_K \subset \Sigma_K$. The fundamental group of $\Sigma_{K,q}$ (a free group on $2g(K) + q$ generators) is generated by these elements.

Lemma 5. *Let* $X_2 = \big[S^1 \times (S^3 \setminus \nu(K \cup K_{p,q}))\big]$. *Then* $\pi_1(X_2 \setminus \nu\Sigma_g)$ *is isomorphic to the group*

$$\frac{\langle x \rangle \, * \, \pi_1(S^3 \setminus \nu(K \cup K_{p,q}))}{\{[x,a_i] = [x,b_i] = [x,c_j] = 1 \mid i = 1, \dots, g(K) \text{ and } j = 1, \dots, q\}}, \tag{3.6}$$

where $\langle x \rangle$ *is the free group on one generator, and* $a_1, b_1, \dots, a_{g(K)}, b_{g(K)}, c_1, \dots, c_q$ *are the generators of* $\pi_1(\Sigma_{K,q})$.

Proof. Note that X_2 is the total space of a fiber bundle:

$$\begin{array}{ccc} \Sigma_{K,q} & \longrightarrow & S^1 \times (S^3 \setminus \nu(K \cup K_{p,q})) \\ & & \Big\downarrow {\scriptstyle \Pi = \mathrm{id} \times \pi} \\ & & S^1 \times \mu(K) \end{array}$$

Suppose $X_2 \cap \Sigma_g = \Pi^{-1}(t_0)$ for some point $t_0 \in S^1 \times \mu(K)$. Then we can write $X_2 \cap \nu\Sigma_g = \Pi^{-1}(\mathbb{D}_{t_0})$ for some small disk $\mathbb{D}_{t_0} \subset S^1 \times \mu(K)$. It follows that $X_2 \setminus \nu\Sigma_g$ is the total space of the restricted bundle:

$$\begin{array}{ccc} \Sigma_{K,q} & \longrightarrow & \big[S^1 \times (S^3 \setminus \nu(K \cup K_{p,q}))\big] \setminus \Pi^{-1}(\mathbb{D}_{t_0}) \\ & & \Big\downarrow {\scriptstyle \Pi|} \\ & & [S^1 \times \mu(K)] \setminus \mathbb{D}_{t_0} \end{array}$$

The new base $[S^1 \times \mu(K)] \setminus \mathbb{D}_{t_0}$ is a torus with a disk removed and hence homotopy equivalent to a wedge of two circles. The generator x corresponding to the S^1 factor in $S^1 \times \mu(K)$ now no longer commutes with $\mu(K)$, but x still commutes with the generators of $\pi_1(\Sigma_{K,q})$ since the map Π had trivial monodromy in the S^1 direction. Presentation (3.6) follows immediately. $\square$

Now we are ready to prove (3.4). The intersection of $X_1 \setminus \nu\Sigma_g$ and $X_2 \setminus \nu\Sigma_g$ is given by the 3-torus $\partial\nu F = X_1 \cap X_2$ minus its intersection with $\nu\Sigma_g$. This intersection is given by the neighborhood of a copy of the meridional circle $\mu(F)$. We have in fact $\partial\nu F \setminus \nu\Sigma_g = \mu(F) \times (F \setminus \nu\{\mathrm{pt}\})$, so that

$$\pi_1\big((X_1 \setminus \nu\Sigma_g) \cap (X_2 \setminus \nu\Sigma_g)\big) \;=\; \mathbb{Z}[\mu(F)] \oplus (\mathbb{Z}[\gamma_1] * \mathbb{Z}[\gamma_2]),$$

where γ_1 and γ_2 form a homotopy basis for $\pi_1(F)$. The generators $\mu(F)$, γ_1 and γ_2 are identified, respectively, with the classes of $\lambda(K)$, $\mu(K)$ and S^1 through the gluing map in (2.1). By using the fact that $X_1 \setminus \nu\Sigma_g$ is simply connected (Lemma 4), we deduce that

$$\begin{aligned} \pi_1\big(E(2)_K \setminus \nu(\Sigma_g \cup T_{p,q})\big) &= \pi_1\big((X_1 \setminus \nu\Sigma_g) \cup (X_2 \setminus \nu\Sigma_g)\big) \\ &= \frac{\pi_1\big(S^3 \setminus \nu(K \cup K_{p,q})\big)}{\mu(K) = 1, \lambda(K) = 1}, \end{aligned}$$

where the last equality comes from observing that the elements of $\pi_1(X_2 \setminus \nu\Sigma_g)$ identified with the elements of $\pi_1((X_1 \setminus \nu\Sigma_g) \cap (X_2 \setminus \nu\Sigma_g))$ become trivial. This accounts for the nullhomotopy of generators $x = [S^1]$, $\mu(K)$ and $\lambda(K)$ appearing in group (3.6).

We are left, therefore, with the exercise of computing the fundamental group of the exterior of a link given by a knot K and its (p,q)-cable, and then quotienting by the relations that $\lambda(K)$ and $\mu(K)$ are trivial.

In principle, we could use the existence of the fibration of $S^3 \setminus \nu(K \cup K_{p,q})$, but this requires an explicit knowledge of the monodromy of the fibration. Another approach is much more viable. First, it is useful to keep in mind the construction of $S^3 \setminus \nu(K \cup K_{p,q})$ obtained by splicing $S^3 \setminus \mathsf{n}K$ and $S^3 \setminus \nu(H \cup C_{p,q})$. Now we can proceed (mimicking one of the standard computations of the fundamental group of a torus knot exterior) as follows. Presenting the knot and its cable as companion and satellite, we can write

$$S^3 \setminus \nu(K \cup K_{p,q}) \;=\; (V_1 \setminus \nu K_{p,q}) \cup_\Omega (V_2 \setminus \nu K_{p,q}). \tag{3.7}$$

The following three paragraphs explain the terms appearing on the right-hand side of (3.7).

First, V_1 is a 3-manifold with torus boundary that is given by the exterior $S^3 \setminus \mathsf{n}K$ of the knot K union a collar of its boundary, extended in the outward direction. $K_{p,q}$ lies on the boundary of V_1. By carving out the intersection with a tubular neighborhood of $K_{p,q}$, we obtain the manifold $(V_1 \setminus \nu K_{p,q})$ that is a deformation retract of V_1. Recall that we denote by $\mathsf{m}(K)$ and $\mathsf{l}(K)$ the meridian and the longitude of $S^3 \setminus \mathsf{n}K$. We point out that these should not be confused with the meridian and the longitude $\mu(K)$ and $\lambda(K)$ of $S^3 \setminus \nu K$; in particular, they are not homotopic to $\mu(K)$ and $\lambda(K)$ in $S^3 \setminus \nu(K \cup K_{p,q})$.

The 3-manifold V_2 is a solid torus knotted in S^3 as a tubular neighborhood of K, with a neighborhood of the core (νK) removed. $K_{p,q}$ lies on the outer boundary of V_2. By carving out the intersection with a tubular neighborhood of $K_{p,q}$ we obtain $(V_2 \setminus \nu K_{p,q})$ that is, again, a deformation retract of V_2 and in particular is homotopy equivalent to a 2-torus. The generators of $\pi_1(V_2 \setminus \nu K_{p,q})$ are given exactly by $\mu(K)$ and $\lambda(K)$. V_1 and V_2 intersect in a 2-torus, which contains the curve $K_{p,q}$, and which has a natural homotopy basis given by $\mathsf{m}(K) = \mu(K)$ and $\mathsf{l}(K) = \lambda(K)$ (the identification being in $\pi_1(V_1 \cap V_2)$), so that $K_{p,q}$ is homotopic to $\mathsf{m}(K)^q \mathsf{l}(K)^p = \mu(K)^q \lambda(K)^p$.

If we avoided removing the neighborhood of $K_{p,q}$, the intersection of V_1 and V_2 being a 2-torus, the Seifert-Van Kampen theorem would give the obvious identifications $\mathsf{m}(K) = \mu(K)$, $\mathsf{l}(K) = \lambda(K)$ in the resulting 3-manifold (and decomposition (3.7) would simply be a redundant presentation of $S^3 \setminus \nu K$). Instead, the intersection of $(V_1 \setminus \nu K_{p,q})$ and $(V_2 \setminus \nu K_{p,q})$ is an annulus Ω, given by the exterior of $K_{p,q}$ in the torus $V_1 \cap V_2$. Such an annulus deformation-retracts to its core, which is homotopic in $V_1 \cap V_2$ to a parallel copy of $K_{p,q}$, whose image in $\pi_1(V_i \setminus \nu K_{p,q})$ is given, respectively, by $\mathsf{m}(K)^q \mathsf{l}(K)^p$ and by $\mu(K)^q \lambda(K)^p$. Therefore an application of the Seifert-Van Kampen theorem gives

$$\pi_1\big(S^3 \setminus \nu(K \cup K_{p,q})\big) \;\cong\; \frac{\pi_1(V_1) * \big(\mathbb{Z}[\mu(K)] \oplus \mathbb{Z}[\lambda(K)]\big)}{\mathsf{m}(K)^q \mathsf{l}(K)^p = \mu(K)^q \lambda(K)^p}. \tag{3.8}$$

We can now complete our argument. When we quotient by the relations $\lambda(K) = \mu(K) = 1$ in (3.8) we obtain that

$$\pi_1\big(E(2)_K \setminus \nu(\Sigma_g \cup T_{p,q})\big) \;\cong\; \frac{\pi_1\big(V_1\big)}{\mathsf{m}(K)^q \mathsf{l}(K)^p = 1}$$

and considering that V_1 deformation-retracts to $S^3 \setminus \mathsf{n}K$ with $\mathsf{m}(K)$ and $\mathsf{l}(K)$ corresponding to the meridian and the longitude of K respectively, we obtain that

$$\pi_1\big(E(2)_K \setminus \nu(\Sigma_g \cup T_{p,q})\big) \;\cong\; \pi_1(S^3_{q/p}(K)).$$

This finishes the proof of (3.4).

The second step in the proof of (3.1) consists of applying the Seifert-Van Kampen theorem to decomposition (3.3) using (3.4). Note that the fundamental group of $S^3 \setminus \mathsf{n}K$ is generated by $\pi_1(\mathsf{S}_K)$ (a free group $\mathbb{F}_{2g(K)}$ on $2g(K)$ generators a_i, b_i, $i = 1, \dots, g(K)$) and the meridian $\mathsf{m}(K)$, subject to the monodromy relations. The relation $\mathsf{m}(K)^q\mathsf{l}(K)^p = 1$ adds an extra relation between the meridian and the generators of $\pi_1(\mathsf{S}_K)$, since $\mathsf{l}(K) = \prod_{i=1}^{g(K)}[a_i, b_i]$. In sum, we can write (3.4) as

$$\begin{aligned} \pi_1\big(E(2)_K \setminus \nu(\Sigma_g \cup T_{p,q})\big) \;&=\; \langle a_1, \dots, a_{g(K)}, b_1, \dots, b_{g(K)}, \mathsf{m}(K) \mid \\ & f_\# a_i = \mathsf{m}(K)^{-1} a_i \mathsf{m}(K),\ f_\# b_i = \mathsf{m}(K)^{-1} b_i \mathsf{m}(K),\ \mathsf{m}(K)^q \mathsf{l}(K)^p = 1\rangle, \end{aligned} \tag{3.9}$$

where $f_\#$ denotes the monodromy map of the fiber bundle $S^3 \setminus \mathsf{n}K \to \mathsf{m}(K)$.

Let Y denote the intersection of the two halves in (3.3). Precisely, Y is diffeomorphic to the cartesian product $(\Sigma_g \setminus \bigsqcup_{j=1}^q \mathbb{D}_{x_j}) \times S^1$, where $\bigsqcup_{j=1}^q \mathbb{D}_{x_j} = \Sigma_g \cap (\nu T_{p,q})$ is the disjoint union of q disks centered at the intersection points $\{x_1, \dots, x_q\}$ of Σ_g with $T_{p,q}$. Note that $\Sigma_g \setminus \bigsqcup_{j=1}^q \mathbb{D}_{x_j} = \Sigma_{K,q} \cup \Sigma'$, where Σ' is the genus 1 surface with boundary described in the proof of Lemma 4. The loops of the form $* \times S^1$, which are copies of the meridian of Σ_g in D_K, are contractible in each half-component of (3.3), as in the proof of Lemma 4.

Remember now that the surface S_K is contained in $(\Sigma_g \setminus \bigsqcup_{j=1}^q \mathbb{D}_{x_j})$, as the latter is built by gluing to S_K an annulus (see Figure 2) in $S^3 \setminus \nu(H \cup C_{p,q})$ with q disks removed, and then capping off with Σ'. When applied to (3.3), Seifert-Van Kampen theorem gives, as $\pi_1(D_K \setminus \nu\Xi_{p,q})$, the free product of the fundamental groups of the two halves, modulo the identification induced by the image of the generators of $\pi_1(\Sigma_g \setminus \bigsqcup_{j=1}^q \mathbb{D}_{x_j})$.

Lemma 6. *The fundamental group $\pi_1\big(E(2)_K \setminus \nu(\Sigma_g \cup T_{p,q})\big)$ of each half-component of decomposition* (3.3) *is generated by the image of $\pi_1(\Sigma_g \setminus \bigsqcup_{j=1}^q \mathbb{D}_{x_j})$ under the homomorphism induced by the inclusion map.*

Proof. In order to prove this lemma we need to keep track of the representatives of the generators of the fundamental group of each half-component. From presentation (3.9) we see that the generators for both copies of $\pi_1(E(2)_K \setminus \nu(\Sigma_g \cup T_{p,q}))$ denoted by a_i and b_i are contained in the image of $\pi_1(\Sigma_g \setminus \bigsqcup_{j=1}^q \mathbb{D}_{x_j})$ for obvious reasons, resulting in the identification of the generators denoted by the same symbols. The lone remaining generator, for each half-component, is $\mathsf{m}(K)$. Now we can observe that, as an element of $\pi_1(S^3 \setminus (K \cup K_{p,q}))$, $\mathsf{m}(K)$ can be written as a word in the generators of this group, for which we choose the presentation used in the proof of Lemma 5. By construction this word will be the same for both halves of decomposition (3.3). It follows that $\mathsf{m}(K)$, as an element of $\pi_1(S^3 \setminus (K \cup K_{p,q}))$, can be written as a word in $\mu(K)$, the a_i's, the b_i's and in the c_j's. (A more careful study of $\pi_1(S^3 \setminus \nu(H \cup C_{p,q}))$ would enable us to identify this word, but we will not need its explicit form.) When we quotient the group $\pi_1(S^3 \setminus (K \cup K_{p,q}))$ by the relations $\mu(K) = \lambda(K) = 1$ to obtain $\pi_1(E(2)_K \setminus \nu(\Sigma_g \cup T_{p,q}))$, we see that (the image of) $\mathsf{m}(K)$ can be written as a word in the generators of (the image of) $\pi_1(\Sigma_{K,q})$, and *a fortiori* of $\pi_1(\Sigma_g \setminus \bigsqcup_{j=1}^q \mathbb{D}_{x_j})$. As the word is the same on each half of the decomposition, the generators $\mathsf{m}(K)$ of two halves are identified. $\square$

As a consequence of Lemma 6 the free product with amalgamation identifies all the corresponding generators of π_1 of two halves of (3.3) and we get the simple formula

$$\pi_1(D_K \setminus \nu\Xi_{p,q}) \cong \pi_1\big(E(2)_K \setminus \nu(\Sigma_g \cup T_{p,q})\big) \cong \pi_1(S^3_{q/p}(K)),$$

with the latter group presented (using standard notation) in (3.2). This concludes the proof of Theorem 3.

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