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COMPLETE MINIMAL HYPERSURFACES IN THE HYPERBOLIC SPACE $\mathbb{H}^4$ WITH VANISHING GAUSS-KRONECKER CURVATURE

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Abstract. We investigate 3-dimensional complete minimal hypersurfaces in the hyperbolic space $\mathbb{H}^4$ with Gauss-Kronecker curvature identically zero. More precisely, we give a classification of complete minimal hypersurfaces with Gauss-Kronecker curvature identically zero, a nowhere vanishing second fundamental form and a scalar curvature bounded from below.

1. Introduction

In order to study the rigidity of minimal hypersurfaces, Dajczer and Gromoll [6] invented the so called *Gauss parametrization*. As a by-product of this approach they were able to locally describe the minimal hypersurfaces in the $(n+1)$-dimensional space form whenever the rank of the nullity distribution is constant. Almeida and Brito [2] initiated the study of compact minimal hypersurfaces in the unit sphere $\mathbb{S}^4$ with vanishing Gauss-Kronecker curvature. In fact they proved that such compact hypersurfaces are boundaries of tubes of minimal 2-spheres in $\mathbb{S}^4$, provided that the second fundamental form never vanishes. Ramanathan [12] extended this result and allowed points where the second fundamental form is zero. In [7], [8] the authors extended the above results to complete minimal hypersurfaces in the Euclidean space $\mathbb{R}^4$ or in the unit sphere $\mathbb{S}^4$.

The aim of this paper is to study complete minimal hypersurfaces in the 4-dimensional hyperbolic space $\mathbb{H}^4$ with identically zero Gauss-Kronecker curvature. We recall that the Gauss-Kronecker curvature is the product of the principal curvatures. In fact, we deal with minimal hypersurfaces whose second fundamental form is nowhere zero, which is equivalent to the fact that the nullity distribution is one dimensional.

It turns out that such hypersurfaces are closely related to stationary spacelike surfaces in the de Sitter space $\mathbb{S}^4_1$, which is the Lorentzian unit sphere in the flat Lorentzian space $\mathbb{R}^5_1$. More precisely, Dajczer and Gromoll [6] noticed that the unit normal bundle of a stationary spacelike surface in $\mathbb{S}^4_1$ gives rise to a minimal hypersurface in $\mathbb{H}^4$ with vanishing Gauss-Kronecker curvature and nowhere zero second fundamental form (for details see Section 2), and, conversely, any such hypersurface is obtained, at least locally, in this way.

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We are interested in the classification of complete minimal hypersurfaces in $\mathbb{H}^4$ with Gauss-Kronecker curvature identically zero. At first we show that there exists an abundance of such hypersurfaces. To this purpose we focus on a class of stationary spacelike surfaces in $\mathbb{S}_1^4$, namely those with vanishing normal curvature. We establish a correspondence between stationary spacelike surfaces in $\mathbb{S}_1^4$ with identically zero normal curvature and minimal surfaces in umbilical hypersurfaces in the hyperbolic space $\mathbb{H}^4$.

Our examples of complete minimal hypersurfaces in $\mathbb{H}^4$ with vanishing Gauss-Kronecker curvature arise as suspensions of complete minimal surfaces in horospheres or equidistant hypersurfaces of $\mathbb{H}^4$. Then the question whether these are the only examples comes naturally.

We prove that these suspensions are in fact the only complete minimal hypersurfaces in $\mathbb{H}^4$ with identically zero Gauss-Kronecker curvature under the assumptions that the second fundamental form is nowhere zero and the scalar curvature is bounded from below.

The paper is organized as follows: In Section 2 we study stationary spacelike surfaces in the de Sitter space $\mathbb{S}_1^4$, we define the polar map and show that it induces minimal hypersurfaces in $\mathbb{H}^4$ with vanishing Gauss-Kronecker curvature. In particular, we prove some auxiliary results about stationary spacelike surfaces in $\mathbb{S}_1^4$ with identically zero normal curvature. Moreover, we furnish a method to produce complete minimal hypersurfaces in $\mathbb{H}^4$ with identically zero Gauss-Kronecker curvature. Section 3 is devoted to the local theory of minimal hypersurfaces in $\mathbb{H}^4$ with vanishing Gauss-Kronecker curvature. Furthermore, we give some auxiliary results. Finally, in Section 4 we state and prove the main result of this paper.

2. The polar map of stationary surfaces in the de Sitter space

At first we set up our notation. Denote by $\mathbb{R}_1^5$ the real vector space $\mathbb{R}^5$ endowed with the Lorentzian metric tensor $\langle\ ,\ \rangle$ given by

$$\langle x, y\rangle = -x_0 y_0 + \sum_{i=1}^{4} x_i y_i,$$

where $x = (x_0, x_1, x_2, x_3, x_4)$, $y = (y_0, y_1, y_2, y_3, y_4) \in \mathbb{R}^5$. We shall use the Minkowski model for the simply connected *hyperbolic space* of constant sectional curvature -1, which is the hyperquadric

$$\mathbb{H}^4 = \left\{x \in \mathbb{R}_1^5 : \langle x, x\rangle = -1, x_0 > 0\right\}.$$

Moreover, the hyperquadric

$$\mathbb{S}_1^4 = \left\{x \in \mathbb{R}_1^5 : \langle x, x\rangle = 1\right\},$$

is the standard model for the simply connected Lorentzian space form of constant curvature 1, and is called the *de Sitter space*.

Consider a 2-dimensional manifold M^2. An immersion $g : M^2 \to \mathbb{S}_1^4$ is called *spacelike* if the induced metric on M^2 via g is Riemannian, which as usual will be denoted again by $\langle\ ,\ \rangle$. Let $i : \mathbb{S}_1^4 \to \mathbb{R}_1^5$ be the inclusion map. Denote by

$$\begin{aligned}
(i \circ g)^* \left(T\mathbb{R}_1^5\right) &= \left\{(x, w) : x \in M^2, w \in T_{g(x)}\mathbb{R}_1^5\right\}, \\
g^* \left(T\mathbb{S}_1^4\right) &= \left\{(x, w) : x \in M^2, w \in T_{g(x)}\mathbb{S}_1^4\right\}
\end{aligned}$$

the *induced bundles* of $i \circ g$ and g, respectively. The *normal bundle* $\mathcal{N}(g)$ of g is given by

$$\mathcal{N}(g) = \left\{(x, w) \in g^* \left(T\mathbb{S}_1^4\right) : w \perp dg \left(T_x M^2\right)\right\}.$$

Also we denote by $\overline{\nabla}$, $\overset{g}{\nabla}$ the *connections of the induced bundles* of $i \circ g$ and g, respectively, and by $\overset{g}{D}$ the *connection of the normal bundle* of g. Given a vector field η along g and a tangent vector X of M^2, we then have

$$\overline{\nabla}_X \eta = \overset{g}{\nabla}_X \eta - \langle dg(X), \eta \rangle g.$$

The *second fundamental form* II of g is given by the *Gauss formula*

$$II(X, Y) = \overset{g}{\nabla}_X dg(Y) - dg(\nabla_X Y),$$

where X,Y are tangent vector fields of M^2 and ∇ stands for the Levi-Civita connection of the induced metric on M^2. The self-adjoint operator A_η defined by

$$\langle A_\eta X, Y \rangle = \langle II(X, Y), \eta \rangle$$

is called the *shape operator* of g relative to η. Moreover, the *Weingarten formula* is

$$\overset{g}{\nabla}_X \eta = -dg(A_\eta X) + \overset{g}{D}_X \eta.$$

A point $x \in M^2$ is called a *totally geodesic point of* g if and only if $II_x = 0$. If each point of M^2 is a totally geodesic point of g, then g is called a *totally geodesic immersion.* Since $\mathbb{S}_1^4$ is a Lorentz manifold of constant sectional curvature, one can derive the *equations of Codazzi and Ricci*, which are respectively

$$\begin{aligned} (\nabla_X A_\eta) Y + A_{\overset{g}{D}_Y \eta} X &= (\nabla_Y A_\eta) X + A_{\overset{g}{D}_X \eta} Y, \\ \left\langle R^D(X, Y) \eta_1, \eta_2 \right\rangle &= \langle [A_{\eta_1}, A_{\eta_2}] X, Y \rangle, \end{aligned}$$

where X,Y are tangent vector fields of M^2, R^D is the curvature tensor of $\overset{g}{D}$ and η, η_1, η_2 are normal vector fields along g (see for instace [10, Chapter 4]).

Now consider an orthonormal adapted frame field $\{e_1, e_2; e_3, e_4\}$ along g, where e_4 is timelike. Then, we have

$$II(X, Y) = \langle A_3 X, Y \rangle e_3 - \langle A_4 X, Y \rangle e_4,$$

where A_3, A_4 are the shape operators of g with respect to the directions e_3 and e_4. The *mean curvature vector field* H is given by

$$H = \frac{1}{2}(trace A_3) e_3 - \frac{1}{2}(trace A_4) e_4.$$

The immersion g is called *stationary,* whenever $H \equiv 0$. The *Gaussian curvature* K of the induced metric is

$$K = 1 + \det A_3 - \det A_4.$$

We denote by $\{\omega_1, \omega_2\}$ the *dual frame* of $\{e_1, e_2\}$ and by ω_{34} the *connection form of the normal bundle of* g, which is determined by

$$\omega_{34}(X) = -\left\langle \overset{g}{D}_X e_3, e_4 \right\rangle.$$

The *normal curvature* $K^\perp$ of g is given by

$$K^\perp = \left\langle R^D\left(e_1, e_2\right) e_3, e_4\right\rangle = \left\langle \left[A_3, A_4\right] e_1, e_2\right\rangle.$$

We recall that

$$d\omega_{34} = -K^\perp \omega_1 \wedge \omega_2. \tag{2.1}$$

Assume now that M^2 is an oriented, 2-dimensional Riemannian manifold and $g : M^2 \to \mathbb{S}^4_1$ is a stationary isometric immersion. It is well known (cf. [1]) that there exists a holomorphic quadric differential on M^2, the so called *Hopf differential*. A point $x \in M^2$ is a zero of the Hopf differential if and only if $K(x) = 1$ and $K^\perp(x) = 0$. Such a point is called a *superminimal point of* g. The immersion g is called *superminimal* if each point of M^2 is a superminimal point of g. From the holomorphicity of the Hopf differential, it follows that either g is superminimal or the superminimal points are isolated.

Consider the *timelike unit normal bundle* $\mathcal{N}^1(g)$ of g, defined by

$$\mathcal{N}^1(g) = \left\{(x, w) \in \mathcal{N}(g) : \langle w, w\rangle = -1\right\}.$$

Denote by $\pi : \mathcal{N}^1(g) \to M^2$ the projection to the first factor and by $\Psi_g : \mathcal{N}^1(g) \to \mathbb{H}^4$ the projection to the second factor. The map Ψ_g is called the *polar map* associated with g. In the sequel, $\mathcal{N}^1(g)$ is endowed with the metric induced by Ψ_g.

Throughout the paper we follow the above mentioned notation and assume that all manifolds under consideration are connected, unless otherwise stated.

The following proposition was essentially proved by Dajczer and Gromoll in [6, Section 1]. In order to make the paper self-contained here we shall give a proof that fits our exposition. This proposition furnishes a method for producing minimal hypersurfaces in $\mathbb{H}^4$ with Gauss-Kronecker curvature identically zero and establishes the close relation between them and stationary surfaces in $\mathbb{S}^4_1$.

Proposition 2.1. *Let M^2 be a 2-dimensional Riemannian manifold and $g : M^2 \to \mathbb{S}^4_1$ a stationary isometric immersion. Then*

- (i) *The polar map Ψ_g associated with g is regular at $(y, w) \in \mathcal{N}^1(g)$ if and only if the second fundamental form of g is non-singular in the direction w.*
- (ii) *On the open set of its regular points, Ψ_g is a minimal immersion in $\mathbb{H}^4$ with Gauss-Kronecker curvature identically zero and a nowhere vanishing second fundamental form.*
- (iii) *If x is a point on M^2 where the normal curvature of g is not zero, then Ψ_g is regular on the fiber of $\mathcal{N}^1(g)$ over x. Furthermore, if x is not a totally geodesic point of g and $K(x) \geq 1$, then Ψ_g is regular on the fiber of $\mathcal{N}^1(g)$ over x.*

Proof. Choose an adapted orthonormal frame field $\{e_1, e_2; e_3, e_4\}$ along g defined on an open set $U \subset M^2$, where e_4 is timelike. We parametrize $\pi^{-1}(U)$ by $U \times \mathbb{R}$ via the map

$$(x, t) \to \left(x, \sinh t e_3(x) + \cosh t e_4(x)\right).$$

Then $\Psi_g(x, t) = \sinh t e_3(x) + \cosh t e_4(x)$. For the sake of convenience we set $W(x, t) = \sinh t e_3(x) + \cosh t e_4(x)$. Obviously $w = W(y, t_0)$, for some $t_0 \in \mathbb{R}$.

(i) Calculating the differential of Ψ_g at the point (y, w) we have

$$d\Psi_g\left(\frac{\partial}{\partial t}\right) = \cosh t_0 e_3(y) + \sinh t_0 e_4(y)$$

and

$$(2.2)\qquad \begin{aligned} d\Psi_g(X) &= \overline{\nabla}_X W = \overset{g}{\nabla}_X W \\ &= -dg(A_w X) + \overset{g}{D}_X W \\ &= -dg(A_w X) + \omega_{34}(X)(\cosh t_0 e_3(y) + \sinh t_0 e_4(y)) \\ &= -dg(A_w X) + \omega_{34}(X)\, d\Psi_g\left(\frac{\partial}{\partial t}\right), \end{aligned}$$

for each $X \in T_y M^2$. From the above relations it follows that (y, w) is a regular point of Ψ_g if and only if $\det A_w(y) \neq 0$.

(ii) The vector field ξ given by $\xi(x,t) = g(x)$, $(x,t) \in U \times \mathbb{R}$, is a unit normal vector field along Ψ_g. Denote by A_ξ the corresponding shape operator and by $\pi_1 : U \times \mathbb{R} \rightarrow U$, $\pi_2 : U \times \mathbb{R} \rightarrow \mathbb{R}$ the corresponding projection maps. Using the Weingarten formula, we have

$$0 = d\xi\left(\frac{\partial}{\partial t}\right) = -d\Psi_g\left(A_\xi \frac{\partial}{\partial t}\right).$$

Consequently, the Gauss-Kronecker curvature of Ψ_g is identically zero. Moreover using (2.2), we get

$$\begin{aligned} -dg(X) &= -d\xi(X) = d\Psi_g(A_\xi X) \\ &= d\Psi_g(d\pi_1(A_\xi X) + d\pi_2(A_\xi X)) \\ &= -dg(A_w(d\pi_1(A_\xi X))) + \overset{g}{D}_{d\pi_1(A_\xi X)} W + d\Psi_g(d\pi_2(A_\xi X)), \end{aligned}$$

for each $X \in T_y M^2$. So

$$d\pi_1(A_\xi X) = A_w^{-1} X,$$

and Ψ_g has principal curvatures

$$k_1(x,w) = -k_3(x,w) = \frac{1}{\sqrt{-\det A_w(x)}},\ k_2(x,w) = 0.$$

(iii) Suppose that $K^\perp(x) \neq 0$. Then, obviously $\det A_w(x) \neq 0$, for each w on the fiber of $\mathcal{N}^1(g)$ over x. Assume now that x is not a totally geodesic point of g, $K(x) \geq 1$ and that there exists a vector w on the fiber of $\mathcal{N}^1(g)$ over x such that $\det A_w(x) = 0$. Let η be a unit normal vector in the normal bundle of g such that $\langle \eta, w \rangle = 0$. Then, because x is not a totally geodesic point, it follows that $K(x) = 1 + \det A_\eta(x) - \det A_w(x) = 1 + \det A_\eta(x) < 1$, which is a contradiction. This completes the proof. □

Remark 2.2. We shall see in Section 3 that every minimal hypersurface in the hyperbolic space $\mathbb{H}^4$ with Gauss-Kronecker curvature identically zero and nowhere vanishing second fundamental form can be obtained, at least locally, as in Proposition 2.1(ii).

We now focus on the class of stationary spacelike minimal surfaces in $\mathbb{S}_1^4$ with identically zero normal curvature. This class will play a crucial role in the classification of complete minimal hypersurfaces in the hyperbolic space with zero Gauss-Kronecker curvature. We recall here that the totally geodesic submanifolds of $\mathbb{S}_1^4$ arise as intersections of $\mathbb{S}_1^4$ with linear subspaces of $\mathbb{R}_1^5$. The following result is due to Alias and Palmer [1, Proposition 3.5]. For the sake of completeness we give another short proof.

Proposition 2.3. *Let M^2 be a 2-dimensional Riemannian manifold and let $g : M^2 \to \mathbb{S}^4_1$ be a stationary isometric immersion. Then $K^\perp \equiv 0$ if and only if $g\left(M^2\right)$ is contained in a totally geodesic hypersurface L^3 of $\mathbb{S}^4_1$, i.e., there exists a vector w such that $\langle g(x), w\rangle = 0$, for each $x \in M^2$. Moreover,*

- (i) *w is spacelike if and only if $K \geq 1$ and $K \not\equiv 1$,*
- (ii) *w is timelike if and only if $K \leq 1$ and $K \not\equiv 1$,*
- (iii) *w may be chosen to be null if and only if $K \equiv 1$.*

Proof. Denote by M_1 the set of superminimal points of g. We distinguish two cases.

Case 1. Assume that M_1 consists of isolated points only. Consider a non-superminimal point x of g and let $\{e_1, e_2; e_3, e_4\}$ be an adapted orthonormal frame field defined on an open set $U_x \subset M^2 - M_1$ around x, e_4 being timelike. Because $K^\perp \equiv 0$, we may suppose that the shape operators associated with e_3 and e_4 are

$$A_3 \sim \begin{pmatrix} \kappa & 0 \\ 0 & -\kappa \end{pmatrix}, \quad A_4 \sim \begin{pmatrix} \mu & 0 \\ 0 & -\mu \end{pmatrix}.$$

Because $K \neq 1$ on U_x, it follows that $\kappa^2 \neq \mu^2$. Now define the vector fields

$$\overline{e}_3 = \frac{1}{\sqrt{|\mu^2 - \kappa^2|}}\left(\mu e_3 - \kappa e_4\right), \ \overline{e}_4 = \frac{1}{\sqrt{|\mu^2 - \kappa^2|}}\left(\kappa e_3 - \mu e_4\right).$$

Then the shape operators corresponding to the directions $\overline{e}_3$ and $\overline{e}_4$ are

$$\overline{A}_3 = 0 \quad \text{and} \quad \overline{A}_4 \sim \begin{pmatrix} \frac{\kappa^2-\mu^2}{\sqrt{|\mu^2-\kappa^2|}} & 0 \\ 0 & -\frac{\kappa^2-\mu^2}{\sqrt{|\mu^2-\kappa^2|}} \end{pmatrix}.$$

From the Codazzi equation we get,

$$A_{\overset{g}{D}_{e_1}\overline{e}_3} e_2 = A_{\overset{g}{D}_{e_2}\overline{e}_3} e_1,$$

or, equivalently,

$$\overline{\omega}_{34}(e_1)\left(\overline{A}_4 e_2\right) = \overline{\omega}_{34}(e_2)\left(\overline{A}_4 e_1\right),$$

where $\overline{\omega}_{34}$ stands for the connection form on the normal bundle of g with respect to the frame $\{\overline{e}_3, \overline{e}_4\}$. Thus $\overline{\omega}_{34} = 0$, and so the vector field $w = \overline{e}_3$ is constant along g and $\langle g, w\rangle = 0$ on U_x. This means that $g(U_x)$ is contained in a totally geodesic hypersurface L^3 of $\mathbb{S}^4_1$. Note that the normal vector field w satisfies

$$\langle w, w\rangle = \frac{\mu^2 - \kappa^2}{|\mu^2 - \kappa^2|} = \frac{K-1}{|K-1|}.$$

Suppose now that $U_y \subset M^2 - M_1$ is an open set around another point $y \in M^2 - M_1$, such that $U_x \cap U_y \neq \emptyset$ and $g(U_y) \subset \overline{L}^3$, where $\overline{L}^3$ is a totally geodesic hypersurface of $\mathbb{S}^4_1$, with normal vector $\overline{w}$. We claim that $\overline{L}^3 = L^3$. To this purpose, it is enough to prove that w and $\overline{w}$ are linearly dependent. Suppose to the contrary that these are linearly independent. Then g is totally geodesic on $U_x \cap U_y$, a contradiction since $K \neq 1$ on $U_x \cap U_y$. So we deduce that $g\left(M^2\right)$ is contained in a totally geodesic hypersurface L^3 of $\mathbb{S}^4_1$, whose normal vector satisfies

$$\langle w, w\rangle = \frac{K-1}{|K-1|},$$

on $M^2 - M_1$. This completes the proof of parts (i) and (ii).

Case 2. Suppose that $M_1 = M^2$. Then g is superminimal and $K \equiv 1$. Because $K^{\perp} \equiv 0$, around each point we may choose a parallel orthonormal frame field $\{\eta_1, \eta_2\}$ in the normal bundle of g. Let (u, v) be local isothermal coordinates. The complex valued functions,

$$\sigma_i (u, v) = \left\langle A_{\eta_i} \frac{\partial}{\partial u}, \frac{\partial}{\partial u} \right\rangle - \sqrt{-1} \left\langle A_{\eta_i} \frac{\partial}{\partial u}, \frac{\partial}{\partial v} \right\rangle, \quad i = 1, 2,$$

are holomorphic, and their zeroes are precisely the totally geodesic points of g. Hence the set M_0 of totally geodesic points of g either coincides with M^2 or consists of isolated points. In the case where $M_0 = M^2$, $g\left(M^2\right)$ is contained in an appropriate totally geodesic hypersurface of $\mathbb{S}^4_1$ whose normal vector is null. Suppose now that M_0 consists of isolated points. Then the set $M^2 - M_0$ is open and connected. Consider a non-totally geodesic point $x \in M^2$ and let $\{e_1, e_2; e_3, e_4\}$ be an adapted orthonormal frame field defined on a simply connected neighborhood $U \subset M^2 - M_0$ of x, e_4 being timelike. We may suppose that $A_3 = A_4$. Since $K^{\perp} \equiv 0$, from (2.1) it follows that $d\omega_{34} = 0$. Thus there exists a smooth function θ on U such that $\omega_{34} = d\theta$. Define the vector field

$$w = e^{\theta} (e_3 - e_4).$$

For each tangent vector field X of U, we have

$$\begin{aligned} dw(X) &= e^{\theta} X(\theta)(e_3 - e_4) + e^{\theta} \overset{g}{\nabla}_X (e_3 - e_4) \\ &= e^{\theta} X(\theta)(e_3 - e_4) + e^{\theta} \omega_{34}(X)(e_4 - e_3) \\ &= 0. \end{aligned}$$

Therefore w is constant and $\langle g, w \rangle = 0$. Thus, $g(U)$ is contained in a totally geodesic hypersurface L^3 of $\mathbb{S}^4_1$ whose normal vector w satisfies $\langle w, w \rangle = 0$. Arguing as in Case 1 we can prove that $g\left(M^2\right) \subset L^3$. This completes the proof. □

The following proposition provides a way to produce all spacelike stationary surfaces in $\mathbb{S}^4_1$ with normal curvature identically zero. For the sake of convenience, we introduce the following notation. Let $h : M^2 \to Q^3$ be an isometric immersion, where M^2 is a 2-dimensional, oriented Riemannian manifold and Q^3 an umbilical hypersurface of $\mathbb{H}^4$. The orientation N of h gives rise in a natural way to a map $\widehat{h} := N : M^2 \to \mathbb{S}^4_1$ which is called the *associate* of h.

We recall here that the umbilical hypersurfaces of $\mathbb{H}^4$ arise as intersections of $\mathbb{H}^4$ with affine hyperplanes of $\mathbb{R}^5_1$. Moreover, an umbilical hypersurface Q^3 of $\mathbb{H}^4$ has positive, negative or zero sectional curvature if Q^3 is a *geodesic sphere*, an *equidistant hypersurface* or a *horosphere*, respectively.

Proposition 2.4. *Let M^2 be a 2-dimensional, oriented Riemannian manifold.*

(*i*) *If $h : M^2 \to Q^3$ is a minimal isometric immersion without totally geodesic points, where Q^3 is an umbilical hypersurface of $\mathbb{H}^4$, then its associate $\widehat{h} : M^2 \to \mathbb{S}^4_1$ is a spacelike stationary immersion with normal curvature identically zero without totally geodesic points.*

(*ii*) *Conversely, assume that $g : M^2 \to \mathbb{S}^4_1$ is a stationary isometric immersion with normal curvature identically zero without totally geodesic points. Then there exist a vector w and a totally geodesic point free minimal immersion*

$h : M^2 \to Q^3$, where Q^3 is an umbilical hypersurface of $\mathbb{H}^4$, with sectional curvature $K_{Q^3} = -\langle w, w\rangle$ such that $\langle g(x), w\rangle = 0$, for each $x \in M^2$ and $g = \widehat{h}$.

Proof. (i) Let η be a unit normal vector of Q^3 in $\mathbb{H}^4$ and A_η the corresponding shape operator. Obviously, $A_\eta = \alpha I$, for some $\alpha \in \mathbb{R}$. We denote by N the orientation of h and by A_N the corresponding shape operator. For each tangent vector X of M^2, we have

$$d\widehat{h}(X) = -dh(A_N X). \tag{2.3}$$

Hence $\widehat{h}$ is an immersion and the metric $\langle X, Y\rangle_{\widehat{h}} = \langle A_N^2 X, Y\rangle$ induced by $\widehat{h}$ on M^2 is Riemannian, where $\langle\ ,\ \rangle$ stands for the Riemannian metric of M^2. Moreover, the vector fields $\{e_3 := \eta \circ h, e_4 := h\}$ constitute a frame field in the normal bundle of $\widehat{h}$. Denote by $\widehat{A}_3$, $\widehat{A}_4$ the shape operators of $\widehat{h}$ with respect to directions e_3 and e_4. Using the Weingarten formula and (2.3), we get

$$de_3(X) = \overset{\widehat{h}}{\nabla}_X e_3 = -d\widehat{h}\left(\widehat{A}_3 X\right) + \overset{\widehat{h}}{D}_X e_3 = dh\left(A_N \widehat{A}_3 X\right) + \overset{\widehat{h}}{D}_X e_3.$$

Since

$$de_3(X) = d\eta(dh(X)) = -A_\eta(dh(X)) = -\alpha dh(X),$$

it follows that

$$\widehat{A}_3 = -\alpha A_N^{-1} \quad \text{and} \quad \overset{\widehat{h}}{D}_X e_3 = 0. \tag{2.4}$$

Moreover,

$$dh(X) = \overset{\widehat{h}}{\nabla}_X e_4 = -d\widehat{h}\left(\widehat{A}_4 X\right) + \overset{\widehat{h}}{D}_X e_4 = dh\left(A_N \widehat{A}_4 X\right) + \overset{\widehat{h}}{D}_X e_4.$$

Therefore

$$\widehat{A}_4 = A_N^{-1} \quad \text{and} \quad \overset{\widehat{h}}{D}_X e_4 = 0. \tag{2.5}$$

From (2.4) and (2.5) we deduce that $\widehat{h}$ is stationary with normal curvature identically zero.

(ii) Suppose now that $g : M^2 \to \mathbb{S}_1^4$ is a stationary isometric immersion with normal curvature identically zero without totally geodesic points. According to Proposition 2.3, $g\left(M^2\right)$ is contained in a totally geodesic hypersurface of $\mathbb{S}_1^4$ with normal vector w. Without loss of generality, we may assume that $\langle w, w\rangle \le 1$. We distinguish two cases.

Case 1. Assume that w is not null. We set $a := \sqrt{1 - \langle w, w\rangle}$. Because M^2 is oriented, we may choose a global vector field η normal along g such that $\langle \eta, w\rangle = 0$ and $\langle \eta, \eta\rangle = a^2 - 1$. Consider the vector fields

$$e_3 := \frac{a\eta - w}{a^2 - 1}, \quad e_4 := \frac{aw - \eta}{a^2 - 1}.$$

Note that $\{e_3, e_4\}$ is a parallel orthonormal frame field of the normal bundle of g and e_4 is timelike. Moreover, $w = e_3 + ae_4$ and $A_3 = -aA_4$, where A_3 and A_4 are the shape operators of g with respect to the directions e_3 and e_4. Since the second fundamental form of g becomes

$$II(X, Y) = -\langle A_4 X, Y\rangle(ae_3 + e_4),$$

where X, Y are tangent vector fields of M^2, we deduce that A_4 is everywhere nonsingular. Define now the map $h : M^2 \to \mathbb{H}^4$, $h(x) := e_4(x)$, $x \in M^2$. We claim that h satisfies all the desired properties. Indeed, for each tangent vector X of M^2 we have

$$dh(X) = \overset{g}{\nabla}_X e_4 = -dg(A_4 X) + \overset{g}{D}_X e_4 = -dg(A_4 X). \tag{2.6}$$

Therefore h is an immersion and the metric $\langle X, Y \rangle_h = \langle A_4^2 X, Y \rangle$ induced on M^2 by h is Riemannian. The normal bundle of h is spanned by $\{\eta_1 := g, \eta_2 := e_3\}$. Denote by $\widetilde{A}_1$ and $\widetilde{A}_2$ the corresponding shape operators of h in the directions η_1 and η_2. Then the Weingarten formula and (2.6) yield

$$dg(X) = \overset{h}{\nabla}_X \eta_1 = -dh\left(\widetilde{A}_1 X\right) + \overset{h}{D}_X \eta_1 = dg\left(A_4 \widetilde{A}_1 X\right) + \overset{h}{D}_X \eta_1.$$

Hence

$$\widetilde{A}_1 = A_4^{-1} \quad \text{and} \quad \overset{h}{D}_X \eta_1 = 0. \tag{2.7}$$

Moreover,

$$d\eta_2(X) = \overset{h}{\nabla}_X \eta_2 = -dh\left(\widetilde{A}_2 X\right) + \overset{h}{D}_X \eta_2 = dg\left(A_4 \widetilde{A}_2 X\right) + \overset{h}{D}_X \eta_2.$$

Since,

$$d\eta_2(X) = \overset{g}{\nabla}_X e_3 = -dg(A_3 X),$$

we get

$$\widetilde{A}_2 = aI \quad \text{and} \quad \overset{h}{D}_X \eta_2 = 0. \tag{2.8}$$

From (2.7) and (2.8) we deduce that the vector field $w = \eta_2 + ah$ is constant along h and $\langle h, w \rangle = -a$. Therefore, $h\left(M^2\right)$ is contained in an umbilical hypersurface Q^3 of $\mathbb{H}^4$, with sectional curvature $K_{Q^3} = -1 + a^2 = -\langle w, w \rangle$. Furthermore, $h : M^2 \to Q^3$ is a minimal immersion with normal η_1 and $g = \widehat{h}$.

Case 2. Assume now that w is null. According to Proposition 2.3(*iii*) g is superminimal. Because M^2 is oriented we may choose a global null vector field η in the normal bundle of g, such that $\langle \eta, w \rangle = 1/2$. Now define the vector fields

$$e_3 := \eta + w, \quad e_4 := w - \eta.$$

Obviously $2w = e_3 + e_4$ and $A_3 = -A_4$, where A_3 and A_4 are the shape operators of g with respect to the directions e_3 and e_4. Note that $\{e_3, e_4\}$ is a parallel orthonormal frame field of the normal bundle of g and e_4 is timelike. Since the second fundamental form of g becomes

$$II(X, Y) = -\langle A_4 X, Y \rangle (e_3 + e_4),$$

where X, Y are tangent vector fields of M^2, it follows that A_4 is everywhere nonsingular. Now consider the map $h : M^2 \to \mathbb{H}^4$, $h(x) := e_4(x)$, $x \in M^2$. We claim that h is the required map. Indeed, for each tangent vector X of M^2 we have

$$dh(X) = -dg(A_4 X).$$

Therefore, h is an immersion and the metric $\langle X, Y \rangle_h = \langle A_4^2 X, Y \rangle$ induced by h in M^2 is Riemannian. Then the rest of the proof proceeds as in Case 1. □

Let M^2 be an oriented, 2-dimensional Riemannian manifold and $h : M^2 \to Q^3$ an isometric immersion, where Q^3 is an umbilical hypersurface of $\mathbb{H}^4$. Denote by η a unit normal vector field of Q^3 in $\mathbb{H}^4$. Consider the map $F_h : M^2 \times \mathbb{R} \to \mathbb{H}^4$, given by

$$F_h(x,t) = \cosh t h(x) + \sinh t \eta \circ h(x), \quad (x,t) \in M^2 \times \mathbb{R},$$

which is called *the suspension* of h in $\mathbb{H}^4$.

It is clear that in the case where $h : M^2 \to Q^3$ is a minimal isometric immersion without totally geodesic points, then $F_h = \Psi_{\widehat{h}} \circ T$, where T is the diffeomorphism $T : M^2 \times \mathbb{R} \to \mathcal{N}^1(\widehat{h})$ given by $T(x,t) = (x, F_h(x,t))$.

In the following proposition we show that there is an abundance of complete minimal hypersurfaces in $\mathbb{H}^4$ with Gauss-Kronecker curvature identically zero.

Proposition 2.5. *Let $h : M^2 \to Q^3$ be a minimal isometric immersion of a 2-dimensional, oriented Riemannian manifold M^2 into an umbilical hypersurface Q^3 of $\mathbb{H}^4$. Then,*

- (i) *On the open subset of the regular points, the suspension F_h of h is a minimal immersion in $\mathbb{H}^4$ with Gauss-Kronecker curvature identically zero.*
- (ii) *The metric induced on $M^2 \times \mathbb{R}$ by F_h is complete if and only if M^2 is complete and Q^3 is a horosphere or an equidistant hypersurface in $\mathbb{H}^4$.*

Proof. (i) Denote by N a unit normal vector field along h in Q^3 and by A_N the corresponding shape operator of h. Let $A_\eta = \alpha I$, $\alpha \in \mathbb{R}$, denote the shape operator of Q^3 in $\mathbb{H}^4$ with respect to the unit normal vector field η. Then

$$dF_h(\partial/\partial t) = \sinh t h + \cosh t \eta \circ h,$$

and for each tangent vector X of M^2, we have

$$\begin{aligned} dF_h(X) &= \cosh t dh(X) + \sinh t d\eta(dh(X)) \\ &= \cosh t dh(X) - \sinh t A_\eta(dh(X)) \\ &= (\cosh t - \alpha \sinh t) dh(X). \end{aligned}$$

Therefore, the point (x,t) is a regular point of F_h if and only if $\cosh t - \alpha \sinh t \neq 0$. The unit vector field ξ given by $\xi(x,t) = N(x)$, $(x,t) \in M^2 \times \mathbb{R}$, is normal along F_h. Denote by A_ξ the corresponding shape operator. Then $d\xi\left(\frac{\partial}{\partial t}\right) = 0$, and for each tangent vector X of M^2, we get

$$\begin{aligned} dF_h(A_\xi X) &= -d\xi(X) = -dN(X) = dh(A_N X) \\ &= \frac{1}{\cosh t - \alpha \sinh t} dF_h(A_N X). \end{aligned}$$

Hence, the principal curvatures of F_h are

$$k_1(x,t) = -k_3(x,t) = \frac{k(x)}{\cosh t - \alpha \sinh t}, \; k_2(x,t) = 0, \tag{2.9}$$

where k is a principal curvature of h.

(ii) The map F_h is an immersion if and only if $\cosh t - \alpha \sinh t \neq 0$, for each $t \in \mathbb{R}$. This holds if and only if $\alpha^2 \leq 1$. Since the sectional curvature of Q^3 is $K_{Q^3} = -1 + \alpha^2$, we deduce that the map F_h is an immersion if and only if Q^3 is a horosphere or an equidistant hypersurface in $\mathbb{H}^4$. Furthermore, the metric $\langle\ ,\ \rangle_{F_h}$ induced on $M^2 \times \mathbb{R}$ by F_h, is the warped product

$$\langle\ ,\ \rangle_{F_h} = dt^2 + (\cosh t - \alpha \sinh t)^2 \langle\ ,\ \rangle,$$

where $\langle\ ,\ \rangle$ is the Riemannian metric of M^2. Appealing to a result due to Bishop and O'Neill [3, Lemma 7.2], $\langle\ ,\ \rangle_{F_h}$ is complete if and only if $\langle\ ,\ \rangle$ is complete. $\square$

Remark 2.6. The polar map associated with a non-complete stationary surface in $\mathbb{S}^4_1$ may give rise to a complete minimal hypersurface in $\mathbb{H}^4$ with Gauss-Kronecker curvature identically zero and a nowhere vanishing second fundamental form. In fact, consider a 2-dimensional, oriented, complete Riemannian manifold M^2 and suppose that $h : M^2 \to Q^3$ is a minimal isometric immersion without totally geodesic points, where Q^3 is a horosphere or an equidistant hypersurface of $\mathbb{H}^4$. The metric $\langle\ ,\ \rangle_{\widehat{h}}$ induced by $\widehat{h}$ is not complete. Indeed, if $\langle\ ,\ \rangle_{\widehat{h}}$ were complete, then by Myers' Theorem and the fact that its Gaussian curvature $K_{\widehat{h}}$ satisfies

$$K_{\widehat{h}} = 1 - \frac{K_{Q^3}}{K_{Q^3} - K} \geq 1,$$

M^2 would be compact. This is a contradiction, since there are no compact minimal surfaces in simply connected space forms of non-positive sectional curvature. Moreover according to Proposition 2.5, the metric induced on $\mathcal{N}^1(\widehat{h})$ by $\Psi_{\widehat{h}}$ is complete.

Remark 2.7. There are numerous examples of complete minimal hypersurfaces in $\mathbb{H}^4$ with Gauss-Kronecker curvature identically zero and unbounded scalar curvature. Indeed, suppose that Q^3 is a horosphere and $h : M^2 \to Q^3$ is a complete minimal immersion. Then the suspension of h is a complete hypersurface in $\mathbb{H}^4$. According to (2.9) its principal curvatures are

$$k_1(x,t) = -k_3(x,t) = \frac{k(x)}{\cosh t - \sinh t},\ k_2(x,t) = 0.$$

Because $\lim_{t\to\infty} k_1(x,t) = \infty$, it follows that the scalar curvature of the suspension must be unbounded. There are also plenty of complete minimal hypersurfaces in $\mathbb{H}^4$ with Gauss-Kronecker curvature zero and bounded scalar curvature. Indeed, there exist complete minimal surfaces in $\mathbb{H}^3$ with Gaussian curvature bounded from below (cf. [4]). The suspension of such surfaces are minimal hypersurfaces in $\mathbb{H}^4$ with Gauss-Kronecker curvature identically zero and bounded scalar curvature.

3. Local theory of minimal hypersurfaces in $\mathbb{H}^4$ with zero Gauss-Kronecker curvature

Let M^3 be a 3-dimensional, oriented Riemannian manifold and $f : M^3 \to \mathbb{H}^4$ an isometric minimal immersion. Denote by ξ a unit normal vector field along f with corresponding shape operator A and principal curvatures $k_1 \geq k_2 \geq k_3$. The *Gauss-Kronecker curvature* $\mathcal{K}$ of f and the *scalar curvature* τ of M^3 are given by

$$\mathcal{K} = k_1 k_2 k_3,\ \tau = -6 - \left(k_1^2 + k_2^2 + k_3^2\right).$$

Assume now that the second fundamental form of f is nowhere zero and that the Gauss-Kronecker curvature is identically zero. Then the principal curvatures are $k_1 = \lambda$, $k_2 = 0$, $k_3 = -\lambda$, where λ is a smooth positive function on M^3. We can choose locally an orthonormal frame field $\{e_1, e_2, e_3\}$ of principal directions corresponding to $\lambda, 0, -\lambda$. Let $\{\omega_1, \omega_2, \omega_3\}$ and $\{\omega_{ij}\}$, $i, j \in \{1, 2, 3\}$, be the dual frame and the connection forms. Hereafter we make the following convection on the ranges of indices

$$1 \leq i, j, k, \ldots \leq 3,$$

and adopt the method of moving frames. The structure equations are

$$\begin{aligned} d\omega_i &= \sum_j \omega_{ij} \wedge \omega_j, \quad \omega_{ij} + \omega_{ji} = 0, \\ d\omega_{ij} &= \sum_l \omega_{il} \wedge \omega_{lj} - (k_i k_j - 1)\, \omega_i \wedge \omega_j. \end{aligned}$$

Consider the functions

$$u := \omega_{12}(e_3), \quad v := e_2(\log \lambda),$$

which will play a crucial role in the sequel. From the structural equations and the Codazzi equations,

$$\begin{aligned} &e_i(k_j) = (k_i - k_j)\, \omega_{ij}(e_j), \ i \neq j, \\ &(k_1 - k_2)\, \omega_{12}(e_3) = (k_2 - k_3)\, \omega_{23}(e_1) = (k_1 - k_3)\, \omega_{13}(e_2), \end{aligned}$$

we easily get

$$\begin{array}{lll} \omega_{12}(e_1) = v, & \omega_{13}(e_1) = \frac{1}{2} e_3(\log \lambda), & \omega_{23}(e_1) = u, \\ \omega_{12}(e_2) = 0, & \omega_{13}(e_2) = \frac{1}{2} u, & \omega_{23}(e_2) = 0, \\ \omega_{12}(e_3) = u, & \omega_{13}(e_3) = -\frac{1}{2} e_1(\log \lambda), & \omega_{23}(e_3) = -v \end{array} \tag{3.1}$$

and

$$\begin{aligned} &e_2(v) = v^2 - u^2 - 1, \ e_1(u) = e_3(v), \\ &e_2(u) = 2uv, \ e_3(u) = -e_1(v). \end{aligned} \tag{3.2}$$

Furthermore, the above equations yield

$$\begin{aligned} &[e_1, e_2] = -v e_1 + \tfrac{1}{2} u e_3, \ [e_2, e_3] = \tfrac{1}{2} u e_1 + v e_3, \\ &[e_1, e_3] = -\tfrac{1}{2} e_3(\log \lambda)\, e_1 - 2u e_2 + \tfrac{1}{2} e_1(\log \lambda)\, e_3. \end{aligned} \tag{3.3}$$

Lemma 3.1. *The functions u and v are harmonic.*

Proof. Using (3.1), from the definition of the Laplacian we have

$$\begin{aligned} \Delta v &= e_1 e_1(v) + e_2 e_2(v) + e_3 e_3(v) - (\omega_{31}(e_3) + \omega_{21}(e_2))\, e_1(v) \\ &\quad - (\omega_{12}(e_1) + \omega_{32}(e_3))\, e_2(v) - (\omega_{13}(e_1) + \omega_{23}(e_2))\, e_3(v) \\ &= e_1 e_1(v) + e_2 e_2(v) + e_3 e_3(v) - \frac{1}{2} e_1(\log \lambda)\, e_1(v) - 2v e_2(v) \\ &\quad - \frac{1}{2} e_3(\log \lambda)\, e_3(v). \end{aligned}$$

In view of (3.2), we get

$$\begin{aligned} e_1 e_1(v) &= -e_1 e_3(u), \ e_3 e_3(v) = e_3 e_1(u), \\ e_2 e_2(v) &= 2v e_2(v) - 2u e_2(u). \end{aligned}$$

Due to (3.2), (3.3) and the previous relations, we obtain

$$\begin{aligned}\Delta v &= -e_1e_3(u)+e_3e_1(u)+2ve_2(v)-2ue_2(u)\\ &\quad -\frac{1}{2}e_1(\log\lambda)e_1(v)-2ve_2(v)-\frac{1}{2}e_3(\log\lambda)e_3(v)\\ &= \frac{1}{2}e_3(\log\lambda)e_1(u)+2ue_2(u)-\frac{1}{2}e_1(\log\lambda)e_3(u)\\ &\quad -2ue_2(u)-\frac{1}{2}e_1(\log\lambda)e_1(v)-\frac{1}{2}e_3(\log\lambda)e_3(v)\\ &= 0.\end{aligned}$$

In a similar way, we verify that $\Delta u = 0$. □

Lemma 3.2. *Let $\gamma : I \subset \mathbb{R} \to M^3$ be an integral curve of e_2 emanating from $x \in M^3$. Then γ is a geodesic of M^3 and $f \circ \gamma$ is a geodesic of $\mathbb{H}^4$. Moreover,*

$$\frac{1}{\lambda^2\circ\gamma(t)}=\frac{1}{2}\left(a(x)e^{2t}+b(x)+d(x)e^{-2t}\right)$$

and

$$v\circ\gamma(t)=-\frac{a(x)e^{2t}-d(x)e^{-2t}}{a(x)e^{2t}+b(x)+d(x)e^{-2t}},$$

where $a(x)$, $b(x)$, $d(x)$ are real constants depending only on x and $t \in I$.

Proof. By making use of (3.1) we, immediately, obtain $\nabla_{e_2}e_2 = 0$. Thus, γ is a geodesic of M^3 and the Gauss formula implies that $f \circ \gamma$ is a geodesic of $\mathbb{H}^4$. By virtue of (3.2), we easily get

$$e_2e_2e_2\left(\frac{1}{\lambda^2}\right)=4e_2\left(\frac{1}{\lambda^2}\right).$$

Restricting the last equation along γ and integrating, we deduce that

$$\frac{1}{\lambda^2\circ\gamma(t)}=\frac{1}{2}\left(a(x)e^{2t}+b(x)+d(x)e^{-2t}\right),$$

where $a(x)$, $b(x)$, $d(x)$ are real constants. Differentiating, we obtain

$$v\circ\gamma(t)=\frac{d}{dt}(\log\lambda\circ\gamma)(t)=-\frac{a(x)e^{2t}-d(x)e^{-2t}}{a(x)e^{2t}+b(x)+d(x)e^{-2t}},$$

and the proof is finished. □

We are now ready to give the local classification of minimal hypersurfaces in $\mathbb{H}^4$ with Gauss-Kronecker curvature zero and a nowhere vanishing second fundamental form, which is in fact the converse of Proposition 2.1(*ii*).

Proposition 3.3. *Let M^3 be a 3-dimensional, oriented, Riemannian manifold and $f : M^3 \to \mathbb{H}^4$ a minimal isometric immersion with unit normal vector field ξ, Gauss-Kronecker curvature identically zero and nowhere vanishing second fundamental form. Each point $x_0 \in M^3$ has a neighborhood U such that the quotient space V of leaves of the nullity distribution on U is a 2-dimensional differentiable manifold with quotient projection $\pi : U \to V$ and*

(*i*) *there exists a spacelike stationary immersion $g : V \to \mathbb{S}^4_1$ and an isometry $T : U \to \mathcal{N}^1(g)$ such that $g \circ \pi = \xi$ and $f = \Psi_g \circ T$ on U,* (*ii*) *the Gaussian curvature K of the metric induced by g on V and the normal curvature $K^{\perp}$ satisfy*

$$K\circ\pi = 1+\frac{1-u^2-v^2}{\lambda^2},\quad K^{\perp}\circ\pi = -\frac{2u}{\lambda^2}$$

on U.

Proof. Consider a coordinate system (x_1,x_2,x_3) on $U\subset M^3$, around x_0, such that $\frac{\partial}{\partial x_2}=e_2$. Denote by V the quotient space of leaves of the nullity distribution on U and by $\pi: U\to V$ the quotient projection. It is well known that V can be equipped with a structure of a 2-dimensional differentiable manifold which makes π a submersion (cf. [11]).

Our assumptions ensure that the unit normal vector field ξ remains constant along each leaf of the nullity distribution, and so we may define a smooth map $g: V\to\mathbb{S}^4_1$ so that $g\circ\pi=\xi$. We claim that g is a spacelike stationary immersion. Indeed, consider a smooth transversal S to the leaves of the nullity distribution, through a point $x\in U$ such that the frame $\{E_1:=e_1(x), E_3:=e_3(x)\}$ spans T_xS. Because π is a submersion, $\{d\pi(E_1), d\pi(E_3)\}$ constitute a base of $T_{\pi(x)}V$. Note that

$$dg\left(d\pi\left(E_1\right)\right)=-\lambda\left(x\right)df\left(E_1\right)\ \text{ and } dg\left(d\pi\left(E_3\right)\right)=\lambda\left(x\right)df\left(E_3\right).$$

Thus g is a spacelike immersion and $\left\{X_1:=\frac{1}{\lambda(x)}d\pi\left(E_1\right), X_3:=\frac{1}{\lambda(x)}d\pi\left(E_3\right)\right\}$ is an orthonormal base at $\pi(x)$ with respect to the metric induced by g. Let $\{\eta_3,\eta_4\}$ be an orthonormal frame field in the normal bundle of g such that $\eta_3\circ\pi=df(e_2)$ and $\eta_4\circ\pi=f$ on S. Then bearing in mind the Gauss formula and (3.1) we obtain

$$\begin{aligned} d\eta_3\left(X_1\right) &= -\frac{v(x)}{\lambda(x)}df\left(E_1\right)+\frac{u(x)}{\lambda(x)}df\left(E_3\right),\\ d\eta_3\left(X_3\right) &= -\frac{u(x)}{\lambda(x)}df\left(E_1\right)-\frac{v(x)}{\lambda(x)}df\left(E_3\right),\\ d\eta_4\left(X_1\right) &= \frac{1}{\lambda(x)}df\left(E_1\right)\ \text{ and }\ d\eta_4\left(X_3\right)=\frac{1}{\lambda(x)}df\left(E_3\right).\end{aligned}$$

Denote by A_3, A_4 the shape operators of g at $\pi(x)$ corresponding to the normal directions η_3 and η_4. Taking into account the above relations, from Weingarten formulas it follows that at $\pi(x)$ we have

$$A_3\sim\frac{1}{\lambda(x)}\begin{pmatrix} -v(x) & -u(x)\\ -u(x) & v(x)\end{pmatrix},\quad A_4\sim\frac{1}{\lambda(x)}\begin{pmatrix}1&0\\0&-1\end{pmatrix},\tag{3.4}$$

with respect to the orthonormal base $\{X_1,X_3\}$. So the immersion $g: V\to\mathbb{S}^4_1$ is a stationary immersion. Moreover we have $f=\Psi_g\circ T$ on U, where the map $T: U\to\mathcal{N}^1(g)$ is defined by $T(x)=(\pi(x), f(x))$, $x\in U$. By restricting U, if necessary, T is an isometry because $\mathcal{N}^1(g)$ is equipped with the metric induced by Ψ_g.

Part (*ii*) follows immediately from (3.4). □

4. Complete minimal hypersurfaces in $\mathbb{H}^4$ with vanishing Gauss-Kronecker curvature

The purpose of this section is to classify complete minimal hypersurfaces in $\mathbb{H}^4$ with Gauss-Kronecker curvature identically zero and a nowhere zero second fundamental form, under the assumption that the scalar curvature is bounded from below. More precisely, we shall prove the following.

Theorem. *Let M^3 be a 3-dimensional, oriented, complete Riemannian manifold whose scalar curvature is bounded from below and $f : M^3 \to \mathbb{H}^4$ a minimal isometric immersion with Gauss-Kronecker curvature identically zero and nowhere zero second fundamental form. Then there exist a minimal isometric immersion $h : M^2 \to Q^3$, without totally geodesic points, of a complete 2-dimensional oriented Riemannian manifold M^2 into an equidistant hypersurface Q^3 of $\mathbb{H}^4$ and a local isometry $T : M^3 \to \mathcal{N}^1(\widehat{h})$ such that $f = \Psi_{\widehat{h}} \circ T$.*

The proof of our result relies heavily on the well known Generalized Maximum Principle due to Omori and Yau ([9],[13]):

Generalized Maximum Principle. *Let M be a complete Riemannian manifold whose Ricci curvature is bounded from below. If φ is a C^2-function on M bounded from above, then there exists a sequence $\{x_n\}$ of points of M such that*

$$\lim \varphi(x_n) = \sup \varphi, \quad |\nabla \varphi|(x_n) \leq \frac{1}{n} \text{ and } \Delta \varphi(x_n) \leq \frac{1}{n},$$

for each $n \in \mathbb{N}$, where ∇, Δ stand for the gradient and the Laplacian operator.

The following lemma is essentially a consequence of a result proved by Cheng and Yau [5, Theorem 8]. For the reader's convenience we shall give a short proof.

Lemma 4.1. *Let M be a complete Riemannian manifold whose Ricci curvature is bounded from below, and φ a C^2-solution of the differential inequality*

$$\Delta \varphi \geq 2\varphi^2.$$

Then φ is bounded from above and $\sup \varphi = 0$.

Proof. We suppose to the contrary that $\sup \varphi = \infty$. Then there exists a point $x_0 \in M$ such that $\varphi(x_0) \geq 2$. Consider a C^2-positive increasing function $F : \mathbb{R} \to \mathbb{R}$ which for $t \geq 2$ is given by $F(t) = 2\left(1 - t^{-1/2}\right)$. The function $\Phi = F \circ \varphi$ is bounded from above, since $\Phi \leq 2$. Appealing to the Generalized Maximum Principle, we deduce that there exists a sequence $\{x_n\}$ such that

$$\lim \Phi(x_n) = \sup \Phi, \quad |\nabla \Phi|(x_n) \leq \frac{1}{n} \text{ and } \Delta \Phi(x_n) \leq \frac{1}{n},$$

for each $n \in \mathbb{N}$. For n large enough we have $\varphi(x_n) \geq 2$. Hence, estimating at x_n we get

$$|\nabla \Phi|(x_n) = F'(\varphi(x_n))\, |\nabla \varphi|(x_n) = \varphi^{-3/2}(x_n)\, |\nabla \varphi|(x_n) \leq \frac{1}{n} \tag{4.1}$$

and

$$\Delta \Phi(x_n) = \varphi^{-3/2}(x_n)\, \Delta \varphi(x_n) - \frac{3}{2}\varphi^{-5/2}(x_n)\, |\nabla \varphi|^2(x_n) \leq \frac{1}{n}. \tag{4.2}$$

Combining (4.2) with (4.1), and bearing in mind that $\Delta \varphi \geq 2\varphi^2$, we obtain

$$2 - \frac{3}{2}\varphi^{-3}(x_n)\, |\nabla \varphi|^2(x_n) \leq \frac{1}{n}\varphi^{-1/2}(x_n).$$

Letting $n \to \infty$, we get a contradiction. Therefore φ must be bounded from above. Appealing again to the Generalized Maximum Principle, and bearing in mind that $\Delta \varphi \geq 2\varphi^2$, we infer that $\sup \varphi = 0$. $\square$

Proof of the Theorem. Let A be the shape operator associated with a unit normal ξ. Then the principal curvatures of f are $k_1 = \lambda$, $k_2 = 0$, $k_3 = -\lambda$, where λ is a smooth positive function on M^3. It is well known that the nullity distribution $\mathcal{D} = \ker A$ is smooth. We distinguish two cases.

Case 1. We assume that there exists a global unit section e_2 of $\mathcal{D}$. Then the function $v = e_2(\log\lambda)$ is globally defined and smooth. Around each point $x \in M^3$ we may choose a neighborhood U_x of x, vector fields e_1, e_3 such that the orthonormal frame field $\{e_1, e_2, e_3\}$ gives the right orientation of M^3, and $Ae_1 = \lambda e_1$, $Ae_3 = -\lambda e_3$ on U_x. If for another point $\overline{x} \in M^3$ with a corresponding neighborhood $U_{\overline{x}}$ and an orthonormal frame field $\{\overline{e}_1, e_2, \overline{e}_3\}$ chosen as before, we have $U_x \cap U_{\overline{x}} \neq \emptyset$, then either $\overline{e}_1 = e_1$ and $\overline{e}_3 = e_3$ or $\overline{e}_1 = -e_1$ and $\overline{e}_3 = -e_3$ on $U_x \cap U_{\overline{x}}$. Thus $\langle \nabla_{e_3} e_1, e_2\rangle = \langle \nabla_{\overline{e}_3}\overline{e}_1, e_2\rangle$ on $U_x \cap U_{\overline{x}}$ and so the local function u introduced in Section 3 can be extended to a smooth global one.

Our assumptions imply that the Ricci curvature of M^3 is bounded from below. Making use of (3.2) and the harmonicity of u and v (Lemma 3.1), we obtain

$$\begin{aligned} \frac{1}{2}\Delta\left(u^2 + v^2 - 1\right) &= |\nabla u|^2 + |\nabla v|^2 \\ &\geq (e_2(u))^2 + (e_2(v))^2 \\ &= 4u^2v^2 + \left(v^2 - u^2 - 1\right)^2 \\ &\geq \left(u^2 + v^2 - 1\right)^2. \end{aligned}$$

Then, by virtue of Lemma 4.1, we have $\sup\left(u^2 + v^2 - 1\right) = 0$, which implies $u^2 + v^2 \leq 1$.

Claim: $u \equiv 0$. At first we shall prove that $v^2 < 1$. Arguing indirectly, we assume that there exists a point $x_0 \in M^3$ such that $|v(x_0)| = 1$. The harmonicity of v and the maximum principle imply either $v \equiv 1$ or $v \equiv -1$. Then Lemma 3.2 yields $a(x_0) = b(x_0) = 0$ or $b(x_0) = d(x_0) = 0$, respectively, and thus $\lambda^2 \circ \gamma(t)$, $t \in \mathbb{R}$, is unbounded, where γ is the integral curve of e_2 emanating from the point x_0. This contradicts our assumption on the scalar curvature. So $v^2 < 1$. It is obvious from Lemma 3.2 that on each integral curve of e_2, the function v changes sign only once.

Consider, now, the set $v^{-1}(0)$. From (3.2) we have $e_2(v) = v^2 - u^2 - 1 < 0$. Hence 0 is a regular value of v and thus $v^{-1}(0)$ is an oriented and connected 2-dimensional submanifold of M^3. The map $\rho : v^{-1}(0) \times \mathbb{R} \to M^3$ defined by $\rho(x,t) := \exp_x(te_2(x))$, where $\exp_x$ denotes the exponential map of M^3 based on the point $x \in v^{-1}(0)$, is a diffeomorphism. Appealing to Lemma 3.2, we have

$$v \circ \rho(x,t) = -\frac{a(x)e^{2t} - d(x)e^{-2t}}{a(x)e^{2t} + b(x) + d(x)e^{-2t}},$$

where $a(x), b(x), d(x)$ are smooth functions on $v^{-1}(0)$. Since $v \circ \rho(x,0) = 0$, we obtain $a(x) = d(x)$ for each $x \in v^{-1}(0)$. Hence,

$$\frac{1}{\lambda^2 \circ \rho(x,t)} = a(x)\cosh 2t + \frac{b(x)}{2} \tag{4.3}$$

and

$$v \circ \rho(x,t) = -\frac{2a(x)\sinh 2t}{2a(x)\cosh 2t + b(x)} = -a(x)\sinh 2t\ \lambda^2 \circ \rho(x,t). \tag{4.4}$$

From (4.3), (4.4) and in view of $e_2(v) = v^2 - u^2 - 1 < 0$, we deduce that $a(x) > 0$ for each $x \in v^{-1}(0)$. Now consider the function $\phi : v^{-1}(0) \times \mathbb{R} \to \mathbb{R}$, $\phi(x,t) = \tanh t$. Since $d\rho\left(\frac{\partial}{\partial t}\right) = e_2$, we have

$$e_2\left(\phi \circ \rho^{-1}\right) = 1 - \phi^2 \circ \rho^{-1}. \tag{4.5}$$

Differentiating (4.4) with respect to $\frac{\partial}{\partial t}$ and making use of (3.2) and (4.4) we obtain

$$\frac{\phi \circ \rho^{-1}}{1 + \phi^2 \circ \rho^{-1}} = \frac{-v}{1 + u^2 + v^2}. \tag{4.6}$$

Obviously we have $v\left(\phi \circ \rho^{-1}\right) \le 0$. The function $G := u^2 + \left(v + \phi \circ \rho^{-1}\right)^2$ is smooth and bounded from above. Appealing to the Generalized Maximum Principle, there exists a sequence $\{x_n\}$ of points in M^3 such that

$$\lim G(x_n) = \sup G, \ \ |\nabla G|(x_n) \le \frac{1}{n} \text{ and } \Delta G(x_n) \le \frac{1}{n},$$

for each $n \in \mathbb{N}$. Because the functions u, v and $\phi \circ \rho^{-1}$ are bounded, without loss of generality, we may assume that

$$\lim u(x_n) = u_0, \ \ \lim v(x_n) = v_0 \text{ and } \lim \phi \circ \rho^{-1}(x_n) = \phi_0,$$

where u_0, v_0 and ϕ_0 are real numbers. Using the equations (3.2), (4.5) and the harmonicity of u and v, we readily see that

$$\begin{aligned} \frac{1}{2} e_2(G) &= u e_2(u) + \left(v + \phi \circ \rho^{-1}\right)\left(e_2(v) + e_2\left(\phi \circ \rho^{-1}\right)\right) \\ &= 2u^2 v + \left(v + \phi \circ \rho^{-1}\right)\left(v^2 - u^2 - \phi^2 \circ \rho^{-1}\right) \\ &= \left(v - \phi \circ \rho^{-1}\right) G \end{aligned} \tag{4.7}$$

and

$$\begin{aligned} \frac{1}{2}\Delta G &= |\nabla u|^2 + \left(v + \phi \circ \rho^{-1}\right)\Delta\left(\phi \circ \rho^{-1}\right) + \left|\nabla\left(v + \phi \circ \rho^{-1}\right)\right|^2 \\ &\ge 4u^2 v^2 + \left(v + \phi \circ \rho^{-1}\right)\Delta\left(\phi \circ \rho^{-1}\right) \\ &\quad + \left(e_2(v) + e_2\left(\phi \circ \rho^{-1}\right)\right)^2 \\ &= 4u^2 v^2 + \left(v + \phi \circ \rho^{-1}\right)\Delta\left(\phi \circ \rho^{-1}\right) \\ &\quad + \left(v^2 - u^2 - \phi^2 \circ \rho^{-1}\right)^2. \end{aligned} \tag{4.8}$$

Estimating at x_n and letting $n \to \infty$, the equation (4.7) yields

$$(v_0 - \phi_0)\sup G = 0.$$

If $v_0 \ne \phi_0$ we obtain $\sup G = 0$, which proves our claim.

Suppose now that $v_0 = \phi_0$. Then, because of $v\left(\phi \circ \rho^{-1}\right) \le 0$, we get $v_0 = \phi_0 = 0$. Making use of (4.6) and of the harmonicity of the functions u and v, a straightforward computation shows that

$$\frac{1-\phi^2\circ\rho^{-1}}{\left(1+\phi^2\circ\rho^{-1}\right)^2}\nabla\left(\phi\circ\rho^{-1}\right)=\frac{2uv}{\left(1+u^2+v^2\right)^2}\nabla u-\frac{1+u^2-v^2}{\left(1+u^2+v^2\right)^2}\nabla v \tag{4.9}$$

and

$$\begin{aligned}\frac{1-\phi^2\circ\rho^{-1}}{\left(1+\phi^2\circ\rho^{-1}\right)^2}\Delta\left(\phi\circ\rho^{-1}\right)=&\frac{2v\left(1-3u^2+v^2\right)}{\left(1+u^2+v^2\right)^3}\left|\nabla u\right|^2\\&+\frac{4u\left(1+u^2-3v^2\right)}{\left(1+u^2+v^2\right)^3}\left\langle\nabla u,\nabla v\right\rangle+\frac{2v\left(3+3u^2-v^2\right)}{\left(1+u^2+v^2\right)^3}\left|\nabla v\right|^2\\&+2\left(\phi\circ\rho^{-1}\right)\frac{3-\phi^2\circ\rho^{-1}}{\left(1+\phi^2\circ\rho^{-1}\right)^3}\left|\nabla\left(\phi\circ\rho^{-1}\right)\right|^2.\end{aligned} \tag{4.10}$$

Since u, v are bounded harmonic functions and M^3 has Ricci curvature bounded from below, by a result due to Yau [13, Theorem 3″], it follows that the functions $\left|\nabla u\right|^2$ and $\left|\nabla v\right|^2$ are also bounded. Hence, from (4.9) and (4.10) we deduce that the sequence $\left\{\Delta\left(\phi\circ\rho^{-1}\right)(x_n)\right\}$ is bounded. Estimating at x_n and passing to the limit, from (4.8) we get $u_0=0$. So $\sup G=0$, because of $v_0=\phi_0=0$. Thus $G\equiv 0$ and consequently $u\equiv 0$, which completes the proof of our claim.

It can be easily seen that the quotient space M^2 of leaves of the nullity distribution can be identified with the manifold $v^{-1}(0)$ via the diffeomorphism ρ. Hence, M^2 inherits in a natural way the structure of a 2-dimensional manifold that makes the quotient projection $\pi : M^3\to M^2$ a submersion. Thus appealing to Proposition 3.3, there exists a spacelike stationary immersion $g : M^2\to\mathbb{S}^4_1$ and an isometry $T : M^3\to\mathcal{N}^1(g)$, defined by $T(x)=(\pi(x),f(x))$, $x\in M^3$, such that $g\circ\pi=\xi$ and $f=\Psi_g\circ T$. From the second part of Proposition 3.3 it follows that the normal curvature of g is identically zero and the Gaussian curvature K of the metric induced by g satisfies $K>1$. So g has no totally geodesic points. Consequently, by virtue of Propositions 2.3 and 2.4, we deduce that g is the associate of a minimal immersion $h : M^2\to Q^3$, where Q^3 is an equidistant hypersurface of $\mathbb{H}^4$.

Case 2. Assume now that the nullity distribution of f doesn't allow a global unit section. We can pick out two unit vectors $e_2(x)$, $-e_2(x)\in\mathcal{D}(x)$, for each $x\in M^3$. Then we can construct a 2-fold covering space $\widetilde{M}^3$ of M^3 with covering map $\Pi:\widetilde{M}^3\to M^3$ by choosing the two points in $\Pi^{-1}(x)$ to correspond to these vectors. One can easily check that $\widetilde{M}^3$ is a connected and oriented manifold. Now we equip $\widetilde{M}^3$ with the covering metric and consider the isometric immersion $\widetilde{f}:=f\circ\Pi:\widetilde{M}^3\to\mathbb{H}^4$ with unit normal $\widetilde{\xi}:=\xi\circ\Pi$. Obviously $d\Pi$ preserves the principal directions, and the principal curvatures of $\widetilde{f}$ are $\widetilde{k}_1=-\widetilde{k}_3=\widetilde{\lambda}:=\lambda\circ\Pi$, $\widetilde{k}_2=0$. We can readily verify that there exists a global unit vector field $\widetilde{e}_2$ which spans the nullity distribution $\widetilde{\mathcal{D}}$ of $\widetilde{f}$. It is clear that $\widetilde{f}$ satisfies all the assumptions of Case 1. Moreover there exists a deck transformation $a:\widetilde{M}^3\to\widetilde{M}^3$ which is in fact an involution and $\Pi^{-1}(x)=\{\widetilde{x},a(\widetilde{x})\}$, for each $x\in M^3$.

The deck transformation a induces an involution $\widetilde{a}$ on the quotient space $\widetilde{M}^2$ of leaves of $\widetilde{\mathcal{D}}$ in a natural way. Since for each $\widetilde{x}\in\widetilde{M}^3$ there is no integral curve of $\widetilde{e}_2$ joining $\widetilde{x}$ with $a(\widetilde{x})$, the involution $\widetilde{a}$ is fixed point free. We denote by $\widetilde{\pi}:\widetilde{M}^3\to\widetilde{M}^2$ the quotient projection. The quotient space $\widetilde{M}^2/\widetilde{a}$ can be equipped with the structure of a 2-dimensional manifold which makes the projection $\widetilde{\pi}_{\widetilde{a}}:\widetilde{M}^2\to\widetilde{M}^2/\widetilde{a}$

a covering map. The map $q : \widetilde{M}^2/\widetilde{a} \to M^2$ given by

$$q \circ \widetilde{\pi}_{\widetilde{a}} \circ \widetilde{\pi} = \pi \circ \Pi,$$

is well defined and a bijection, where M^2 is the quotient space of leaves of $\mathcal{D}$ and $\pi : M^3 \to M^2$ is the quotient projection. Using the map q we can equip M^2 with the structure of a 2-dimensional differentiable manifold which makes q a diffeomorphism and π a submersion. Thus we may identify $\widetilde{M}^2/\widetilde{a}$ with M^2.

From Case 1, we know that the map $\widetilde{g} : \widetilde{M}^2 \to \mathbb{S}^4_1$, which is induced by $\widetilde{\xi}$, is a spacelike stationary immersion without totally geodesic points. Furthermore, there exists a minimal immersion $\widetilde{h} : \widetilde{M}^2 \to Q^3$, where Q^3 is an equidistant hypersurface of $\mathbb{H}^4$, such that $\widetilde{g}$ coincides with the associate of $\widetilde{h}$ and $\widetilde{f} = \Psi_{\widetilde{g}} \circ \widetilde{T}$, where $\widetilde{T}$ is the isometry given by $\widetilde{T}(\widetilde{x}) = (\widetilde{\pi}(\widetilde{x}), \widetilde{f}(\widetilde{x}))$, $\widetilde{x} \in \widetilde{M}^3$. Bearing in mind Propositions 2.3, 2.4 and taking into account (3.4) we easily see that

$$\widetilde{h} \circ \widetilde{\pi} = \frac{1}{\sqrt{1-\widetilde{v}^2}} \left(\widetilde{v} d\widetilde{f}(\widetilde{e}_2) + \widetilde{f} \right),$$

where $\widetilde{v} = \widetilde{e}_2(\log \widetilde{\lambda})$. Since a is a deck transformation, the maps $g : M^2 \to \mathbb{S}^4_1$ and $h : M^2 \to Q^3$ given by

$$g \circ \widetilde{\pi}_{\widetilde{a}} = \widetilde{g} \quad \text{and} \quad h \circ \widetilde{\pi}_{\widetilde{a}} = \widetilde{h},$$

are well defined. Then g is the associate of h, since $\widetilde{g}$ is the associate of $\widetilde{h}$. Moreover $f = \Psi_{\widehat{h}} \circ T$, where $T : M^3 \to \mathcal{N}^1(\widehat{h})$ is the local isometry given by $T(x) = (\pi(x), f(x))$, $x \in M^3$. This completes the proof. □

Remark 4.2. It should be interesting to know whether a similar classification result can be obtained without the assumption that the scalar curvature is bounded from below.

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