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HÖLDER REGULARITY OF THE NORMAL DISTANCE WITH AN APPLICATION TO A PDE MODEL FOR GROWING SANDPILES

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ABSTRACT. Given a bounded domain Ω in $\mathbb{R}^2$ with smooth boundary, the *cut locus* $\overline{\Sigma}$ is the closure of the set of nondifferentiability points of the distance d from the boundary of Ω. The normal distance to the cut locus, $\tau(x)$, is the map which measures the length of the line segment joining x to the cut locus along the normal direction $Dd(x)$, whenever $x \notin \overline{\Sigma}$. Recent results show that this map, restricted to boundary points, is Lipschitz continuous, as long as the boundary of Ω is of class $C^{2,1}$. Our main result is the global Hölder regularity of τ in the case of a domain Ω with analytic boundary. We will also show that the regularity obtained is optimal, as soon as the set of the so-called *regular conjugate points* is nonempty. In all the other cases, Lipschitz continuity can be extended to the whole domain Ω. The above regularity result for τ is also applied to derive the Hölder continuity of the solution of a system of partial differential equations that arises in granular matter theory and optimal mass transfer.

1. Introduction

It is not unusual in mathematics that an abstract result applies to more concrete problems. This is the case of the present work, where a regularity result on a map that arises in differential geometry gives information on the regularity of solutions of a system of partial differential equations that comes from granular matter theory. The "abstract" result we are referring to is the global Hölder regularity of the so-called *normal distance to the cut locus*. To introduce the problem, let Ω be a bounded connected domain in $\mathbb{R}^2$ with regular boundary. We shall denote by $d(\cdot) : \overline{\Omega} \to \mathbb{R}$ the distance function from the boundary $\partial\Omega$, i.e.

$$d(x) = \min_{y \in \partial\Omega} |y - x| \qquad \forall\, x \in \overline{\Omega}\,.$$

The set of points $x \in \Omega$ at which d is not differentiable will be called the *singular set* of d and denoted by Σ. Its closure is often referred to as the *cut locus*. Now, the *normal distance* to $\overline{\Sigma}$ is defined as

$$(1) \qquad \tau(x) = \begin{cases} \min\left\{t \geq 0 \ : \ x + tDd(x) \in \overline{\Sigma}\right\} & \forall x \in \overline{\Omega}\setminus\Sigma, \\ 0 & \forall x \in \overline{\Sigma}\,. \end{cases}$$

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The regularity of τ when restricted to boundary points can be deduced either from the work of Itoh and Tanaka [11], who investigated the case of C^∞ smooth submanifolds of an n-dimensional smooth manifold, or from the paper of Li and Nirenberg [12], that treats the case of the normal distance in the context of Finsler geometry. In both papers, Lipschitz regularity has been proved for $C^{2,1}$ boundaries in any space dimension and the result is sharp in the sense that neither the regularity of the boundary can be weakened, nor better regularity can be obtained for τ. The regularity of τ on $\partial\Omega$ gives a local regularity result up to the cut locus, as it can be easily seen by describing τ in the interior of Ω as the composition of the map restricted to boundary points with the projection map. A natural question is which kind of global regularity we can expect. It is easy to prove that τ is continuous in $\overline{\Omega}$, as soon as Ω is of class C^2 (see [4, Lemma 2.14]). On the other hand, examining the epigraph of a parabola, one realizes that in this case τ is at most Hölder continuous with exponent 2/3 (see Example 2.1 below).

In this work we prove that for smooth boundaries, the "parabola" is the worst case, in the sense that the normal distance to the cut locus is at least Hölder continuous with exponent 2/3 in the whole set Ω. Also, we will show that the loss of Lipschitz regularity does not occur at singular points, but around the set of the so-called *regular conjugate points.* As it is well known, such an exceptional set is in the *closure* of the singular set and is defined as

$$\Gamma = \{x \in \Omega \backslash \Sigma \ : \ d(x)\kappa(x) = 1\},$$

where $\kappa(x)$ denotes the curvature of $\partial\Omega$ at the projection point of x.

Let us be more precise about our regularity request upon the boundary of Ω: for smooth boundaries we intend "as smooth as possible", that is, analytic. Indeed our argument strongly relies on the analysis of the cut locus of Ω, that we want to have the structure of a geometric graph, i.e. a connected set made of a finite numbers of edges and vertices. To our best knowledge, this structure is guaranteed only in the case of analytic boundaries. Moreover, for less regular boundaries, even of class C^∞, no precise structural theorems for the cut locus are available. Of course, the regularity result we obtain for analytic boundaries could hold true also for less regular domains. Nevertheless, the result is optimal as regards the regularity obtained, as the parabola case shows.

A brief description of the techniques we use is now in order. At the beginning, our approach relies on the analysis of [2] and [1], describing the propagation of singularities of semi-concave functions, together with the structural theorems on the cut locus of [6], which permit us to estimate the local Lipschitz constant of τ in the set $\Omega \setminus (\Sigma \cup \mathcal{S})$, where

$$\mathcal{S} := \bigcup_{x_0 \in \Gamma} \{x_0 - tDd(x_0) \ : \ t \in [0, d(x_0)]\}.$$

Afterwards, we make use of the local expansion in power series of the boundary to obtain a fine description of the "explosion speed" of this local Lipschitz constant with respect to the distance from the set $\Sigma \cup \mathcal{S}$. This is the crucial part of the proof, which enables us to conclude the Hölder continuity of τ by means of a simple regularity lemma.

The second part of this work concerns the application to a problem in granular matter theory. Indeed, the system of partial differential equations

$$(2)\qquad \begin{cases} -\mathrm{div}(vDu) = f & \text{in } \Omega, \\ v \geq 0,\ |Du| \leq 1 & \text{in } \Omega, \\ |Du| - 1 = 0 & \text{in } \{v > 0\}, \end{cases}$$

complemented with the conditions

$$(3)\qquad \begin{cases} u \geq 0, & \text{in } \Omega, \\ u \equiv 0 & \text{on } \partial\Omega, \end{cases}$$

can be viewed as a model for the equilibrium configuration that may occur to a growing sandpile generated by a stationary source $f \geq 0$ onto a table $\Omega \subset \mathbb{R}^2$ (see [3] or [10]). We observe that problem (2)–(3) also appears, with different requests on data, in the so-called Monge–Kantorovich mass transfer problem (see [8]).

In [4], the first two authors of this paper obtained a representation formula for the (unique) solution of problem (2)–(3) by defining a suitable notion of solutions. In particular, they found that, in the case when $\Omega \subset \mathbb{R}^2$ is a bounded domain with boundary of class $\mathcal{C}^2$ and $f \geq 0$ is a continuous function in Ω, the unique *continuous* solution of system (2)–(3) is given by the pair (d, v_f), where d is the distance function from $\partial\Omega$, $v_f = 0$ on $\overline{\Sigma}$ and

$$(4)\qquad v_f(x) = \begin{cases} \displaystyle\int_0^{\tau(x)} f(x + tDd(x)) \frac{1 - (d(x) + t)\kappa(x)}{1 - d(x)\kappa(x)}\, dt & \forall x \in \overline{\Omega}\backslash\overline{\Sigma}, \\ 0 & \forall x \in \overline{\Sigma}. \end{cases}$$

Here, $\overline{\Sigma}$ is the cut locus of Ω, $\kappa(x)$ denotes the curvature of $\partial\Omega$ at the projection point of $x \notin \Sigma$ and $\tau(\cdot)$ is the normal distance to $\overline{\Sigma}$. Such a representation formula was then extended to general n-dimensional problems in [5].

Since the regularity of the distance function is well known, we will focus on the analysis of the regularity of v_f only. We will show that v_f is a Hölder continuous function on Ω under the assumptions that f is Lipschitz continuous in Ω and Ω is a bounded domain of $\mathbb{R}^2$ with analytic boundary, different from a disk. The proof follows essentially the argument used for τ, apart from the initial estimate for the local Lipschitz constant of v_f (far away from conjugate points), which is obtained directly from the representation formula (4).

The paper is organized as follows. Sections 2 and 3 are concerned with notations and preliminary results. Section 4 is devoted to the proof of the regularity result on τ. Finally, Section 5 contains our application to the regularity of the solution of system (2)–(3).

2. Preliminaries

In this section we introduce the basic notations used in this paper and recall some preliminary results, mainly concerning properties of the distance function and of its singular sets. The reader is referred, e.g., to [4] for details and proofs.

In what follows we denote by $\langle\cdot,\cdot\rangle$ and $|\cdot|$ the Euclidean scalar product and norm in $\mathbb{R}^2$, respectively. For any $x \in \mathbb{R}^2$ and $r > 0$, $B_r(x)$ stands for the open ball with center x and radius r. For any pair $x, y \in \mathbb{R}^2$ we denote by $]x, y[$ and $[x, y]$, respectively, the open and closed line segment of extreme points x and y.

For any given set $K \subset \mathbb{R}^2$ we define $\mathrm{diam} K = \sup\{|y - x| \ :\ x, y \in K\}$.

For any measurable set $A \subset \mathbb{R}^2$ and any bounded measurable function $u : A \to \mathbb{R}$ the quantity $\|u\|_{\infty,A}$ stands for the essential supremum of u in A.

Let Ω be a bounded domain in $\mathbb{R}^2$ with analytic boundary $\partial\Omega$, different from a disk. We shall denote by $d(\cdot) : \overline{\Omega} \to \mathbb{R}$ the distance function from the boundary $\partial\Omega$, i.e.

$$d(x) = \min_{y \in \partial\Omega} |y - x| \qquad \forall\, x \in \overline{\Omega}\,.$$

The set of points $x \in \Omega$ at which d is not differentiable will be called the *singular set* of d and denoted by Σ. Such a set is also referred to as the *ridge*. Introducing the projection $\Pi(x)$ of x onto $\partial\Omega$ in the usual way,

$$\Pi(x) = \{y \in \partial\Omega \mid |y - x| = d(x)\}, \qquad \forall\, x \in \overline{\Omega},$$

it turns out that Σ can be viewed as the set of points x at which $\Pi(x)$ is not a singleton. All points $x \in \Omega \setminus \Sigma$ will be called *regular*.

Hereafter, for any $y \in \partial\Omega$, we denote by $\kappa(y)$ the curvature of $\partial\Omega$ at y under the sign convention $\kappa \geq 0$ if Ω is convex. Also, we will label in the same way the extension of κ to $\Omega \setminus \Sigma$ given by

$$\kappa(x) = \kappa(\Pi(x)) \qquad \forall\, x \in \Omega \setminus \Sigma\,.$$

We recall that for any $x \in \Omega$ and for any $y \in \Pi(x)$ we have

$$d(x)\kappa(y) \leq 1. \tag{5}$$

Now, let us introduce the sets of *regular conjugate points* and *singular conjugate points*—respectively Γ and $\tilde{\Gamma}$—as

$$\Gamma = \{x \in \Omega\backslash\Sigma \ : \ d(x)\kappa(x) = 1\,\}\ ,$$

$$\tilde{\Gamma} = \{x \in \Sigma \ : \ d(x)\kappa(y) = 1 \ \text{for some } y \in \Pi(x)\,\}\ .$$

Notice that a point $x \in \Omega\backslash\Sigma$ belongs to Γ if and only if

$$\Pi(x) = \left\{x - \frac{1}{\kappa(x)}\, Dd(x)\right\}\ .$$

Under the above assumptions on Ω, the following properties hold true:

- $\overline{\Sigma} \subset \Omega$ and $\overline{\Sigma} = \Sigma \cup \Gamma$;
- if $x \in \Omega$ is regular and not conjugate, then $d(x)\kappa(x) < 1$ and
$$D^2 d(x) = -\,\frac{\kappa(x)}{1 - \kappa(x)d(x)}\; q \otimes q \tag{6}$$
where q is any unit vector such that $\langle q, Dd(x)\rangle = 0$.

In the above equation, $p \otimes q$ stands for the tensor product of two vectors $p, q \in \mathbb{R}^2$, defined as $(p \otimes q)(x) = p\,\langle q, x\rangle\ ,\ \forall x \in \mathbb{R}^2$.

Let u be a Lipschitz continuous function in some domain $\omega \subset \mathbb{R}^2$. As usual, the super- and subdifferentials of u at some point $x \in \omega$ are the sets

$$D^+u(x) = \left\{p \in \mathbb{R}^2 \mid \limsup_{h \to 0} \frac{u(x+h) - u(x) - \langle p, h\rangle}{|h|} \leq 0\right\}$$

and

$$D^-u(x) = \left\{p \in \mathbb{R}^2 \mid \liminf_{h \to 0} \frac{u(x+h) - u(x) - \langle p, h\rangle}{|h|} \geq 0\right\}\ .$$

The set of limiting gradients of u at x is defined as follows:

$$D^*u(x) = \{p \in \mathbb{R}^2 \mid \exists x_n \to x,\ \exists Du(x_n) \to p\}\ .$$

In the case of the distance function, it turns out that

$$D^*d(x) = \left\{ \frac{x-y}{|x-y|} \mid y \in \Pi(x) \right\},$$

or, equivalently,

$$\Pi(x) = x - d(x)D^*d(x), \tag{7}$$

and $D^+d(x) = \mathrm{co}D^*d(x)$.

In the sequel we will also set

$$\Sigma^1 = \{x \in \Sigma \ : \ \dim D^+d(x) = 1\}$$

and

$$\Sigma^2 = \{x \in \Sigma \ : \ \dim D^+d(x) = 2\}.$$

Following [4], we introduce the map

$$\tau(x) = \begin{cases} \min\left\{t \geq 0 \ : \ x + tDd(x) \in \overline{\Sigma}\right\} & \forall x \in \overline{\Omega}\backslash\overline{\Sigma}, \\ 0 & \forall x \in \overline{\Sigma}\,, \end{cases} \tag{8}$$

sometimes called the *maximal retraction length of* Ω *onto* $\overline{\Sigma}$ or *normal distance* to $\overline{\Sigma}$, which plays a major role in our analysis. If Ω has a $C^{2,1}$ boundary, then τ has the following properties:

(i) τ is continuous in $\overline{\Omega}$;
(ii) τ is Lipschitz continuous on $\partial\Omega$;
(iii) τ is locally Lipschitz continuous in $\Omega \setminus \overline{\Sigma}$.

Clearly, property (iii) above follows from (ii) and from the smoothness of the projection onto $\partial\Omega$. Property (ii) is proven in [11] for boundaries of class C^∞, and in [12] for the $C^{2,1}$ case in $\mathbb{R}^N$. Property (i) is easy to check; see [4]. A simple proof of (ii) for the case $N = 2$, of interest to this paper, is given in the Appendix of [4].

Our aim is to show that, under suitable hypotheses on $\partial\Omega$, τ is a Hölder continuous function in $\overline{\Omega}$. In our analysis we will exclude the case when Ω is a disk. For suppose that $\Omega = B_R(0)$, for some $R > 0$. Then τ is trivially Lipschitz continuous in $\overline{B}_R(0)$, since $\tau(x) = |x|$ for $x \in \overline{B}_R(0)$. However, when Ω is not a disk, Lipschitz continuity may fail, even if the boundary is very smooth, as the next example shows.

Example 2.1 (The parabola case)**.** In the cartesian plane consider the set

$$\Omega := \{(x, y) \in \mathbb{R}^2 \ : \ y > x^2\},$$

whose boundary is a parabola with vertex $(0, 0)$. The graph of the map $s \mapsto s^2$ is a regular parametrization of the boundary and the vector

$$N(s) = \frac{1}{\sqrt{1+4s^2}} \begin{pmatrix} -2s, \\ 1 \end{pmatrix}$$

is the inward unit normal to $\partial\Omega$ at the point (s, s^2). By the symmetry of $\partial\Omega$ with respect to the vertical axis we deduce that $\overline{\Sigma}$ must be contained in such an axis. Moreover, an easy calculation shows that for any $s \neq 0$, the line through (s, s^2) with direction $N(s)$ intersects the vertical axis in the point $(0, s^2 + 1/2)$. Hence,

$$\overline{\Sigma} = \{(0, y) \ : \ y \geq 1/2\}$$

and

$$\tau((s, s^2)) = \frac{1}{2}\sqrt{1+4s^2}.$$

Taking into account that the curvature at the point (s, s^2) is given by

$$\kappa((s, s^2)) = \frac{2}{(1+4s^2)^{3/2}},$$

we deduce that the unique conjugate point of Ω is $(0, 1/2)$, which is also regular.

Let us prove that the map τ cannot be Lipschitz continuous in the whole set Ω by showing that for any a small enough

$$|\tau((a, 1/2)) - \tau((0, 1/2))| \geq M|(a, 1/2) - (0, 1/2)|^{2/3},$$

for some constant $M > 0$.

For any fixed $a \in (-\sqrt{2}/2, 0) \cup (0, \sqrt{2}/2)$, the unique projection on the boundary of $(a, 1/2)$ is the point (s_a, s_a^2) where s_a satisfies $s_a = \frac{a^{1/3}}{2^{1/3}}$. Indeed, for any s, the line through (s, s^2) with direction $N(s) = Dd((s, s^2))$ has equation

$$y - s^2 = -\frac{1}{2s}(x - s).$$

Hence, $(a, 1/2)$ belongs to this line if and only if $s^2 = \frac{a}{2s}$, i.e., $s_a = \frac{a^{1/3}}{2^{1/3}}$. We deduce that

$$\begin{aligned}
\tau((a, 1/2)) &= |(a, 1/2) - (0, s_a^2 + 1/2)| = \left(a^2 + \frac{a^{4/3}}{2^{4/3}}\right)^{1/2} \\
&= a^{2/3}\left(a^{2/3} + \frac{1}{2^{4/3}}\right)^{1/2} \geq \frac{1}{2^{2/3}}\, a^{2/3},
\end{aligned}$$

which proves the claim.

This example shows that even in the case of analytic boundaries, τ cannot be Lipschitz continuous around a regular conjugate point. Indeed, as we will see in Theorem 4.1, the only obstruction to Lipschitz regularity is the presence of conjugate points. On the other hand, such points necessarily occur in the case of simply connected domains with analytic boundary, different from a disk (see Proposition 3.1 below). We conclude this section with the definition of geometric graph.

Definition 2.2. We call a geometric graph any closed connected set which consists of a finite number of disjoint vertices and edges, where a vertex is a point in $\mathbb{R}^2$ and an edge is a regular curve with finite length whose limits of tangents at the end points exist.

3. The cut locus of analytic sets

In this section we collect together some known and new results on $\overline{\Sigma}$ in the case of analytic boundary $\partial\Omega$.

The main motivation for this strong requirement on $\partial\Omega$ is that the knowledge of the structure of $\overline{\Sigma}$ is essential in the analysis of the regularity of the maximal retraction length τ, and only in the case of analytic boundaries a complete description is available.

The following result can be deduced from [6] or [14].

Proposition 3.1. *Let Ω be a bounded domain with analytic boundary, different from a disk. Then Γ, $\tilde{\Gamma}$ and Σ^2 are finite sets. Moreover, $\overline{\Sigma}$ is a geometric graph. The edges of the graph are real analytic curves and the vertices are precisely the points of $\Gamma \cup \tilde{\Gamma} \cup \Sigma^2$. The number of analytic arcs starting from a vertex equals the number of projections of the vertex onto the boundary.*

If also Ω is a simply connected domain, then Γ is nonempty.

Moreover, the following proposition from [4] (see also [1] for general results on this subject) holds true in the case of a C^2 boundary $\partial\Omega$.

Proposition 3.2. *Let $x_0 \in \Sigma$, and let p_0, q_0 be two distinct limiting gradients at x_0 such that the segment $[p_0, q_0]$ is a face of $D^+d(x_0)$. Let n_0 be a nonzero vector satisfying*

$$\langle p, n_0\rangle \le \langle p_0, n_0\rangle = \langle q_0, n_0\rangle \qquad \forall\, p \in D^+d(x_0)\,.$$

Then, there exist a number $\eta > 0$ and a Lipschitz arc $\zeta : [0, \eta] \to \Omega$ such that

$$\zeta(0) = x_0\,, \qquad \dot{\zeta}(0) = -n_0\,, \qquad \zeta(s) \in \Sigma \quad \forall\, s \in [0, \eta]\,. \tag{9}$$

Moreover, $\zeta(s_n) \in \Sigma^1$ for some sequence $s_n \downarrow 0$, and

$$D^+d(\zeta(s_n)) = [p_n, q_n] \qquad \forall n \ge 0 \tag{10}$$

where $p_n \to p_0$ and $q_n \to q_0$ as $n \to \infty$.

Finally, in the case of propagation from a regular conjugate point, we have the following result.

Lemma 3.3. *Let Ω be a bounded domain with analytic boundary and x_0 a regular conjugate point of the distance function. Then the analytic singular arc propagating from x_0 coincides in a suitable neighborhood of x_0 with the unique solution of the differential inclusion*

$$\begin{cases} \dot{\zeta}(s) \in D^+d(\zeta(s)), \\ \zeta(0) = x_0. \end{cases} \tag{11}$$

Proof. For any starting point $x_0 \in \Omega$, the existence of a global solution of (11) is a classical result in the theory of differential inclusions. The fact that such a solution is unique is a consequence of the semiconcavity property of the distance function (see, e.g., [1]). The solution of (11) is at least Lipschitz continuous as a consequence of the inclusion $D^+d(x) \subseteq \overline{B}_1(0)$ for any $x \in \Omega$. So let us denote by $\zeta(\cdot)$ the unique global Lipschitz solution of (11) with $x_0 \in \Gamma$. We will first prove that, at least for small times, arc $\zeta(\cdot)$ cannot consist of regular points only. Indeed, if there exists $s_0 > 0$ such that $\zeta(s) \notin \Sigma$ for any $s \in (0, s_0)$, the differential inclusion reduces to the equation $\dot{\zeta}(s) = Dd(\zeta(s))$ for $s \in (0, s_0)$. Moreover, being Γ finite in the case of analytic boundary, we can suppose that $\zeta(s) \notin \Gamma$ for all $s \in (0, s_0)$. Hence differentiating the equation we obtain

$$\ddot{\zeta}(s) = D^2d(\zeta(s))\dot{\zeta}(s) = D^2d(\zeta(s))Dd(\zeta(s)) = 0, \qquad s \in (0, s_0).$$

But then $\zeta(s) = x_0 + sDd(x_0)$, $Dd(\zeta(s)) = Dd(x_0)$ for any $s \in (0, s_0)$ and we have $d(\zeta(s)) = d(x_0) + s = 1/\kappa(x_0) + s$, i.e.

$$d(\zeta(s))\kappa(\zeta(s)) = 1 + s\kappa(x_0) > 1,$$

against (5). Hence we have proven that there exists a sequence $\{s_k\}$ converging to 0 such that $\zeta(s_k)$ is singular. Without loss of generality we can suppose that

$\zeta(s_k) \in \Sigma^1$ for all $k \in \mathbb{N}$, since Σ^2 is finite by Proposition 3.1. From the upper semicontinuity of the superdifferential $D^+d(\cdot)$ we get that there exists $\delta > 0$ such that $D^+d(x) \subseteq Dd(x_0) + \frac{1}{2}B_1(0)$ for any $x \in B_\delta(x_0)$; since $|\zeta(s_k) - x_0| \leq s_k$ we deduce that $0 \notin D^+d(\zeta(s_k))$ for k sufficiently large. In light of Proposition 3.2 we then have that $\zeta(\cdot)$ is locally singular around each point $\zeta(s_k)$. For any k set

$$\sigma_k := \sup\{t \geq 0 \ : \ \zeta(s_k + t) \in \Sigma \cap B_\delta(x_0)\}.$$

In order to complete the proof we need to show that σ_k does not shrink to 0 as $k \to \infty$. So, suppose by contradiction that $\sigma_k \to 0$ as $k \to \infty$. By definition, $\zeta(s_k + \sigma_k)$ is either a regular conjugate point or $|\zeta(s_k + \sigma_k) - x_0| = \delta$. In the latter case, $\delta = |\zeta(s_k + \sigma_k) - x_0| \leq s_k + \sigma_k$ and then $\sigma_k \geq \delta/2$ for k large. The former case is excluded by Proposition 3.1, because the number of regular conjugate points is finite in the case of analytic boundaries.

Hence, we have found a singular Lipschitz arc propagating from x_0; such an arc must coincide with the unique analytic arc with vertex x_0 given by Proposition 3.1. □

Remark 3.4. Collecting together the previous results, we can say that if $\Omega \subset \mathbb{R}^2$ is a bounded domain with analytic boundary, different from a disk, then for any $x_0 \in \overline{\Sigma}$ there exist exactly m analytic singular arcs starting from x_0, where m is the number of elements of $D^*d(x_0)$, say $D^*d(x_0) = \{p_1, \ldots, p_m\}$. When x_0 is singular, the initial directions of these arcs are given by the opposite of the unit outward normal vectors to the exposed faces of $D^+d(x_0)$. More precisely, for any $p_i \neq p_j$ such that $[p_i, p_j] \subset \partial D^+d(x_0)$ let n_{ij} be defined by

$$\max_{p \in D^+d(\bar{x})} \langle p, n_{ij} \rangle = \langle p_i, n_{ij} \rangle = \langle p_j, n_{ij} \rangle.$$

Then, $-n_{ij}$ gives the initial direction of a singular arc starting from x_0. In the case when x_0 is regular and conjugate, the initial direction of the unique singular arc starting from x_0 is $Dd(x_0)$. Moreover, being Σ^2 finite, any analytic singular arc ζ starting from a point $x_0 \in \overline{\Sigma}$ is locally made of points of Σ^1 only. Hence, $D^+d(\zeta(s)) = [p(s), q(s)]$, with $p(s),\ q(s) \in D^*d(\zeta(s))$. Also, when $x_0 \in \Sigma$, Proposition 3.2 gives that there exist $\delta_0 > 0$ and $s_0 > 0$ such that

$$\mathrm{diam}(D^+d(\zeta(s)) = |p(s) - q(s)| \geq \delta_0, \qquad \forall s \in (0, s_0).$$

Finally, as a consequence of the fact that $\Gamma \cup \tilde{\Gamma}$ is finite, we deduce the following property. For any $x_0 \in \Gamma \cup \tilde{\Gamma}$, let $\mathcal{S}(x_0)$ be the line segment $[x_0, x_0 - d(x_0)p_0]$, where $p_0 = Dd(x_0)$ if x_0 is a regular point and $p_0 \in D^*d(x_0)$ satisfies $d(x_0)\kappa(x_0 - d(x_0)p_0) = 1$ if x_0 is singular. Then, there exists an open cone $\mathcal{C}_0$, with apex x_0 and symmetry axis containing the segment $\mathcal{S}(x_0)$ such that $\mathcal{C}_0 \cap \Sigma = \emptyset$. Such a property will be useful in the sequel when we study the behaviour of τ and of v_f, respectively, on the sets

$$\Sigma \cup \left(\bigcup_{x_0 \in \Gamma} \mathcal{S}(x_0) \right), \qquad \Sigma \cup \left(\bigcup_{x_0 \in \tilde{\Gamma} \cup \Gamma} \mathcal{S}(x_0) \right).$$

4. Regularity of the maximal retraction length of Ω

The main result of this section is the proof of the Hölder continuity of τ in the whole set Ω. We already know that τ is locally Lipschitz continuous in $\Omega \setminus \overline{\Sigma}$, but, as the next theorem will show, the (local) Lipschitz constant of τ explodes near the set of regular conjugate points, if this set is nonempty. On the other hand, Example 2.1 gives an indication on how things go in the case of a regular boundary, suggesting the idea that a local analysis near the set of regular conjugate points can produce the right estimates to get the Hölder regularity of τ.

The formal proof turns out to be quite long, so that it must be divided into several steps. As a first step, we will provide an estimate for the local Lipschitz constant of τ in the set $\Omega \setminus \mathcal{S}$, where

$$\mathcal{S} := \bigcup_{x_0 \in \Gamma} \{x_0 - tDd(x_0) \ : \ t \in [0, d(x_0)]\}.$$

Afterwards, we will compare it with the distance to the set $\Sigma \cup \mathcal{S}$. This is the crucial part of the proof, where we will make use of local coordinates for the boundary $\partial\Omega$ and the set $\overline{\Sigma}$. At the very end, we will be able to conclude the Hölder continuity of τ by means of a simple regularity lemma.

Let us start with the local Lipschitz estimate. We recall that our standing assumption is the following:

$$\Omega \text{ is a bounded domain of } \mathbb{R}^2$$
$$\text{with analytic boundary, different from a disk.}$$

We will omit the above assumption in the sequel.

Theorem 4.1. *Set* $\mathcal{S} := \bigcup_{x_0 \in \Gamma} \{x_0 - tDd(x_0) \ : \ t \in [0, d(x_0)]\}$. *Then, for any* $x \in \Omega \setminus (\Sigma \cup \mathcal{S})$ *there exists an open ball* $B_r(x)$, $r = r(x) > 0$, *such that for all* $y \in B_r(x)$,

$$\tau(y) \le \tau(x) + C(x)|x - y|, \tag{12}$$

where

$$C(x) = 2\left(1 + \frac{1 - (d(x) + \tau(x))\kappa(x)}{1 - d(x)\kappa(x)} \frac{1}{\delta(\bar{x})}\right), \tag{13}$$

$\bar{x} = x + \tau(x)Dd(x)$ *is the singular point corresponding to* x *and*

$$\delta(\bar{x}) = \min\left\{|p - q| \ : \ p, q \in D^*d(\bar{x}),\ [p, q] \subseteq \partial D^+d(\bar{x})\right\}. \tag{14}$$

Proof. Fix any $x \in \Omega \setminus (\Sigma \cup \mathcal{S})$ and set $\bar{x} = x + \tau(x)Dd(x)$. Moreover, let $e_2 = Dd(x)$ and e_1 such that $\{e_1, e_2\}$ is a positively oriented orthonormal basis of $\mathbb{R}^2$. We will first prove the theorem in the case $\bar{x} \in \Sigma^1$. Under this assumption, there exists a limiting gradient p such that $D^+d(\bar{x}) = [e_2, p]$. Moreover, by Propositions 3.1 and 3.2 there exists an analytic singular arc (except maybe for the point $\bar{x}$), say $\zeta(\cdot)$, passing through $\bar{x}$ with direction $-n$, where n is defined by

$$\max_{q \in D^+d(\bar{x})} \langle q, n\rangle = \langle p, n\rangle = \langle e_2, n\rangle$$

and $\langle n, e_1\rangle < 0$. As a matter of fact, there are exactly two nonzero vectors satisfying the previous equality, both orthogonal to $p - e_2$ and then having opposite direction. So, let us locally represent the arc $\zeta(\cdot)$ above as

$$\zeta(s) = \bar{x} - ns + \mathrm{o}(s), \tag{15}$$

where $s \in (-s_0, s_0)$ and $s_0 > 0$. Now, take $r > 0$ sufficiently small such that the ball $B_r(x)$ is contained in $\Omega \setminus (\Sigma \cup \mathcal{S})$ and consider any point $y \in B_r(x)$. Define the continuous map $\phi : B_r(x) \times (-s_0, s_0) \to \mathbb{R}$ by

$$\phi(y, s) := \langle \zeta(s) - y, \mathcal{R}Dd(y)\rangle, \tag{16}$$

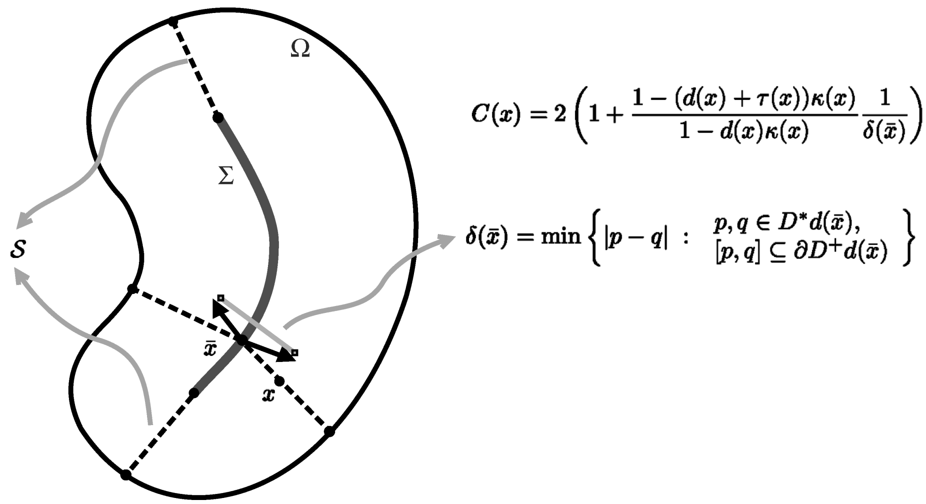

FIGURE 1. The behaviour of $C(x)$

where $\mathcal{R}$ is the rotation matrix

$$\mathcal{R} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{17}$$

Recalling (6), we have for any $y \in B_r(x)$

$$\begin{aligned} Dd(y) &= Dd(x) + D^2d(x)(y-x) + \mathrm{o}(|y-x|) \\ &= e_2 - \frac{\kappa(x)}{1-\kappa(x)d(x)}\,(e_1 \otimes e_1)\,(y-x) + \mathrm{o}(|y-x|). \end{aligned}$$

Then,

$$\begin{aligned} &\phi(y,s) = \left\langle x-y, -e_1 - \frac{\kappa(x)}{1-\kappa(x)d(x)}\langle y-x, e_1\rangle e_2 + \mathrm{o}(|y-x|)\right\rangle \\ &+\left\langle \tau(x)e_2 - ns + \mathrm{o}(s), -e_1 - \frac{\kappa(x)}{1-\kappa(x)d(x)}\langle y-x, e_1\rangle e_2 + \mathrm{o}(|y-x|)\right\rangle \\ &= \langle y-x, e_1\rangle \left(1 - \frac{\tau(x)\kappa(x)}{1-\kappa(x)d(x)}\right) + \frac{\kappa(x)}{1-\kappa(x)d(x)}\langle y-x, e_1\rangle\langle y-x, e_2\rangle \\ &+\langle n, e_1\rangle s + \frac{\kappa(x)s}{1-\kappa(x)d(x)}\langle y-x, e_1\rangle\langle n, e_2\rangle + \mathrm{o}(s) + \mathrm{o}(|y-x|). \end{aligned} \tag{18}$$

Hence, there exist $\bar{s} = \bar{s}(x) > 0$ and $r = r(x) > 0$ such that

$$\phi(y,\bar{s}) < 0, \qquad \phi(y,-\bar{s}) > 0, \qquad \forall\ y \in B_r(x). \tag{19}$$

Therefore, we conclude that for any $y \in B_r(x)$ we can find some $s_y \in (-\bar{s}, \bar{s})$ such that $\phi(y, s_y) = 0$, i.e.

$$\zeta(s_y) = y + \rho_y Dd(y), \qquad \text{for some } \rho_y \in \mathbb{R}. \tag{20}$$

Notice that $s_y \to 0$ as $y \to x$. So, $\zeta(s_y) \to \bar{x} = x + \tau(x)Dd(x)$ as $y \to x$ and then $\rho_y \to \tau(x)$ as $y \to x$. Possibly reducing again r we can then assume that $\rho_y > 0$ for any $y \in B_r(x)$. Let us estimate s_y. Since $s_y \to 0$ as $y \to x$, we have that

$\frac{\kappa(x)}{1-\kappa(x)d(x)} s_y \langle y-x, e_1\rangle\langle n, e_2\rangle = \mathrm{o}(|y-x|)$, so that

$$0 = s_y\langle n, e_1\rangle + \langle y-x, e_1\rangle \left(\frac{1-(\tau(x)+d(x))\kappa(x)}{1-\kappa(x)d(x)}\right) + \mathrm{o}(s_y) + \mathrm{o}(|y-x|) \tag{21}$$

and then

$$s_y + \mathrm{o}(s_y) = -\frac{\langle y-x, e_1\rangle}{\langle n, e_1\rangle}\left(\frac{1-(\tau(x)+d(x))\kappa(x)}{1-\kappa(x)d(x)}\right) + \mathrm{o}(|y-x|), \tag{22}$$

which gives

$$s_y = -\frac{\langle y-x, e_1\rangle}{\langle n, e_1\rangle}\left(\frac{1-(\tau(x)+d(x))\kappa(x)}{1-\kappa(x)d(x)}\right) + \mathrm{o}(|y-x|). \tag{23}$$

Using (23) we can actually estimate ρ_y, which is an upper bound for $\tau(y)$. Indeed,

$$\begin{aligned}
\rho_y &= \langle \zeta(s_y) - y, Dd(y)\rangle \\
&= \left\langle x + \tau(x)e_2 - ns_y - y, e_2 - \frac{\kappa(x)}{1-\kappa(x)d(x)}\langle y-x, e_1\rangle e_1\right\rangle \\
&\quad + \mathrm{o}(s_y) + \mathrm{o}(|y-x|) \\
&= -\langle y-x, e_2\rangle + \tau(x) - s_y\langle n, e_2\rangle + \frac{\kappa(x)s_y}{1-\kappa(x)d(x)}\langle y-x, e_1\rangle\langle n, e_1\rangle \\
&\quad + \mathrm{o}(s_y) + \mathrm{o}(|y-x|) \\
&= \tau(x) - \langle y-x, e_2\rangle - \left(\frac{1-(\tau(x)+d(x))\kappa(x)}{1-\kappa(x)d(x)}\right)\frac{\langle n, e_2\rangle}{\langle n, e_1\rangle}\langle y-x, e_1\rangle \\
&\quad + \mathrm{o}(|y-x|) \\
&\le \tau(x) + \left(2 + \frac{1-(\tau(x)+d(x))\kappa(x)}{1-\kappa(x)d(x)}\frac{|\langle n, e_2\rangle|}{|\langle n, e_1\rangle|}\right)|y-x|, \quad \forall y \in B_r(x),
\end{aligned}$$

provided we take r small enough. Now, recalling that n is orthogonal to $p - e_2$, we deduce that

$$\begin{aligned}
|\langle n, e_1\rangle| &= \frac{|n|}{|p-e_2|}\,|\langle \mathcal{R}(p-e_2), e_1\rangle| = \frac{|n|}{|p-e_2|}\,|\langle p-e_2, e_2\rangle| \\
&= \frac{|n|}{|p-e_2|}\,(1-\langle p, e_2\rangle) = \frac{|n|}{2}|p-e_2|.
\end{aligned}$$

Therefore,

$$\tau(y) \le \tau(x) + 2\left(1 + \frac{1-(\tau(x)+d(x))\kappa(x)}{1-\kappa(x)d(x)}\right)\frac{1}{|p-e_2|}|y-x|, \tag{24}$$

which is the desired inequality, since in this case $\delta(\bar{x}) = |p-e_2|$.

Next, let us suppose that $\bar{x} \in \Sigma^2$. By Remark 3.4 we already know that $D^*d(\bar{x})$ is finite. Then, there exist two limiting gradients, say $p_1, p_2 \neq e_2$, such that $[p_i, e_2]$ $(i = 1, 2)$ is an exposed face of $D^+d(\bar{x})$, that is, $[p_i, e_2] \subset \partial D^+d(\bar{x})$. Moreover, there exist two analytic (except for the starting point) arcs propagating from $\bar{x}$ with initial direction given by the opposite of the unit outward normals n_1 and n_2 to the faces $[p_1, e_2]$ and $[p_2, e_2]$ of $D^+d(\bar{x})$ respectively. We claim that

$$e_2 = \lambda_1 n_1 + \lambda_2 n_2 \tag{25}$$

for suitable numbers $\lambda_1, \lambda_2 > 0$. Indeed, the normal cone to $D^+d(\bar{x})$ at $\bar{x}$ is generated by $\{n_1, n_2\}$. Since e_2 belongs to such a cone, $e_2 = \lambda_1 n_1 + \lambda_2 n_2$ with $\lambda_1, \lambda_2 \ge 0$.

If $\lambda_1 = 0$, then $\lambda_2 = 1$ and $e_2 = n_2$. Therefore, $\langle p_2, n_2\rangle = \langle e_2, n_2\rangle = 1$, which implies $p_2 = n_2 = e_2$ in contrast with the definition of p_2. So, $\lambda_1 > 0$. Similarly, $\lambda_2 > 0$; our claim is thus proved. Now, taking into account that e_1 and e_2 are mutually orthogonal, we have $0 = \lambda_1\langle n_1, e_1\rangle + \lambda_2\langle n_2, e_1\rangle$. So, either $\langle n_1, e_1\rangle < 0$ and $\langle n_2, e_1\rangle > 0$ or vice versa. Suppose $\langle n_1, e_1\rangle < 0$. Then the arc

$$\zeta(s) = \begin{cases} \bar{x} - n_1 s + \mathrm{o}(s), & \text{for } s \in [0, s_0), \\ \bar{x} + n_2 s + \mathrm{o}(s), & \text{for } s \in (-s_0, 0) \end{cases} \tag{26}$$

is the local representation of the singular arc mentioned above. By repeating the argument of the case $\bar{x} \in \Sigma^1$, we obtain that there exists a ball $B_r(x)$ such that, for any $y \in B_r(x)$,

$$\tau(y) \le \tau(x) + 2\left(1 + \frac{1 - (\tau(x) + d(x))\kappa(x)}{1 - \kappa(x)d(x)}\right)\frac{1}{\min\{|p_1 - e_2|, |p_2 - e_2|\}}|y - x|.$$

The general inequality is now a straightforward consequence of the previous computations. □

Lemma 4.2. *For any ball B compactly embedded in $\Omega \setminus \mathcal{S}$ there exists a positive constant δ_B such that*

$$\delta(\bar{x}) \ge \delta_B \qquad \textit{for any} \qquad x \in B,$$

where $\bar{x} = x + \tau(x)Dd(x)$.

Proof. First of all, $\delta(\cdot)$ is strictly positive on $\Omega \setminus \mathcal{S}$ (see Remark 3.4). Moreover, let $\overline{B} \subset \Omega \setminus \mathcal{S}$ be any ball and suppose, by contradiction, that there exists a sequence $\{x_k\} \subset B$ such that $\delta(\bar{x}_k) \to 0$ as $k \to \infty$. We can assume, without loss of generality, that $\bar{x}_k \in \Sigma^1$. Hence, for any k, there exist $p_k, q_k \in D^*d(\bar{x}_k)$ with $[p_k, q_k] = \partial D^+ d(\bar{x}_k)$ and $\delta(\bar{x}_k) = |p_k - q_k| \to 0$ as $k \to \infty$. Consider now the projections y_k and z_k corresponding to p_k and q_k respectively, that is, $y_k = \bar{x}_k - d(\bar{x}_k)p_k$ and $z_k = \bar{x}_k - d(\bar{x}_k)q_k$. Then,

$$\begin{aligned} \frac{y_k - z_k}{|y_k - z_k|} &= -d(\bar{x}_k)\frac{(p_k - q_k)}{|y_k - z_k|} = -b_\Omega(\bar{x}_k)\frac{(Db_\Omega(y_k) - Db_\Omega(z_k))}{|y_k - z_k|} \\ &= -b_\Omega(\bar{x}_k)D^2 b_\Omega(z_k + \lambda_k(y_k - z_k)) \cdot \frac{(y_k - z_k)}{|y_k - z_k|}, \end{aligned} \tag{27}$$

for some $\lambda_k \in (0,1)$, where $b_\Omega(\cdot)$ denotes the signed distance from $\partial\Omega$,

$$b_\Omega(x) = \begin{cases} d_{\partial\Omega}(x) & \text{if } x \in \overline{\Omega} \\ \\ -d_{\partial\Omega}(x) & \text{if } x \in \mathbb{R}^2 \setminus \overline{\Omega}\,. \end{cases}$$

Choosing appropriate subsequences, still called $\{x_k\}$, $\{p_k\}$, $\{q_k\}$, we can suppose that $x_k \to x_0 \in \Omega \setminus \mathcal{S}$, $p_k, q_k \to e_0 \in D^*d(\bar{x}_0)$ and $\frac{y_k - z_k}{|y_k - z_k|} \to \theta_0$ as $k \to \infty$. Thus $\bar{x}_k \to \bar{x}_0$ and, passing to the limit in (27), we obtain

$$\theta_0 = -d(\bar{x}_0)D^2 d(\hat{x}_0) \cdot \theta_0, \tag{28}$$

where $\hat{x}_0 \in \partial\Omega$ is the limiting point of both y_k and z_k. Recalling the structure of the Hessian matrix $D^2 d(\hat{x}_0)$ (see (6)), we conclude that $d(\bar{x}_0)\kappa(\hat{x}_0) = 1$. Therefore, $\bar{x}_0$ belongs to $\tilde{\Gamma} \cup \Gamma$. But $\bar{x}_0$ cannot be a regular conjugate point because $x_0 \notin \mathcal{S}$ by construction; on the other hand, $\bar{x}_0$ cannot be a singular point either, for otherwise $\{\bar{x}_k\}$ would be a sequence of singular points approaching x_0 with

$$\operatorname{diam}(D^+d(\bar{x}_k)) = |p_k - q_k| \to 0, \qquad \text{as } k \to \infty,$$

in contrast with Remark 3.4 on the structure of $\overline{\Sigma}$. This contradiction proves the assertion of the lemma. □

Proposition 4.3. *The map τ is locally Lipschitz continuous on the set $\Omega \setminus \mathcal{S}$. Moreover, τ is differentiable a.e. in $\Omega \setminus (\Sigma \cup \mathcal{S})$ and*

$$|\nabla\tau(x)| \le C(x) \qquad x \in \Omega \setminus (\Sigma \cup \mathcal{S}) \quad a.e.,$$

where $C(x)$ is given by (13).

Proof. We will first prove that for any $x \in \Omega \setminus (\Sigma \cup \mathcal{S})$ we have

$$(29) \qquad |p| \le C(x) \qquad \forall\, p \in \partial_P\tau(x) \quad \forall\, x \in \Omega \setminus (\Sigma \cup \mathcal{S})\,,$$

where $\partial_P\tau(x)$ denotes the proximal subgradient of τ at x. Indeed, recall that a vector $p \in \mathbb{R}^2$ belongs to $\partial_P\tau(x)$ if and only if there exist numbers $\sigma, \eta > 0$ such that

$$\tau(y) \ge \tau(x) + \langle p, y - x\rangle - \sigma|y - x|^2 \qquad \forall\, y \in B_\eta(x)\,,$$

see [7, Theorem 2.5, p. 33]. Now, combine the above inequality with (12) to obtain

$$\langle p, y - x\rangle \le C(x)|y - x| + \sigma|y - x|^2$$

whenever $|y - x| < \min\{r, \eta\}$. The last inequality implies (29).

Now, we note that $C(\cdot)$ is locally bounded on $\Omega \setminus (\Sigma \cup \mathcal{S})$ by Lemma 4.2 and the inequality

$$\frac{1 - (\tau(x) + d(x))\kappa(x)}{1 - \kappa(x)d(x)} \le 1 + \operatorname{diam}(\Omega) \max_{x \in \partial\Omega} [\kappa(x)]_-,$$

where $[\kappa(x)]_- := \max\{0, -\kappa(x)\}$. Owing to [7, Theorem 7.3, p. 52], τ is locally Lipschitz in $\Omega \setminus (\Sigma \cup \mathcal{S})$. Thus, τ is also differentiable a.e. on such a set. Moreover, whenever τ is differentiable in a point x, for any y in a suitable ball around x, we can write

$$\tau(y) - \tau(x) = \langle \nabla\tau(x), y - x\rangle + \mathrm{o}(|y - x|).$$

Hence, if we take $y := x + \rho\frac{\nabla\tau(x)}{|\nabla\tau(x)|}$, with $\rho > 0$ small enough, and we substitute it in the above equality, we have, again by (12),

$$\rho|\nabla\tau(x)| \le C(x)\rho + \mathrm{o}(\rho).$$

This last inequality readily implies

$$|\nabla\tau(x)| \le C(x) \qquad x \in \Omega \setminus (\Sigma \cup \mathcal{S}) \quad \text{a.e.}$$

In order to complete the proof, we need to bound $C(x)$ from above when x approaches Σ. The expression of $C(x)$ given by (13) for any $x \in \Omega \setminus (\Sigma \cup \mathcal{S})$ is meaningful also on the set of singular points, provided we define

$$C(x) = 2\left(1 + \frac{1}{\delta(x)}\right), \qquad \text{for all} \quad x \in \Sigma.$$

Taking into account Lemma 4.2 we easily deduce the local Lipschitz continuity of τ on $\Omega \setminus \mathcal{S}$. □

Remark 4.4. At this point of our reasoning it is important to stress again that the loss of Lipschitz regularity for τ depends on the presence of conjugate points only. When Ω is a bounded domain with no conjugate points (both regular and singular) and $\mathcal{C}^{2,1}$ boundary, then it can be shown that the results obtained so far still hold true. In particular, it turns out that τ is Lipschitz continuous on the whole set $\overline{\Omega}$. Indeed, if Ω has no conjugate points, as in the case of an annulus, it can be proven that any $x \in \Omega$ has a finite number of projections onto $\partial\Omega$, which is one of the main properties we need in the proof of those results. On the other hand, if Ω is a simply connected domain with analytic boundary, different from a disk, then the set of regular conjugate points is nonempty, so that we cannot avoid the loss of regularity they produce.

As a consequence of the previous remark, in what follows we will assume that the set of regular conjugate points Γ is nonempty.

The second step of our argument is to estimate $C(x)$ in (13) in terms of $d_{\tilde{\mathcal{S}}}(x)$, which is the distance of x from the set $\tilde{\mathcal{S}} := \mathcal{S} \cup \Sigma$. Aiming at this, we need some deeper results on the behaviour of the singular arcs starting from a regular conjugate point. In what follows we choose the reference system so that the regular conjugate point coincides with the point $x_0 = (0, r)$, $r > 0$, being $(0, 0)$ its projection on the boundary. Moreover, we locally represent $\partial\Omega$ as the graph of an analytic function $\alpha : (-s_0, s_0) \to \mathbb{R}$, $0 < s_0 < r$, such that $\alpha(0) = 0$, $\alpha'(0) = 0$ and $\alpha''(0) = \frac{1}{r}$. We claim that there exist $n \geq 2$, $a = a(n) > 0$ and an analytic function $b(\cdot)$ in $(-s_0, s_0)$ satisfying

$$b(s) = \sum_{i \geq 2n+1} b_i s^i, \tag{30}$$

such that

$$\alpha(s) = \left[r - \left(r^2 - s^2\right)^{1/2}\right] - as^{2n} + b(s) \qquad \forall s \in (-s_0, s_0). \tag{31}$$

Indeed, one of the main properties of analytic boundaries is that the curvature κ has a maximum at the projection of a conjugate point (see [6, Theorem 3.1]). More precisely, given a local representation of the boundary as above, the curvature, whose expression is

$$\kappa(s) := \kappa((s, \alpha(s)) = \frac{\alpha''(s)}{(1 + \alpha'(s)^2)^{3/2}}, \tag{32}$$

satisfies $\kappa(s) \leq \kappa(0)$ for any s in a neighborhood of 0. In particular, being κ analytic and nonconstant (Ω is not a disk), we obtain that there exists $n \geq 2$ such that, for any $1 \leq m \leq 2n - 3$,

$$\kappa^{(m)}(0) = 0$$

and

$$\kappa^{(2n-2)}(0) < 0.$$

Writing the above relations in terms of the derivatives of α, and taking into account that $\beta(s) := r - \left(r^2 - s^2\right)^{1/2}$ is a local representation of the circle of centre $(0, r)$ and radius r (the unique analytic curve with constant curvature $1/r$), we obtain that the difference $\alpha(s) - \beta(s)$ is not identically zero and its Taylor expansion at 0 is of the form $-as^{2n} + b(s)$, where a is $\frac{1}{(2n)!}$ times the difference between the $2n$-th derivatives of the functions $\beta(s)$ and of $\alpha(s)$ at $s = 0$, and $b(s)$ is the remainder of the difference of the Taylor expansions in 0 of α and β. Being $b(s)$ of the form $\sum_{i\geq 2n+1} b_i s^i$, we will say that it is a series of valuation $\mathrm{Val}(b) \geq 2n+1$, meaning that the first index with nonzero coefficient is $2n+1$.

Our next lemma provides a description of the singular arc starting from x_0 with respect to the boundary parameter s.

Lemma 4.5. *There exist $\varepsilon > 0$ and two analytic functions $t : (-\varepsilon, \varepsilon) \to \mathbb{R}$ and $\rho : (-\varepsilon, \varepsilon) \to \mathbb{R}$, with $t(0) = 0$, $\rho(0) = r$ and*

$$\begin{cases} t(s) = s + o(s), \\ \rho(s) = r + 2nar^2 s^{2n-2} + o(s^{2n-2}), \end{cases} \tag{33}$$

such that for any $s \in (0, \varepsilon)$

$$A(s) + \rho(s)\ \nu(s) = A(-t(s)) + \rho(s)\ \nu(-t(s)), \tag{34}$$

where

$$A(s) = (s, \alpha(s))$$

is a point on the boundary of Ω and

$$\nu(s) := \left(\frac{-\alpha'(s)}{(1+\alpha'(s)^2)^{1/2}}, \frac{1}{(1+\alpha'(s)^2)^{1/2}} \right)$$

is the inner unit normal to the boundary at the boundary point $A(s)$.

Moreover, there exist $\eta > 0$ such that

$$\Sigma \cap B_\eta((0,r)) = \{\xi(s) \mid s \in (0,\varepsilon)\},$$

where

$$\xi(s) = A(s) + \rho(s)\ \nu(s).$$

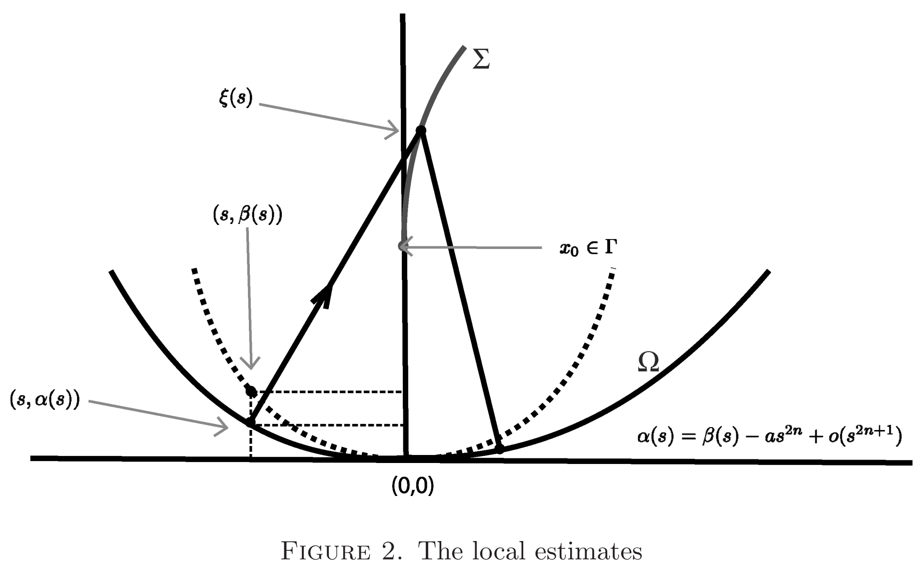

Figure 2. The local estimates

Proof. Our first step is to find, for any $s > 0$ sufficiently small, numbers $t < 0$ and $\rho > 0$ satisfying

$$s + \rho \left[\frac{-\alpha'(s)}{(1+\alpha'(s)^2)^{1/2}} \right] = -t + \rho \left[\frac{-\alpha'(-t)}{(1+\alpha'(-t)^2)^{1/2}} \right], \tag{35}$$

$$\alpha(s) + \frac{\rho}{(1+\alpha'(s)^2)^{1/2}} = \alpha(-t) + \frac{\rho}{(1+\alpha'(-t)^2)^{1/2}}. \tag{36}$$

Since $\alpha''(0) > 0$, if we choose s and t sufficiently small, we have $\alpha'(-t) < 0$ and $\alpha'(s) > 0$. Hence, (35) gives

$$\rho = -\frac{s+t}{\dfrac{\alpha'(-t)}{(1+\alpha'(-t)^2)^{1/2}} - \dfrac{\alpha'(s)}{(1+\alpha'(s)^2)^{1/2}}}. \tag{37}$$

Now we want to simplify this expression by using (31). Since the map $x \mapsto \frac{x}{(1+x^2)^{1/2}}$ is analytic, for any x and y we can write

$$\frac{y}{(1+y^2)^{1/2}} = \frac{x}{(1+x^2)^{1/2}} + \frac{y-x}{(1+x^2)^{3/2}} + C(y-x),$$

where $\mathrm{Val}(C) \geq 2$. Substituting $x = \beta'(s)$ and $y = \alpha'(s) = \beta'(s) - 2nas^{2n-1} + b'(s)$ in the above expression, we obtain

$$\frac{\alpha'(s)}{(1+\alpha'(s)^2)^{1/2}} = \frac{\beta'(s)}{(1+\beta'(s)^2)^{1/2}} + \frac{-2nas^{2n-1} + b'(s)}{(1+\beta'(s)^2)^{3/2}} + C_1(s),$$

where $\mathrm{Val}(C_1) \geq 2(2n-1)$. Moreover,

$$\beta'(s) = \frac{s}{(r^2-s^2)^{1/2}},$$

$$\frac{\beta'(s)}{(1+\beta'(s)^2)^{1/2}} = \frac{s}{(r^2-s^2)^{1/2}} \cdot \left[1 + \frac{s^2}{r^2-s^2} \right]^{-1/2} = \frac{s}{r},$$

while

$$\frac{1}{(1+\beta'(s)^2)^{3/2}} = \left[1 + \frac{s^2}{r^2-s^2} \right]^{-3/2} = \frac{(r^2-s^2)^{3/2}}{r^3} = 1 + C_2(s),$$

with $\mathrm{Val}(C_2) \geq 2$. So,

$$\frac{\alpha'(s)}{(1+\alpha'(s)^2)^{1/2}} = \frac{s}{r} - 2nas^{2n-1} + C_3(s),$$

where $\mathrm{Val}(C_3) \geq 2n$, since $\mathrm{Val}(b') \geq 2n$. Substituting the last expression into (37), we then deduce that

$$\begin{aligned} \rho = \rho(s,t) &= -\frac{s+t}{-\dfrac{s+t}{r} + 2na(t^{2n-1} + s^{2n-1}) + C_3(s) - C_3(-t)} \\ &= \frac{r}{1 - 2nar\dfrac{t^{2n-1}+s^{2n-1}}{s+t} + r\dfrac{C_3(-t) - C_3(s)}{s+t}}. \end{aligned} \tag{38}$$

Notice that $\frac{t^{2n-1}+s^{2n-1}}{s+t}$ is a polynomial of degree $2n-2$, while

$$C_3(s,t) := \frac{C_3(-t) - C_3(s)}{s+t}$$

is analytic of valuation greater than or equal to $2n-1$.

We will now try to solve (36), which is equivalent to

$$\frac{\alpha(-t)-\alpha(s)}{\rho}+\frac{1}{\left(1+\alpha'(-t)^2\right)^{1/2}}-\frac{1}{\left(1+\alpha'(s)^2\right)^{1/2}}=0. \tag{39}$$

Reasoning as above we find that

$$\frac{1}{(1+\alpha'(s)^2)^{1/2}}=\frac{1}{(1+\beta'(s)^2)^{1/2}}-\frac{\beta'(s)}{(1+\beta'(s)^2)^{3/2}}(-2nas^{2n-1}+b'(s))+C_4(s),$$

with $\mathrm{Val}(C_4)\geq 2(2n-1)$. On the other hand,

$$\frac{\beta'(s)}{(1+\beta'(s)^2)^{3/2}}=\frac{(r^2-s^2)}{r^2}\cdot\frac{s}{r}=\frac{s}{r}+C_5(s)$$

where $\mathrm{Val}(C_5)\geq 3$. Thus,

$$\frac{1}{(1+\alpha'(s)^2)^{1/2}}=\left(1-\frac{s^2}{r^2}\right)^{1/2}+\frac{2na}{r}s^{2n}+C_6(s),$$

where $\mathrm{Val}(C_6)\geq 2n+1$. Using the previous computations and taking into account the expression of ρ in (38), we can finally estimate (39) as

$$\begin{aligned}0=&\left(\frac{1}{r}-2na\frac{t^{2n-1}+s^{2n-1}}{s+t}+C_3(s,t)\right)\\ &\cdot\left(r-\left(r^2-t^2\right)^{1/2}-at^{2n}+b(-t)-r+\left(r^2-s^2\right)^{1/2}+as^{2n}-b(s)\right)\\ &+\left(1-\frac{t^2}{r^2}\right)^{1/2}+\frac{2na}{r}t^{2n}+C_6(-t)-\left(1-\frac{s^2}{r^2}\right)^{1/2}-\frac{2na}{r}s^{2n}-C_6(s),\end{aligned} \tag{40}$$

that is,

$$\begin{aligned}0=&(2na-a)(t^{2n}-s^{2n})\\ &-2nar\frac{t^{2n-1}+s^{2n-1}}{s+t}\left(\left(r^2-s^2\right)^{1/2}-\left(r^2-t^2\right)^{1/2}\right)+M(s,t),\end{aligned} \tag{41}$$

with $\mathrm{Val}(M)\geq 2n+1$. Furthermore,

$$\left(\left(r^2-s^2\right)^{1/2}-\left(r^2-t^2\right)^{1/2}\right)=\frac{t^2-s^2}{2r}+P(s,t),\qquad \mathrm{Val}(P)\geq 4.$$

This gives

$$(2na-a)(t^{2n}-s^{2n})-na(t^{2n}-s^{2n})-na(s^{2n-1}t-st^{2n-1})+Q(s,t)=0, \tag{42}$$

being $Q(s,t)$ an analytic function of valuation $\mathrm{Val}(Q)\geq 2n+1$ of the form

$$Q(s,t)=\sum_{k\geq 2n+1}q_k(s^k-(-1)^k t^k). \tag{43}$$

Now, let us set $u:=\dfrac{t}{s}$. Since $s>0$, then (42) becomes

$$(na-a)(u^{2n}-1)-na(u-u^{2n-1})+\frac{1}{s^{2n}}Q(s,su)=0. \tag{44}$$

Exploiting the structure of Q in (43) we see that $\frac{1}{s^{2n}}Q(s,su)$ is equal to $sR(s,u)$, where R is an analytic function of valuation $\mathrm{Val}(R)\geq 2n+1$. So, at the end of these computations we can say that finding $t(\cdot)$ and $\rho(\cdot)$ that verify (33) and (34) is equivalent to finding, for any s sufficiently small, some $u=u(s)$ which solves (44). To this end, we apply the Implicit Function Theorem to the analytic function

$$\phi(s,u)=(na-a)(u^{2n}-1)-na(u-u^{2n-1})+sR(s,u)$$

at the point $(\bar{s}, \bar{u}) = (0, 1)$. Since

$$\phi(0,1) = 0 \quad \text{and} \quad \frac{\partial \phi}{\partial u}(0,1) = 2na(n-1) - na(1-2n+1) = 4na(n-1) \neq 0,$$

the existence of $t(\cdot)$ and $\rho(\cdot)$ is proven.

Now, let us recover the local representation of Σ in terms of the above maps. Since $\Gamma \cup \Sigma^2$ is finite and $(0, r) \notin \Sigma^2$, we can find some $\eta > 0$ such that

$$\Sigma \cap B_\eta((0,r)) \subset \Sigma^1, \quad \overline{\Sigma} \cap B_\eta((0,r)) \subset \Sigma^1 \cup \{(0,r)\}. \tag{45}$$

Moreover, by Lemma 3.3, there is an analytic arc $\zeta : [0, \varepsilon_0) \to \mathbb{R}^2$ such that $\zeta(0) = (0, r)$ and $\Sigma \cap B_\eta((0,r)) = \{\zeta(r) \mid r \in (0, \varepsilon_0)\}$. Possibly reducing ε, we can suppose that for any $s \in (0, \varepsilon)$

$$\begin{aligned} &A(s) + \tau(A(s))\ \nu(s) \in B_\eta((0,r)), \\ &A(-t(s)) + \tau(A(-t(s)))\ \nu(-t(s)) \in B_\eta((0,r)). \end{aligned} \tag{46}$$

Then, for any $s \in (0, \varepsilon)$ there exist θ_s and $\tilde{\theta}_s$ satisfying, respectively,

$$A(s) + \tau(A(s))\ \nu(s) = \zeta(\theta_s)$$

and

$$A(-t(s)) + \tau(A(-t(s)))\ \nu(-t(s)) = \zeta(\tilde{\theta}_s).$$

Suppose that $\theta_s < \tilde{\theta}_s$. Then, $\zeta(\theta_s)$ belongs to the interior of the Jordan curve delimited by the segments $\left[A(-t(s)), \zeta(\tilde{\theta}_s)\right]$, $\left[p(s), \zeta(\tilde{\theta}_s)\right]$ and the curve joining $p(s)$ and $A(t(s))$ on the graph of α, being $p(s)$ the other projection of $\zeta(\tilde{\theta}_s)$. On the other hand, $A(s)$ does not belong to the interior of this curve on the graph of α because otherwise the point $A(s) + \rho(s)\ \nu(s)$ would lie on the segment $\left[A(-t(s)), \zeta(\tilde{\theta}_s)\right]$ and have two projections, namely $A(s)$ and $A(-t(s))$; a contradiction. Since $A(s)$ does not belong to the interior of the curve $(p(s), A(-t(s)))$, we have that either $A(s) = p(s)$ or

$$|\zeta(\theta_s) - p(s)| < |\zeta(\theta_s) - A(s)|,$$

which is again a contradiction. Hence, $\theta_s \geq \tilde{\theta}_s$. By the same argument, $\tilde{\theta}_s \geq \theta_s$. Therefore, $\theta_s = \tilde{\theta}_s$ and $\tau(A(s)) = \tau(A(-t(s))) = \rho(s)$. □

Lemma 4.6. *There exists $\varepsilon > 0$ such that for any $s \in (-\varepsilon, \varepsilon)$ the curvature of $\partial\Omega$ at the point $A(s) = (s, \alpha(s))$ is given by*

$$\kappa(s) = \frac{1}{r} - 2n(n-1)as^{2n-2} + o(s^{2n-2}). \tag{47}$$

Moreover, $\kappa(s) \geq \frac{1}{2r}$ for any $s \in (-\varepsilon, \varepsilon)$.

Proof. Fix $\varepsilon > 0$ as in Lemma 4.5. Using (32), (31) and arguing as in the previous lemma, we have

$$\begin{aligned} \kappa(s) \quad &= \quad \left(\beta''(s) - 2n(n-1)as^{2n-2} + \mathrm{o}(s^{2n-2})\right) \\ &\cdot \quad \left(\frac{1}{(1+\beta'(s)^2)^{3/2}} + \frac{3\beta'(s)}{(1+\beta'(s)^2)^{5/2}} 2nas^{2n-1} + \mathrm{o}(s^{2n-1})\right), \end{aligned}$$

where

$$\beta'(s) = \frac{s}{(r^2 - s^2)^{1/2}}, \quad \beta''(s) = \frac{r^2}{(r^2 - s^2)^{3/2}}$$

and

$$\frac{1}{(1+\beta'(s)^2)^{3/2}} = \frac{(r^2-s^2)^{3/2}}{r^3}.$$

Substituting β', β'' in the expression of $\kappa(s)$ we easily obtain

$$\kappa(s) = \frac{1}{r} - 2n(n-1)as^{2n-2} + \mathrm{o}(s^{2n-2})$$

and, possibly reducing ε, $\kappa(s) \geq \frac{1}{2r}$. □

Now, we proceed to estimate $C(x)$, given in (13), with respect to $d_{\tilde{\mathcal{S}}}(x)$, where $\tilde{\mathcal{S}} := \mathcal{S} \cup \Sigma$ and

$$\mathcal{S} := \bigcup_{x_0 \in \Gamma} \{x_0 - tDd(x_0) \ : \ t \in [0, d(x_0)]\}.$$

For any $h_0 > 0$ sufficiently small set $\mathcal{S}_{h_0} := \{x \in \Omega \mid d_{\mathcal{S}}(x) < h_0\}$. By Proposition 4.3 we deduce that, outside $\mathcal{S}_{h_0}$, the L^∞ norm of $C(x)$ is bounded by some constant C_{h_0}. Hence, for any $x \in \Omega \setminus (\mathcal{S}_{h_0} \cup \Sigma)$ we have that

$$C(x) \leq C_{h_0} \frac{d_{\tilde{\mathcal{S}}}(x)^{\frac{1}{2n-1}}}{d_{\tilde{\mathcal{S}}}(x)^{\frac{1}{2n-1}}} \leq C_{h_0} \frac{\operatorname{diam}(\Omega)^{\frac{1}{2n-1}}}{d_{\tilde{\mathcal{S}}}(x)^{\frac{1}{2n-1}}}.$$

It remains to estimate $C(x)$ in $\mathcal{S}_{h_0} \setminus \tilde{\mathcal{S}}$.

Lemma 4.7. *Let x_0 be a regular conjugate point. For any $h_0 > 0$ let $\mathcal{S}_{h_0}(x_0)$ be the connected component of $\mathcal{S}_{h_0}$ containing x_0. Then, for h_0 sufficiently small and for any $x \in \mathcal{S}_{h_0}(x_0) \setminus \tilde{\mathcal{S}}$, we have*

$$C(x) \leq \frac{K}{d_{\tilde{\mathcal{S}}}(x)^{\frac{1}{2n-1}}}, \tag{48}$$

where n is the integer given in (31) *and K is a constant depending on h_0 and Ω only.*

Proof. To begin with, let us fix the coordinates so that $x_0 = (0, r)$, $r > 0$, and $(0, 0)$ is the projection of x_0 onto $\partial\Omega$. Moreover, let $\partial\Omega$ be represented, in a neighborhood of $(0, 0)$, by the graph of an analytic function $\alpha(\cdot)$, defined in $(-s_0, s_0)$, such that $\alpha(0) = 0$, $\alpha'(0) = 0$ and $\alpha''(0) = \frac{1}{r}$. Let us denote again

$$A(s) = (s, \alpha(s)) \quad \text{and} \quad \nu(s) := \left(\frac{-\alpha'(s)}{(1+\alpha'(s)^2)^{1/2}}, \frac{1}{(1+\alpha'(s)^2)^{1/2}} \right).$$

Now, take $\varepsilon > 0$ as in Lemma 4.5. Choose h_0 sufficiently small such that the projection onto $\partial\Omega$ of any $x \in \mathcal{S}_{h_0}(x_0) \setminus (\Sigma \cup \mathcal{S})$ is given by $A(s_x)$ for some $s_x \in (-\varepsilon, \varepsilon)$. Actually $s_x \neq 0$ because $x \notin \mathcal{S}$ and $\mathcal{S}_{h_0}(x_0) \setminus (\Sigma \cup \mathcal{S})$ is a two-connected-components set, contained in the disjoint union of the sets $\mathcal{C}^-$ and $\mathcal{C}^+$, where

$$\mathcal{C}^+ = \bigcup_{s \in (0,\varepsilon)} T_s, \qquad \mathcal{C}^- = \bigcup_{s \in (-\varepsilon, 0)} T_s$$

and

$$T_s :=]A(s); A(s) + \tau(A(s))Dd(A(s))[.$$

Let us fix our attention on the connected component

$$\mathcal{S}^+ := (\mathcal{S}_{h_0}(x_0) \setminus (\Sigma \cup \mathcal{S})) \cap \mathcal{C}^+.$$

In light of Lemma 4.5, we have

$$\Sigma \cap \mathcal{S}_{h_0}(x_0) = \{\xi(s) : s \in (0, \varepsilon)\},$$

where

$$\xi(s) = A(s) + \rho(s)\nu(s)$$

and $\rho(s) = r + 2nar^2 s^{2n-2} + \mathrm{o}(s^{2n-2})$. Hence, for any $s \in (0, \varepsilon)$,

$$T_s =]A(s), \xi(s)[\,.$$

Let us first estimate $C(\cdot)$ on any of these rays. Fix a ray T_s, with $s \in (0, \varepsilon)$. Then, from (13) we have that for any $x \in T_s$

$$C(x) = 2\left(1 + \frac{1 - d(\xi(s))\kappa(s)}{1 - d(x)\kappa(s)} \frac{1}{\delta(\xi(s))}\right),$$

where $\kappa(s)$ stands for the curvature at the boundary point $A(s)$. Since s is fixed, in order to estimate $C(x) d_{\tilde{\mathcal{S}}}(x)^{\frac{1}{2n-1}}$ on T_s let us consider first the ratio

$$\frac{d_{\tilde{\mathcal{S}}}(x)^{\frac{1}{2n-1}}}{1 - d(x)\kappa(s)}. \tag{49}$$

We claim that for $x \in T_s$ an upper bound for $d_{\tilde{\mathcal{S}}}(x)$ is given by

$$\frac{(d(\xi(s)) - d(x))\, |A(s) - A(-t(s))|}{d(\xi(s))}, \tag{50}$$

with $t(s)$ as in Lemma 4.5. Indeed, $\xi(s) \in \Sigma^1$; so, consider the other projecting line from $\xi(s)$, which is $T_{-t(s)}$, and the point $\tilde{x}$ on this line satisfying the condition $d(x) = d(\tilde{x})$; then we have

$$d_{\tilde{\mathcal{S}}}(x) \le |x - \tilde{x}| = \frac{(d(\xi(s)) - d(x))\, |A(s) - A(-t(s))|}{d(\xi(s))}.$$

Hence, the ratio in (49) is bounded from above by

$$\sup_{x \in T_s} \frac{(d(\xi(s)) - d(x))^{\frac{1}{2n-1}}\, |A(s) - A(-t(s))|^{\frac{1}{2n-1}}}{d(\xi(s))^{\frac{1}{2n-1}} (1 - d(x)\kappa(s))}. \tag{51}$$

On the other hand, since $d(\cdot)$ is linearly increasing on T_s, the above supremum is attained at the (unique) point $x \in T_s$ satisfying

$$d(x) = \frac{2n-1}{2n-2} d(\xi(s)) - \frac{1}{(2n-2)\kappa(s)}.$$

In conclusion, for any $x \in T_s$,

$$\begin{aligned} C(x) d_{\tilde{\mathcal{S}}}(x)^{\frac{1}{2n-1}} &\le 2\mathrm{diam}(\Omega)^{\frac{1}{2n-1}} \\ &+ \frac{2\left(\left(-\frac{1}{2n-2} d(\xi(s)) + \frac{1}{(2n-2)\kappa(s)}\right) |A(s) - A(-t(s))|\right)^{\frac{1}{2n-1}} (1 - d(\xi(s))\kappa(s))}{d(\xi(s))^{\frac{1}{2n-1}} \left(\frac{2n-1}{2n-2} - \frac{2n-1}{2n-2} d(\xi(s))\kappa(s)\right) \delta(\xi(s))} \\ &= 2\mathrm{diam}(\Omega)^{\frac{1}{2n-1}} \\ &+ \frac{2(2n-2)^{\frac{2n-2}{2n-1}}}{2n-1} \frac{[1 - \kappa(s) d(\xi(s))]^{\frac{1}{2n-1}}\, |A(s) - A(-t(s))|^{\frac{1}{2n-1}}}{(\kappa(s) d(\xi(s)))^{\frac{1}{2n-1}} \delta(\xi(s))}. \end{aligned} \tag{52}$$

In order to finish the estimate of $C(x)d_{\tilde{\mathcal{S}}}(x)^{\frac{1}{2n-1}}$ it remains to bound from above the last term in (52)–call it E_s–on $(0,\varepsilon)$. We will complete the reasoning in three steps.

Step 1: *Estimate of* $\delta(\xi(s))$ *and* $|A(s)-A(-t(s))|$.

Since $\xi(s)\in\Sigma^1$, we have by (7)

$$\delta(\xi(s))=\frac{|A(s)-A(-t(s))|}{d(\xi(s))}. \tag{53}$$

Moreover, Lemma 4.5 gives that

$$|A(s)-A(-t(s))|=2s+\mathrm{o}(s). \tag{54}$$

Also, $|A(s)-A(-t(s))|\geq s$.

Step 2: *Estimate of* $1-d(\xi(s))\kappa(s)$.

Recalling that $d(\xi(s))=\rho(s)$, with ρ given by (33) and that κ satisfies (47), we easily derive

(55)

$$\begin{aligned}
&1-d(\xi(s))\kappa(s)\\
=\;&1-\left[r+2nar^2s^{2n-2}+\mathrm{o}(s^{2n-2})\right]\left[\tfrac{1}{r}-2n(n-1)as^{2n-2}+\mathrm{o}(s^{2n-2})\right]\\
=\;&[2n(2n-1)-2n]ras^{2n-2}+\mathrm{o}(s^{2n-2})\\
=\;&4n(n-1)ras^{2n-2}+\mathrm{o}(s^{2n-2}).
\end{aligned}$$

Step 3: *Estimate of* E_s.

If we collect together (53), (54) and (55), and take into account the bounds $\kappa(s)\geq\frac{1}{2r}$ from Lemma 4.6 and $r\leq d(\xi(s))\leq\mathrm{diam}(\Omega)/2$, which follows from the identity $d(\xi(s))=\rho(s)$ and from (33) with s small enough, we readily obtain

$$\begin{aligned}
E_s&\leq\frac{2}{(k(s)d(\xi(s)))^{\frac{1}{2n-1}}}\frac{[4n(n-1)ras^{2n-2}+\mathrm{o}(s^{2n-2})]^{\frac{1}{2n-1}}d(\xi(s))}{|A(s)-A(-t(s))|^{\frac{2n-2}{2n-1}}}\\
&\leq\left([4n(n-1)a]^{\frac{1}{2n-1}}\mathrm{diam}(\Omega)^{\frac{2n}{2n-1}}\right)(1+\mathrm{o}(1_s)).
\end{aligned}$$

Possibly reducing again ε (and then h_0) we get that

$$C(x)d_{\tilde{\mathcal{S}}}(x)^{\frac{1}{2n-1}}\leq 2\mathrm{diam}(\Omega)^{\frac{1}{2n-1}}+2[4n(n-1)a]^{\frac{1}{2n-1}}\mathrm{diam}(\Omega)^{\frac{2n}{2n-1}}, \tag{56}$$

for any $x\in(\mathcal{S}_{h_0}(x_0)\setminus(\Sigma\cup\mathcal{S}))\cap\mathcal{C}^+$, which is an upper bound with an absolute constant, depending on Ω and on the conjugate point x_0. Since the above estimate can be proven with the same reasoning on $(\mathcal{S}_{h_0}(x_0)\setminus(\Sigma\cup\mathcal{S}))\cap\mathcal{C}^-$, the result is complete. □

Corollary 4.8. *There exist a constant $C>0$ and an integer $\bar{n}\geq 2$ such that for any $x\in\Omega\setminus\tilde{\mathcal{S}}$ we have*

$$C(x)\leq\frac{C}{d_{\tilde{\mathcal{S}}}(x)^{\frac{1}{2\bar{n}-1}}}. \tag{57}$$

Proof. This is an immediate consequence of the previous lemma and of the finiteness of the set of regular conjugate points Γ. In particular, $\bar{n}$ is the smallest n arisen in the previous local estimates. □

Roughly speaking, the previous result is an estimate of the "explosion speed" of the local Lipschitz constant of τ when approaching the set $\tilde{\mathcal{S}}$. In the following computations, it will be important to know also the behaviour of τ when restricted to $\tilde{\mathcal{S}}$.

Lemma 4.9. *The restriction of τ to $\tilde{\mathcal{S}}$ is Lipschitz continuous.*

Proof. By definition τ is zero on the (closure of the) ridge set and linear, with rate 1, on any ray $[x_0, x_0 - d(x_0)Dd(x_0)]$, when $x_0 \in \Gamma$. So, in order to find the (global) Lipschitz constant of τ on the set $\tilde{\mathcal{S}}$, it suffices to estimate $|\tau(x) - \tau(y)|$ in the case when

$$x \in [x_0, x_0 - d(x_0)Dd(x_0)] =: \mathcal{S}(x_0)$$

and

$$y \in [y_0, y_0 - d(y_0)Dd(y_0)] =: \mathcal{S}(y_0)$$

for some $x_0, y_0 \in \Gamma$, $x_0 \neq y_0$ and when

$$x \in [x_0, x_0 - d(x_0)Dd(x_0)] =: \mathcal{S}(x_0) \qquad \text{and} \qquad y \in \Sigma.$$

In the former case we have

$$\begin{aligned} |\tau(x) - \tau(y)| &= ||x - x_0| - |y - y_0|| \leq |x - y| + |y_0 - x_0| \\ &\leq |x - y| \left(1 + \frac{\operatorname{diam}(\Omega)}{\min_{x_0 \neq y_0 \in \Gamma} \operatorname{dist}(\mathcal{S}(x_0), \mathcal{S}(y_0))}\right), \end{aligned}$$

since the set of conjugate points is finite and $\operatorname{dist}(\mathcal{S}(x_0), \mathcal{S}(y_0)) > 0$ for $x_0 \neq y_0$. In the latter case, the special structure of the ridge set for an analytic boundary, and in particular the finiteness of the set of conjugate points, guarantees that there exists a cone $\mathcal{C}_0$, with apex x_0, semi–vertex angle $\theta_0 = \theta(x_0) > 0$ and symmetry axis containing the segment $\mathcal{S}(x_0)$ such that $\mathcal{C}_0 \cap \Sigma = \emptyset$. Hence, $|x - y| > d_{\mathcal{C}_0}(x) = |x - x_0| \sin\theta_0$, which gives

$$|\tau(x) - \tau(y)| = |\tau(x)| = |x - x_0| < \frac{1}{\sin\theta_0}|x - y|.$$

Defining γ as the maximum of $1/\sin\theta(x_0)$ over all $x_0 \in \Gamma$, the Lipschitz constant of τ over $\tilde{\mathcal{S}}$ is given by

$$L := \max\left\{\gamma, 1 + \frac{\operatorname{diam}(\Omega)}{\min_{x_0 \neq y_0 \in \Gamma} \operatorname{dist}(\mathcal{S}(x_0), \mathcal{S}(y_0))}\right\}. \tag{58}$$

□

Summarizing the properties of the map τ obtained so far, we can say that in the case of a bounded domain with analytic boundary, different from a disk:

(1) τ is locally Lipschitz continuous on $\Omega \setminus \tilde{\mathcal{S}}$ and almost everywhere

$$|\nabla\tau(x)| \leq \frac{C}{d_{\tilde{\mathcal{S}}}(x)^{\frac{1}{2\bar{n}-1}}},$$

where $C > 0$ and $\bar{n} \in \mathbb{N}$ are the ones of Corollary 4.8.

(2) τ is continuous on Ω.

(3) τ is Lipschitz continuous on $\tilde{\mathcal{S}}$ and $\tilde{\mathcal{S}}$ has empty interior.

Now we are going to show that (1)–(3) are enough to conclude the Hölder continuity of τ on the whole set Ω. Aiming at this, we need another technical lemma.

Lemma 4.10. *Let $\phi : [0, 1] \to \mathbb{R}$ be a locally Lipschitz continuous function on $(0, 1)$ such that for some $\alpha \in (0, 1)$ $|\phi'(t)| \leq \frac{C}{t^\alpha}$ almost everywhere and ϕ is continuous on $[0, 1]$. Then $|\phi(1) - \phi(0)| \leq \frac{C}{1-\alpha}$.*

Proof. For any $s \in (0,1)$

$$|\phi(1) - \phi(s)| \leq \left| \int_s^1 \phi'(u)\, du \right| \leq \int_s^1 \frac{C}{u^\alpha}\, du = \frac{C}{1-\alpha} \left[u^{1-\alpha}\right]_s^1 \leq \frac{C}{1-\alpha}.$$

Letting $s \to 0^+$ and using the continuity of ϕ the previous inequality gives the result. □

Theorem 4.11. *Let Ω be a bounded domain with analytic boundary, different from a disk. Suppose also that the set of regular conjugate points is nonempty. Then τ is Hölder continuous in Ω with exponent $\frac{2\bar{n}-2}{2\bar{n}-1}$, being $\bar{n}$ as in Corollary* 4.8. *In particular, the map τ is at least $2/3$-Hölder continuous.*

Proof. Since $\tilde{\mathcal{S}}$ has empty interior and τ in continuous on Ω, it is enough to show that there exists some constant $C' > 0$ such that

$$|\tau(x) - \tau(y)| \leq C'|x-y|^{\frac{2\bar{n}-2}{2\bar{n}-1}} \qquad \forall x, y \in \Omega \setminus \tilde{\mathcal{S}}. \tag{59}$$

We distinguish two cases.

Case 1: Assume that $\max\{d_{\tilde{\mathcal{S}}}(x), d_{\tilde{\mathcal{S}}}(y)\} \leq 2|x-y|$. Then

$$|\tau(x) - \tau(y)| \leq |\tau(x) - \tau(x_1)| + |\tau(x_1) - \tau(y_1)| + |\tau(y_1) - \tau(y)|,$$

where x_1 and y_1 belong to the projection set of x and y on $\tilde{\mathcal{S}}$ respectively. Now set

$$\phi(s) := \tau(x_1 + s(x - x_1)), \qquad \text{for } s \in [0,1].$$

Since $x_1 + s(x-x_1) \notin \tilde{\mathcal{S}}$ for $s \in (0,1]$ and $\phi'(s) = \nabla\tau(x_1 + s(x-x_1)) \cdot (x - x_1)$ almost everywhere, we have by property (1). above that

$$|\phi'(s)| \leq \frac{C|x-x_1|}{d_{\tilde{\mathcal{S}}}(x_1 + s(x-x_1))^{\frac{1}{2\bar{n}-1}}} \qquad \text{a.e. } s \in (0,1],$$

where $C > 0$ and $\bar{n} \in \mathbb{N}$ are the ones of Corollary 4.8. Also notice that ϕ is continuous on $[0,1]$ because τ is continuous on Ω and that $d_{\tilde{\mathcal{S}}}(x_1 + s(x-x_1)) = s|x-x_1|$. Hence we can apply Lemma 4.10 to ϕ, obtaining

$$|\tau(x) - \tau(x_1)| = |\phi(1) - \phi(0)| \leq \frac{C(2\bar{n}-1)}{2\bar{n}-2}|x-x_1|^{\frac{2\bar{n}-2}{2\bar{n}-1}} \leq 2C|x-x_1|^{\frac{2\bar{n}-2}{2\bar{n}-1}}. \tag{60}$$

Arguing in the same way for y we get $|\tau(y) - \tau(y_1)| \leq 2C|y-y_1|^{\frac{2\bar{n}-2}{2\bar{n}-1}}$. Moreover, being τ Lipschitz continuous of constant L on $\tilde{\mathcal{S}}$ (see (58)), then

$$|\tau(x) - \tau(y)| \leq 2C\left[|x-x_1|^{\frac{2\bar{n}-2}{2\bar{n}-1}} + |y-y_1|^{\frac{2\bar{n}-2}{2\bar{n}-1}}\right] + L|y_1 - x_1|.$$

By assumption $|x-x_1| = d_{\tilde{\mathcal{S}}}(x) \leq 2|x-y|$ and $|y-y_1| = d_{\tilde{\mathcal{S}}}(y) \leq 2|x-y|$. Thus $|y_1 - x_1| \leq |y-y_1| + |x-y| + |x-x_1| \leq 5|x-y|$. Therefore, setting

$$C' := 2 \cdot 2^{\frac{2\bar{n}-2}{2\bar{n}-1}}C + 5L\mathrm{diam}(\Omega)^{\frac{1}{2\bar{n}-1}},$$

we conclude that $|\tau(x) - \tau(y)| \leq C'|x-y|^{\frac{2\bar{n}-2}{2\bar{n}-1}}$ in the above hypotheses.

Case 2: Suppose now that $\max\{d_{\tilde{\mathcal{S}}}(x), d_{\tilde{\mathcal{S}}}(y)\} > 2|x-y|$. Without loss of generality we can assume that $d_{\tilde{\mathcal{S}}}(x) > 2|x-y|$. Then for any $z \in [x,y]$ we have

$$d_{\tilde{\mathcal{S}}}(z) \geq d_{\tilde{\mathcal{S}}}(x) - |z-x| \geq 2|x-y| - |y-x| = |y-x|.$$

Hence the map $\phi(s) := \tau(x + s(y-x))$ is well defined and satisfies

$$|\phi'(s)| \leq \frac{C|x-y|}{d_{\tilde{\mathcal{S}}}(x+s(y-x))^{\frac{1}{2\bar{n}-1}}} \leq \frac{C|x-y|}{|x-y|^{\frac{1}{2\bar{n}-1}}} = C|x-y|^{\frac{2\bar{n}-2}{2\bar{n}-1}}$$

almost everywhere. Hence

$$|\tau(x) - \tau(y)| = |\phi(1) - \phi(0)| \leq C|x-y|^{\frac{2\bar{n}-2}{2\bar{n}-1}}.$$

Since $C' > C$, (59) is proven. □

5. An application to granular matter theory: Regularity results

In this section we will apply the results obtained so far for the normal distance to the cut locus to investigate the regularity of the solutions of a boundary value problem that arises in the framework of granular matter theory.

The equilibrium configuration that may occur to a growing sandpile generated by a stationary source $f \geq 0$ onto a table $\Omega \subset \mathbb{R}^2$ can be described by the system of partial differential equations

$$\begin{cases} -\mathrm{div}(vDu) = f & \text{in } \Omega, \\ v \geq 0,\ |Du| \leq 1 & \text{in } \Omega, \\ |Du| - 1 = 0 & \text{in } \{v > 0\}, \end{cases} \tag{61}$$

complemented with the conditions

$$\begin{cases} u \geq 0, & \text{in } \Omega, \\ u \equiv 0 & \text{on } \partial\Omega. \end{cases} \tag{62}$$

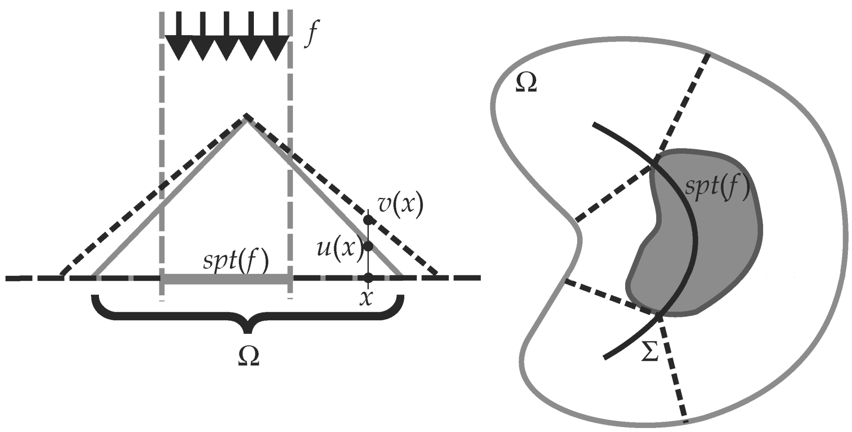

Figure 3. The stationary model

The model comes essentially from the work of Hadeler and Kuttler [10], built on previous work by Boutreux and de Gennes [3]. The physical model presents $u(x)$ and $v(x)$, respectively, as the heights of the *standing* and *rolling layers* at a point $x \in \Omega$. Indeed, u represents the amount of matter that remains at rest, while v describes matter moving down along the surface of the standing layer and falling from the table when the base of the heap touches the boundary of Ω. For physical reasons, the slope of the standing layer cannot exceed a given constant—typical of the matter under consideration—that we normalize to 1. Consequently, the standing layer must vanish on the boundary of the table. So, $|Du| \leq 1$ in Ω and $u = 0$ on $\partial\Omega$. Also, in the region where v is positive, the standing layer has to be "maximal", for otherwise more matter would roll down there to rest. On the other hand, the rolling layer results from transporting matter along the surface of the standing layer at a speed that is assumed proportional to the slope Du, with constant equal to 1.

A correct representation formula for the (pointwise) solution of (61)–(62) is provided in [10] in 1 space dimension.

Recently, the first two authors of this paper [4] have obtained the 2-dimensional representation formula for the solution of problem (61)–(62), starting from the physical considerations of Hadeler and Kuttler [10]. Due to the lack of regularity of the solutions of the eikonal equation $|Du| = 1$ and of the conservation law $-\mathrm{div}(vDu) = f$, the solutions of problem (61)–(62) are taken to be understood in the following sense.

A pair (u, v) of *continuous* functions in Ω is a solution of problem (61)–(62) if

(a) $u = 0$ on $\partial\Omega$, $\|Du\|_{\infty,\Omega} \leq 1$, and u is a viscosity solution of

$$|Du| = 1 \quad \text{in} \quad \{x \in \Omega \ : \ v(x) > 0\}$$

(b) $v \geq 0$ in Ω and, for every test function $\phi \in \mathcal{C}_c^\infty(\Omega)$,

$$\int_\Omega v(x)\langle Du(x), D\phi(x)\rangle dx = \int_\Omega f(x)\phi(x)dx\,.$$

The paper [4] provides a complete description of system (61)–(62) in the plane, with an explicit formula for its solutions and a uniqueness result. Indeed, in the case when $\Omega \subset \mathbb{R}^2$ is a bounded domain with boundary of class $\mathcal{C}^2$ and $f \geq 0$ is a continuous function in Ω, it is proven that the unique solution of system (61)–(62) is the pair (d, v_f), where d is the distance function from $\partial\Omega$, $v_f = 0$ on $\overline{\Sigma}$ and

$$v_f(x) = \begin{cases} \displaystyle\int_0^{\tau(x)} f(x + tDd(x))\frac{1-(d(x)+t)\kappa(x)}{1-d(x)\kappa(x)}\,dt & \forall x \in \overline{\Omega}\backslash\overline{\Sigma}, \\ 0 & \forall x \in \overline{\Sigma}, \end{cases} \tag{63}$$

Here, $\kappa(x)$ denotes the curvature of $\partial\Omega$ at the projection point of x and $\tau(\cdot)$ is the normal distance to $\overline{\Sigma}$.

The regularity of the distance function, depending on the boundary regularity of Ω, is well known. Hence, we will analyze the regularity of the map v_f only. Our main result is that v_f is a Hölder continuous function on Ω under the standing assumptions that

$$f \text{ is a Lipschitz continuous function in } \Omega \tag{64}$$

and

$$\begin{gathered}\Omega \text{ is a bounded domain of } \mathbb{R}^2 \\ \text{with analytic boundary, different from a disk.}\end{gathered}$$

Remark 5.1. As in the case of the maximal retraction length of Ω onto $\overline{\Sigma}$, we exclude a priori the case when Ω is a disk, because otherwise v_f is Lipschitz continuous as soon as f is. Indeed, let $\Omega = B_R(0)$, for some $R > 0$. Then,

$$d(x) = R - |x|, \qquad \tau(x) = |x|, \qquad \kappa(x) = 1/R, \qquad \forall x \in \overline{B}_R(0).$$

Hence, for any choice of $x,\, y \in \overline{B}_R(0) \setminus \{0\}$ with $|x| \geq |y|$ we have

$$\begin{aligned} |v_f(y) - v_f(x)| &= \left| \int_0^{\tau(y)} f(y + tDd(y)) \frac{1 - (d(y)+t)\kappa(y)}{1 - d(y)\kappa(y)}\, dt \right. \\ &\qquad \left. - \int_0^{\tau(x)} f(x + tDd(x)) \frac{1 - (d(x)+t)\kappa(x)}{1 - d(x)\kappa(x)}\, dt \right| \\ &\leq \int_0^{|y|} \frac{1 - (d(x)+t)\kappa(x)}{1 - d(x)\kappa(x)} \left| f(y + tDd(y)) - f(x + tDd(x)) \right|\, dt \\ &\quad + \int_0^{|y|} f(y + tDd(y)) \left| \frac{1 - (d(y)+t)\kappa(y)}{1 - d(y)\kappa(y)} - \frac{1 - (d(x)+t)\kappa(x)}{1 - d(x)\kappa(x)} \right|\, dt \\ &\quad + \int_{|y|}^{|x|} f(x + tDd(x)) \frac{1 - (d(x)+t)\kappa(x)}{1 - d(x)\kappa(x)}\, dt \\ &=: I_1 + I_2 + I_3; \end{aligned}$$

but,

$$\begin{aligned} |f(y + tDd(y)) - f(x + tDd(x))| &\leq \|f\|_{Lip} \left| x - t\tfrac{x}{|x|} - y + t\tfrac{y}{|y|} \right| \\ &\leq \|f\|_{Lip} \left\{ |x - y| + t \left| \tfrac{|x|y - |y|x}{|x||y|} \right| \right\}, \\ &\leq \|f\|_{Lip} |x - y| \left\{ 1 + t \left(\tfrac{1}{|y|} + \tfrac{1}{|x|} \right) \right\} \end{aligned}$$

$$\left| \frac{1 - (d(y)+t)\kappa(y)}{1 - d(y)\kappa(y)} - \frac{1 - (d(x)+t)\kappa(x)}{1 - d(x)\kappa(x)} \right| = t \left| \frac{|y| - |x|}{|x||y|} \right|$$

and

$$\frac{1 - (d(x)+t)\kappa(x)}{1 - d(x)\kappa(x)} = \frac{|x| - t}{|x|} \leq 1.$$

Hence,

$$I_1 \leq \|f\|_{Lip} |x - y| \left\{ |y| + \frac{|y|^2}{2} \left(\frac{1}{|y|} + \frac{1}{|x|} \right) \right\} \leq \|f\|_{Lip} (R + 1) |x - y|,$$

$$I_2 \leq \|f\|_\infty \frac{|y|^2}{2} \left| \frac{|y| - |x|}{|x||y|} \right| \leq \frac{\|f\|_\infty}{2} |x - y|$$

and

$$I_3 \leq \|f\|_\infty (|x| - |y|) \leq \|f\|_\infty |x - y|,$$

which gives the Lipschitz continuity of v_f in $\overline{B}_R(0)$.

We also remark that if Ω is not a disk, Lipschitz continuity may fail, as it can be seen by considering the parabola case together with the choice $f \equiv 1$ in $\overline{\Omega}$.

In what follows, we will denote by $\tilde{\Sigma}$ the set

$$\tilde{\Sigma} = \Sigma \cup \left(\bigcup_{x \in \partial\Omega,\, \tau(x)\kappa(x) = 1} [x, x + \tau(x)Dd(x)] \right). \tag{65}$$

The precise regularity statement for v_f is the following.

Theorem 5.2. *Assume that f is a Lipschitz continuous function and that Ω is a bounded domain of $\mathbb{R}^2$ with analytic boundary, different from a disk. Then v_f is a Hölder continuous function with exponent $\frac{1}{2m-1}$ for some suitable $m \in \mathbb{N}$, $m \geq 2$ depending on the geometry of $\partial\Omega$.*

Proof. The regularity result on v_f will be proven by an argument similar to the one used to prove the Hölder continuity of the normal distance τ. We will then divide the proof in several steps, aiming at the following goals:

Step 1. We prove that for all $x \in \Omega \setminus \tilde{\Sigma}$ there exists a ball $B_r(x) \subset \Omega \setminus \tilde{\Sigma}$, with $r = r(x) > 0$, such that for any $y \in B_r(x)$

$$v_f(y) - v_f(x) \leq C \left(1 + \frac{\tau(x)^2}{(1 - d(x)\kappa(x))^3} + \frac{1}{d_{\tilde{\Sigma}}(x)^{\frac{1}{2\bar{n}-1}}} \right) |x - y|, \tag{66}$$

where $C > 0$ depends on f and Ω only, and $\bar{n} \in \mathbb{N}$ is the integer that appears in Theorem 4.1.

Step 2. We show that for all $x \in \Omega \setminus \tilde{\Sigma}$

$$\frac{\tau(x)^2}{(1 - d(x)\kappa(x))^3} \leq \frac{C}{d_{\tilde{\Sigma}}(x)^{\frac{2m-2}{2m-1}}} \tag{67}$$

for some constant $C > 0$ independent of x and some $m \in \mathbb{N}$, $2 \leq \bar{n} \leq m$. In this way we can rewrite (66) as

$$v_f(y) - v_f(x) \leq \frac{C}{d_{\tilde{\Sigma}}(x)^{\frac{2m-2}{2m-1}}} |x - y|, \tag{68}$$

where $C > 0$ is some constant independent of x.

Step 3. We show that v_f is differentiable almost everywhere in $\Omega \setminus \tilde{\Sigma}$, with

$$|\nabla v_f(x)| \leq \frac{C}{d_{\tilde{\Sigma}}(x)^{\frac{2m-2}{2m-1}}}, \qquad \text{a.e.} \tag{69}$$

Step 4. We prove the Lipschitz continuity of v_f when restricted to $\tilde{\Sigma}$.

Step 5. We conclude the proof as in Theorem 4.1 by applying Lemma 4.10.

Let us begin our argument.

Step 1. Consider any $x \in \Omega \setminus \tilde{\Sigma}$ and set $\eta(x) := \frac{\kappa(x)}{1 - d(x)\kappa(x)}$, so that we can write $\frac{1-(d(x)+t)\kappa(x)}{1-d(x)\kappa(x)} = 1 - t\eta(x)$ in (63). Notice that

$$1 - t\eta(x) \leq 1 + \operatorname{diam}(\Omega) \max_{x \in \partial\Omega} [\kappa(x)]_-, \qquad \forall\, x \in \Omega \setminus \tilde{\Sigma},\, t \in [0, \tau(x)) \tag{70}$$

and

$$|\eta(x)|\tau(x) \leq \max \left\{ 1; \operatorname{diam}(\Omega) \max_{x \in \partial\Omega} [\kappa(x)]_- \right\}, \qquad \forall\, x \in \Omega \setminus \tilde{\Sigma}, \tag{71}$$

where $[\kappa(x)]_- := \max\{0, -\kappa(x)\}$. Now, for any $x \in \Omega \setminus \tilde{\Sigma}$ choose $r > 0$ such that $B_r(x) \subset \Omega \setminus \tilde{\Sigma}$ and

$$\begin{cases} |\eta(y)| \leq |\eta(x)| + 1, \\ \dfrac{1}{1 - d(y)\kappa(y)} \leq \dfrac{2}{1 - d(x)\kappa(x)}, \end{cases} \qquad \text{for all } y \in B_r(x). \tag{72}$$

Suppose first that $\tau(y) \le \tau(x)$. Thus, for any $y \in B_r(x)$

$$
\begin{aligned}
v_f(y) - v_f(x) = & \int_0^{\tau(y)} f(y + tDd(y))(1 - t\eta(y))\, dt \\
& - \int_0^{\tau(x)} f(x + tDd(x))(1 - t\eta(x))\, dt \\
\le & \int_0^{\tau(y)} [f(y + tDd(y))(1 - t\eta(y)) - f(x + tDd(x))(1 - t\eta(x))]\, dt \\
\le & \int_0^{\tau(x)} |(1 - t\eta(x))\,(f(y + tDd(y)) - f(x + tDd(x)))|\, dt \\
& + \|f\|_\infty \frac{\tau(x)^2}{2} |\eta(y) - \eta(x)| \\
=: & I_1 + I_2. \qquad (73)
\end{aligned}
$$

Observe that (70) and (72) give

$$
\begin{aligned}
I_1 & \le \|f\|_{Lip} \int_0^{\tau(x)} (1 - t\eta(x))\,[|x - y| + t|Dd(x) - Dd(y)|]\, dt \\
& \le \|f\|_{Lip}(1 + \mathrm{diam}(\Omega) \max_{x \in \partial\Omega} [\kappa(x)]_-)\tau(x)|x - y| \\
& + \|f\|_{Lip}(1 + \mathrm{diam}(\Omega) \max_{x \in \partial\Omega} [\kappa(x)]_-) \frac{\tau(x)^2}{2} |Dd(x) - Dd(y)|,
\end{aligned}
$$

where

$$
\begin{aligned}
|Dd(x) - Dd(y)| & = \left| \int_0^1 D^2 d(y + t(x - y)) \cdot (x - y)\, dt \right| \\
& = \left| \int_0^1 -\eta(x_t)(\mathcal{R}Dd(x_t) \otimes \mathcal{R}Dd(x_t)) \cdot (x - y)\, dt \right| \\
& \le \int_0^1 |\eta(x_t)|\, dt\, |x - y| \\
& \le (|\eta(x)| + 1)|x - y|, \qquad (74)
\end{aligned}
$$

(in the above inequalities x_t stands for $y + t(x - y)$). Applying (71), we can write

$$I_1 \le C_1 |x - y|$$

where C_1 is a constant depending on the data f and Ω, but independent of the choice of x. So it remains to estimate I_2. We have

$$
\begin{aligned}
\tau(x)^2|\eta(x) - \eta(y)| & = \tau(x)^2 \frac{|\kappa(x) - \kappa(y) - (d(x) - d(y))\kappa(x)\kappa(y)|}{(1 - d(x)\kappa(x))(1 - d(y)\kappa(y))} \\
& \le \frac{\tau(x)^2|\kappa(x) - \kappa(y)|}{(1 - d(x)\kappa(x))(1 - d(y)\kappa(y))} + \frac{\tau(x)^2|d(x) - d(y)|\, |\kappa(x)\kappa(y)|}{(1 - d(x)\kappa(x))(1 - d(y)\kappa(y))} \\
& =: I_{21} + I_{22}.
\end{aligned}
$$

Now, exploiting (71), (72) and (74) we get

$$
\begin{aligned}
\frac{\tau(x)^2|\kappa(x)\kappa(y)|}{(1 - d(x)\kappa(x))(1 - d(y)\kappa(y))} & = \tau(x)^2|\eta(x)\eta(y)| \\
& \le \tau(x)^2|\eta(x)|(1 + |\eta(x)|) \le C_{22},
\end{aligned}
$$

and then

$$I_{22} \le C_{22}|x-y|, \tag{75}$$

where C_{22} is a constant depending on Ω and independent of x. On the other hand, denoting by $\|\kappa\|_{Lip}$ the Lipschitz constant of κ over $\partial\Omega$, we have

$$\begin{aligned} I_{21} &= \tau(x)^2 \frac{|\kappa(x)-\kappa(y)|}{(1-d(x)\kappa(x))(1-d(y)\kappa(y))} \\ &= \tau(x)^2 \frac{|\kappa(x-d(x)Dd(x))-\kappa(y-d(y)Dd(y))|}{(1-d(x)\kappa(x))(1-d(y)\kappa(y))} \\ &\le \tau(x)^2 \frac{\|\kappa\|_{Lip}\,(2|x-y|+\mathrm{diam}(\Omega)|Dd(x)-Dd(y)|)}{(1-d(x)\kappa(x))(1-d(y)\kappa(y))} \\ &\le 2\tau(x)^2 \frac{\|\kappa\|_{Lip}\,(2|x-y|+\mathrm{diam}(\Omega)(|\eta(x)|+1)|x-y|)}{(1-d(x)\kappa(x))^2} \\ &\le C_{21}\frac{\tau(x)^2}{(1-d(x)\kappa(x))^3}|x-y|, \end{aligned}$$

where $C_{21} = C_{21}(\Omega)$ is independent of x. Summarizing the previous computations we can write

$$v_f(y) - v_f(x) \le \tilde{C}_1\left(1 + \frac{\tau(x)^2}{(1-d(x)\kappa(x))^3}\right)|x-y|, \tag{76}$$

where $\tilde{C}_1$ is a constant depending on the data f and Ω, but independent of the choice of x. If we consider the case of $\tau(y) > \tau(x)$, it is easy to see that inequality (73) becomes

$$\begin{aligned} v_f(y) - v_f(x) = &\int_0^{\tau(y)} f(y+tDd(y))(1-t\eta(y))\,dt \\ &- \int_0^{\tau(x)} f(x+tDd(x))(1-t\eta(x))\,dt \\ \le &\left|\int_0^{\tau(x)} [f(y+tDd(y))(1-t\eta(y)) - f(x+tDd(x))(1-t\eta(x))]\,dt\right| \\ &+ \left|\int_{\tau(x)}^{\tau(y)} f(y+tDd(y))(1-t\eta(y))\,dt\right| \\ \le &\int_0^{\tau(x)} |(1-t\eta(x))\,(f(y+tDd(y)) - f(x+tDd(x)))|\,dt \\ &+\ \|f\|_\infty \frac{\tau(x)^2}{2}|\eta(y)-\eta(x)| + \int_{\tau(x)}^{\tau(y)} f(y+tDd(y))(1-t\eta(y))\,dt \\ =: &\ I_1 + I_2 + I_3, \end{aligned} \tag{77}$$

where I_1 and I_2 are exactly the same as in (73). Moreover,

$$I_3 \le (1+\mathrm{diam}(\Omega)\max_{x\in\partial\Omega}[\kappa(x)]_-)\|f\|_\infty(\tau(y)-\tau(x)).$$

Hence, by Theorem 4.1, Corollary 4.8 and inclusion $\tilde{\mathcal{S}} \subset \tilde{\Sigma}$ we deduce that

$$I_3 \le \frac{C_3}{d_{\tilde{\mathcal{S}}}(x)^{\frac{1}{2\bar{n}-1}}}|x-y| \le \frac{C_3}{d_{\tilde{\Sigma}}(x)^{\frac{1}{2\bar{n}-1}}}|x-y|, \tag{78}$$

for some constants $C_3 > 0$, $\bar{n} \in \mathbb{N}$ which depend on the data f and Ω, but are independent of the choice of x. Therefore if $\tau(y) > \tau(x)$ we have

$$(79)\quad v_f(y) - v_f(x) \leq \tilde{C}_2 \left(1 + \frac{\tau(x)^2}{(1 - d(x)\kappa(x))^3} + \frac{1}{d_{\tilde{\Sigma}}(x)^{\frac{1}{2\bar{n}-1}}} \right) |x - y|, \qquad \tilde{C}_2 > 0.$$

The first step is then concluded with the choice of $C = \tilde{C}_2$ in (66).

Step 2. Let us now estimate

$$E(x) := \frac{\tau(x)^2}{(1 - d(x)\kappa(x))^3}.$$

We can restrict our attention to a neighborhood of the set of conjugate points, since we can globally bound $E(x)$ on the complement of such a set. In order to do so, we first give some local estimates around each conjugate point. Since the set of conjugate points (regular and singular) is finite, the local estimates suffice to derive a global one. Let us start by considering the case of a regular conjugate point x_0. Once again suppose that $x_0 = (0, r)$, $r > 0$, and $\Pi(x_0) = \{(0, 0)\}$. Moreover, suppose that $\partial\Omega$ is locally the graph of an analytic function $\alpha : (-s_0, s_0) \to \mathbb{R}$ such that $\alpha(0) = 0$, $\alpha'(0) = 0$ and $\alpha''(0) = \frac{1}{r}$ and let $n = n(x_0) \geq 2$ be the integer such that representation (31) holds true. We claim that for some h_0 sufficiently small there exists a constant $C_0 > 0$ such that

$$(80)\qquad d_{\tilde{\Sigma}}(x) \leq C_0 \tau(x) s, \qquad \forall\, x \in B_{h_0}(x_0) \setminus \tilde{\Sigma},$$

where s is the parameter defining the projection $(s, \alpha(s))$ of the point x. Indeed, let $t(s)$ and $\xi(s)$ be as in Lemma 4.5, and y be the point on the line segment $[(-t(s), \alpha(-t(s))); \xi(s)]$ satisfying $|y - \xi(s)| = |x - \xi(s)|$. Then, from an elementary geometric argument,

$$\frac{|y - x|}{((s + t(s))^2 + (\alpha(s) - \alpha(-t(s)))^2)^{1/2}} = \frac{\tau(x)}{d(\xi(s))}.$$

Recalling that, by definition, $d(\xi(s)) \geq d(x_0)$ and that

$$\left((s + t(s))^2 + (\alpha(s) - \alpha(-t(s)))^2\right)^{1/2} = 2s + \mathrm{o}(s),$$

we get

$$|y - x| \leq C_0 \tau(x) s, \qquad \forall\, x \in B_{h_0}(x_0) \setminus \tilde{\Sigma},$$

for some $C_0 > 0$, provided that s is sufficiently small and a fortiori for h_0 small enough. Thus (80) follows, since segment $[x, y]$ intersects $\tilde{\Sigma}$ and then $d_{\tilde{\Sigma}}(x) \leq |y - x|$. Now, let us conclude the estimate of $E(x)$ in the set $B_{h_0}(x_0) \setminus \tilde{\Sigma}$. We distinguish two cases: $\tau(x) > |\xi(s) - x_0|$ and $\tau(x) \leq |\xi(s) - x_0|$. In what follows we will assume (eventually reducing h_0) that $\kappa(x) \geq \frac{\kappa(x_0)}{2} = \frac{1}{2r}$ in $B_{h_0}(x_0) \setminus \tilde{\Sigma}$, since κ is a continuous function on $\Omega \setminus \Sigma$. Suppose first that $\tau(x) > |\xi(s) - x_0|$. Then, taking into account that $\xi(s) = \left(\mathrm{o}(s^{2n-2}), r + 2nr^2 a s^{2n-2} + \mathrm{o}(s^{2n-2})\right)$, we have

$$(81)\qquad |\xi(s) - x_0| = 2nr^2 a s^{2n-2} + \mathrm{o}(s^{2n-2}) \leq C s^{2n-2}$$

provided h_0 is small enough. Thus, since $1-d(x)\kappa(x) \geq \tau(x)\kappa(x)$ (because $(\tau(x)+d(x))\kappa(x) \leq 1$),

$$\begin{aligned} d_{\tilde{\Sigma}}(x)^\alpha \frac{\tau(x)^2}{(1-d(x)\kappa(x))^3} &\leq \frac{C_0^\alpha \tau(x)^{2+\alpha} s^\alpha}{(\tau(x)\kappa(x))^3} \\ &\leq \frac{C_1 s^\alpha}{|\xi(s)-x_0|^{1-\alpha}} \leq C_2 s^{\alpha(2n-1)-2n+2}. \end{aligned} \tag{82}$$

On the other hand, when $\tau(x) \leq |\xi(s)-x_0|$, taking into account that by (55)

$$1-d(\xi(s))\kappa(\xi(s)) = 4n(n-1)ras^{2n-2} + \mathrm{o}(s^{2n-2}) \geq \tilde{C}s^{2n-2}$$

(for h_0 is sufficiently small and some $\tilde{C}>0$), we have

$$\begin{aligned} d_{\tilde{\Sigma}}(x)^\alpha \frac{\tau(x)^2}{(1-d(x)\kappa(x))^3} &\leq \frac{C_0^\alpha \tau(x)^{2+\alpha} s^\alpha}{(1-d(\xi(s))\kappa(\xi(s)))^3} \\ &\leq \frac{C_3 s^\alpha |\xi(s)-x_0|^{\alpha+2}}{s^{3(2n-2)}} \leq C_4 s^{\alpha(2n-1)-2n+2}. \end{aligned} \tag{83}$$

In both inequalities (82) and (83) the exponent $\alpha = \frac{2n-2}{2n-1} < 1$ guarantees that there is some $h_0 > 0$ and a constant $C > 0$ such that

$$E(x) \leq \frac{C}{d_{\tilde{\Sigma}}(x)^{\frac{2n-2}{2n-1}}}, \qquad \forall\, x \in B_{h_0}(x_0) \setminus \tilde{\Sigma}. \tag{84}$$

Now, let us estimate $E(x)$ near any singular conjugate point $\tilde{x}_0$. Recall that $\tilde{x}_0$ is singular and conjugate if $\tau(y)\kappa(y) = 1$ for some $y \in \Pi(\tilde{x}_0)$. Hence it is easy to see that $E(x)$ can explode only if x approaches $\tilde{x}_0$ near the projecting line on y. But then the local behaviour of the boundary $\partial\Omega$ around y is not different from the regular conjugate point case. In particular, the local representation of $\partial\Omega$ via an analytic map α holds true as before. Moreover, take any analytic curve starting from $\tilde{x}_0$ with direction $Dd(y)$ and let Σ^* be the union of the trace of such a curve with the line segment $[\tilde{x}_0; y]$. Then, $d_{\tilde{\Sigma}}(x) \leq d_{\Sigma^*}(x)$ for any x near the projecting line on y, because the singular arcs starting from $\tilde{x}_0$ have initial directions that are transversal to $Dd(y)$. Notice that if $\tilde{x}_0$ was a regular conjugate point, then $d_{\Sigma^*}(x)$ would coincide with the distance $d_{\tilde{\Sigma}}(x)$. Therefore, the previous inequality and (82)–(83) give

$$d_{\tilde{\Sigma}}(x)^{\frac{2n-2}{2n-1}} \frac{\tau(x)^2}{(1-d(x)\kappa(x))^3} \leq d_{\Sigma^*}(x)^{\frac{2n-2}{2n-1}} \frac{\tau(x)^2}{(1-d(x)\kappa(x))^3} \leq C, \tag{85}$$

where $n = n(\tilde{x}_0) \geq 2$ is the integer that appears in (31). Now, being $\Gamma \cup \tilde{\Gamma}$ a finite set, the maximum $m \geq 2$ of all integers n selected in the previous computations fits for the estimates (84) and (85) in a suitable neighborhood of $\tilde{\Sigma}$. Moreover, $2 \leq \bar{n} \leq m$, where $\bar{n}$ is the integer that appears in (66). Hence,

$$\frac{1}{2\bar{n}-1} \leq \frac{2\bar{n}-2}{2\bar{n}-1} \leq \frac{2m-2}{2m-1},$$

which means that actually the estimate (66) reads

$$v_f(y) - v_f(x) \leq \tilde{C}\left(1 + \frac{1}{d_{\tilde{\Sigma}}(x)^{\frac{2m-2}{2m-1}}}\right)|x-y|, \qquad \text{for some } \tilde{C} > 0. \tag{86}$$

Since $d_{\tilde{\Sigma}}(x) \leq \mathrm{diam}(\Omega)$, we finally get (68) by taking $C = \tilde{C}\left(\mathrm{diam}(\Omega)^{\frac{2m-2}{2m-1}} + 1\right)$.

Step 3. As in Proposition 4.3 it suffices to prove that for any $x \in \Omega \setminus \tilde{\Sigma}$ we have

$$|p| \leq \frac{C}{d_{\tilde{\Sigma}}(x)^{\frac{2m-2}{2m-1}}} \qquad \forall\, p \in \partial_P v_f(x) \quad \forall\, x \in \Omega \setminus \tilde{\Sigma}, \tag{87}$$

where $\partial_P v_f(x)$ denotes the proximal subgradient of v_f at x. By definition, a vector $p \in \mathbb{R}^2$ belongs to $\partial_P v_f(x)$ if and only if there exist numbers $\sigma, \eta > 0$ such that

$$v_f(y) \geq v_f(x) + \langle p, y - x \rangle - \sigma |y - x|^2 \qquad \forall\, y \in B_\eta(x)\,;$$

see [7, Theorem 2.5, p. 33]. Now, combine the above inequality with (68) to obtain

$$\langle p, y - x \rangle \leq \frac{C}{d_{\tilde{\Sigma}}(x)^{\frac{2m-2}{2m-1}}} |y - x| + \sigma |y - x|^2$$

whenever $|y - x| < \min\{r, \eta\}$. The last inequality implies (87). The differentiability almost everywhere of v_f in $\Omega \setminus \tilde{\Sigma}$ and (69) follow from (84), as in Proposition 4.3.

Step 4. Let us prove the Lipschitz continuity of v_f on $\tilde{\Sigma}$. By definition, $v_f \equiv 0$ on $\overline{\Sigma}$. Moreover, by [4, Proposition 3.1] and [4, Proposition 4.2], for any $x \in \Omega \setminus \overline{\Sigma}$ and $\theta \in (0, \tau(x))$, we have

$$v_f(x) \leq \|f\|_\infty \left[1 + \max_{x \in \partial\Omega} [\kappa(x)]_-\right] \tau(x) =: K_- \tau(x) \tag{88}$$

and

$$\begin{aligned} & v_f(x) - \frac{1 - (d(x) + \theta)\kappa(x)}{1 - d(x)\kappa(x)} v_f(x + \theta Dd(x)) \\ & = \int_0^\theta f(x + tDd(x)) \frac{1 - (d(x) + t)\kappa(x)}{1 - d(x)\kappa(x)}\, dt. \end{aligned}$$

Hence

(i) if $x \in \,]x_0, x_0 + \tau(x_0)Dd(x_0)]$ for some $x_0 \in \partial\Omega$ such that $\kappa(x_0)\tau(x_0) = 1$ and $y = x_0 + \tau(x_0)Dd(x_0)$, then

$$|v_f(x) - v_f(y)| = |v_f(x)| \leq K_- \tau(x) = K_- |x - y|; \tag{89}$$

(ii) if $x, y \in \,]x_0, x_0 + \tau(x_0)Dd(x_0)]$ for some $x_0 \in \partial\Omega$ such that $\kappa(x_0)\tau(x_0) = 1$, then we can suppose without loss of generality that $\tau(x) > \tau(y)$, obtaining

$$\begin{aligned} & |v_f(x) - v_f(y)| = |v_f(x) - v_f(x + |x - y| Dd(x))| \\ & = v_f(x + |x - y| Dd(x)) \left| 1 - \frac{1 - (d(x) + |x - y|)\kappa(x)}{1 - d(x)\kappa(x)} \right| \\ & + \int_0^{|x-y|} f(x + tDd(x)) \frac{1 - (d(x) + t)\kappa(x)}{1 - d(x)\kappa(x)}\, dt \\ & \leq K_- \tau(y) \frac{|x - y| \kappa(x)}{1 - d(x)\kappa(x)} + K_- |x - y| \qquad (90) \\ & \leq K_- |x - y| \frac{\tau(x)\kappa(x)}{1 - d(x)\kappa(x)} + K_- |x - y| \\ & \leq K_- \left(1 + \frac{\operatorname{diam}(\Omega)}{2} \max_{x \in \partial\Omega} [\kappa(x)]_-\right) |x - y| =: L_1 |x - y|; \end{aligned}$$

(iii) if

$$x \in \ [x_0, x_0 + \tau(x_0)Dd(x_0)] =: \mathcal{S}(x_0)$$

and

$$y \in\ [y_0, y_0 + \tau(y_0)Dd(y_0)] =: \mathcal{S}(y_0)$$

for some $x_0 \neq y_0 \in \partial\Omega$ with $\kappa(x_0)\tau(x_0) = \kappa(y_0)\tau(y_0) = 1$, then

$$\begin{aligned} |v_f(x) - v_f(y)| &\leq |v_f(x)| + |v_f(y)| \leq K_-\mathrm{diam}(\Omega) \\ &\leq \frac{K_-\mathrm{diam}(\Omega)}{\min_{x_0 \neq y_0 \in \Gamma\cup\tilde{\Gamma}} \mathrm{dist}(\mathcal{S}(x_0), \mathcal{S}(y_0))}|x-y| =: L_2|x-y|; \end{aligned} \tag{91}$$

(iv) if $x \in \mathcal{S}(x_0)$ for some $x_0 \in \partial\Omega$ with $\kappa(x_0)\tau(x_0) = 1$ and $y \in \Sigma$, we have to distinguish two subcases.

If $x_0 + \tau(x_0)Dd(x_0) \in \Gamma$, then the Lipschitz continuity of τ on $\Sigma \cup \mathcal{S}$ (see Lemma 4.9) gives

$$|v_f(x) - v_f(y)| = |v_f(x)| \leq K_-\tau(x) \leq K_-L|x-y| =: L_3|x-y|. \tag{92}$$

On the other hand, when $x_0 + \tau(x_0)Dd(x_0) \in \tilde{\Gamma}$, as in estimate (92) we find

$$|v_f(x) - v_f(y)| = |v_f(x)| \leq K_-\tau(x) = K_-|x - x_0|,$$

but we cannot directly apply Lemma 4.9, since it does not cover the case of segments starting from singular conjugate points. However, since Ω has finitely many conjugate points, we can affirm again (as in the regular conjugate points case) that there exists a cone $\tilde{\mathcal{C}_0}$, with apex $x_0 + \tau(x_0)Dd(x_0)$, semi–vertex angle $\tilde{\theta}_0 = \tilde{\theta}(x_0 + \tau(x_0)Dd(x_0)) > 0$ and symmetry axis containing the segment $\mathcal{S}(x_0)$, such that $\tilde{\mathcal{C}_0} \cap \Sigma = \emptyset$. Hence, $|x-y| > d_{\tilde{\mathcal{C}_0}}(x) = |x-x_0| \sin\tilde{\theta}_0$, which gives

$$|v_f(x) - v_f(y)| = |v_f(x)| \leq K_-\tau(x) = K_-|x - x_0| < \frac{K_-}{\sin\tilde{\theta}_0}|x-y|.$$

Defining $\tilde{\gamma}$ as the maximum of $1/\sin\tilde{\theta}_0$ over all $\tilde{\theta}_0$ related to singular conjugate points, we then obtain that

$$|v_f(x) - v_f(y)| = |v_f(x)| \leq K_-\tilde{\gamma}|x-y| =: L_4|x-y|. \tag{93}$$

The (global) Lipschitz continuity of v over all $\tilde{\Sigma}$ follows by taking as Lipschitz constant of v_f the maximal constant L_i, $i = 1, \dots, 4$, shown in the above inequalities (90)–(93).

Step 5. We proceed as in the proof of Theorem 4.1. Since $\tilde{\Sigma}$ has empty interior and v_f in continuous on Ω, it is enough to show that there exists some constant $C' > 0$ such that

$$|v_f(x) - v_f(y)| \leq C'|x-y|^{\frac{1}{2m-1}} \qquad \forall x, y \in \Omega \setminus \tilde{\Sigma}. \tag{94}$$

We distinguish two cases.

Case 1: Assume that $\max\{d_{\tilde{\Sigma}}(x), d_{\tilde{\Sigma}}(y)\} \leq 2|x-y|$. Then

$$|v_f(x) - v_f(y)| \leq |v_f(x) - v_f(x_1)| + |v_f(x_1) - v_f(y_1)| + |v_f(y_1) - v_f(y)|,$$

where x_1 and y_1 belong to the projection set of x and y on $\tilde{\Sigma}$ respectively. Now set

$$\phi(s) := v_f(x_1 + s(x - x_1)), \qquad \text{for } s \in [0,1].$$

Since $x_1 + s(x - x_1) \notin \tilde{\Sigma}$ for $s \in (0, 1]$ and $\phi'(s) = \nabla v_f(x_1 + s(x - x_1)) \cdot (x - x_1)$ almost everywhere, we have by (69) that

$$|\phi'(s)| \leq \frac{C|x-x_1|}{d_{\tilde{\mathcal{S}}}(x_1 + s(x-x_1))^{\frac{2m-2}{2m-1}}} \qquad \text{a.e. } s \in (0,1],$$

where $C > 0$ and $m \in \mathbb{N}$ are the ones of step 3. Moreover, $d_{\tilde{\Sigma}}(x_1 + s(x - x_1)) = s|x - x_1|$ and ϕ is continuous on $[0, 1]$ because v_f is continuous on Ω. Hence we can apply Lemma 4.10 to ϕ, obtaining

$$|v_f(x) - v_f(x_1)| = |\phi(1) - \phi(0)| \leq C(2m - 1)|x - x_1|^{\frac{1}{2m-1}}. \tag{95}$$

Arguing in the same way for y we get $|v_f(y) - v_f(y_1)| \leq C(2m - 1)|y - y_1|^{\frac{1}{2m-1}}$. Moreover, being v_f Lipschitz continuous–say of constant L–on $\tilde{\mathcal{S}}$ by step 4, then

$$|v_f(x) - v_f(y)| \leq C(2m - 1)\left[|x - x_1|^{\frac{1}{2m-1}} + |y - y_1|^{\frac{1}{2m-1}}\right] + L|y_1 - x_1|.$$

By assumption $|x - x_1| = d_{\tilde{\Sigma}}(x) \leq 2|x - y|$ and $|y - y_1| = d_{\tilde{\Sigma}}(y) \leq 2|x - y|$. Thus $|y_1 - x_1| \leq |y - y_1| + |x - y| + |x - x_1| \leq 5|x - y|$. Therefore, setting

$$C' := 2(2m - 1) \cdot 2^{\frac{1}{2m-1}} C + 5L\mathrm{diam}(\Omega)^{\frac{2m-2}{2m-1}},$$

we conclude that $|v_f(x) - v_f(y)| \leq C'|x - y|^{\frac{1}{2m-1}}$ in the above hypotheses.

Case 2: Suppose now that $\max\{d_{\tilde{\Sigma}}(x), d_{\tilde{\Sigma}}(y)\} > 2|x - y|$. Without loss of generality we can assume that $d_{\tilde{\Sigma}}(x) > 2|x - y|$. Then for any $z \in [x, y]$ we have

$$d_{\tilde{\Sigma}}(z) \geq d_{\tilde{\Sigma}}(x) - |z - x| \geq 2|x - y| - |y - x| = |y - x|.$$

Hence the map $\phi(s) := v_f(x + s(y - x))$ is well defined and satisfies

$$|\phi'(s)| \leq \frac{C|x - y|}{d_{\tilde{\Sigma}}(x + s(y - x))^{\frac{2m-2}{2m-1}}} \leq \frac{C|x - y|}{|x - y|^{\frac{2m-2}{2m-1}}} = C|x - y|^{\frac{1}{2m-1}}$$

almost everywhere. Hence

$$|v_f(x) - v_f(y)| = |\phi(1) - \phi(0)| \leq C|x - y|^{\frac{1}{2m-1}}.$$

Since $C' > C$, (94) is proven. □

Remark 5.3. As a concluding remark, we observe once again that when Ω is a bounded domain with no conjugate points (both regular and singular) and $\mathcal{C}^{2,1}$ boundary, then v_f is Lipschitz continuous on the whole set $\overline{\Omega}$, as long as f is, because of the Lipschitz continuity of τ (see Remark 4.4).

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