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DIFFERENTIAL EQUATIONS AND RECURSION RELATIONS FOR LAGUERRE FUNCTIONS ON SYMMETRIC CONES

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ABSTRACT. We obtain the differential equation and recurrence relations satisfied by the Laguerre functions $l_{\mathbf{m}}^{\nu}$ on an arbitrary symmetric cone Ω.

1. INTRODUCTION

It is well known that the classical Laguerre polynomials L_n^α for $\alpha > -1$ may be defined [1] by

$$L_n^\alpha(x) = (\alpha+1)_n \sum_{k=0}^{n} \binom{n}{k} \frac{1}{(\alpha+1)_k}(-x)^k, \tag{1.1}$$

and they form an orthogonal basis for $L^2(\mathbb{R}^+, e^{-x}x^\alpha dx)$. It follows that the classical Laguerre functions l_n^α defined by

$$l_n^\alpha(x) = e^{-x}L_n^\alpha(2x) \tag{1.2}$$

are an orthogonal basis for $L^2(\mathbb{R}^+, x^\alpha dx)$. Moreover, they satisfy the differential equation

$$x\frac{d^2 l_n^\alpha(x)}{dx^2} + (\alpha+1)\frac{dl_n^\alpha(x)}{dx} - xl_n^\alpha(x) = -(2n+\alpha+1)l_n^\alpha(x), \tag{1.3}$$

and the following recurrence relations:

$$x\frac{d^2 l_n^\alpha(x)}{dx^2} + (2x+\alpha+1)\frac{dl_n^\alpha(x)}{dx} + (x+\alpha+1)l_n^\alpha(x) = -2(n+\alpha)l_{n-1}^\alpha(x), \tag{1.4}$$

$$x\frac{d^2 l_n^\alpha(x)}{dx^2} - (2x-\alpha-1)\frac{dl_n^\alpha(x)}{dx} + (x-\alpha-1)l_n^\alpha(x) = -2(n+1)l_{n+1}^\alpha(x). \tag{1.5}$$

As in [2], we refer to the differential operators on the left of (1.4) and (1.5) as the annihilation and creation operators, respectively. By (1.4) and (1.5),

$$2x\frac{dl_n^\alpha(x)}{dx} + (\alpha+1)l_n^\alpha(x) = (n+1)l_{n+1}^\alpha(x) - (n+\alpha)l_{n-1}^\alpha(x). \tag{1.6}$$

By (1.3) - (1.5), we have

$$xl_n^\alpha(x) = (n + \frac{\alpha+1}{2})l_n^\alpha(x) - \frac{n+\alpha}{2}l_{n-1}^\alpha(x) - \frac{n+1}{2}l_{n+1}^\alpha(x). \tag{1.7}$$

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It is also well known that the definitions (1.1) and (1.2) of the Laguerre polynomials and Laguerre functions have been generalized to symmetric cones (see (2.8) and (2.9) below). [12] discusses the differential equations satisfied by the Laguerre polynomials and Laguerre functions on the symmetric cones of positive definite matrices over $\mathbb{C}$ and $\mathbb{R}$. Using an approach of Lie group representations, [4] generalizes (1.6) to an arbitrary symmetric cone. Using computations of matrices, [2] generalizes (1.3) - (1.5) to the cones of positive definite matrices over $\mathbb{C}$, or positive definite Hermitian matrices.

Using the method of Jordan algebras, we generalize in this paper all relations (1.3) - (1.7) to all symmetric cones, which include the cones of positive definite matrices over $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, the Lorentz cones, and the exceptional cone of 3×3 positive definite matrices over the Cayley algebra. Hence, this paper is a generalization of [2] and [12] to a general setting. Moreover, our method is simpler, and uses much less notation than in [4] and [2]. Finally, we also obtain the recurrence relations involving the *first-order* differential annihilation and creation operators (see Theorem 4.5 below). As [13] indicated, our method for symmetric cones cannot be used for Laguerre polynomials with general multiplicities d. Such a more general case will be studied in the next paper.

This paper is organized into four sections, as follows. In Section 2, we review the definitions and structures of Jordan algebras and symmetric cones, and the definition of Laguerre functions on these cones. In Section 3, we define the gradient of a $\mathbb{C}$-valued function as well as a *V-valued* function f on a Euclidean Jordan algebra V. We review some recurrence formulas for spherical polynomials $\Phi_{\mathbf{m}}$. We also obtain some gradient formulas for some functions. In Section 4, we obtain our main results of this paper, the differential equations and recursion formulas for Laguerre functions on symmetric cones.

2. Laguerre functions on symmetric cones

In this section, we review the structure of Jordan algebras and symmetric cones, and the definition of Laguerre functions on these cones, that are needed in this paper. We refer to [7] for details.

Let V be a simple Euclidean Jordan algebra, denote by n its dimension as a real vector space, denote its rank by r, and let e be the identity element in V. The interior Ω of the subset of all elements x^2 where $x \in V$ is an irreducible symmetric cone. Any irreducible symmetric cone is isomorphic to a cone of this kind, and any symmetric cone is the direct product of irreducible symmetric cones. Fixing a Jordan frame $\{c_1, \dots, c_r\}$ in V, we have $e = \sum_{j=1}^r c_j$. Denoting by $\circ$ the Jordan product in V, V has the following subspaces: $V_j = \{x \in V : c_j \circ x = x\}$ and $V_{jk} = \{x \in V : c_j \circ x = \frac{1}{2}x \text{ and } c_k \circ x = \frac{1}{2}x\}$. Then $V_j = \mathbb{R}c_j$ for $j = 1, \dots, r$ are 1-dimensional subalgebras of V, while the subspaces V_{jk} for $j, k = 1, \dots, r$ with $j < k$ all have a common dimension d. Then, V has the Pierce decomposition

$$V = \left(\sum_{j=1}^{r} \oplus V_j\right) \oplus \left(\sum_{j<k} \oplus V_{jk}\right), \tag{2.1}$$

which is the orthogonal direct sum. It follows that $n = r + \frac{d}{2}r(r-1)$.

The trace in V is the linear functional

$$\operatorname{tr} x = \langle x | e \rangle,$$

where $\langle\cdot|\cdot\rangle$ is the inner product in V. The characteristic function ψ of Ω is defined by

$$\psi(x) = \int_\Omega e^{-\langle x|y\rangle} dy$$

for all $x \in \Omega$, and the Koecher norm function Δ is given by

$$\Delta(x) = c\,\psi(x)^{-\frac{r}{n}},$$

where c is a constant determined by the normalization $\Delta(e) = 1$. If $\Delta(x) \neq 0$, then x is invertible, and there is a V-valued polynomial Q of degree $r-1$ such that $x^{-1} = \Delta(x)^{-1}Q(x)$.

Let G be the connected component of the identity in the automorphism group $G(\Omega)$. Then G acts transitively on Ω, and $\Omega \cong G/K$ where K is the stability group of the identity element e in Ω; i.e., K consists of all $k \in G$ such that $k \cdot e = e$. For any $x \in V$, there is $k \in K$ and $\xi_1, \ldots, \xi_r \in \mathbb{R}$ such that

$$x = k \cdot (\xi_1 c_1 + \cdots + \xi_r c_r). \tag{2.2}$$

We refer to (2.2) as the polar decomposition of x. When x is written in the form (2.2),

$$\operatorname{tr}(x) = \sum_{j=1}^{r} \xi_r \quad \text{and} \quad \Delta(x) = \prod_{j=1}^{r} \xi_r. \tag{2.3}$$

For $j = 1, \ldots, r$, let $E_j = c_1 + \cdots + c_j$, and set $J_j = \{x \in V : E_j \circ x = x\}$. Denote by P_j the orthogonal projection of V onto the subalgebra J_j, and define

$$\Delta_j(x) = \delta_j(P_j x)$$

for $x \in V$, where δ_j denotes the Koecher norm function for J_j. Then Δ_j is a polynomial on V that is homogeneous of degree j. Let $\lambda = (\lambda_1, \ldots, \lambda_r) \in \mathbb{C}^r$, and define the function Δ_λ on V by

$$\Delta_\lambda(x) = \Delta(x)^{\lambda_r} \prod_{j=1}^{r-1} \Delta_j(x)^{\lambda_j - \lambda_{j+1}}. \tag{2.4}$$

In particular, when $\lambda_j = m_j$ are integers for all $j = 1, \ldots, r$ and

$$m_1 \geq \cdots \geq m_r \geq 0, \tag{2.5}$$

$\mathbf{m} = (m_1, \ldots, m_r)$ is called a partition, and (2.4) defines a polynomial function $\Delta_\mathbf{m}$ on V that is homogeneous of degree $|\mathbf{m}| = m_1 + \cdots + m_r$. For each partition $\mathbf{m}$, the spherical polynomial of weight $\mathbf{m}$ on V may be defined by

$$\Phi_\mathbf{m}(x) = \int_K \Delta_\mathbf{m}(k \cdot x) dk.$$

It follows that $\Phi_\mathbf{m}$ is a K-invariant homogeneous polynomial of degree $|\mathbf{m}|$. By analytic continuation to the complexification $V^\mathbb{C}$ of V, tr, Δ, and $\Phi_\mathbf{m}$ can be extended to polynomial functions on $V^\mathbb{C}$.

The gamma function Γ_Ω for the cone Ω is defined on $\mathbb{C}^r$ by

$$\Gamma_\Omega(\lambda) = \int_\Omega e^{-\operatorname{tr} x} \Delta_\lambda(x) \Delta(x)^{-\frac{n}{r}} dx \tag{2.6}$$

whenever the integral converges absolutely. By [7, VII.1.1], in the range

$$\operatorname{Re} \lambda_j > (j-1)\frac{d}{2}$$

of the variable λ, the integral (2.6) converges absolutely, and Γ_Ω is evaluated in terms of the classical gamma function as

$$\Gamma_\Omega(\lambda) = (2\pi)^{\frac{1}{2}(n-r)} \prod_{j=1}^{r} \Gamma(\lambda_j - (j-1)\frac{d}{2}). \tag{2.7}$$

For $\lambda \in \mathbb{C}^r$ and $\mathbf{m}$ any partition we define

$$[\lambda]_{\mathbf{m}} = \frac{\Gamma_\Omega(\lambda + \mathbf{m})}{\Gamma_\Omega(\lambda)}.$$

For $\alpha \in \mathbb{C}$ and nonnegative integer j the classical Pochhammer symbol $(\alpha)_j$ is defined by

$$(\alpha)_j = \frac{\Gamma(\alpha + j)}{\Gamma(\alpha)} = \alpha(\alpha+1)\cdots(\alpha+j-1).$$

It follows from (2.7) that

$$[\lambda]_{\mathbf{m}} = \prod_{j=1}^{r} \left(\lambda_j - (j-1)\frac{d}{2}\right)_{m_j}.$$

The function $\Phi_{\mathbf{m}}(e+x)$ is a K-invariant polynomial of degree $|\mathbf{m}|$, and hence has an expansion

$$\Phi_{\mathbf{m}}(e+x) = \sum_{|\mathbf{n}| \leq |\mathbf{m}|} \binom{\mathbf{m}}{\mathbf{n}} \Phi_{\mathbf{n}}(x),$$

where $\binom{\mathbf{m}}{\mathbf{n}}$ is the generalized binomial coefficients. For $\nu \in \mathbb{R}$, the generalized Laguerre polynomials are defined by

$$L^{\nu}_{\mathbf{m}}(x) = [\nu]_{\mathbf{m}} \sum_{|\mathbf{n}| \leq |\mathbf{m}|} \binom{\mathbf{m}}{\mathbf{n}} \frac{1}{[\nu]_{\mathbf{n}}} \Phi_{\mathbf{n}}(-x), \tag{2.8}$$

and the generalized Laguerre functions by

$$l^{\nu}_{\mathbf{m}}(x) = e^{-\mathrm{tr}(x)} L^{\nu}_{\mathbf{m}}(2x). \tag{2.9}$$

Let $T_\Omega = V + i\Omega$ be the Siegel *upper* half plane in $V^{\mathbb{C}}$. By [7, Proposition XV.4.2], for

$$\nu > (r-1)\frac{d}{2} \tag{2.10}$$

and $z \in T_\Omega$,

$$\int_\Omega e^{i\langle z|x\rangle} l^{\nu}_{\mathbf{m}}(x) \Delta(x)^{\nu - \frac{n}{r}} dx = \Gamma_\Omega(\mathbf{m}+\nu)\Delta(e - iz)^{-\nu} \Phi_{\mathbf{m}}((z-ie)(z+ie)^{-1}), \tag{2.11}$$

where $(z+ie)^{-1}$ is the inverse of $z+ie$ in $V^{\mathbb{C}}$.

In this paper, we assume condition (2.10), which is also the assumption in [2], [4], and [12]. In the classical case, (2.10) becomes $\alpha > -1$, as discussed in the Introduction.

By [7, XIII.1], define

$$L^2_\nu(\Omega) = L^2(\Omega, \Delta(2u)^{\nu - \frac{n}{r}} du),$$

and $H^2_\nu(T_\Omega)$ as the space of holomorphic functions F on T_Ω such that

$$||F||^2_\nu = \int_{T_\Omega} |F(z)|^2 \Delta(y)^{\nu - \frac{2n}{r}} dx\, dy < \infty,$$

where $z = x + iy$. If f belongs to $L^2_\nu(\Omega)$, then the function F,

$$F(z) = (2\pi)^{-\frac{n}{2}} \int_\Omega e^{i\langle z|s\rangle} f(s)\Delta(2s)^{\nu-\frac{n}{r}} ds, \tag{2.12}$$

belongs to $H^2_\nu(T_\Omega)$, and the map $\mathcal{L}_\nu : f \mapsto F$ is a linear isomorphism from $L^2_\nu(\Omega)$ onto $H^2_\nu(T_\Omega)$. It follows from (2.11) and (2.12) that

$$\mathcal{L}_\nu l^\nu_{\mathbf{m}} = [\nu]_{\mathbf{m}} \Delta\left(\frac{z+ie}{2i}\right)^{-\nu} \Phi_{\mathbf{m}}((z-ie)(z+ie)^{-1}).$$

3. The gradient of a function on V and the recurrence relations for $\Phi_{\mathbf{m}}$

In this section, we define the gradient for a $\mathbb{C}$-valued and a *V-valued* function f on a simple Euclidean Jordan algebra V, and obtain some gradient results for some functions. We also review some recurrence formulas for the spherical polynomials $\Phi_{\mathbf{m}}$.

Let $f : V \to \mathbb{R}$ be a differentiable function; i.e., all directional derivatives D_u, $u \in V$, exist. For $s \in V$, we define the gradient $\triangledown f(s) \in V$ of f by the formula

$$\langle \triangledown f(s)|u\rangle = D_u f(s) = \frac{d}{dt} f(s+tu)\bigg|_{t=0}. \tag{3.1}$$

For a $\mathbb{C}$-valued function $f = f_1 + if_2$, we define $\triangledown f = \triangledown f_1 + i\triangledown f_2$. For $z = x+iy \in V^{\mathbb{C}}$, we define $D_z = D_x + iD_y$. Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of V, $s = \sum_{\alpha=1}^n s_\alpha e_\alpha \in V^{\mathbb{C}}$, and $u = \sum_{\alpha=1}^n u_\alpha e_\alpha \in V^{\mathbb{C}}$. By (3.1),

$$\langle \triangledown f(s)|u\rangle = \sum_{\alpha=1}^n \frac{\partial f(s)}{\partial s_\alpha} \bar{u}_\alpha,$$

and

$$\triangledown f(s) = \sum_{\alpha=1}^r \frac{\partial f(s)}{\partial s_\alpha} e_\alpha. \tag{3.2}$$

It is easy to see that (3.2) is independent of the choice of an orthonormal basis $\{e_1, \ldots, e_n\}$ of V so that (3.2) is an equivalent definition of $\triangledown f(s)$.

Moreover, a function $f : V \to V$ may be expressed by

$$f(s) = \sum_{\alpha=1}^n f_\alpha(s) e_\alpha \tag{3.3}$$

under an orthonormal basis $\{e_1, \ldots, e_n\}$ of V. Define $\triangledown f$ by

$$\triangledown f(s) = \sum_{\alpha,\beta=1}^n \frac{\partial f_\alpha(s)}{\partial s_\beta} e_\alpha \circ e_\beta. \tag{3.4}$$

It is also easy to see that the right side of (3.4) is independent of the choice of an orthonormal basis $\{e_1, \ldots, e_n\}$, and $\triangledown f(s)$ is well defined. Alternatively, one may define $\triangledown f(s)$ as $\sum_{\alpha,\beta=1}^n \frac{\partial f_\alpha(s)}{\partial s_\beta} e_\alpha \otimes e_\beta$ in the tensor product $V \otimes V$. However, we use (3.4) to keep $\triangledown f(s) \in V$, a simpler space. By a simple computation, the product rule of differentiation

$$\operatorname{tr}(\triangledown(f(s) \circ g(s))) = \operatorname{tr}((\triangledown f(s)) \circ g(s)) + \operatorname{tr}(f(s) \circ \triangledown g(s)) \tag{3.5}$$

holds for V-valued functions f and g, where V is a Euclidean Jordan algebra. When f is a $\mathbb{C}$-valued function, and the product is the scalar product, (3.5) also holds.

By (3.2) and (3.4), if f is a $\mathbb{C}$-valued function on V, then

$$\nabla\nabla f(s) = \sum_{\alpha,\beta=1}^{n} \frac{\partial^2 f(s)}{\partial s_\alpha \partial s_\beta} e_\alpha \circ e_\beta.$$

Lemma 3.1.

$$\nabla s = \frac{n}{r} e. \tag{3.6}$$

Proof. By (3.4)

$$\nabla s = \sum_{\alpha,\beta=1}^{n} \frac{\partial s_\alpha}{\partial s_\beta} e_\alpha \circ e_\beta.$$

Since $\frac{\partial s_\alpha}{\partial s_\beta} = \delta_{\alpha\beta}$,

$$\nabla s = \sum_{\alpha=1}^{n} e_\alpha \circ e_\alpha = \sum_{\alpha=1}^{n} e_\alpha^2. \tag{3.7}$$

Let $\{c_1, \ldots, c_r\}$ be a Jordan frame of V and decompose V into the orthogonal direct sum (2.1). Then an orthonormal basis $\{e_1, \ldots, e_n\}$ of V may be formed by adding elements in V_{jk} to this Jordan frame. By [7, Proposition IV.1.4(i)], if $e_\alpha \in V_{jk}$, then

$$e_\alpha^2 = e_\alpha \circ e_\alpha = \frac{1}{2}(c_j + c_k). \tag{3.8}$$

It follows from (3.7), (3.8), and the fact $c_j^2 = c_j$ that

$$\nabla s = \sum_{\alpha=1}^{n} e_\alpha^2 = \sum_{j=1}^{r} c_j + \frac{n-r}{r} \sum_{j=1}^{r} c_j = \frac{n}{r} \sum_{j=1}^{r} c_j = \frac{n}{r} e.$$

□

Lemma 3.2. *For $\nu \in \mathbb{R}$, and an invertible element $s \in V^{\mathbb{C}}$,*

$$\nabla(\Delta(s)^\nu) = \nu \Delta(s)^\nu s^{-1}. \tag{3.9}$$

Proof. If $s \in V$, then s has the spectral decomposition; i.e., there is a Jordan frame $\{c_1, \ldots, c_r\}$ and $s_1, \ldots, s_r \in \mathbb{R}$ such that $s = \sum_{j=1}^{r} s_j c_j$. If s is invertible, then $s_j \neq 0$ for $j = 1, \ldots, r$ and $s^{-1} = \sum_{j=1}^{r} s_j^{-1} c_j$. We complete the system $c_1, \ldots, c_r$ to an orthonormal basis $\{e_\alpha\}$ by adding elements belonging to V_{jk}. Since $\Delta(s)^\nu$ is K-invariant, it follows from [7, Lemma VI.4.3] that $\frac{\partial \Delta(s)^\nu}{\partial s_\alpha} = 0$ if e_α belongs to $V_{jk}(j < k)$. By (3.2) and (2.3),

$$\nabla(\Delta(s)^\nu) = \sum_{\alpha=1}^{n} \frac{\partial(\Delta(s)^\nu)}{\partial s_\alpha} e_\alpha = \sum_{j=1}^{r} \frac{\partial(\Delta(s)^\nu)}{\partial s_j} c_j = \sum_{j=1}^{r} \frac{\nu \Delta(s)^\nu}{s_j} c_j = \nu \Delta(s)^\nu s^{-1}.$$

Hence, the lemma is proved for $s \in V$. By analytic continuation, the lemma also holds for $s \in V^{\mathbb{C}}$. □

Lemma 3.3. *For $\nu \in \mathbb{R}$, and an invertible element $s \in V^{\mathbb{C}}$,*

$$\nabla(s\Delta(s)^{\nu-\frac{n}{r}}) = \nu\Delta(s)^{\nu-\frac{n}{r}}e, \tag{3.10}$$

and

$$\nabla\nabla(s\Delta(s)^{\nu-\frac{n}{r}}) = \nu\nabla(\Delta(s)^{\nu-\frac{n}{r}}e) = \nu\nabla(\Delta(s)^{\nu-\frac{n}{r}}). \tag{3.11}$$

Proof. By (3.5), (3.6), and (3.9),

$$\begin{aligned}\nabla(s\Delta(s)^{\nu-\frac{n}{r}}) &= \Delta(s)^{\nu-\frac{n}{r}}\nabla s + s\circ\nabla(\Delta(s)^{\nu-\frac{n}{r}}) \\ &= \Delta(s)^{\nu-\frac{n}{r}}\frac{n}{r}e + s\circ(\nu-\frac{n}{r})\Delta(s)^{\nu-\frac{n}{r}}s^{-1} = \nu\Delta(s)^{\nu-\frac{n}{r}}e,\end{aligned}$$

which is (3.10). (3.11) follows from (3.10), (3.2), (3.4), and the fact that e is the identity element of V. □

Let $\mathbf{m} = (m_1, \ldots, m_r)$ be a partition. Define

$$\mathbf{m} + \gamma_j = (m_1, \ldots, m_{j-1}, m_j + 1, m_{j+1}, \ldots, m_r), \tag{3.12}$$

and

$$\mathbf{m} - \gamma_j = (m_1, \ldots, m_{j-1}, m_j - 1, m_{j+1}, \ldots, m_r)$$

whenever they do not violate condition (2.5) [14]. [10] computed the binomial coefficient

$$\binom{\mathbf{m}}{\mathbf{m}-\gamma_j} = (m_j + \frac{d}{2}(r-j))\prod_{k\neq j}\frac{m_j - m_k + \frac{d}{2}(k-j-1)}{m_j - m_k + \frac{d}{2}(k-j)}.$$

We adopt the notation

$$C_{\mathbf{m}}(j) = \prod_{k\neq j}\frac{m_k - m_j - \frac{d}{2}(k+1-j)}{m_k - m_j - \frac{d}{2}(k-j)}, \tag{3.13}$$

denote $\mathrm{tr}\,(\nabla)f(s) = \mathrm{tr}\,(\nabla f(s))$, $\mathrm{tr}\,(s\nabla)f(s) = \mathrm{tr}\,(s\circ\nabla f(s))$, and $\mathrm{tr}\,(s^2\nabla)f(s) = \mathrm{tr}\,(s^2\circ\nabla f(s))$. By (3.12), $\gamma_1 = (1, 0, \ldots, 0)$. It is known that $\Phi_{\gamma_1}(s) = \frac{1}{r}\mathrm{tr}(s)$. The following recurrence formulas for the spherical polynomials $\Phi_{\mathbf{m}}$, some of which involve the gradient, can be found in [11].

Lemma 3.4. *Let $\mathbf{m}$ be a partition and $s \in V^{\mathbb{C}}$. Then,*

$$\mathrm{tr}\,(s)\,\Phi_{\mathbf{m}}(s) = \sum_{j=1}^{r} c_{\mathbf{m}}(j)\,\Phi_{\mathbf{m}+\gamma_j}(s), \tag{3.14}$$

$$\mathrm{tr}(\nabla)\,\Phi_{\mathbf{m}}(s) = \sum_{j=1}^{r}\binom{\mathbf{m}}{\mathbf{m}-\gamma_j}\Phi_{\mathbf{m}-\gamma_j}(s), \tag{3.15}$$

and

$$\mathrm{tr}\,(s^2\nabla)\,\Phi_{\mathbf{m}}(s) = \sum_{j=1}^{r}(m_j - (j-1)\frac{d}{2})\,c_{\mathbf{m}}(j)\,\Phi_{\mathbf{m}+\gamma_j}(s). \tag{3.16}$$

Since $\mathrm{tr}\,(s\nabla)$ is the Euler operator, and $\Phi_{\mathbf{m}}$ is a homogeneous polynomial of degree $|\mathbf{m}|$ on $V^{\mathbb{C}}$, we have

$$\mathrm{tr}(s\nabla)\Phi_{\mathbf{m}}(s) = \sum_{\alpha=1}^{n}s_\alpha\frac{\partial\Phi_{\mathbf{m}}(s)}{\partial s_\alpha} = |\mathbf{m}|\Phi_{\mathbf{m}}(s). \tag{3.17}$$

4. Differential equation and recursion relations for $l^\nu_{\mathbf{m}}$

In this section, we generalize the main result of [2], Theorem 5.1, from the symmetric cones of Hermitian matrices to arbitrary symmetric cones. This generalization is carried out in Theorems 4.3 and 4.4. Theorem 4.2 is a new recurrence relation, and Theorem 4.5 gives recurrence relations involving the *first-order* differential annihilation and creation operators. Theorem 4.1 recovers the main result of [4], Theorem 7.9, though our method is much simpler.

Theorem 4.1. *The Laguerre functions are related by the following recurrence relations:*

$$\begin{aligned}&\operatorname{tr}(2s\nabla+\nu e)l^\nu_{\mathbf{m}}(s)\\ &=\sum_{j=1}^r C_{\mathbf{m}}(j)l^\nu_{\mathbf{m}+\gamma_j}(s)-\sum_{j=1}^r(\mathbf{m}_j-1+\nu-(j-1)\frac{d}{2})\binom{\mathbf{m}}{\mathbf{m}-\gamma_j}l^\nu_{\mathbf{m}-\gamma_j}(s).\end{aligned}\tag{4.1}$$

Proof. By (3.5), (3.10), and the fact that every $s\in\Omega$ is invertible,

$$\begin{aligned}&\operatorname{tr}(\nabla(s\,l^\nu_{\mathbf{m}}(s)\Delta(s)^{\nu-\frac{n}{r}}))\\ &=\operatorname{tr}(s\Delta(s)^{\nu-\frac{n}{r}}\circ\nabla l^\nu_{\mathbf{m}}(s))+l^\nu_{\mathbf{m}}(s)\operatorname{tr}(\nabla(s\Delta(s)^{\nu-\frac{n}{r}}))\\ &=\Delta(s)^{\nu-\frac{n}{r}}\operatorname{tr}(s\nabla+\nu e)\,l^\nu_{\mathbf{m}}(s)\end{aligned}$$

and

$$\Delta(s)^{\nu-\frac{n}{r}}\operatorname{tr}(2s\nabla+\nu e)\,l^\nu_{\mathbf{m}}(s)=\operatorname{tr}(2\nabla(s\,l^\nu_{\mathbf{m}}(s)\,\Delta(s)^{\nu-\frac{n}{r}}))-\nu r\,l^\nu_{\mathbf{m}}(s)\,\Delta(s)^{\nu-\frac{n}{r}}.\tag{4.2}$$

(2.11) may be viewed as a Laplace transform of $l^\nu_{\mathbf{m}}(x)\,\Delta(x)^{\nu-\frac{n}{r}}$. Considering this transform and its inverse transform [5, (3.1.1) and (3.1.2)], it follows from (4.2) that

$$\begin{aligned}&\int_\Omega e^{i\langle z|s\rangle}\,\Delta(s)^{\nu-\frac{n}{r}}\operatorname{tr}(2s\nabla+\nu e)\,l^\nu_{\mathbf{m}}(s)\,ds\\ &=\operatorname{tr}(-2z\nabla-\nu e)\,(\Gamma_\Omega(\mathbf{m}+\nu)\,\Delta(e-iz)^{-\nu}\,\Phi_{\mathbf{m}}((z-ie)(z+ie)^{-1})).\end{aligned}\tag{4.3}$$

By (3.9),

$$\begin{aligned}&\operatorname{tr}(z\nabla)\,\Delta(e-iz)^{-\nu}\\ &=\operatorname{tr}(z\circ(-\nu)\,\Delta(e-iz)^{-\nu}\,(-i)(e-iz)^{-1})\\ &=-\frac{\nu}{2}\,\Delta(e-iz)^{-\nu}\operatorname{tr}(e+(z-ie)(z+ie)^{-1}).\end{aligned}\tag{4.4}$$

Denote

$$w=(z-ie)(z+ie)^{-1},\tag{4.5}$$

and ∇_w as the gradient with respect to w. By (4.5), (3.1), and a simple calculation,

$$\begin{aligned}\operatorname{tr}(z\nabla)\,\Phi_{\mathbf{m}}((z-ie)(z+ie)^{-1})&=\operatorname{tr}(2iz\circ(z+ie)^{-2}\nabla_w)\,\Phi_{\mathbf{m}}(w)\\ &=\frac{1}{2}\operatorname{tr}((e-w^2)\nabla_w)\,\Phi_{\mathbf{m}}(w).\end{aligned}\tag{4.6}$$

By (3.5), (4.4) - (4.6),

$$\begin{aligned}
&\operatorname{tr}(-2z\nabla-\nu e)\,(\Delta(e-iz)^{-\nu}\,\Phi_{\mathbf{m}}((z-ie)(z+ie)^{-1}))\\
&=\Phi_{\mathbf{m}}((z-ie)(z+ie)^{-1})\operatorname{tr}(-2z\nabla)\,\Delta(e-iz)^{-\nu}\\
&+\Delta(e-iz)^{-\nu}\operatorname{tr}(-2z\nabla)\,\Phi_{\mathbf{m}}((z-ie)(z+ie)^{-1})\\
&-\nu r\,\Delta(e-iz)^{-\nu}\,\Phi_{\mathbf{m}}((z-ie)(z+ie)^{-1})\\
&=\Delta(e-iz)^{-\nu}\operatorname{tr}(\nu w+(w^2-e)\nabla_w)\Phi_{\mathbf{m}}(w). \qquad (4.7)
\end{aligned}$$

By (4.3), (4.7), and (3.14) - (3.16), we have

$$\begin{aligned}
&\int_\Omega e^{i\langle z|s\rangle}\,\Delta(s)^{\nu-\frac{n}{r}}\operatorname{tr}(2s\nabla+\nu e)\,l^\nu_{\mathbf{m}}(s)\,ds\\
&=\Gamma_\Omega(\mathbf{m}+\nu)\,\Delta(e-iz)^{-\nu}\Big\{-\sum_{j=1}^r\binom{\mathbf{m}}{\mathbf{m}-\gamma_j}\,\Phi_{\mathbf{m}-\gamma_j}(w)\\
&+\sum_{j=1}^r(m_j+\nu-(j-1)\frac{d}{2})\,C_{\mathbf{m}}(j)\,\Phi_{\mathbf{m}+\gamma_j}(w)\Big\}. \qquad (4.8)
\end{aligned}$$

By (2.7),

$$\Gamma_\Omega(\mathbf{m}+\nu)=\Gamma_\Omega(\mathbf{m}+\nu-\gamma_j)(m_j-1+\nu-(j-1)\frac{d}{2}), \qquad (4.9)$$

and

$$\Gamma_\Omega(\mathbf{m}+\nu+\gamma_j)=\Gamma_\Omega(\mathbf{m}+\nu)(m_j+\nu-(j-1)\frac{d}{2}). \qquad (4.10)$$

By (4.8) - (4.10), and the fact that $\mathcal{L}_\nu$ is a linear isomorphism from $L^2_\nu(\Omega)$ onto $H^2_\nu(T_\Omega)$, we have (4.1). □

Theorem 4.2. *The Laguerre functions are related by the following recurrence relations:*

$$\begin{aligned}
&\operatorname{tr}(s)l^\nu_{\mathbf{m}}(s)=(|\mathbf{m}|+\frac{\nu r}{2})\,l^\nu_{\mathbf{m}}(s)-\frac{1}{2}\sum_{j=1}^r C_{\mathbf{m}}(j)\,l^\nu_{\mathbf{m}+\gamma_j}(s)\\
&-\frac{1}{2}\sum_{j=1}^r(\mathbf{m}_j-1+\nu-(j-1)\frac{d}{2})\binom{\mathbf{m}}{\mathbf{m}-\gamma_j}\,l^\nu_{\mathbf{m}-\gamma_j}(s). \qquad (4.11)
\end{aligned}$$

Proof. Similar to (4.4) and (4.6), we have

$$\operatorname{tr}(i\nabla)\,\Delta(e-iz)^{-\nu}=\frac{\nu}{2}\,\Delta(e-iz)^{-\nu}\operatorname{tr}(w-e), \qquad (4.12)$$

and

$$\operatorname{tr}(i\nabla)\,\Phi_{\mathbf{m}}((z-ie)(z+ie)^{-1})=\frac{1}{2}\operatorname{tr}((e-2w+w^2)\nabla_w)\,\Phi_{\mathbf{m}}(w), \qquad (4.13)$$

where w is given by (4.5). By (3.5), (3.14) - (3.17), (4.12), and (4.13),

$$\begin{aligned}
&-i\mathrm{tr}\,(\nabla)\,(\Delta(e-iz)^{-\nu}\,\Phi_{\mathbf{m}}((z-ie)(z+ie)^{-1}))\\
&=\Delta(e-iz)^{-\nu}\left\{(|\mathbf{m}|+\frac{\nu r}{2})\,\Phi_{\mathbf{m}}(w)-\frac{1}{2}\sum_{j=1}^{r}\binom{\mathbf{m}}{\mathbf{m}-\gamma_j}\,\Phi_{\mathbf{m}-\gamma_j}(w)\right.\\
&\left.-\frac{1}{2}\sum_{j=1}^{r}(m_j+\nu-(j-1)\frac{d}{2})\,C_{\mathbf{m}}(j)\,\Phi_{\mathbf{m}+\gamma_j}(w)\right\}.
\end{aligned}\tag{4.14}$$

Similar to (4.3),

$$\begin{aligned}
&\int_\Omega e^{i\langle z|s\rangle}\mathrm{tr}\,(s)\,l_{\mathbf{m}}^{\nu}(s)\,\Delta(s)^{\nu-\frac{n}{r}}\,ds\\
&\qquad=-i\mathrm{tr}\,(\nabla)\,(\Gamma_\Omega(\mathbf{m}+\nu)\,\Delta(e-iz)^{-\nu}\,\Phi_{\mathbf{m}}((z-ie)(z+ie)^{-1})).
\end{aligned}\tag{4.15}$$

By (4.9), (4.10), (4.14), and (4.15), we have (4.11). □

Theorem 4.3. *The Laguerre function $l_{\mathbf{m}}^{\nu}$ satisfies the following differential equation:*

$$\mathrm{tr}\,(-s\nabla\nabla-\nu\nabla+s)\,l_{\mathbf{m}}^{\nu}(s)=(\nu r+2|\mathbf{m}|)\,l_{\mathbf{m}}^{\nu}(s).\tag{4.16}$$

Proof. By (3.5), (3.10), and (3.11), we have

$$\begin{aligned}
&\mathrm{tr}\,(\nabla\nabla(s\,l_{\mathbf{m}}^{\nu}(s)\,\Delta(s)^{\nu-\frac{n}{r}}))\\
&=\Delta(s)^{\nu-\frac{n}{r}}\,\mathrm{tr}\,(s\nabla\nabla l_{\mathbf{m}}^{\nu}(s))+2\mathrm{tr}\,((\nabla l_{\mathbf{m}}^{\nu}(s))\circ(\nabla(s\Delta(s)^{\nu-\frac{n}{r}})))\\
&\quad+l_{\mathbf{m}}^{\nu}(s)\mathrm{tr}\,(\nabla\nabla(s\Delta(s)^{\nu-\frac{n}{r}}))\\
&=\Delta(s)^{\nu-\frac{n}{r}}\,\mathrm{tr}\,(s\nabla\nabla l_{\mathbf{m}}^{\nu}(s))+2\nu\Delta(s)^{\nu-\frac{n}{r}}\mathrm{tr}\,(\nabla l_{\mathbf{m}}^{\nu}(s))+\nu l_{\mathbf{m}}^{\nu}(s)\,\mathrm{tr}\,(\nabla)\,\Delta(s)^{\nu-\frac{n}{r}},
\end{aligned}\tag{4.17}$$

and

$$\begin{aligned}
&\mathrm{tr}\,(\nabla(l_{\mathbf{m}}^{\nu}(s)\,\Delta(s)^{\nu-\frac{n}{r}}))\\
&=\mathrm{tr}\,(\Delta(s)^{\nu-\frac{n}{r}}\,\nabla l_{\mathbf{m}}^{\nu}(s)+l_{\mathbf{m}}^{\nu}(s)\,\nabla(\Delta(s)^{\nu-\frac{n}{r}}))\\
&=\Delta(s)^{\nu-\frac{n}{r}}\,\mathrm{tr}\,(\nabla l_{\mathbf{m}}^{\nu}(s))+l_{\mathbf{m}}^{\nu}(s)\,\mathrm{tr}\,(\nabla)\,\Delta(s)^{\nu-\frac{n}{r}}.
\end{aligned}\tag{4.18}$$

By (4.17) and (4.18),

$$\begin{aligned}
&\mathrm{tr}\,(-\nabla\nabla(s\,l_{\mathbf{m}}^{\nu}(s)\,\Delta(s)^{\nu-\frac{n}{r}})+\nu\nabla(l_{\mathbf{m}}^{\nu}(s)\,\Delta(s)^{\nu-\frac{n}{r}})+s\,l_{\mathbf{m}}^{\nu}(s)\,\Delta(s)^{\nu-\frac{n}{r}})\\
&=\Delta(s)^{\nu-\frac{n}{r}}\,\mathrm{tr}\,(-s\nabla\nabla-\nu\nabla+s)\,l_{\mathbf{m}}^{\nu}(s).
\end{aligned}\tag{4.19}$$

By (2.11) and (4.19),

$$\begin{aligned}
&\int_\Omega e^{i\langle z|s\rangle}\,\Delta(s)^{\nu-\frac{n}{r}}\,\mathrm{tr}\,(-s\nabla\nabla-\nu\nabla+s)\,l_{\mathbf{m}}^{\nu}(s)\,ds\\
&=-i\,\mathrm{tr}\,((z^2+e)\nabla+\nu z)\,(\Gamma_\Omega(\mathbf{m}+\nu)\,\Delta(e-iz)^{-\nu}\,\Phi_{\mathbf{m}}((z-ie)(z+ie)^{-1})).
\end{aligned}\tag{4.20}$$

By (4.5), (3.1), and a calculation, $(z^2+e)\nabla=2iw\nabla_w$, and by (3.17),

$$\begin{aligned}
&\mathrm{tr}\,((z^2+e)\nabla)\,\Phi_{\mathbf{m}}((z-ie)(z+ie)^{-1})=2i\,\mathrm{tr}(w\nabla_w)\,\Phi_{\mathbf{m}}(w)\\
&=2i|\mathbf{m}|\,\Phi_{\mathbf{m}}(w)=2i|\mathbf{m}|\,\Phi_{\mathbf{m}}((z-ie)(z+ie)^{-1}).
\end{aligned}\tag{4.21}$$

Similar to (4.4), we have

$$\mathrm{tr}\,((z^2+e)\nabla)\,\Delta(e-iz)^{-\nu}=-\nu\Delta(e-iz)^{-\nu}\,\mathrm{tr}\,(z)+i\nu r\,\Delta(e-iz)^{-\nu}.\tag{4.22}$$

By (4.20) - (4.22),

$$\int_\Omega e^{i\langle z|s\rangle}\,\Delta(s)^{\nu-\frac{n}{r}}\,\mathrm{tr}\,(-s\nabla\nabla-\nu\nabla+s)\,l^\nu_{\mathbf{m}}(s)\,ds$$
$$=(\nu r+2|\mathbf{m}|)\,\Gamma_\Omega(\mathbf{m}+\nu)\,\Delta(e-iz)^{-\nu}\,\Phi_{\mathbf{m}}((z-ie)(z+ie)^{-1}).$$

Then the theorem follows from (2.11). □

Theorem 4.4. *The Laguerre functions are related by the following recurrence relations:*

$$\frac{1}{2}\,\mathrm{tr}\,(s\nabla\nabla+\nu\nabla+2s\nabla+s+\nu e)\,l^\nu_{\mathbf{m}}(s)=-\sum_{j=1}^{r}(\mathbf{m}_j-1+\nu-(j-1)\frac{d}{2})\binom{\mathbf{m}}{\mathbf{m}-\gamma_j}\,l^\nu_{\mathbf{m}-\gamma_j}(s),$$

and

$$\frac{1}{2}\,\mathrm{tr}\,(-s\nabla\nabla-\nu\nabla+2s\nabla-s+\nu e)l^\nu_{\mathbf{m}}(s)=\sum_{j=1}^{r}C_{\mathbf{m}}(j)l^\nu_{\mathbf{m}+\gamma_j}(s).$$

Proof. The theorem follows directly from (4.1), (4.11), (4.16), and a simple computation. □

Similarly, it follows from (4.1) and (4.11) that

Theorem 4.5. *The Laguerre functions are related by the following recurrence relations:*

$$\mathrm{tr}\,(s\nabla+s-\frac{|\mathbf{m}|}{r}e)\,l^\nu_{\mathbf{m}}(s)=-\sum_{j=1}^{r}(\mathbf{m}_j-1+\nu-(j-1)\frac{d}{2})\binom{\mathbf{m}}{\mathbf{m}-\gamma_j}\,l^\nu_{\mathbf{m}-\gamma_j}(s),$$

and

$$\mathrm{tr}\,(s\nabla-s+(\frac{|\mathbf{m}|}{r}+\nu)e)l^\nu_{\mathbf{m}}(s)=\sum_{j=1}^{r}C_{\mathbf{m}}(j)l^\nu_{\mathbf{m}+\gamma_j}(s).$$

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